Infrared-Finite Algorithms in QED II. The Expansion of the Groundstate of an Atom Interacting with the Quantized Radiation Field

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Abstract

In this paper, we present an explicit and constructive algorithm enabling us to calculate the groundstate and the groundstate energy of a nonrelativistic atom minimally coupled to the quantized radiation field up to an error of arbitrary finite order in the feinstructure constant. Because of infrared divergences, which invalidate a straightforward Taylor expansion, an iterative construction is employed to remove the infrared cutoff in photon momentum space and to produce a convergent algorithm.

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I Description of the Problem and Summary of Main Results

In this paper we pursue our program to control the infrared behaviour of nonrelativistic QED. We construct an explicit and constructive infrared-finite algorithm for the groundstate and the groundstate energy of a nonrelativistic atom interacting with the quantized radiation field, determining these quantities up to error terms of arbitrary finite order in the finite structure constant. In a previous paper, we have presented, using an iterative scheme, a convergent expansion for the groundstate and the groundstate energy, assuming that the feinstructure constant $\alpha \sim 1/137$ is a sufficiently small positive number. Starting from this result, we will derive an explicit representation for the groundstate and the groundstate energy in the form of finite sums of expressions computable in terms of finitely many convergent explicit integrals and an error term bounded above by $const \cdot \alpha^N$, for any given $N > 0$. In a forthcoming paper, we will present a mathematically precise justification of Bohr's frequency condition in atomic spectroscopy, by constructing an asymptotic expansion of the scattering amplitudes for Rayleigh scattering of light at an atom [2].

In this paper, an atom is described as a quantum-mechanical bound state consisting of a static, positively charged, pointlike nucleus surrounded by electrons. The electrons are described as nonrelativistic, pointlike quantum-mechanical particles with electric charge $-e$ and spin $\frac{1}{2}$, as originally proposed by Pauli. They are bound to the nucleus by the electrostatic Coulomb field and interact with the soft modes of the quantized electromagnetic field.

To keep our exposition as simple as possible, we consider a hydrogen atom consisting of a single, static proton of charge e surrounded by only one electron. The spin of the electron then turns out to be an inessential complication, and, for simplicity, we neglect the Zeeman coupling of the magnetic moment of the electron to the quantized magnetic field. It would, however, not be difficult to include the Zeeman term in our analysis. Through this paper we follow the notation and convention of [1]. For the convenience of the reader, we recall the main definitions.

The Hilbert space of pure state vectors of the system just described is given by

$$\mathcal{H} \; := \; \mathcal{H}_{el} \otimes \mathcal{F} \, , \tag{I.1}$$

where $\mathcal{H}_{el} = L^2(\mathbb{R}^3)$ is the Hilbert space appropriate for the description of a single electron, disregarding its spin, and $\mathcal{F}$ is the Fock space used to describe the states of the transverse modes of the quantized electromagnetic field, i.e., the *photons*. More explicitly,

$$\mathcal{F} \ := \ \bigoplus_{N=0}^{\infty} \mathcal{F}^{(N)} \, , \qquad \mathcal{F}^{(0)} = \mathbb{C}\,\Omega \, , \tag{I.2}$$

where Ω is the vacuum vector, i.e., the state of the electromagnetic field without any excited modes, and

$$\mathcal{F}^{(N)} \ := \ \mathcal{S}_N \bigotimes_{j=1}^{N} \mathfrak{h} \, , \qquad N \geq 1 \, , \tag{I.3}$$

where the Hilbert space $\mathfrak{h}$ of state vectors of a single photon is

$$\mathfrak{h} \ := \ L^2[\mathbb{R}^3 \times \mathbb{Z}_2] \, , \tag{I.4}$$

with photon momentum space given by $\mathbb{R}^3$, and $\mathbb{Z}_2$ is accounting for the two independent transverse polarizations, or helicities, of a photon. In Eq. (I.2), $\mathcal{S}_N$ denotes the orthogonal projection onto the subspace of $\bigotimes_{j=1}^{N} \mathfrak{h}$ of totally symmetric N-photon wave functions, in accordance with the fact that photons satisfy Bose-Einstein statistics. Thus, $\mathcal{F}^{(N)}$ is the subspace of $\mathcal{F}$ of state vectors for configurations of exactly N photons. It is convenient to represent the Hilbert space $\mathcal{H}$ as the space of square-integrable wave functions on the electron position space $\mathbb{R}^3$ with values in the photon Fock space $\mathcal{F}$, i.e.,

$$\mathcal{H} \ \cong \ L^2\Big[\mathbb{R}^3 \, ; \, \mathcal{F}\Big] \, . \tag{I.5}$$

The dynamics of the system is generated by the Hamiltonian

$$H(\alpha^{\frac{1}{2}}) \ := \ \Big(- i\vec{\nabla}_{\vec{x}} + \alpha^{3/2} \vec{A}(\alpha \vec{x}) \Big)^2 \ - \ V(x) \ + \ \check{H} \, . \tag{I.6}$$

In Eq. (I.6), $\vec{\nabla}_{\vec{x}}$ denotes the gradient with respect to the electron position $\vec{x} \in \mathbb{R}^3$, $\alpha \cong 1/137$ is the feinstructure constant, $\vec{A}(\vec{x})$ denotes the vector potential of the transverse modes of the quantized electromagnetic field in the *Coulomb gauge*,

$$\vec{\nabla}_{\vec{x}} \cdot \vec{A}(\vec{x}) \ = \ 0 \, , \tag{I.7}$$

V is the Coulomb potential of electrostatic attraction of the electron to the nucleus. The general assumption on V for which our analysis works is the following:

Hypothesis 1. *The form domain of V includes the form domain $\mathrm{H}^1(\mathbb{R}^3)$ of $-\Delta$ and, for any $\varepsilon > 0$, there exists a constant $b_\varepsilon < \infty$, such that*

$$\pm V \ \leq \ \varepsilon \, (-\Delta) \ + \ b_\varepsilon \cdot \mathbf{1} \tag{I.8}$$

on $\mathrm{H}^1(\mathbb{R}^3)$*. Moreover,* $\lim_{|x|\to\infty} V(x) = 0$*, and* $e_{el} := \inf \sigma(H_{el}) < 0$*, where* $H_{el} := -\Delta + V$*, is an isolated eigenvalue of multiplicity one with corresponding normalized eigenvector* $\varphi_{el} \in \mathcal{H}_{el}$*, i.e.,*

$$\Delta e_{el} \ := \ \inf\{\sigma(H_{el} - e_{el}) \cap \mathbb{R}^+\} > 0 \, . \tag{I.9}$$

$\check{H}$ is the Hamiltonian of the quantized, free electromagnetic field. This operator is given by

$$\check{H} \ := \ \sum_{\lambda=\pm} \int d^3k \; a^*(\vec{k},\lambda)\, |\vec{k}|\, a(\vec{k},\lambda) \, , \tag{I.10}$$

where $a^*(\vec{k},\lambda)$ and $a(\vec{k},\lambda)$ are the usual photon creation- and annihilation operators, which satisfy the canonical commutation relations

$$[a^*(\vec{k},\lambda)\,,\, a^*(\vec{k}',\lambda')] \ = \ [a(\vec{k},\lambda)\,,\, a(\vec{k}',\lambda')] \ = \ 0 \, , \tag{I.11}$$

$$[a(\vec{k},\lambda)\,,\, a^*(\vec{k}',\lambda')] \ = \ \delta_{\lambda\lambda'}\, \delta(\vec{k} - \vec{k}') \, , \tag{I.12}$$

$$a(\vec{k},\lambda)\,\Omega \ = \ 0 \, , \tag{I.13}$$

for all $\vec{k}, \vec{k}' \in \mathbb{R}^3$ and $\lambda, \lambda' \in \mathbb{Z}_2 \equiv \{\pm\}$.

The vector potential, $\vec{A}(\vec{x})$, in the Coulomb gauge is given by

$$\vec{A}(\vec{x}) \ := \tag{I.14}$$
$$\frac{1}{(2\pi)^{3/2}} \sum_{\lambda=\pm} \int \frac{d^3k}{\sqrt{2\,|\vec{k}|}}\, \Lambda(\vec{k}) \left\{ \vec{\varepsilon}(\vec{k},\lambda) e^{-i\vec{k}\cdot\vec{x}} a^*(\vec{k},\lambda) \ + \ \vec{\varepsilon}(\vec{k},\lambda)^* e^{i\vec{k}\cdot\vec{x}} a(\vec{k},\lambda) \right\} ,$$

where $\Lambda(\vec{k})$ is the characteristic function of the ball $\{\vec{k} \in \mathbb{R}^3 \mid |\vec{k}| \le \kappa\}$ (or a nonnegative, smooth approximation thereof), and $\vec{\varepsilon}(\vec{k},+)$, $\vec{\varepsilon}(\vec{k},-)$ are photon polarization vectors, i.e., two unit vectors in $\mathbb{C} \otimes \mathbb{R}^3$ satisfying

$$\vec{\varepsilon}(\vec{k},\lambda)^* \cdot \vec{\varepsilon}(\vec{k},\mu) \ = \ \delta_{\lambda\mu} \, , \qquad \vec{k} \cdot \vec{\varepsilon}(\vec{k},\lambda) \ = \ 0 \, , \tag{I.15}$$

for $\lambda, \mu = \pm$. The equation $\vec{k} \cdot \vec{\varepsilon}(\vec{k},\lambda) = 0$ expresses the Coulomb gauge condition.

The function $\Lambda(\vec{k})$ insures that modes of the electromagnetic field corresponding to wave vectors $\vec{k}$ with $|\vec{k}| \ge \kappa$ do not interact with the electron; i.e., Λ represents an *ultraviolet cutoff* that will be kept fixed throughout our analysis. The vector potential defined in (I.14) is thus cut off in the ultraviolet.

In the following, we simplify our notation by setting

$$k \ := \ (\vec{k},\lambda) \, , \quad \omega(k) \ \equiv \ |k| \ := \ |\vec{k}| \, , \quad \text{and} \quad \int f(k)\, dk \ := \ \sum_{\lambda=\pm} \int f(\vec{k},\lambda)\, d^3k \, , \tag{I.16}$$

for any integrable (possibly vector-valued) function f.

Next, we summarize the plan and the main results of this paper.

In the remainder of this section, we recall the main ideas of the *iterated analytic perturbation method*, introduced in [7], and some basic notation used for the construction in [1] of the groundstate of the Hamiltonian $H(\alpha^{\frac{1}{2}})$. It is proven that the groundstate is unique, provided the feinstructure constant α is small enough, depending on the value of the ultraviolet cutoff κ. (For a discussion of related results in the literature we refer to [1]). The main feature of the analysis in [1] is the achievement of an *explicit, inductive construction* of $(E_{\rm gs} \equiv E_{\rm gs}(\alpha), \phi_{\rm gs} \equiv \phi_{\rm gs}(\alpha^{\frac{1}{2}}))$, based on a decomposition of the terms in the Hamiltonian $H \equiv H(\alpha^{\frac{1}{2}})$ that describe the interactions between the electron and the photons into a sum of terms with the property that at least one photon creation- or annihilation operator is localized in a shell

$$\Big\{k \;\Big|\; \kappa\beta^{n+1} \leq |k| \leq \kappa\beta^n\Big\}\,, \quad \kappa \geq 1\,, \quad 0 < \beta < \kappa^{-1}\,, \quad n \in \mathbb{N}_0 = \{0,1,2,\dots\}\,, \tag{I.17}$$

of photon momentum space. In the present paper we set $\beta := \alpha$ which is compatible with [1]. Our inductive construction yields a sequence $(E_n, \phi_n)_{n=0}^{\infty}$ of approximate groundstate energies and groundstates constructed from Hamiltonians H_n with an infrared cutoff at $|k| = \kappa\alpha^n$. More precisely, the expression for H_n is obtained from the one for H by replacing the cutoff Λ in Definition (I.14) of the vector potential $\vec{A}(\vec{x})$ by the characteristic function of $\{\vec{k} \in \mathbb{R}^3 \mid \kappa\alpha^n < |\vec{k}| \leq \kappa\}$. It has been shown in [1] that

$$\lim_{n\to\infty} E_n = E_{\rm gs}\,, \qquad \lim_{n\to\infty} \phi_n = \phi_{\rm gs}\,. \tag{I.18}$$

These results and some underlying technical tools are collected in Sect. I.2 and will be widely used in the re-expansion of $\phi_{\rm gs}$ in $\alpha^{\frac{1}{2}}$. To outline the general ideas underlying our re-expansion algorithm is the purpose of Sect. I.3. We include this section for the convenience of the reader, to ease the reading of Sects. II and III, in which the actual construction of the re-expanded groundstate vector is carried out.

In Sect. II, we start by exploiting the infrared asymptotic freedom of the model: As follows from [1], we have

$$|E_n - E_{\rm gs}| \;\leq\; \mathcal{O}(\alpha^N)\,, \tag{I.19}$$
$$\|\phi_n - \phi_{\rm gs}\| \;\leq\; \mathcal{O}(\alpha^N)\,, \tag{I.20}$$

for any finite $N > 0$, provided $n \geq const \cdot N$, because the interaction term in the Hamiltonian H behaves as it was *irrelevant* in the *infrared*, when restricted to a spectral region close to the groundstate energy.

We then introduce some intermediate objects at all scales m, with $0 \leq m \leq const \cdot N$; namely truncated ground states, $(\phi_m)^T$, and truncated ground-state energy shifts, $(E^m_{m+1})^T$. They allow us to compute the quantities we are interested in up to an arbitrary chosen accuracy in α by performing a finite (because of the truncation) number of operations in the scale-dependent resolvents of the Hamiltonians H_n and on interactions terms.

In Sect. III, we present the asymptotic expansion of ϕ_{gs} and E_{gs}. It is derived from a re-expansion of the finitely many scale-dependent quantities, defining the truncated groundstates and groundstate energy shifts in terms of finitely many bare quantities, (convergent integrals in resolvents of H_0 and combinations of $\vec{p} = -i\vec{\nabla}_{\vec{x}}$ and of $e^{\pm i\vec{k}\cdot\alpha\vec{x}}$). In Corollary III.7 we formulate our final result, which we restate below.

Main Result

For any N and for α in a fixed interval $(0, \bar{\alpha}(2N-1))$, where $\bar{\alpha}(2N-1)$ is N-dependent, $E_{\mathrm{gs}} \equiv E_{\mathrm{gs}}(\alpha)$ and $\phi_{\mathrm{gs}} \equiv \phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})$ have expansions of the form

$$E_{\mathrm{gs}}(\alpha) = E_0 + \sum_{\ell=3}^{N} \varepsilon_\ell(\alpha)\,\alpha^\ell + o(\alpha^N) \tag{I.21}$$

$$\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}}) = \phi_0 + \sum_{\ell=3}^{2N} \varphi_\ell(\alpha)\,\alpha^{\ell/2} + o(\alpha^N) \tag{I.22}$$

with

$$\lim_{\alpha\to 0} \alpha^\delta |\varepsilon_\ell(\alpha)| = 0 \qquad \forall \delta > 0 \tag{I.23}$$

$$\lim_{\alpha\to 0} \alpha^\delta \|\varphi_\ell(\alpha)\| = 0 \qquad \forall \delta > 0 \tag{I.24}$$

for arbitrary $N = 1, 2, 3, \ldots$. In expansions (I.21) and (I.22), $\varepsilon_\ell(\alpha)$ and $\varphi_\ell(\alpha)$ are computable in terms of finitely many convergent integrals, for any $3 \leq \ell < \infty$.

Eqs. (I.23), (I.24) account for the possible appearance of powers of $\ln[1/\alpha]$ ("infrared logarithms"). We expect that such infrared logarithms are not artifact of our algorithm, but are an expression of infrared divergences in naive perturbation theory: The quantities $E_{\mathrm{gs}}(\alpha)$ and $\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})$ are not analytic nor even smooth at $\alpha^{\frac{1}{2}} = 0$, but derivatives in $\alpha^{\frac{1}{2}}$ of sufficiently high order of these quantities diverge as $\alpha \to 0$.

To our knowledge, a rigorous asymptotic expansion *up to any order in α* of the groundstate and the groundstate energy in nonrelativistic QED has never been provided in the literature before. An expansion to leading order follows from the proof of the existence of the groundstate contained in [4]; (see also [3] for an approach to the problem by renormalization group analysis).

We stress that the expansion of the groundstate is crucial in constructing similar expansions of physical quantities, such as scattering amplitudes of Rayleigh scattering. The expansion of the groundstate energy is a byproduct that is less significant. Technical improvements in the calculation of some high order contributions to the groundstate energy have been provided in [5],[6].

I.1 Notation

We introduce some more notation useful throughout this paper. Given an operator-valued function $F : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{B}(\mathcal{H}_{el})$, we write

$$a^*(F) \; := \; \int F(k) \otimes a^*(k) dk \tag{I.25}$$

$$a(F) \; := \; \int F(k)^* \otimes a(k) dk. \tag{I.26}$$

Thereby the velocity operator $\vec{v}$ can be written as

$$\vec{v} \; := \; -i\vec{\nabla}_x \, + \, a^*(\vec{G}) \, + \, a(\vec{G}) \, , \tag{I.27}$$

where $\vec{G} : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{B}(\mathcal{H}_{el})^3$ are the multiplication operators defined by

$$\vec{G}(k) \; := \; \frac{\alpha^{3/2}}{(2\pi)^{3/2}} \, \frac{\Lambda(k)}{\sqrt{2\,|k|}} \, e^{-i\alpha \vec{k}\cdot\vec{x}} \, \vec{\varepsilon}(k); \tag{I.28}$$

moreover, in terms of the velocity operator, the Hamiltonian assumes the simple form

$$H(\alpha^{\frac{1}{2}}) \; = \; \vec{v}^{\,2} \, - \, V(x) \, + \, \check{H}. \tag{I.29}$$

We know recall the multiscale structure underlying our construction of the groundstate of $H \equiv H(\alpha^{\frac{1}{2}})$.
For $\alpha < \kappa^{-1}$, where κ^{-1} is the ultraviolet cutoff in Eq.(I.14), we define a decreasing sequence $(\sigma_n)_{n=0}^{\infty}$ of energy scales by setting

$$\sigma_n \; := \; \kappa \, \alpha^n \, . \tag{I.30}$$

Note, in passing, that, for all $n \in \mathbb{N}_0$, we have that $\sigma_{n+1} \leq \kappa\alpha < 1$.
We factorize Fock space $\mathcal{F} = \mathcal{F}(\mathfrak{h})$ into tensor products by introducing suitable subspaces corresponding to the one-photon Hilbert space corresponding to different energy scales:

$$\mathfrak{h}_n \; := \; L^2\Big[\mathcal{K}_n\Big] \quad \text{and} \quad \mathfrak{h}_n^m \; := \; L^2\Big[\mathcal{K}_n^m\Big] \, , \tag{I.31}$$

where

$$\text{for } 0 \leq n \leq \infty: \quad \mathcal{K}_n \ := \ \Big\{ (\vec{k}, \tau) \in \mathbb{R}^3 \times \mathbb{Z}_2 \ \Big| \ \sigma_n \leq \omega(\vec{k}) \Big\} \,, \tag{I.32}$$

$$\text{for } 0 \leq m < n \leq \infty: \quad \mathcal{K}_n^m \ := \ \Big\{ (\vec{k}, \tau) \in \mathbb{R}^3 \times \mathbb{Z}_2 \ \Big| \ \sigma_n \leq \omega(\vec{k}) < \sigma_m \Big\} \,. \tag{I.33}$$

Note that $\mathcal{K}_n^0 \subset \mathcal{K}_n$ is a proper subset. For integers $1 \leq m < n < \ell \leq \infty$, we have the disjoint decomposition $\mathcal{K}_\ell = \mathcal{K}_m \cup \mathcal{K}_n^m \cup \mathcal{K}_\ell^n$ and hence the direct sum

$$\mathfrak{h}_\ell \ \cong \ \mathfrak{h}_m \oplus \mathfrak{h}_n^m \oplus \mathfrak{h}_\ell^n \,, \tag{I.34}$$

which gives rise to the isomorphism

$$\mathcal{F}_\ell \ \cong \ \mathcal{F}_m \otimes \mathcal{F}_n^m \otimes \mathcal{F}_\ell^n \,, \tag{I.35}$$

with $\mathcal{F}_n := \mathcal{F}(\mathfrak{h}_n)$ and $\mathcal{F}_n^m := \mathcal{F}(\mathfrak{h}_n^m)$. In particular, for any $n \in \mathbb{N}$,

$$\mathcal{F} \ = \ \mathcal{F}_\infty \ \cong \ \mathcal{F}_n \otimes \mathcal{F}_{n+1}^n \otimes \mathcal{F}_\infty^{n+1} \,. \tag{I.36}$$

The interaction is cut into slices corresponding to ever lower energy scales, we make use of the operators

$$\vec{G}_n(k) \ := \ \mathbf{1}\Big(\sigma_n \leq |k|\Big) \, \vec{G}(k) \,, \quad \text{and} \quad \vec{G}_n^m(k) \ := \ \mathbf{1}\Big(\sigma_n \leq |k| < \sigma_m\Big) \, \vec{G}(k) \,, \tag{I.37}$$

for all $k \in \mathbb{R}^3 \times \mathbb{Z}_2$ and $m, n \in \mathbb{N}_0$, with $m < n$. Note that $\vec{G} = \sum_{n=0}^\infty \vec{G}_{n+1}^n$ and that $\vec{G}_n$ is the coupling function of the interaction regularized in the infrared region ($|k| \geq \sigma_n$). We set

$$\mathcal{H}_n \ := \ \mathcal{H}_{el} \otimes \mathcal{F}_n \quad \text{and} \quad \mathcal{H}_n^m \ := \ \mathcal{H}_{el} \otimes \mathcal{F}_n^m \,, \tag{I.38}$$

and for energy-scale indices $m, n \in \mathbb{N}_0$, $m < n$, we define the velocity operator $\vec{v}_n$, the field energy operators $\check{H}_n$, $\check{H}_n^m$, and the Hamiltonian H_n by

$$\vec{v}_n \ := \ -i\vec{\nabla}_x \, + \, a^*(\vec{G}_n) \, + \, a(\vec{G}_n) \,, \tag{I.39}$$

$$\check{H}_n \ := \ \int \mathbf{1}(\sigma_n \leq |k|) \, \omega(k) \, a^*(k) a(k) \, dk \,, \tag{I.40}$$

$$\check{H}_n^m \ := \ \int \mathbf{1}(\sigma_n \leq |k| < \sigma_m) \, \omega(k) \, a^*(k) a(k) \, dk \,, \tag{I.41}$$

$$H_n \ := \ \vec{v}_n^{\,2} \, - \, V(x) \, + \, \check{H}_n \,, \tag{I.42}$$

as operators on $\mathcal{H}_n$ and $\mathcal{H}_n^m$, respectively. We introduce the groundstate energy on scale n and groundstate energy differences

$$E_n \ := \ \inf \sigma(H_n) \quad \text{and} \quad E_n^m \ := \ E_m - E_n \,. \tag{I.43}$$

We define the positive operators H_n^+ and $\widetilde{H}_n^+$ on $\mathcal{H}_n$ and $\mathcal{H}_{n+1}$, respectively, by

$$\begin{aligned} H_n^+ &:= H_n - E_n , && \text{(I.44)} \\ \widetilde{H}_n^+ &:= H_n^+ \otimes \mathbf{1}_{n+1}^n + \mathbf{1}_n \otimes \check{H}_{n+1}^n , && \text{(I.45)} \end{aligned}$$

where we denote the identity operator on $\mathcal{H}_n$ and on $\mathcal{F}_n^m$ by $\mathbf{1}_n$ and $\mathbf{1}_n^m$, respectively. Definitions (I.44) and (I.45) are meant to exploit regular perturbation theory in the step from the Hamiltonian H_n to the Hamiltonian H_{n+1}, at successive energy scales, in next equation (I.48), where $\widetilde{H}_n^+$ plays the role of the unperturbed operator. We identify $\vec{v}_n$ with $\vec{v}_n \otimes \mathbf{1}_{n+1}^n$ acting on $\mathcal{H}_{n+1}$. Note that, for $n \in \mathbb{N}_0$,

$$\vec{v}_{n+1} = \vec{v}_n + a^*(\vec{G}_{n+1}^n) + a(\vec{G}_{n+1}^n) . \tag{I.46}$$

Similarly, given $\psi_n \in \mathcal{H}_n$, we define a vector

$$\tilde{\psi}_n := \psi_n \otimes \Omega_{n+1}^n \in \mathcal{H}_{n+1} , \tag{I.47}$$

where Ω_n and Ω_n^m denote the vacuum vectors in $\mathcal{F}_n$ and $\mathcal{F}_n^m$, respectively. With these notations, we have that

$$H_{n+1}^+ = \widetilde{H}_n^+ + W_{n+1}^n + E_{n+1}^n , \tag{I.48}$$

where

$$\begin{aligned} W_{n+1}^n &:= (\vec{v}_{n+1})^2 - (\vec{v}_n)^2 && \text{(I.49)} \\ &= 2\,a^*(\vec{G}_{n+1}^n)\cdot\vec{v}_n + 2\,\vec{v}_n\cdot a(\vec{G}_{n+1}^n) + \Big(a^*(\vec{G}_{n+1}^n) + a(\vec{G}_{n+1}^n)\Big)^2 \\ &= 2\,a^*(\vec{G}_{n+1}^n)\cdot\vec{v}_n + 2\,\vec{v}_n\cdot a(\vec{G}_{n+1}^n) + a^*(\vec{G}_{n+1}^n)\cdot a^*(\vec{G}_{n+1}^n) \\ &\quad + a(\vec{G}_{n+1}^n)\cdot a(\vec{G}_{n+1}^n) + 2\,a^*(\vec{G}_{n+1}^n)\cdot a(\vec{G}_{n+1}^n) + \Big\|\vec{G}_{n+1}^n\Big\|^2 , \end{aligned}$$

with $\Big\|\vec{G}_{n+1}^n\Big\|^2 := \int |\vec{G}_{n+1}^n(k)|^2\,dk$. In Eq. (I.48), we make use of the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}(x) = 0$, which implies that

$$a^*(\vec{G}_{n+1}^n)\cdot\vec{v}_n = \vec{v}_n\cdot a^*(\vec{G}_{n+1}^n) \quad \text{and} \quad a(\vec{G}_{n+1}^n)\cdot\vec{v}_n = \vec{v}_n\cdot a(\vec{G}_{n+1}^n) . \tag{I.50}$$

I.2 Preliminary results and outline of strategy

Here, we describe some results derived in [1], concerning the construction of the groundstate of $H \equiv H(\alpha^{\frac{1}{2}})$, that will be used again and again in the re-expansion procedure. A key ingredient used in our construction of the groundstate of H is the simple identity

$$2\,\vec{v} \;=\; i\,[H\,,\,\vec{x}], \tag{I.51}$$

which implies that the interaction term in H, which is *marginal* in the *infrared* region (in the sense of power counting), is actually *infrared-irrelevant* on the subspace of all those states where the electron is bound to the nucleus.

In the next proposition, the main results are the gap estimates (I.52), (I.53) and the Neumann series expansion (I.57) of the spectral projections on the groundstates of the Hamiltonians H_n.

Proposition I.1. *Assume Hypothesis 1. Then there exist constants* $0 < C' \leq C$, *with* $C \geq 4$ *such that, for all* $\alpha < \frac{1}{2C}$, E_n *is an eigenvalue of multiplicity one. Moreover*

$$\inf\Big[\sigma(H_n^+)\setminus\{0\}\Big] \;=:\; \mathrm{gap}_n \;\geq\; [1-\frac{3}{4}C\alpha]\,\sigma_n\,, \tag{I.52}$$

$$\inf\Big[\sigma(\widetilde{H}_n^+)\setminus\{0\}\Big] \;=:\; \widetilde{\mathrm{gap}}_n \;=\; \sigma_{n+1}\,, \tag{I.53}$$

and

$$\sup_{z\in\Gamma_{n+1}}\Big\|\Big(\frac{1}{\widetilde{H}_n^+ - z}\Big)^{1/2}\Big(-W_{n+1}^n\Big)\Big(\frac{1}{\widetilde{H}_n^+ - z}\Big)^{1/2}\Big\| \;\leq\; C\,\alpha\,, \tag{I.54}$$

where $\Gamma_{n+1} := \{\frac{1}{4}\sigma_{n+1}e^{i\vartheta}\in\mathbb{C}|\vartheta\in[0,2\pi)\}$. *The spectral projections* $\widetilde{P}_n := |\frac{\tilde{\phi}_n}{\|\tilde{\phi}_n\|}\rangle\langle\frac{\tilde{\phi}_n}{\|\tilde{\phi}_n\|}|$ *and* $P_{n+1} := |\frac{\phi_{n+1}}{\|\phi_{n+1}\|}\rangle\langle\frac{\phi_{n+1}}{\|\phi_{n+1}\|}|$, *where* $\tilde{\phi}_n$ *and* ϕ_{n+1} *are groundstate vectors of* $\widetilde{H}_n$ *and* H_{n+1}, *respectively, correspond to*

$$\widetilde{P}_n \;=\; \frac{-1}{2\pi i}\int_{\Gamma_{n+1}}\frac{dz_{n+1}}{\widetilde{H}_n^+ - z_{n+1}}\,, \tag{I.55}$$

and

$$P_{n+1} \;=\; \frac{-1}{2\pi i}\int_{\Gamma_{n+1}}\frac{dz_{n+1}}{H_{n+1}-E_{n+1}-z_{n+1}}\,, \tag{I.56}$$

respectively, with $\Gamma_n := \{\frac{1}{4}\sigma_n e^{i\vartheta}\in\mathbb{C}|\vartheta\in[0,2\pi)\}$. *Their difference has a norm convergent series expansion*

$$P_{n+1}-\widetilde{P}_n \;=\; \frac{-1}{2\pi i}\sum_{\nu=1}^{\infty}\int_{\Gamma_{n+1}}(-1)^\nu\,Y_{n+1}^{(\nu)}(z_{n+1})\,dz_{n+1}\,, \tag{I.57}$$

where

$$Y_{n+1}^{(\nu)}(z) := \frac{1}{\widetilde{H}_n^+ - z}\Big(W_{n+1}^n\,\frac{1}{\widetilde{H}_n^+ - z}\Big)^\nu\,, \tag{I.58}$$

and

$$\|P_{n+1}-\widetilde{P}_n\| \;\leq\; C'\,\alpha^{(n+2)/2}\,. \tag{I.59}$$

The eigenvalue $E_\infty \equiv E_{gs}$ *is non degenerate.*

I.3 Outline of the Re-Expansion Algorithm

Our goal is to derive an explicit approximation of $\frac{\phi_{gs}}{\|\phi_{gs}\|} := \lim_{n\to\infty} \frac{\phi_n}{\|\phi_n\|}$ up to an error of order $o(\alpha^N)$. We may as well consider the vector ϕ_{2N-1}, because

$$\left\| \frac{\phi_{2N-1}}{\|\phi_{2N-1}\|} - \frac{\phi_{gs}}{\|\phi_{gs}\|} \right\| \leq \mathcal{O}(\alpha^{N+\frac{1}{2}}) . \tag{I.60}$$

The bound (I.60) follows from Proposition I.1. Up to normalization, the vector $\frac{\phi_{2N-1}}{\|\phi_{2N-1}\|}$ is given by the product $P_{2N-1} \cdots P_1$ of the projections $P_{2N-1}, \ldots, P_1$ applied to the groundstate, ϕ_0, of the bare Hamiltonian H_0^+, i.e., there is $\lambda_{2N-1} \in \mathbb{C} \setminus \{0\}$ such that

$$\frac{\phi_{2N-1}}{\|\phi_{2N-1}\|} = \lambda_{2N-1} \cdot P_{2N-1} \cdots P_1 \, \phi_0. \tag{I.61}$$

A more precise version of Formula (I.61) is given in the next section, where it is shown that the vector $P_{2N-1} \cdots P_1 \, \phi_0$ is an approximate expression for ϕ_{gs}, up to a remainder term of order $o(\alpha^N)$. From Proposition I.1, more precisely from Estimate (I.54), we infer that the expansion in Eq. (I.57) can be truncated at a finite order in such a way that the remainder term is $o(\alpha^N)$. Thus, the vector

$$\phi^T_{2N-1} := P^T_{2N-1} \cdots P^T_1 \phi_0 , \tag{I.62}$$

where

$$P^T_m := \frac{-1}{2\pi i} \sum_{j=0}^{N} \int_{\Gamma_m} \frac{1}{\widetilde{H}^+_{m-1} - z_m} \left[(-W_m^{m-1}) \, \frac{1}{\widetilde{H}^+_{m-1} - z_m} \right]^j dz_m , \tag{I.63}$$

is an approximate expression for ϕ_{gs}, up to a remainder term of order $o(\alpha^N)$. Note that the truncation in the definition of P^T_m only depends on N.

From now on, we focus our attention on the analysis of the finitely many resolvents appearing on the right side of (I.63).

Our derivation of explicit finite expressions for the vector

$$\phi^T_m := P^T_m \cdots P^T_1 \, \phi_0 \tag{I.64}$$

relies on three operations (i), (ii), and (iii), to be iterated a finite number of times:

(i) The photon creation- and annihilation operators are Wick-ordered, shell by shell.

(ii) The identity operator, $\mathbf{1}_n$, in the space $\mathcal{H}_n$ is decomposed into the sum $P_n + P_n^\perp$.

(iii) Two slightly different, truncated Neumann series expansions, which we call $\mathcal{A}$ and $\mathcal{B}$, are used to re-expand the contributions associated to $P_n^{\perp}$ coming from operation ii).

As a first step, we expand the resolvents

$$\frac{1}{\widetilde{H}^{+}_{m-1} - z_m} \tag{I.65}$$

appearing in the truncated projection P_m^T in (I.64) until only resolvents of the Hamiltonian H^{+}_{m-2} are left. We then put the expressions obtained in a form that enables us to iterate the operation, until only resolvents of the bare Hamiltonian H_0^{+} are left.

This can be accomplished if we take into account that the operator (I.65) is applied to a vector containing only a finite, N-dependent, but α-independent, number of photons with momenta in the shell $\mathcal{K}_m^{m-1}$; in particular, they are at most $2N$ because of the truncation in P_m^T and the fact that the slice interaction W_m^{m-1} is quadratic in the creation and annihilation operators. After Wick-ordering of these photon creation- and annihilation operators in the shell $\mathcal{K}_m^{m-1}$, the original resolvent (I.65) is replaced by a finite, N-dependent number – at most $N!$ – of terms containing resolvents of the form

$$\frac{1}{H^{+}_{m-1} - z_m + \sum_j |k_j|} \tag{I.66}$$

applied to a vector in $\mathcal{H}_{m-1}$, where $\sum_j |k_j|$ is a finite sum of photon energies $|k_j|$, with a number of terms depending only on N; in fact they are at most $2N$. Note that

$$\textstyle\sum_j |k_j| \ \in \ J_m \ := \ \{0\} \cup [\sigma_m \ , \ \ 2N \cdot \sigma_{m-1}) \ , \tag{I.67}$$

$$\tfrac{1}{4}\sigma_m \ \leq \ \Big| \textstyle\sum_j |k_j| - z_m \Big| \ \leq \ (2N+1)\sigma_{m-1} \ . \tag{I.68}$$

The key idea underlying the re-expansion of ϕ_{gs} is to split the operator (I.66) into two pieces, namely,

$$\tfrac{1}{H^{+}_{m-1} - z_m + \sum_j |k_j|} \, P_{m-1} \ = \ \tfrac{P_{m-1}}{-z_m + \sum_j |k_j|} \ , \tag{I.69}$$

$$\tfrac{1}{H^{+}_{m-1} - z_m + \sum_j |k_j|} \, P^{\perp}_{m-1} \ . \tag{I.70}$$

The term in (I.69) is a multiple of P_{m-1}, the factor of proportionality being an explicit number. Since

$$\begin{aligned} P_{m-1} - P^T_{m-1} \ &= \\ = \frac{-1}{2\pi i} \sum_{j=N+1}^{\infty} \int_{\Gamma_{m-1}} &\frac{1}{\widetilde{H}^{+}_{m-2} - z_{m-1}} \left[(-W^{m-2}_{m-1}) \, \frac{1}{\widetilde{H}^{+}_{m-2} - z_{m-1}} \right]^j dz_{m-1} \ , \end{aligned} \tag{I.71}$$

we may approximate P_{m-1} by P^T_{m-1} up to a remainder of order $o(\alpha^N)$. The term in (I.70) is analyzed by using the Neumann series expansion below, which we call of type $\mathcal{A}$:

$$\begin{aligned} &\frac{1}{H^+_{m-1} - z_m + \sum_j |k_j|} P^\perp_{m-1} \qquad (I.72)\\ &= P^\perp_{m-1} \frac{1}{\widetilde{H}^+_{m-2} - z_m + \sum_j |k_j|} P^\perp_{m-1} + P^\perp_{m-1} \frac{1}{\widetilde{H}^+_{m-2} - z_m + \sum_j |k_j|} P^\perp_{m-1} \\ &\quad \times \sum_{j=1}^{\infty} \Big[\big(- W^{m-2}_{m-1} - E^{m-2}_{m-1} \big) \frac{1}{\widetilde{H}^+_{m-2} - z_m + \sum_j |k_j|} P^\perp_{m-1} \Big]^j . \end{aligned}$$

This expansion converges because the absolute value of the energy shift E^{m-2}_{m-1} is very small, namely of order $\mathcal{O}(\alpha^{m+1})$ and because

$$\Big\| \frac{1}{\widetilde{H}^+_{m-2} - z_m + \sum_j |k_j|} P^\perp_{m-1} \Big\| \ \leq \ \mathcal{O}(\sigma^{-1}_{m-1}) \,, \qquad (I.73)$$

thanks to the presence of the orthogonal projection $P^\perp_{m-1}$. These two facts and Estimate (I.54) imply that the j^{th} term in the series expansion in (I.72) is bounded by $(C\alpha)^j$, for some constant $C \in \mathbb{R}^+$. Thus, the series expansion (I.72) is convergent and can be approximated up to terms of order $o(\alpha^N)$ by its truncation at order $j = N + 1$, where we replace $P^\perp_{m-1}$ by $\mathbf{1} - P^T_{m-1}$.

As a result of the previous operations (the decomposition (I.69)-(I.70), the truncation (I.71) of the series expansion of P^T_{m-1}, and the Neumann series expansion $\mathcal{A}$ in (I.72)), the resolvent (I.65) is represented by a polynomial in the following operators:

- the resolvents

$$\frac{1}{\widetilde{H}^+_{m-2} - z_m + \sum_j |k_j|} \,, \qquad \frac{1}{\widetilde{H}^+_{m-2} - z_{m-1}} \,, \qquad (I.74)$$

- the slice interaction W^{m-2}_{m-1}, and
- the energy shift E^{m-2}_{m-1}.

Returning to the expression (I.64), we then Wick-order the photon operators corresponding to photon momenta in the shell $\mathcal{K}^{m-2}_{m-1}$. This yields again finitely many terms and the resolvents in Eq. (I.74) are replaced by resolvents of the form

$$\frac{1}{H^+_{m-2} - z_{m-1} + \sum_i |q_i|} \,, \qquad (I.75)$$

$$\frac{1}{H^+_{m-2} - z_m + \sum_j |k_j| + \sum_i |q_i|} \,, \qquad (I.76)$$

where the photon momenta q_i all belong to the shell $\mathcal{K}_{m-1}^{m-2}$ and of terms, in $\sum_i |q_i|$, is bounded by an $N-$dependent number, thanks to the truncation in P_{m-1}^T. The resolvent in (I.75) has already the desired form that can be traced back to (I.66), but the one in (I.76) doesn't. As before, we decompose the resolvents (I.75), (I.76) into two pieces, using the projections P_{m-2} and $P_{m-2}^\perp$. This yields terms proportional to the projection P_{m-2}, with an explicit factor of proportionality, and terms of the form

$$\frac{1}{H_{m-2}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp , \qquad \frac{1}{H_{m-2}^+ - z_m + \sum_j |k_j| + \sum_i |q_i|} P_{m-2}^\perp . \tag{I.77}$$

The first term in Eq.(I.77) is analyzed by using the Neumann series expansion of type $\mathcal{A}$, see (I.72), with $m-1$ replaced by $m-2$. The second term in Eq. (I.77) is treated by applying the following expansion, which we call of type $\mathcal{B}$:

$$\begin{aligned}
&\frac{1}{H_{m-2}^+ - z_m + \sum_j |k_j| + \sum_i |q_i|} P_{m-2}^\perp = \\
&= P_{m-2}^\perp \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp \\
&\quad + P_{m-2}^\perp \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp \qquad\qquad (\text{I.78}) \\
&\quad \times \sum_{j=1}^{N} \Big[\Big(- W_{m-2}^{m-3} - E_{m-2}^{m-3} \Big) \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp \\
&\qquad + \Big(z_m - z_{m-1} + \sum_j |k_j| \Big) \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp \Big]^j \\
&\quad + \frac{1}{H_{m-2}^+ - z_{m-1} + \sum_j |k_j| + \sum_i |q_i|} P_{m-2}^\perp \qquad\qquad (\text{I.79}) \\
&\quad \times \Big[\Big(- W_{m-2}^{m-3} - E_{m-2}^{m-3} \Big) \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp \\
&\qquad + \Big(z_m - z_{m-1} + \sum_j |k_j| \Big) \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1} + \sum_i |q_i|} P_{m-2}^\perp \Big]^{N+1} .
\end{aligned}$$

In this truncated expansion, the remainder term (I.79) is proven to be of order $o(\alpha^N)$ with respect to the original expression in (I.77). To see this, we use the bounds on the shift of the integration variable, $z_m - z_{m-1}$, and on the sum $\sum_j |k_j|$: Both quantities are of order, at most, σ_{m-1}, the first one by definition of z_m and z_{m-1}, and the second one because of the fact that the number of terms in $\sum_j |k_j|$ is finite and independent of α. The replacement of the integration variable z_m by the variable z_{m-1} is possible because of Estimate (I.53) on the spectral gap. One then expands in the energies $|k_j|$, which are bounded by $const \cdot \sigma_{m-1}$. The norms of the remaining resolvents in (I.79) are bounded by $const \cdot \sigma_{m-2}^{-1}$ thanks to $P_{m-2}^{\perp}$. Finally we replace $P_{m-2}^{\perp}$ by $\mathbf{1}_{m-2} - P_{m-2}^{T}$ in the truncated expansion (I.78).

Note a new feature of the expansion of type $\mathcal{B}$, as compared to an expansion of type $\mathcal{A}$: We do not control the growth of the coefficient c_N in $\|(I.79)\| \leq c_N \, \alpha_{N+1}$, as $N \to \infty$, and so we do not assert *convergence* of the series of type $\mathcal{B}$ - it is only asymptotic.

Hence, starting with a resolvent of the form given in Eq. (I.66), and then performing the operations just described we end up with a polynomial in the following operators:

- the resolvents

$$\frac{1}{\widetilde{H}_{m-3}^{+} - z_{m-1} + \sum_i |q_i|}, \qquad \frac{1}{\widetilde{H}_{m-3}^{+} - z_{m-2}}, \tag{I.80}$$

- the slice interaction W_{m-2}^{m-3},
- the energy shift E_{m-2}^{m-3},
- the differences $z_m - z_{m-1}$ of integration variables,
- sums of single-photon energies $\sum_j |k_j|$.

We proceed by Wick-ordering the photon creation- and annihilation operators in the shell $\mathcal{K}_{m-2}^{m-3}$ in every term obtained so far, starting from the vector in Eq. (I.64), using (I.63), and then expanding the resolvents as described above. Two types of resolvents result:

$$\frac{1}{H_{m-3}^{+} - z_{m-1} + \sum_i |q_i| + \sum_{i'} |q'_{i'}|}, \frac{1}{H_{m-3}^{+} - z_{m-2} + \sum_{i'} |q'_{i'}|} \tag{I.81}$$

where the new sum, $\sum_{i'} |q'_{i'}|$, corresponds to photon momenta, $q'_{i'}$, in the shell $\mathcal{K}_{m-2}^{m-3}$, and the number of terms is bounded by an N-dependent, but α-independent integer.

After inserting the partition of unity, $\mathbf{1}_{m-3} = P_{m-3} + P_{m-3}^{\perp}$, we arrive at the expressions in (I.77), but with $m-2$ replaced by $m-3$. Thus, our re-expansion procedure, based on truncated Neumann expansions of type $\mathcal{A}$ and $\mathcal{B}$ and on Wick-ordering, can be iterated.
By applying the re-expansion procedure, scale by scale, to all the resolvents appearing in the truncated projections of Expression (I.62), we eventually end

up with an expansion of the vector in (I.62) involving only "bare" resolvents, i.e., resolvents of the form

$$\frac{1}{H_0^+ - z_2 + \sum_i |q_i| + \sum_{i'} |q'_{i'}|} \quad , \frac{1}{H_0^+ - z_1 + \sum_{i'} |q'_{i'}|}, \tag{I.82}$$

with momenta $\{q_i\}$, $\{q'_{i'}\}$ belonging to the shells $\mathcal{K}_2^1$ and $\mathcal{K}_1^0$, respectively.

As we will see in Sections II and III, the main difficulty in the inductive proof of convergence of the re-expansion procedure is related to determining the energy shifts E_m^n more explicitly. In re-expanding ϕ_{2N-1}, all the energy shifts up to scale $2N-1$ appear in our formulas. The energy shifts can be expressed in terms of the groundstate vectors on scales up to $2N-2$. Explicit expressions for the energy shifts E_m^n are obtained from the re-expansions of $\{\phi_i | i = 1, ..., 2N-2\}$, up to remainder terms of $o(\alpha^N)$.

The groundstate vector, fully re-expanded up to order α^N, is obtained in Corollary III.7 for α in an interval $(0, \bar{\alpha})$ where $\bar{\alpha}$ is N-independent. In the interval $(0, \bar{\alpha}(2N-1))$, where $\bar{\alpha}(2N-1)$ is N-dependent (see Theorem III.4), the re-expanded groundstate vector is of the following form: It can be written as an α-independent and only N-dependent sum of expressions of the type

$$\int_{\Gamma_{2N-1}} dz_{2N-1} \prod_{j_{2N-2}=1}^{\bar{j}(2N-2)} \int_{\Gamma_{2N-2}} dz_{2N-2}^{j_{2N-2}} \cdots \prod_{j_3=1}^{\bar{j}(3)} \int_{\Gamma_3} dz_3^{j_3} \prod_{j_2=1}^{\bar{j}(2)} \int_{\Gamma_2} dz_2^{j_2} \prod_{j_1=1}^{\bar{j}(1)} \int_{\Gamma_1} dz_1^{j_1} A \varphi_{el} \otimes \Omega \tag{I.83}$$

where φ_{el} is the groundstate of the Coulomb Hamiltonian H_{el}, and the numbers, $\bar{j}(m)$, $1 \le m \le 2N-1$, of contour integrations are α-independent;
The symbol A denotes an integral in all the photon momenta shells down to $\mathcal{K}_{2N-1}^{2N-2}$

$$A = \int_{\mathcal{K}_{2N-1}^{2N-2}} \prod_{l_{2N-1}=1}^{\bar{l}(2N-1)} dk_{l_{2N-1}} \int_{\mathcal{K}_{2N-2}^{2N-3}} \prod_{l_{2N-2}=1}^{\bar{l}(2N-2)} dk_{l_{2N-2}} \cdots \int_{\mathcal{K}_1^0} \prod_{l_1=1}^{\bar{l}(1)} dk_{l_1} B \tag{I.84}$$

with $\bar{l}(2N-1), .., \bar{l}(1)$ finite and α-independent, $\int_{\mathcal{K}_{2N-1}^{2N-2}}, ..., \int_{\mathcal{K}_1^0}$ multiple integrals;
The operator B is a tensor product of operators

$$B = D(k, z) \otimes C(k) \tag{I.85}$$

where:

- $C(k)$ is applied to the Fock space $\mathcal{F}_\infty^0$ and is the product of a finite and α-independent number of photon creation operators, contained in the range $\mathcal{K}_{2N-1}^0$; here k stands for all the momenta (and helicities) of the photon creation operators;

- $D(k,z)$ is applied to $\mathcal{H}_{el} \otimes \mathcal{F}_0$ and is the product of a finite and α-independent number of factors written below and properly combined; for details see Definition III.3 and Theorem III.4. In $D(k,z)$, k stands for all the momenta (and helicities) integration variables.

  The factors include:

  * The bare resolvents and the projection on the vector $\varphi_{el} \otimes \Omega$

$$P_0 \tag{I.86}$$

$$\frac{1}{H_0^+ - z_1 + \sum_i |q_i|} P_0^\perp \tag{I.87}$$

$$\frac{1}{H_0^+ - z_2 + \sum_i |q_i| + \sum_{i'} |q'_{i'}|} P_0^\perp \tag{I.88}$$

  where $\sum_i q_i$ and $\sum_{i'} q'_{i'}$ are sums of photon momenta in $\mathcal{K}_1^0$ and $\mathcal{K}_2^1$ respectively, the number of terms being α-independent;

  * the bare vertices

$$\vec{p} \cdot \frac{\vec{\varepsilon}(k_i)}{\sqrt{2|k_i|}} e^{-i\alpha \vec{k}_i \cdot \vec{x}} \tag{I.89}$$

$$\frac{\vec{\varepsilon}(k_i)}{\sqrt{2|k_i|}} \cdot \frac{\vec{\varepsilon}(k_j)}{\sqrt{2|k_j|}} e^{-i\alpha \vec{k}_i \cdot \vec{x}} e^{-i\alpha \vec{k}_j \cdot \vec{x}} \quad , \tag{I.90}$$

$$\frac{\vec{\varepsilon}(k_i)}{\sqrt{2|k_i|}} \cdot \frac{\vec{\varepsilon}(k_j)^*}{\sqrt{2|k_j|}} e^{-i\alpha \vec{k}_i \cdot \vec{x}} e^{i\alpha \vec{k}_j \cdot \vec{x}}$$

  and their adjoints, k_i, k_j are photon momenta in the range $\mathcal{K}_{2N-1}^0$;

  * the c-numbers operators

$$\frac{1}{\sum_i |q_i| - z_1} \mathbf{1}_0 \tag{I.91}$$

$$\frac{1}{\sum_{i_m} |q_{i_m}| + \sum_{i_{m-1}} |q_{i_{m-1}}| - z_m} \mathbf{1}_0 \, ; \; \frac{1}{\sum_{i_m} |q_{i_m}| - z_m} \mathbf{1}_0 \tag{I.92}$$

$$\sum_{i_{2N-1}} |k_{i_{2N-1}}| \mathbf{1}_0, ..., \sum_{i_3} |k_{i_3}| \mathbf{1}_0 \, ; (z_{m-1} - z_m) \mathbf{1}_0$$

$$\langle W_{2N-1}^{2N-2-1} \rangle_{\Omega_{2N-1}} \mathbf{1}_0, ..., \langle W_1^0 \rangle_{\Omega_1} \mathbf{1}_0; \; (E_{m'}^{m'-1})_\sharp^T \mathbf{1}_0 \tag{I.93}$$

  where:

  The number of terms in all the sums is α-independent, q_{i_m} is a photon momentum in the shell $\mathcal{K}_m^{m-1}$ with $2 \leq m \leq 2N-1$;

finally, $\langle W_{2N-1}^{2N-2}\rangle_{\Omega_{2N-1}}$ is the vacuum expectation value of the operator W_{2N-1}^{2N-2}, and the symbol $(E_{m'}^{m'-1})_\sharp^T$, $1 \leq m' \leq 2N-2$, stands for an explicit approximate expression of the energy shift $E_{m'}^{m'-1}$ which will be defined iteratively in Definition III.3, where a more precise definition of a fully re-expanded vector is given.

We now briefly summarize the mix of ingredients in our construction and we comment on the final result concerning the structure of the infrared singularity.
The main source of ingredients for our expansion is the scaling analysis in [1] inspired by the analogous one in [7]. The estimate in (I.60) implicitly provides a huge resummation of terms in a formal Taylor expansion. At fixed N, this result enables us to basically work with finitely many photons, the number of them being only N-dependent and uniform in α, for α small enough. This manifestly appears in the truncation of the groundstate projection in Eq. (I.63), as well as in the fact that, shell by shell, we must Wick-order only finitely many photon creation- and annihilation operators, and in the truncated expansions of type $\mathcal{A}$ and $\mathcal{B}$.
Our expansion scheme also relies on the gap estimates scale by scale, still related to the *asymptotic freedom* of the model displayed by the results in [1]. This information plays a crucial role in all the expansions as well as in the decomposition of the resolvents, scale by scale, into a part proportional to the groundstate projection and an orthogonal one, which enables us to separate an explicit and main contribution - corresponding to the parallel part - from a smaller one that can be further expanded - corresponding to the orthogonal one.
The relevance of the structural properties controlled in [1] is evident in the details of the expansions (I.21) and (I.22), contained in Corollary III.7. For a fixed order N, we split an N-independent interval $(0, \bar{\alpha})$ into a lower interval $(0, \bar{\alpha}(M = 2N-1))$ and into an upper one $[\bar{\alpha}(M = 2N-1), \bar{\alpha})$ (discussed in the Remark after Corollary III.7), where $\bar{\alpha}(M = 2N-1)$ is an N-dependent positive number. In the lower interval, the expansion is controlled by the algorithm. In the upper one we can basically use plain perturbation theory, since we have an infrared cut-off that only depends on N but the gap in the spectrum of the Hamiltonian is α-independent. Now - also for the purpose of comparison with [5], [6], where moderate high order contributions to the groundstate energy have been calculated - it is clear that the difficult task is to extend the upper region down to $\alpha = 0$, for large values of N. Here we encounter the true nature of the infrared singularity. For large N, even introducing an α-dependent infrared cut-off, plain perturbation theory is not a-priori well defined. Nevertheless, we can cover the lower interval and study

the asymptotic behavior of the groundstate and of the groundstate energy as $\alpha \to 0$, *up to any order* N, thanks to the a-priori results deriving from the scaling analysis and thanks to the shell by shell re-expansion procedure which is somehow guided by the non-perturbative method of constructing the groundstate.

In the remaining part of this Section, we explain the reason that $\ln[1/\alpha]$ terms may, in principle, occur in the asymptotic expansions in Eqs. (I.21) and (I.22).

This is possible by isolating some particular string of operators contained in Expression (I.83), for instance the string:

$$\cdots \vec{p} \cdot a(\vec{G}_m^{m-1}) \frac{1}{H_0^+ - z_m} (E_m^{m-1})_\sharp^T \frac{1}{H_0^+ - z_m} \vec{p} \cdot a^*(\vec{G}_m^{m-1}) \cdots \tag{I.94}$$

For the expression above we assume - it is actually the case for some terms when m is large enough - that:

When the variable z_m is integrated out, the integration around the pole $z_m = 0$ is not vanishing;

Except the contraction with initial and final point in the string, no other contraction of photon operators crosses the resolvents.

Then, inserting the complete set of states corresponding to the eigenstates of the unperturbed Hamiltonian, H_0^+, between $\vec{p} \cdot a(\vec{G}_m^{m-1})$ and $\vec{p} \cdot a^*(\vec{G}_m^{m-1})$, we consider the contribution of the projector associated to the unperturbed groundstate. The computation of the contraction of the creation- and annihilation operators gives the following result (up to a multiplicative coefficient)

$$(E_m^{m-1})_\sharp^T \int_{\mathcal{K}_m^{m-1}} dk \frac{1}{|k|^3}, \tag{I.95}$$

where the integral in k yields a $\ln[1/\alpha]$ term but some power of α is always contained in $(E_m^{m-1})_\sharp^T$, by construction. At this point, two important remarks are necessary:

1) Because of the shell-slicing in the photon momenta space no singularity in $\alpha = 0$ can appear as output of the algorithm. This is the content of Theorem III.4 in Section III. More explicitly, concerning the string in Eq. (I.94), it is clear that more blocks of the type

$$(E_m^{m-1})_\sharp^T \frac{1}{H_0^+ - z_m} \tag{I.96}$$

inside the string produce more powers of α, because the inverse power due to the integral in the photons momenta shell is compensated by the numerator;

2) The logarithmic terms cannot sum up to yield some power of α, for instance through the identity

$$e^{-\ln 1/\alpha} = \alpha, \tag{I.97}$$

because, if we fix a power of α which multiplies logarithmic terms, the terms of this type cannot be infinitely many. In fact, starting from a scale n with n large enough, the new terms are of higher order in α, due to the estimate in Eq.(I.60). On the other hand, for a fixed scale, the number of terms is finite because of the truncation procedure.

The dependence on α of the coefficients $\varepsilon_l(\alpha)$ and of the vectors $\varphi_l(\alpha)$, in Eqs. (I.21) and (I.22), respectively, is also due to the Coulomb interaction, through the rescaling of the unit of the electron position underlying the definition of Hamiltonian (I.6), see [1]. This dependence is however not singular as $\alpha \to 0$.

II Truncated groundstate vectors and truncated energy shifts

Starting from Proposition I.1 a sequence of unnormalized groundstate vectors, denoted by $\{\phi_n\}$, $\{\tilde{\phi}_n\}$, is recursively defined by

$$\phi_{n+1} := P_{n+1}\tilde{\phi}_n \tag{II.1}$$

with $\tilde{\phi}_n := \phi_n \otimes \Omega_{n+1}^n$, $\tilde{\phi}_0 := \phi_0 \otimes \Omega_1^0$ and $\phi_0 := \varphi_{el} \otimes \Omega_0$, with $\|\phi_0\| = 1$.
By Proposition I.1, there exist positive constants C' and δ, independent of α and of $n, m \in \mathbb{N}$, such that

$$\|\phi_m - \phi_{n-1} \otimes \Omega_m^{n-1}\| \leq C'\alpha^{\frac{n+1}{2}} \qquad \text{for} \quad m \geq n \tag{II.2}$$

and

$$\|\phi_n\| \geq \delta. \tag{II.3}$$

Estimates (II.2), (II.3) enable us to carry out the first important step in the re-expansion procedure of the (unnormalized) groundstate vector ϕ_{gs}

$$\phi_{\text{gs}} := \lim_{n\to\infty} \phi_n \otimes \Omega_\infty^n. \tag{II.4}$$

If an error term of order $o(\alpha^N)$ is tolerated, it suffices to re-expand the vector ϕ_{2N-1}. By telescoping

$$\begin{aligned} \phi_{2N-1} &= P_{2N-1}\tilde{\phi}_{2N-2} - \tilde{\phi}_{2N-2} && \text{(II.5)} \\ &+ \sum_{j=2}^{2N-1} (P_{2N-j}\tilde{\phi}_{2N-j-1} - \tilde{\phi}_{2N-j-1}) \otimes \Omega_{2N-1}^{2N-j} && \text{(II.6)} \\ &+ \tilde{\phi}_0 \otimes \Omega_{2N-1}^1 && \text{(II.7)} \end{aligned}$$

and similarly

$$E_{2N-1} = E_0 - \sum_{j=1}^{2N-1} E_{2N-j}^{2N-j-1}. \tag{II.8}$$

Each difference in (II.5) and (II.6) will be given as a series expansion in the spirit of Proposition I.1. The next step will consist in keeping only a finite, N-dependent, number of terms in each of these series expansions, with a remainder term of order $o(\alpha^N)$.

Remark

Throughout our paper, the symbol $o(\alpha^N)$ stands for a positive quantity bounded by $const \cdot \alpha^{N+\epsilon}$, for any $\epsilon > 0$, where the multiplicative constant is only uniform in α, unless otherwise specified. The symbol $\mathcal{O}(\alpha^N)$ is defined similarly.

Proposition I.1 enables us to propose a meaningful and useful definition of truncated vectors:

Definition II.1. *For α which fulfills the assumptions of Proposition I.1, the truncated vectors*

$$(\phi_m)^T \ , \ (\tilde{\phi}_m)^T \qquad\qquad 0 \leq m \leq M = 2N - 1 \tag{II.9}$$

are defined recursively starting from

$$(\phi_0)^T := \phi_0 \ , \ (\tilde{\phi}_0)^T := \tilde{\phi}_0 \tag{II.10}$$

according to the formulas

$$(\phi_m)^T \ := \ (P_m)^T (\tilde{\phi}_{m-1})^T \tag{II.11}$$
$$(\tilde{\phi}_m)^T \ := \ (\phi_m)^T \otimes \Omega_{m+1}^m \tag{II.12}$$

where

$$(P_m)^T := -\frac{1}{2\pi i} \sum_{j=0}^{N} \int_{\Gamma_m} dz_m \frac{1}{\widetilde{H}_{m-1}^{+} - z_m} \Big[(-W_m^{m-1}) \frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big]^j . \tag{II.13}$$

The definition is meaningful because

$$\|(\phi_m)^T\| \geq \Big[\frac{1 - 2C\alpha}{1 - C\alpha}\Big]^m \|\phi_0\| \tag{II.14}$$

thanks to the estimate in Eq. (I.54).

Under the assumptions of Definition II.1, the following key properties are trivially fulfilled

$$\|(\phi_m)^T - \phi_m\| \leq o(\alpha^N) \tag{II.15}$$

and

$$|((\tilde{\phi}_m)^T \, , \, (\phi_{m+1})^T)| \geq \Big[\frac{1 - 2C\alpha}{1 - C\alpha}\Big]^{2m+1} \|\phi_0\|^2 . \tag{II.16}$$

Next, taking into account the identity

$$E_m^{m-1} = E_{m-1} - E_m = -\frac{(W_m^{m-1} \tilde{\phi}_{m-1}, \phi_m)}{(\tilde{\phi}_{m-1}, \phi_m)}, \tag{II.17}$$

truncated energy shifts are defined by $(E_m^{m-1})^T := \Delta\mathcal{E}\Big((\phi_m)^T, (\tilde{\phi}_{m-1})^T\Big)$ where

$$\Delta\mathcal{E}\Big((\phi_m)^T, (\tilde{\phi}_{m-1})^T\Big) := -\frac{(W_m^{m-1} (\tilde{\phi}_{m-1})^T \ , (\phi_m)^T)}{((\tilde{\phi}_{m-1})^T \ , (\phi_m)^T)} . \tag{II.18}$$

The truncated energy shifts will play an important role in our re-expansion. This hinges on the estimates in next lemma which control their very fast decreasing rate in α for large m.

Lemma II.2. *Under the assumptions of Definition II.1, the estimates below hold true:*

$$|E_m^{m-1}| \leq \mathcal{O}(\alpha^{2+m}) \tag{II.19}$$

and

$$|E_m^{m-1} - (E_m^{m-1})^T| \leq o(\alpha^{\frac{1}{2}+m+N}) \tag{II.20}$$

with $1 \leq m \leq M = 2N - 1$.

Proof.
Proof of Inequality (II.19)
We start from the denominator in expression (II.17):

$$|(\tilde{\phi}_{m-1}\ \phi_m)| = |(\tilde{\phi}_{m-1}, P_m \tilde{\phi}_{m-1})| \geq \delta' \|\tilde{\phi}_{m-1}\|^2 \tag{II.21}$$

for some $\delta' > 0$, independent of α and $m \in \mathbb{N}$. This follows from Proposition I.1. To estimate the numerator in (II.17), we again use the series expansion of P_m:

$$\begin{aligned} & (W_m^{m-1}\tilde{\phi}_{m-1}\ , \phi_m) = \\ = \ & (W_m^{m-1}\tilde{\phi}_{m-1}\ , \tilde{\phi}_{m-1}) & \text{(II.22)} \\ & -\frac{1}{2\pi i}\sum_{j=1}^{\infty}\int_{\Gamma_m} dz_m \Big(W_m^{m-1}\tilde{\phi}_{m-1}\ , & \text{(II.23)} \\ & \qquad , \frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big[(-W_m^{m-1})\ \frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big]^j \tilde{\phi}_{m-1}\Big). \end{aligned}$$

Next we bound the two terms, (II.22) and (II.23):
Term (II.22):

$$(W_m^{m-1}\tilde{\phi}_{m-1}\ , \tilde{\phi}_{m-1}) = \frac{\alpha^3}{3(2\pi)^2}(\sigma_{m-1}^2 - \sigma_m^2)\|\tilde{\phi}_{m-1}\|^2; \tag{II.24}$$

Term (II.23): We must bound the absolute value of the scalar product inside the contour integral as follows

$$\begin{aligned} & \Big|\Big(W_m^m\tilde{\phi}_{m-1}\ , \frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big[(-W_m^{m-1})\ \frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big]^j \tilde{\phi}_{m-1}\Big)\Big| \leq \\ \leq \ & \Big\|\Big(\frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big)^{\frac{1}{2}} W_m^{m-1}\tilde{\phi}_{m-1}\Big\|^2 \times & \text{(II.25)} \\ & \times\Big\|\Big(\frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big)^{\frac{1}{2}} W_m^{m-1}\Big(\frac{1}{\widetilde{H}_{m-1}^{+} - z_m}\Big)^{\frac{1}{2}}\Big\|^{j-1}\Big|\frac{1}{z_m}\Big|. \end{aligned}$$

Using Estimate (I.54) in Proposition I.1, the absolute value of the sum in Eq. (II.23) is seen to be bounded by

$$\frac{1}{1-C\alpha}\sup_{z_m\in\Gamma_m}\|(\widetilde{H}^+_{m-1}-z_m)^{-\frac{1}{2}}W_m^{m-1}\tilde{\phi}_{m-1}\|^2. \tag{II.26}$$

Our next step consists in proving that

$$\sup_{z_m\in\Gamma_m}\|(\widetilde{H}^+_{m-1}-z_m)^{-\frac{1}{2}}W_m^{m-1}\tilde{\phi}_{m-1}\|^2\leq\mathcal{O}(\alpha^{2+m}) \tag{II.27}$$

which proves inequality (II.19) claimed in the statement of the Lemma. The vector inside the norm on the left side of (II.27) can be split as $\varphi_1+\varphi_2+\varphi_3$, where

$$\begin{aligned}
\varphi_1 &:= (\widetilde{H}^+_{m-1}-z_m)^{-\frac{1}{2}}\vec{v}_{m-1}\cdot a^*(\vec{G}_m^{m-1})\tilde{\phi}_{m-1} && \text{(II.28)}\\
&= (\frac{\alpha}{2\pi})^{\frac{3}{2}}\int_{\mathcal{K}_m^{m-1}}\frac{dk}{\sqrt{2\,|k|}}a^*(k)(\widetilde{H}^+_{m-1}-z_m+|k|)^{-\frac{1}{2}}\times && \text{(II.29)}\\
&\qquad\times\vec{\varepsilon}(k)\cdot\vec{v}_{m-1}e^{-i\alpha\vec{k}\cdot\vec{x}}\tilde{\phi}_{m-1}\\
\varphi_2 &:= (\widetilde{H}^+_{m-1}-z_m)^{-\frac{1}{2}}a^*(\vec{G}_m^{m-1})\cdot a^*(\vec{G}_m^{m-1})\tilde{\phi}_{m-1} && \text{(II.30)}\\
&= (\frac{\alpha}{2\pi})^3\int_{\mathcal{K}_m^{m-1}}\int_{\mathcal{K}_m^{m-1}}\frac{dk\,dq}{\sqrt{2\,|k|\,|q|}}\vec{\varepsilon}(k)\cdot\vec{\varepsilon}(q)\times && \text{(II.31)}\\
&\qquad\times a^*(k)a^*(q)(\widetilde{H}^+_{m-1}-z_m+|k|+|q|)^{-\frac{1}{2}}e^{-i\alpha(\vec{k}+\vec{q})\cdot\vec{x}}\tilde{\phi}_{m-1}\\
\varphi_3 &:= \frac{\alpha^3}{3(2\pi)^2}(\sigma^2_{m-1}-\sigma^2_m)(\widetilde{H}^+_{m-1}-z_m)^{-\frac{1}{2}}\tilde{\phi}_{m-1} && \text{(II.32)}
\end{aligned}$$

The corresponding norms are:

$$\begin{aligned}
\|\varphi_1\|^2 = (\frac{\alpha}{2\pi})^3\int_{\mathcal{K}_m^{m-1}}\frac{dk}{2|\vec{k}|^2}(\vec{\varepsilon}(k)\cdot\vec{v}_{m-1}e^{-i\alpha\cdot\vec{x}}\tilde{\phi}_{m-1}\,, && \text{(II.33)}\\
,\frac{|k|}{|\widetilde{H}^+_{m-1}-z_m+|k||}\vec{\varepsilon}(k)\cdot\vec{v}_{m-1}e^{-i\alpha\vec{k}\cdot\vec{x}}\tilde{\phi}_{m-1})
\end{aligned}$$

which is bounded by $C_1\alpha^{m+2}$, C_1 a numerical coefficient, as follows from the inequalities

$$\begin{aligned}
(\sum_{l=1}^{l=3}\|v^l_{m-1}e^{-i\alpha\vec{k}\cdot\vec{x}}\phi_{m-1}\|^2)^{\frac{1}{2}} &=: \|\vec{v}_{m-1}e^{-i\alpha\vec{k}\cdot\vec{x}}\phi_{m-1}\|\\
&= \|e^{-i\alpha\vec{k}\cdot\vec{x}}(\vec{v}_{m-1}-\alpha\vec{k})\phi_{m-1}\| && \text{(II.34)}\\
&\leq \alpha\kappa\|\phi_{m-1}\|+\|\vec{v}_{m-1}\phi_{m-1}\| && \text{(II.35)}
\end{aligned}$$

and

$$\Big\| \frac{|k|}{\widetilde{H}^{+}_{m-1} - z_m + |k|} \Big\| \leq \frac{4}{3} \tag{II.36}$$

for $\kappa > |k| \geq \sigma_m$, $z_m \in \Gamma_m$.
Next,

$$\begin{aligned} \|\varphi_2\|^2 &= 2(\frac{\alpha}{2\pi})^6 \int_{\mathcal{K}^{m-1}_m} \int_{\mathcal{K}^{m-1}_m} \frac{dk\, dq}{2\,|k|^2\,|q|} (\vec{\varepsilon}(k) \cdot \vec{\varepsilon}(q))^2 \times \\ &\quad \times \Big\| \frac{|k|^{\frac{1}{2}}}{|\widetilde{H}^{+}_{m-1} - z_m + |k| + |q||^{\frac{1}{2}}} e^{-i\alpha(\vec{k}+\vec{q})\cdot\vec{x}} \phi_{m-1} \Big\|^2 \end{aligned} \tag{II.37}$$

which is bounded by

$$\begin{aligned} &\frac{2}{3}(\frac{\alpha}{2\pi})^6 \Big(\int_{\mathcal{K}^{m-1}_m} \int_{\mathcal{K}^{m-1}_m} \frac{dk\, dq}{2\,|k|^2\,|q|} (\vec{\varepsilon}(k) \cdot \vec{\varepsilon}(q))^2 \Big) \times \\ &\quad \times \sup_{|k| \geq \sigma_{n+1}|} \| \frac{|k|}{\widetilde{H}^{+}_{m-1} - z_m + |k|} \| \leq C_2 \alpha^{3+3m} \end{aligned} \tag{II.38}$$

and

$$\begin{aligned} \|\varphi_3\|^2 &= \| \frac{\alpha^3}{3(2\pi)^2} (\sigma^2_{m-1} - \sigma^2_m)(\widetilde{H}^{+}_{m-1} - z_m)^{-\frac{1}{2}} \phi_{m-1} \|^2 \\ &\leq C_3 \alpha^{2+3m}. \end{aligned} \tag{II.39}$$

with C_2 and C_3 numerical coefficients.
Thus the bound (II.27) is proved.
Proof of Inequality (II.20)
According to (II.17), (II.18), the difference

$$E^{m-1}_m - (E^{m-1}_m)^T \tag{II.40}$$

corresponds to

$$-\frac{(W^{m-1}_m \tilde{\phi}_{m-1}\ , \phi_m)}{(\tilde{\phi}_{m-1}\ , \phi_m)} + \frac{(W^{m-1}_m (\tilde{\phi}_{m-1})^T\ , (\phi_m)^T)}{((\tilde{\phi}_{m-1})^T\ , \phi^T_m)} \tag{II.41}$$

$$= -\frac{(W^{m-1}_m \tilde{\phi}_{m-1}\ , \phi_m)[((\tilde{\phi}_{m-1})^T\ , (\phi_m)^T) - (\tilde{\phi}_{m-1}\ , \phi_m)]}{((\tilde{\phi}_{m-1})^T\ , (\phi_m)^T)(\tilde{\phi}_{m-1}\ , \phi_m)} \tag{II.42}$$

$$-\frac{(W^{m-1}_m \tilde{\phi}_{m-1}\ , \phi_m) - (W^{m-1}_m (\tilde{\phi}_{m-1})^T\ , (\phi_m)^T)}{((\tilde{\phi}_{m-1})^T\ , (\phi_m)^T)} \tag{II.43}$$

The term in Eq. (II.42) is easily controlled by a quantity $o(\alpha^{2+m+N})$ because of Estimate (II.15), because of the first part of the lemma and the inequality in Eq. (II.16). In order to bound the term in Eq. (II.43), the denominator is controlled as in (II.42). Concerning the numerator, we first rewrite it as follows

$$\begin{aligned} & (W_m^{m-1}(\tilde{\phi}_{m-1} - (\tilde{\phi}_{m-1})^T)\ , \phi_m) && \text{(II.44)} \\ + \ & (W_m^{m-1}(\tilde{\phi}_{m-1})^T\ , \phi_m - (\phi_m)^T). && \text{(II.45)} \end{aligned}$$

We then notice that, by steps similar to those in the first part of the lemma, the absolute value of the first scalar product, (II.44), can be controlled in terms of

$$\begin{aligned} & \frac{1}{1-C\alpha} \sup_{z_m \in \Gamma_m} \|(\widetilde{H}^+_{m-1} - z_m)^{-\frac{1}{2}} W_m^{m-1} \tilde{\phi}_{m-1}\| \times \\ \times \sup_{z_m \in \Gamma_m} & \|(\widetilde{H}^+_{m-1} - z_m)^{-\frac{1}{2}} W_m^{m-1} (\tilde{\phi}_{m-1} - (\tilde{\phi}_{m-1})^T)\| \leq o(\alpha^{2+m+N}). \quad \text{(II.46)} \end{aligned}$$

This is possible because of (II.15) and provided

$$\|\vec{v}_{m-1}(\phi_{m-1} - (\phi_{m-1})^T)\| \leq o(\alpha^N), \tag{II.47}$$

which is easy to prove, by construction of $(\phi_{m-1})^T$.
For the scalar product (II.45), the bound

$$\|\vec{v}_{m-1}(\phi_{m-1})^T\| \leq \mathcal{O}(1) \tag{II.48}$$

ensures the desired result, (II.20), after applying the Schwarz inequality. □

From Lemma II.2, we easily derive the following corollary.

Corollary II.3. *Assume that* $\Delta\mathcal{E}\left(\psi_m^{appr.}, \tilde{\psi}_{m-1}^{appr.}\right)$ *is well defined for the vectors belonging to the sequences* $\{\psi_m^{appr.}\}$ *and* $\{\tilde{\psi}_m^{appr.} := \psi_{m-1}^{appr.} \otimes \Omega_m^{m-1}\}$*, respectively, with* $\psi_m^{appr.} \in \mathcal{H}_m$ *and* $|(\tilde{\psi}_{m-1}^{appr.}, \psi_m^{appr.})| > \delta > 0$ *where* δ *is* α*-independent. Assume that inequalities as in Eqs. (II.15),(II.46) and (II.48) are fulfilled when* $(\phi_m)^T$ *is replaced by* $\psi_m^{appr.}$*, then*

$$|E_m^{m-1} - \Delta\mathcal{E}\left(\psi_m^{appr.}, \tilde{\psi}_{m-1}^{appr.}\right)| \leq o(\alpha^{\frac{1}{2}+m+N}). \tag{II.49}$$

III Re-expansion

The truncated vectors $\{(\phi_m)^T : 1 \leq m \leq 2N-1\}$ are written in terms of a finite number of resolvents of the operators $\{H_m^+ : 0 \leq m \leq 2N-2\}$ and interaction terms $\{W_{m'}^{m'-1} : 1 \leq m' \leq 2N-1\}$, where N is the order in α up to which we intend to re-expand. The re-expansion consists in expressing the resolvents of the operators $\{H_m^+ : 0 \leq m \leq 2N-2\}$ in the original definition (Definition II.1) of $\{(\phi_m)^T\}$ in terms of sums of product of finitely many resolvents of H_0^+, boson creation operators, the operators $\vec{p}$ and $e^{\pm i\vec{k}\cdot\alpha\vec{x}}$, and explicitly computable quantities, up to an error of order α^N. The dependence of each vector $(\phi_m)^T$ on the previous one, $(\phi_{m-1})^T$, as displayed in Definition II.1, links the re-expansion of $(\phi_{2N-1})^T$ to all the re-expanded expressions for $\{(\phi_m)^T : 1 \leq m \leq 2N-2\}$. The strategy to cope with all these features, which has been previously outlined, is developed in detail in this Section. To begin with, we provide some further necessary notations. We assume that α is small enough to ensure all the results stated in the previous sections.

III.1 Notations and definitions

We use planar polar coordinates to parameterizes our integrations. Each z_m, $m \geq 1$, is written as $\rho_m e^{i\theta}$ with $\rho_m = |z_m|$ over the contour integral Γ_m. For $m \geq 2$, we also define

$$z_{m-1}^{(m)} := z_m^{(m)} + \rho_{m-1}e^{i\theta} - \rho_m e^{i\theta} \tag{III.1}$$

where $z_m^{(m)} \equiv z_m$; the superscript (m) in $z_{m-1}^{(m)}$ is a label to keep track of the scale of the original integration variable, z_m, from which it derives in the re-expansion to be described. The general definition is:

$$z_{m-j-1}^{(m)} := z_{m-j}^{(m)} + \rho_{m-j-1}e^{i\theta} - \rho_{m-j}e^{i\theta}. \tag{III.2}$$

for $0 \leq j \leq m-2$. The basic objects featured in our re-expansion come in two varieties:

(i) Resolvents of the Hamiltonians H_{m-2}^+ and $\widetilde{H}_{m-3}^+$

$$R_{m-2}^+(z_{m-1}^{(m')} - S_{l(m-1)}(k)) := \frac{1}{H_{m-2}^+ - z_{m-1}^{(m')} + S_{l(m-1)}(k)} \tag{III.3}$$

$$R_{m-2}^+(z_m^{(m')} - S_{l(m),l(m-1)}(k)) := \frac{1}{H_{m-2}^+ - z_m^{(m')} + S_{l(m),l(m-1)}(k)} \tag{III.4}$$

$$\widetilde{R}_{m-3}^+(z_{m-2}^{(m')}) := \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-2}^{(m')}} \tag{III.5}$$

$$\widetilde{R}_{m-3}^+(z_{m-1}^{(m')} - S_{l(m-1)}(k)) := \frac{1}{\widetilde{H}_{m-3}^+ - z_{m-1}^{(m')} + S_{l(m-1)}(k)} \tag{III.6}$$

with $m' \geq m$, $m \geq 2$ in Eqs. (III.3) and (III.4), $m \geq 3$ in Eqs. (III.5) and (III.6), where

$$S_{l(m)}(k)) := \sum_{i=1}^{l(m)} |k_i(m)| \qquad k = (k_1(m), \cdots, k_{lm}(m)) \tag{III.7}$$

$$\begin{aligned} S_{l(m),l(m-1)}(k) \quad := \quad & \sum_{i=1}^{l(m)} |k_i(m)| + \sum_{i}^{l(m-1)} |k_i(m-1)| \\ & k = (k_1(m), \cdots, k_{lm}(m), k_1(m-1), \cdots, k_{lm-1}(m-1)) \end{aligned} \tag{III.8}$$

and our notation for the photon momenta, $k_i(m)$, $k_i(m-1)$, are referred to the momentum-space shells $\mathcal{K}_m^{m-1}$, $\mathcal{K}_{m-1}^{m-2}$, respectively:

$$\sigma_m \leq |k_i(m)| \leq \sigma_{m-1} \quad , \quad \sigma_{m-1} \leq |k_i(m-1)| \leq \sigma_{m-2}. \tag{III.9}$$

The non-negative numbers $l(m)$, $l(m-1)$ of terms in the sums (III.7), (III.8) are bounded and α-independent; $l(m) = 0$, $l(m-1) = 0$ mean no sums. The lower bounds in Eq. (III.9) and our bounds on the energy gaps in $\sigma(H_{m-2}^{+})$, $\sigma(\widetilde{H}_{m-3}^{+})$ above the groundstate energies (see Eqs. (I.52), (I.53)) ensure that the operators (III.3),(III.4),(III.5),(III.6) are bounded.

(ii) The scale dependent interaction vertexes

$$\Upsilon_m^{+}(k_i(m')) \quad := \quad \vec{v}_m \cdot \frac{\vec{\varepsilon}(k_i(m'))}{\sqrt{2|k_i(m')|}} e^{-i\alpha \vec{k}_i(m') \cdot \vec{x}} \tag{III.10}$$

$$\Upsilon_m^{-}(k_i(m')) \quad := \quad (\Upsilon_m^{+}(k_i(m'))^{*} \tag{III.11}$$

$$\begin{aligned} \Gamma^{+,+}(k_i(m'), k_j(m'')) \quad & = \quad (\Gamma^{-,-}(k_i(m'), k_j(m'')))^{*} \\ & := \quad \frac{\vec{\varepsilon}(k_i(m'))}{\sqrt{2|k_i(m')|}} \cdot \frac{\vec{\varepsilon}(k_j(m''))}{\sqrt{2|k_j(m'')|}} e^{-i\alpha \vec{k}_i(m') \cdot \vec{x}} e^{-i\alpha \vec{k}_j(m'') \cdot \vec{x}} \end{aligned} \tag{III.12}$$

$$\begin{aligned} \Gamma^{+,-}(k_i(m'), k_j(m'')) \quad & = \quad (\Gamma^{-,+}(k_i(m'), k_j(m'')))^{*} \\ & := \quad \frac{\vec{\varepsilon}(k_i(m'))}{\sqrt{2|k_i(m')|}} \cdot \frac{(\vec{\varepsilon}(k_j(m'')))^{*}}{2|k_j(m'')|} e^{-i\alpha \vec{k}_i(m') \cdot \vec{x}} e^{+i\alpha \vec{k}_j(m'') \cdot \vec{x}} \end{aligned} \tag{III.13}$$

with $m \geq 0$, $m', m'' \geq 0$.

III.2 Truncation map

As outlined in Section I, the strategy of the re-expansion is to split the resolvents (III.3) and (III.4) into two parts, respectively: a parallel and an orthogonal one to the spectral projection P_{m-2}. Since, as an operator, the parallel part is just the projection P_{m-2} times an explicit number

$$\begin{aligned} R^{+\,\|}_{m-2}(z^{(m')}_{m-1} - S_{l(m-1)}(k)) &:= R^{+}_{m-2}(z^{(m')}_{m-1} - S_{l(m-1)}(k))P_{m-2} \\ &= (S_{l(m-1)}(k) - z^{(m')}_{m-1})^{-1}P_{m-2} \end{aligned} \tag{III.14}$$

$$\begin{aligned} R^{+\,\|}_{m-2}(z^{(m')}_{m} - S_{l(m),l(m-1)}(k)) &:= R^{+}_{m-2}(z^{(m')}_{m} - S_{l(m),l(m-1)}(k))P_{m-2} \\ &= (S_{l(m),l(m-1)}(k) - z^{(m')}_{m})^{-1}P_{m-2} \end{aligned} \tag{III.15}$$

our expansion for the ground state vector $(\phi_{m'})^T$, re-expanded on scale $m-2$, $2 \leq m \leq m' \leq 2N-1$, will contain, as far as resolvents and projections are concerned, products of the following elementary operators:

$$P_{m-2} \tag{III.16}$$

$$\begin{aligned} &R^{+}_{m-2}(z^{(m')}_{m-1} - S_{l(m-1)}(k))P^{\perp}_{m-2} = \\ &\qquad =: R^{+\,\perp}_{m-2}(z^{(m')}_{m-1} - S_{l(m-1)}(k)) \end{aligned} \tag{III.17}$$

$$\begin{aligned} &R^{+}_{m-2}(z^{(m')}_{m} - S_{l(m),l(m-1)}(k))P^{\perp}_{m-2} = \\ &\qquad =: R^{+\,\perp}_{m-2}(z^{(m')}_{m} - S_{l(m),l(m-1)}(k)). \end{aligned} \tag{III.18}$$

We now specify the expansions $\mathcal{A}$ and $\mathcal{B}$ and to this purpose we define a map τ^N, the "truncation map" acting on the operators (III.16), (III.17) and (III.18) according to the rules below. After an application of τ^N, the Hamiltonian appearing in the new expressions is $\widetilde{H}^{+}_{m-3}$. As intermediate elementary operators appearing in the re-expansion we encounter the following ones:

$$\widetilde{P}_{m-3} \tag{III.19}$$

$$\widetilde{R}^{+}_{m-3}(z^{(m-2)}_{m-2})\widetilde{P}^{\perp}_{m-3} =: \widetilde{R}^{+\,\perp}_{m-3}(z^{(m-2)}_{m-2}) \tag{III.20}$$

$$\begin{aligned} &\widetilde{R}^{+}_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k))\widetilde{P}^{\perp}_{m-3} = \\ &\qquad =: \widetilde{R}^{+\,\perp}_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k)). \end{aligned} \tag{III.21}$$

Definition of the map τ^N

In the next definition (i), we just rephrase the expansion in Eq. (I.63) according to the new notations. The definition (ii) follows from (i) by defining the truncation of the orthogonal projection, $P^{\perp}_{m-2}$, as $\mathbf{1}_{m-2} - \tau^N(P_{m-2})$.

(i)

$$\begin{aligned}
&\tau^N(P_{m-2}) := \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (III.22)\\
= \;& \widetilde{P}_{m-3} - \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\Gamma_{m-2}} dz^{(m-2)}_{m-2}\Big((-z_{m-2})^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(z_{m-2})\Big)\times\\
&\times\Big[-W^{m-3}_{m-2}\Big((-z_{m-2})^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(z_{m-2})\Big)\Big]^j
\end{aligned}$$

(ii)

$$\begin{aligned}
&\tau^N(P^{\perp}_{m-2}) := \mathbf{1}_{m-2} - \tau^N(P_{m-2}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad (III.23)\\
= \;& \mathbf{1}_{m-2} - \widetilde{P}_{m-3} + \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\Gamma_{m-2}} dz^{(m-2)}_{m-2}\Big((-z_{m-2})^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(z_{m-2})\Big)\times\\
&\times\Big[-W^{m-3}_{m-2}\Big((-z_{m-2})^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(z_{m-2})\Big)\Big]^j
\end{aligned}$$

Now, at point iii) and iv) below, we specify the truncation of the expansion of type $\mathcal{A}$ and $\mathcal{B}$, respectively.
(iii) By iteratively using the resolvent equation

$$\begin{aligned}
R^{+}_{m-2}(\zeta) &= \widetilde{R}^{+}_{m-3}(\zeta) \qquad\qquad\qquad\qquad\qquad\qquad (III.24)\\
&\quad + R^{+}_{m-2}(\zeta)\Big(-W^{m-3}_{m-2} - E^{m-3}_{m-2}\Big)\widetilde{R}^{+}_{m-3}(\zeta)
\end{aligned}$$

where $\zeta := z^{(m')}_{m-1} - S_{l(m-1)}(k)$, we can easily write the expansion

$$\begin{aligned}
R^{+}_{m-2}(\zeta)P^{\perp}_{m-2} &= P^{\perp}_{m-2}R^{+}_{m-2}(\zeta)P^{\perp}_{m-2} && \text{(III.25)}\\
&= P^{\perp}_{m-2}\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2} + && \text{(III.26)}\\
&\quad + \Big(P^{\perp}_{m-2}\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2}\Big)\times && \text{(III.27)}\\
&\qquad \times\sum_{j=1}^{N}\Big[\Big(-W^{m-3}_{m-2} - E^{m-3}_{m-2}\Big)\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2}\Big]^j\\
&\quad + R^{+}_{m-2}(\zeta)P^{\perp}_{m-2}\times && \text{(III.28)}\\
&\qquad \times[\Big(-W^{m-3}_{m-2} - E^{m-3}_{m-2}\Big)\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2}\Big]^{N+1}
\end{aligned}$$

Finally, starting from the expressions in (III.26) and (III.27), we define our truncation of the orthogonal part of the resolvent in Eq. (III.25). The definition is

$$
\begin{aligned}
&\tau^N(R^{+\,\perp}_{m-2}(\zeta) := \qquad\qquad (III.29)\\
= \;& \Big[\tau^N(P^{\perp}_{m-2})\Big((-\zeta)^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(\zeta)\Big)\tau^N(P^{\perp}_{m-2})\Big]\\
&+\Big[\tau^N(P^{\perp}_{m-2})\Big((-\zeta)^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(\zeta)\Big)\tau^N(P^{\perp}_{m-2})\Big] \times\\
&\quad \times \sum_{j=1}^{N}\Big[\Big(-W^{m-3}_{m-2} - (E^{m-3}_{m-2})^T\Big)\times\\
&\quad\quad \times\Big((-\zeta)^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(\zeta)\Big)\tau^N(P^{\perp}_{m-2})\Big]^j.
\end{aligned}
$$

Notice that the expression above is written in terms of the elementary operators (III.19), (III.20) and (III.21); in particular, the operator $\widetilde{P}^{\perp}_{m-3}$, arising from the truncated expansion $\tau^N(P^{\perp}_{m-2})$, is always attached to a resolvent; hence either it yields a term of type $\widetilde{R}^{+\,\perp}_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k))$ or it is not present at all.
(iv) Analogously, we start from the equality below

$$
\begin{aligned}
&R^{+}_{m-2}(\zeta')P^{\perp}_{m-2} = \qquad\qquad (III.30)\\
= \;& P^{\perp}_{m-2}\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2} \qquad\qquad (III.31)\\
&+\Big(P^{\perp}_{m-2}\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2}\Big)\times \qquad\qquad (III.32)\\
&\quad\times\sum_{j=1}^{N}\begin{pmatrix} -W^{m-3}_{m-2} - E^{m-3}_{m-2} \\ +z^{(m')}_{m} - z^{(m')}_{m-1} + S_{l(m)}(k)\end{pmatrix}\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2}\Big]^j\\
&+R^{+}_{m-2}(\zeta')P^{\perp}_{m-2}\times \qquad\qquad (III.33)\\
&\quad\times\begin{pmatrix} -W^{m-3}_{m-2} - E^{m-3}_{m-2} \\ +z^{(m')}_{m} - z^{(m')}_{m-1} + S_{l(m)}(k)\end{pmatrix}\widetilde{R}^{+}_{m-3}(\zeta)P^{\perp}_{m-2}\Big]^{N+1}
\end{aligned}
$$

where $\zeta' := z^{(m')}_{m} - S_{l(m),l(m-1)}(k)$ and where we recall that $S_{l(m)}(k)$ and $S_{l(m),l(m-1)}(k)$ are defined in Eqs. (III.7) and (III.8), respectively.
We define our truncation of the orthogonal part of the resolvent in (III.30), by inserting $(E^{m-3}_{m-2})^T$ in place of E^{m-3}_{m-2} and $\tau^N(P^{\perp}_{m-2})$ in place of $P^{\perp}_{m-2}$, in expressions (III.31) and (III.32):

$$
\begin{aligned}
&\tau^N(R^{+\,\perp}_{m-2}(\zeta')) := \qquad\qquad (III.34)\\
= \;& \Big[\tau^N(P^{\perp}_{m-2})\Big((-\zeta)^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(\zeta)\Big)\tau^N(P^{\perp}_{m-2})\Big]\\
&+\Big[\tau^N(P^{\perp}_{m-2})\Big((-\zeta)^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(\zeta)\Big)\tau^N(P^{\perp}_{m-2})\Big] \times\\
&\quad \times \sum_{j=1}^{N}\Big[\Big(-W^{m-3}_{m-2} - (E^{m-3}_{m-2})^T + z^{(m')}_{m} - z^{(m')}_{m-1} + S_{l(m)}(k)\Big)\times\\
&\quad \times\Big((-\zeta)^{-1}\widetilde{P}_{m-3} + \widetilde{R}^{+\,\perp}_{m-3}(\zeta)\Big)\tau^N(P^{\perp}_{m-2})\Big]^j
\end{aligned}
$$

In Theorem III.4, by basically combining the map τ^N and the Wick-ordering, we recover the elementary operators (III.16),(III.17) and (III.18), on scale $m-3$. Before implementing this program, it is however necessary to check that, by applying our map τ^N, only terms of, at most, order $o(\alpha^N)$ are neglected; i.e.,

$$\begin{aligned} &\|(Id-\tau^N)(P_{m-2})\| \\ \leq\ & C\alpha^{N+1}\cdot\|P_{m-2}\| \end{aligned} \tag{III.35}$$

$$\begin{aligned} &\|(Id-\tau^N)(R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m-1)}(k))\| \\ \leq\ & C\alpha^{N+\frac{1}{2}}\cdot\|R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m-1)}(k))\| \end{aligned} \tag{III.36}$$

$$\begin{aligned} &\|(Id-\tau^N)(R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m),l(m-1)}(k))\| \\ \leq\ & C\alpha^{N+\frac{1}{2}}\cdot\|R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m),l(m-1)}(k))\| \end{aligned} \tag{III.37}$$

where Id is the identity map and C an α-independent constant. Concerning the projection P_{m-2}, Property (III.35) has been proven in Proposition I.I. For the orthogonal part of the resolvent, expanded at point (iii) and (iv), we need the following two lemmata.

Lemma III.1. *For $m'\geq m-1\geq 2$*

$$\begin{aligned} &\|(Id-\tau^N)(R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m-1)}(k))\| \\ \leq\ & C\alpha^{N+\frac{1}{2}}\cdot\|R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m-1)}(k))\|. \end{aligned} \tag{III.38}$$

Proof. By the Neumann series expansion in Eqs. (III.26), (III.27) and (III.28), Expression (III.29) corresponds to expression (III.25), with $\tau^N(P^{\perp}_{m-2})$, $\tau^N(P_{m-2})$ and $(E^{m-3}_{m-2})^T$ replaced by $P^{\perp}_{m-2}$,P_{m-2} and E^{m-3}_{m-2}, respectively, up to the remainder term

$$\begin{aligned} &R^{+}_{m-2}(z^{(m')}_{m-1}-S_{l(m-1)}(k))P^{\perp}_{m-2}\times \\ &\times\Big[\Big(-W^{m-3}_{m-2}-E^{m-3}_{m-2}\Big)\widetilde{R}^{+}_{m-3}(z^{(m')}_{m-1}-S_{l(m-1)}(k))P^{\perp}_{m-2}\Big]^{N+1}. \end{aligned} \tag{III.39}$$

We first prove that the norm of (III.39) is less than

$$C\alpha^{N+\frac{1}{2}}\cdot\|R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m-1)}(k))\| \tag{III.40}$$

for an α-independent constant C. Later on, we will show that the truncations $\tau^N(P^{\perp}_{m-2})$,$\tau^N(P_{m-2})$ and $(E^{m-3}_{m-2})^T$ yield a remainder of the same order as (III.40).

To control Expression (III.39), we re-write it in terms of a product of operator blocks:

$$\Big((R^+_{m-2}(u)\Big)^{\frac{1}{2}} P^\perp_{m-2} \times \tag{III.41}$$
$$\times \Big[(R^+_{m-2}(u))^{\frac{1}{2}} P^\perp_{m-2}\Big(- W^{m-3}_{m-2} - E^{m-3}_{m-2}\Big)(\widetilde{R}^+_{m-3}(u))^{\frac{1}{2}}\Big] \times \tag{III.42}$$
$$\times \Big[(\widetilde{R}^+_{m-3}(u))^{\frac{1}{2}} P^\perp_{m-2}\Big(- W^{m-3}_{m-2} - E^{m-3}_{m-2}\Big)(\widetilde{R}^+_{m-3}(u))^{\frac{1}{2}}\Big]^N \times \tag{III.43}$$
$$\times (\widetilde{R}^+_{m-3}(u))^{\frac{1}{2}} P^\perp_{m-2} \tag{III.44}$$

with $u = z^{(m')}_{m-1} - S_{l(m-1)}(k)$. Now because of the expansion

$$P^\perp_{m-2} \;=\; \widetilde{P}^\perp_{m-3} + \frac{1}{2\pi i}\sum_{j=1}^{\infty}\int_{\Gamma_{m-2}} dz^{(m-2)}_{m-2}\, \widetilde{R}^+_{m-3}(z^{(m-2)}_{m-2}) \times \tag{III.45}$$
$$\times \Big[- W^{m-3}_{m-2} \widetilde{R}^+_{m-3}(z^{(m-2)}_{m-2})\Big]^j$$

we can provide, by usual methods, a bound of order, at most, α for the norm of the operator block in (III.42) and for the norm of each of the N blocks in (III.43); in this power counting it is important to take into account that a factor $\alpha^{\frac{1}{2}}$ is gained because of the projection $P^\perp_{m-2}$ in the operator block in front of each of them. In the end, we easily get the bound in Eq. (III.40).
To complete our proof, we must show that the expression

$$P^\perp_{m-2}\widetilde{R}^+_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k))P^\perp_{m-2} + \tag{III.46}$$
$$+ \;\; \Big[P^\perp_{m-2}\widetilde{R}^+_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k))P^\perp_{m-2}\Big] \times$$
$$\times \sum_{j=1}^{N} \Big[\Big(- W^{m-3}_{m-2} - E^{m-3}_{m-2}\Big)\widetilde{R}^+_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k))P^\perp_{m-2}\Big]^j$$

differs from (III.29) by a remainder term whose norm is controlled by the bound in Eq. (III.40). Concerning the replacement of E^{m-3}_{m-2} by $(E^{m-3}_{m-2})^T$, we use the estimate (II.20) proved in Lemma II.2. For the truncations of $P^\perp_{m-2}$, the result follows because the truncation τ^N amounts to neglecting

$$\sum_{j=N+1}^{\infty}\int_{\Gamma_{m-2}} dz^{(m-2)}_{m-2}\, \widetilde{R}^+_{m-3}(z^{(m-2)}_{m-2})[-W^{m-2}_{m-3}\widetilde{R}^+_{m-3}(z^{(m-2)}_{m-2})]^j \tag{III.47}$$

in the expansion of $P^\perp_{m-2}$ carried out in Eqs. (III.45). When estimating the norms of all the factors in the remainder, we find that either the norm of at least one factor in (III.42), (III.43) is, at most, $\mathcal{O}(\alpha^{N+\frac{1}{2}})$ or we gain an extra-factor α^{N+1} in blocks of type (III.44). $\square$

In the next lemma, we provide the norm bound in Eq. (III.37).

Lemma III.2. *For $m' \geq m \geq 3$*

$$\begin{aligned} & \|(Id-\tau^N)(R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m),l(m-1)}(k))\| \\ \leq \quad & C\alpha^{N+\frac{1}{2}} \cdot \|R^{+,\perp}_{m-2}(z^{(m')}_{m-1}-S_{l(m),l(m-1)}(k))\|. \end{aligned} \tag{III.48}$$

Proof. The proof is analogous to Lemma III.2, since the two more terms to be added to the numerator, $-W^{m-2}_{m-3}-E^{m-2}_{m-3}$, in the operator blocks (III.42) and (III.43) are small enough, namely

$$|z^{(m')}_{m}-z^{(m')}_{m-1}| \leq \mathcal{O}(\alpha^{m-1}) \tag{III.49}$$

by construction, and

$$|S_{l(m)}(k)| \leq \mathcal{O}(\alpha^{m-1}), \tag{III.50}$$

because, by definition, $S_{l(m)}(k)=\sum_{i=1}^{l(m)}|k_i(m)|$, with $l(m)$ an α-independent number. □

Remark
Notice that the results in Lemmata III.1 and III.2 are still valid if $(E^{m-3}_{m-2})^T$ is replaced by any other approximation, $(E^{m-3}_{m-2})^{appr.}$, of E^{m-3}_{m-2} satisfying the property $|E^{m-3}_{m-2}-(E^{m-3}_{m-2})^{appr.}| \leq o(\alpha^{m+N-\frac{3}{2}})$.

III.3 Fully re-expanded vectors

In this paragraph we present the full re-expansion of the approximated ground state vector $(\phi_m)^T$.

Definition III.3. *We say that the α-dependent vector $(\phi_m)^T$, $1 \leq m \leq M = 2N-1$, can be re-expanded up to order $2N$, in $\alpha^{\frac{1}{2}}$, in the interval $\alpha \in (0, \bar{\alpha}(m))$, where $\bar{\alpha}(m) > 0$, if it can be split as*

$$(\phi_m)^T = (\phi_m)^T_\sharp + \xi^N \tag{III.51}$$

where:
i) $\|(\phi_m)^T_\sharp\| \geq \delta > 0$, with δ an α-independent constant;
ii) $\|\xi^N\| \leq o(\alpha^N)$;
iii) $(\phi_m)^T_\sharp$ is of the form given below.
The vector $(\phi_m)^T_\sharp$ can be explicitly computed in terms of bare quantities. It consists of a finite, N-dependent, but α-independent number of expressions

whose norm is, at most, of order $\mathcal{O}(1)$ *in* α. *They are of the type*

$$\int_{\Gamma_2} \alpha^{m-2} dz_2^{(m)} \prod_{j=1}^{\bar{j}(m-1)} \int_{\Gamma_2} \alpha^{m-3} dz_2^{(m-1),j} \times ... \tag{III.52}$$
$$... \times \prod_{j=1}^{\bar{j}(3)} \int_{\Gamma_2} \alpha dz_2^{(3),j} \prod_{j=1}^{\bar{j}(2)} \int_{\Gamma_2} dz_2^{(2),j} \prod_{j=1}^{\bar{j}(1)} \int_{\Gamma_1} dz_1^{(1),j} A\varphi_{el} \otimes \Omega$$

where:
$\int_{\Gamma_2} \alpha^{m-3} dz_2^{(m-1),j} (= \int_{\Gamma_{m-1}} dz_{m-1}^{(m-1),j})$ *denotes a contour integral with path* Γ_2 *in the integration variable* $z_2^{(m-1)}$ *(original variable* z_{m-1}*).We note that the numbers,* $\bar{j}(l)$, *of contour integrations are* α*-independent;*
The symbol A *denotes the integral*

$$A = \int_{\mathcal{K}_m^{m-1}} \prod_{l=1}^{\bar{l}(m)} dk_l(m) \int_{\mathcal{K}_{m-1}^{m-2}} \prod_{l=1}^{\bar{l}(m-1)} dk_l(m-1) ... \int_{\mathcal{K}_1^0} \prod_{l=1}^{\bar{l}(1)} dk_l(1)\ B \tag{III.53}$$

with $\bar{l}(m), .., \bar{l}(1)$ *finite and* α*-independent,* $\int_{\mathcal{K}_m^{m-1}}, ..., \int_{\mathcal{K}_1^0}$ *multiple integrals;*
The operator B *can be written as*

$$B = D(k, z) \otimes C(k) \tag{III.54}$$

where:

- $C(k)$ *is the product of a finite and* α*-independent number of photon creation operators, contained in the range* $\mathcal{K}_m^0$*; here* k *stands for all the momenta (and helicities) of the photon creation operators;*

- $D(k, z)$ *is the product of a finite and* α*-independent number of factors belonging to three groups,* $\mathcal{G}_0^1$, $\mathcal{G}_0^2$ *and* $\mathcal{G}_0^3$, *specified below; here the argument* k *denotes all the momenta (and helicities) integration variables and the argument* z *denotes all the contour integration variables in Expression (III.52).*

  $\mathcal{G}_0^1$

$$P_0 \tag{III.55}$$
$$R_0^{+\perp}(z_1^{(1)} - S_{l(1)}(k)) \tag{III.56}$$
$$R_0^{+\perp}(z_2^{(m')} - S_{l(1),l(2)}(k)) \tag{III.57}$$

where $0 \leq l(1) \leq \bar{l}(1)$, $0 \leq l(2) \leq \bar{l}(2)$, $2 \leq m' \leq m$*;*

$\mathcal{G}_0^2$

$$\Upsilon_0^{\pm}(k_i(m')) \qquad , \qquad \Gamma^{\pm,\pm}(k_i(m'),k_j(m'')) \tag{III.58}$$

with $1 \leq m', m'' \leq m$, $1 \leq i \leq \bar{l}(m')$, $1 \leq j \leq \bar{l}(m'')$*;*

$\mathcal{G}_0^3$

$$\frac{1}{S_{l(1)}(k) - z_1^{(1)}}\mathbf{1}_0 \tag{III.59}$$

$$\frac{1}{S_{l(m''),\bar{l}(m''-1)}(k) - \alpha^{m''-2} z_2^{(m'')}}\mathbf{1}_0\,;\ \frac{1}{S_{l(m'')}(k) - \alpha^{m''-2} z_2^{(m'')}}\mathbf{1}_0 \tag{III.60}$$

$$S_{l(m)}(k)\mathbf{1}_0, ..., S_{l(3)}(k)\mathbf{1}_0\,;\ ;\ (z_{m''-1}^{(m')} - z_{m''}^{(m')})\mathbf{1}_0 \tag{III.61}$$

$$\langle W_m^{m-1}\rangle_{\Omega_m}\mathbf{1}_0, ..., \langle W_1^0\rangle_{\Omega_1}\mathbf{1}_0\ (E_{m'''}^{m'''-1})_\sharp^T\mathbf{1}_0$$

where $2 \leq m'' \leq m' \leq m$, $0 \leq l(m'') \leq \bar{l}(m'')$, $0 \leq l(m''-1) \leq \bar{l}(m''-1)$, *and the symbol* $(E_{m'''}^{m'''-1})_\sharp^T$, $1 \leq m''' \leq m-1$, *stands for an explicit expression which consists of a finite and* α*-independent number of integrals involving only bare quantities according to the formula*

$$(E_{m'''}^{m''''-1})_\sharp^T := -\frac{(W_{m'''}^{m''''-1}(\tilde{\phi}_{m'''-1})_\sharp^T\ ,(\phi_{m'''})_\sharp^T)}{((\tilde{\phi}_{m'''-1})_\sharp^T\ ,(\phi_{m'''})_\sharp^T)} \tag{III.62}$$

$$(\tilde{\phi}_{m'''-1})_\sharp^T := (\phi_{m'''-1})_\sharp^T \otimes \Omega_{m'''}^{m'''-1} \tag{III.63}$$

where it is assumed that all the vectors in $\{(\phi_j)^T)|j = 1, ..., m-1\}$ *can be fully re-expanded in* $(0, \bar{\alpha}(m))$ *and* $|((\tilde{\phi}_{j-1})_\sharp^T\ ,(\phi_j)_\sharp^T)| > \delta > 0$*, where* δ *is an* α*-independent constant.*

Remark

The definition of $(\phi_m)_\sharp^T$ depends only on the vectors $(\phi_{m'})_\sharp^T$ with $1 \leq m' < m$.

III.3.1 Explicit truncation map $\tau_\sharp^N$ and main theorem

In the next theorem we prove that the vector $(\phi_{2N-1})^T$ can be fully re-expanded and the related energy shift has an explicit expression up to order N in α. These results, combined with (II.15), (II.20), yield the expansions (I.21) and (I.22) announced in Section I, for α in an N-dependent neighborhood of 0. Before stating the theorem, we must define the *explicit truncation map* $\tau_\sharp^N$ which is used in the proof.

Definition of the map $\tau_\sharp^N$

The map $\tau_{\sharp}^{N}$ derives from τ^{N} defined in Section III.2 by replacing the variable $(E_m^{m-1})^T$ by the explicit quantity $(E_m^{m-1})_{\sharp}^T$ (see Definition III.3), $1 \le m \le M = 2N-1$, which will turn out to be well defined and fulfilling the property (see Theorem III.4)

$$| \, (E_m^{m-1})_{\sharp}^T - (E_m^{m-1})^T \, | \le o(\alpha^{m+\frac{1}{2}+N}). \qquad \text{(III.64)}$$

Therefore Lemmata III.1 and III.2 hold for $\tau_{\sharp}^{N}$ as well; see the Remark after Lemma III.2. In the case of the operators $\{P_m | \, m = 1, ..., 2N-1\}$, the two maps, τ^N and $\tau_{\sharp}^{N}$, coincide.

Theorem III.4. *For any $N \in \mathbb{N}$, the vector $(\phi_{M=2N-1})^T$ can be fully re-expanded, up to the order $2N$, in $\alpha^{\frac{1}{2}}$, for α in the interval $(0, \bar{\alpha}(M))$, where $\bar{\alpha}(M) > 0$. In the same interval, $(E_M^{M-1})_{\sharp}^T$ is well defined and fulfills the constraint*

$$|(E_M^{M-1})^T - (E_M^{M-1})_{\sharp}^T| \le o(\alpha^{3N-\frac{1}{2}}). \qquad \text{(III.65)}$$

Proof. The proof is by induction on a finite set $\{m \,|1 \le m \le 2N-1\}$. We assume that certain properties are fulfilled for $1 \le q \le m-1$, and we show that they hold true for $q = m$.
Inductive hypotheses
For $1 \le q \le m-1$:
H1) The vector $(\phi_q)^T$ can be fully re-expanded up to the order $2N$ in $\alpha^{\frac{1}{2}}$ in an interval $(0, \bar{\alpha}(m-1))$ where $\bar{\alpha}(m-1) > 0$, i.e., a vector $(\phi_q)_{\sharp}^T$ exists and has all the properties specified in Definition III.3;
H2) For $\alpha \in (0, \bar{\alpha}(m-1))$, the truncated energy shifts are well defined and fulfill the property

$$(E_q^{q-1})^T = (E_q^{q-1})_{\sharp}^T + o(\alpha^{q+\frac{1}{2}+N}); \qquad \text{(III.66)}$$

H3) Given the natural number a such that $2 \le q \le a \le 2N-1$, in an interval $\alpha \in (0, \bar{\alpha}_{\psi_q})$, where $\bar{\alpha}(q-1) \ge \bar{\alpha}_{\psi_q} > 0$, it is possible to fully re-expand a vector $\psi_q(\alpha)$, defined below, up to the order $2N$ in $\alpha^{\frac{1}{2}}$, i.e., we adapt Definition III.3 to the vector ψ_q in a natural way, by dropping condition i) (in Definition III.3) to include vectors ψ_q whose norm is not bounded away from 0 uniformly in α. The definition of the vector ψ_q is:

$$\begin{aligned}\psi_q \;\; &:= \prod_{p'=q}^{a} \Big(\prod_{j=1}^{\bar{j}(p')} \int_{\Gamma_q} \alpha^{p'-q} dz_q^{(p'),j} \Big) \prod_{j=1}^{\bar{j}(q-1)} \int_{\Gamma_{q-1}} dz_{q-1}^{(q-1),j} \times \qquad \text{(III.67)} \\ &\times \prod_{p=q}^{a+1} \Big(\int_{\mathcal{K}_{p-1}^{p-2}} \prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1) \Big) D_{q-2}(k,z) (\phi_{q-2})^T \otimes C_a^{q-2}(k) \Omega_a^{q-2}\end{aligned}$$

where
i) $\bar{j}(p'), \bar{j}(q-1), \bar{i}(p-1)$ are α-independent natural numbers;
ii) $C_a^{q-2}(k)$ is a finite product of an α-independent number of photon creation operators with momenta contained in the range $\mathcal{K}_a^{q-2}$;
iii) $D_{q-2}(k,z)$ is a finite product of an α-independent number of factors among the three groups below and with the constraints given by two properties, $\mathcal{P}1$ and $\mathcal{P}2$, discussed later.
Groups

- $\mathcal{G}^1_{q-2}$

$$P_{q-2} \tag{III.68}$$
$$R^{+\perp}_{q-2}(z^{(q-1)}_{q-1} - S_{l(q-1)}(k)) \tag{III.69}$$
$$R^{+\perp}_{q-2}(z^{(p')}_{q} - S_{l(q),\bar{l}(q-1)}(k)) \tag{III.70}$$

where $q \leq p' \leq a$, and $0 \leq l(q) \leq \bar{l}(q)$, $0 \leq l(q-1) \leq \bar{l}(q-1)$ with $\bar{l}(q)$ and $\bar{l}(q-1)$ α-independent numbers.

- $\mathcal{G}^2_{q-2}$

$$\Upsilon^{\pm}_{q-2}(k_i(q')) \qquad , \qquad \Gamma^{\pm,\pm}(k_i(q'), k_j(q'')) \tag{III.71}$$

where $q-1 \leq q', q'' \leq a$, $1 \leq i \leq \bar{l}(q')$, $1 \leq j \leq \bar{l}(q'')$ with $\bar{l}(q')$, $\bar{l}(q'')$ α-independent numbers.

- $\mathcal{G}^3_{q-2}$

$$\frac{1}{S_{l(q-1)}(k) - z^{(q-1)}_{q-1}} \mathbf{1}_{q-2} \tag{III.72}$$
$$\frac{1}{S_{l(p'),l(p'-1)}(k) - \alpha^{p'-q} z^{(p')}_{q}} \mathbf{1}_{q-2} \, ; \; \frac{1}{S_{l(p')}(k) - \alpha^{p'-q} z^{(p')}_{q}} \mathbf{1}_{q-2} \tag{III.73}$$
$$\{\xi^{q-2}_{q-1} \mathbf{1}_{q-2}\}, ..., \{\xi^{a-1}_{a} \mathbf{1}_{q-2}\} \tag{III.74}$$

where $q \leq p' \leq a$ and each of the families $\{\xi^{q-2}_{q-1} \mathbf{1}_{q-2}\}, ..., \{\xi^{a-1}_{a} \mathbf{1}_{q-2}\}$ obey $|\xi^{j-2}_{j-1}| \leq \mathcal{O}(\alpha^{j-1})$ and $\#\{\xi^{j-2}_{j-1} \mathbf{1}_{q-2}\} \leq L(N)$ where $L(N)$ is an α-independent number. The c-numbers are supposed to be explicit, i.e. they can be computed in terms of bare quantities.

Properties $\mathcal{P}1$,$\mathcal{P}2$

$\mathcal{P}1$ Next to each operator $g^2_{q-2} \in \mathcal{G}^2_{q-2}$, there is an operator $g^1_{q-2} \in \mathcal{G}^1_{q-2}$ on the left and on the right side.

$\mathcal{P}2$ The following estimate of the norm of the vector (III.67) is, at most, $\mathcal{O}(1)$ in α

$$\Big\| \prod_{p'=q}^{a} \Big(\prod_{j=1}^{\bar{j}(p')} \int_{\Gamma_q} \alpha^{p'-q} dz_q^{(p'),j} \Big) \prod_{j=1}^{\bar{j}(q-1)} \int_{\Gamma_{q-1}} dz_{q-1}^{(q-1),j} \times \tag{III.75}$$

$$\times \prod_{p=q}^{a+1} \Big(\int_{\mathcal{K}_{p-1}^{p-2}} \prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1) \Big) D_{q-2}(k,z)(\phi_{q-2})^T \otimes C_a^{q-2}(k)\Omega_a^{q-2} \Big\|$$

$$\leq \prod_{p'=q}^{a} \Big(\prod_{j=1}^{\bar{j}(p')} \alpha^{p'-q} |\Gamma_q^j| \Big) \prod_{j=1}^{\bar{j}(q-1)} |\Gamma_{q-1}^j| \times \tag{III.76}$$

$$\sup_z \Big\{ \prod_{p=q}^{a+1} \Big(\int_{\mathcal{K}_{p-1}^{p-2}} \prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1) \Big) \prod_{p=q}^{a+1} \Big(\int_{\mathcal{K}_{p-1}^{p-2}} \prod_{i=1}^{\bar{i}(p-1)} dk_i'(p-1) \Big) \times$$

$$\times \sum_{\pi} \Delta_\pi(k',k) \| D_{q-2}^*(k',z) D_{q-2}(k,z) \| \cdot \| (\phi_{q-2})^T \|^2 \Big\}^{\frac{1}{2}}$$

where:

* π stands for each pairing of the photon creation- and annihilation operators in $[C_a^{q-2}]^*(k')[C_a^{q-2}](k)$ and $\Delta_\pi(k',k)$ is the corresponding product of delta-functions and Kronecker's symbols;
* $|\Gamma_q^j| = 2\pi\alpha^q$ is the length of the j-th contour path;
* $\sup_z$ is the sup in the integration variables z running over the contours $\{\Gamma_q^j\}$, $\{\Gamma_{q-1}^j\}$.
  We warn the reader that, even if we use the same symbol, not all the k, k' integration variables are necessarily paired, in general the pairings $\{\pi\}$ are related to a subset of them, those ones contained in $C_a^{q-2*}(k')C_a^{q-2}(k)$ (see the comment below Eq. (III.54)).

Now we specify the estimate of $\|D_{q-2}(k',z)^* D_{q-2}(k,z)\|$ starting from the constraints on the structure of $D_{q-2}(k,z)$. Let $D_{q-2}(k,z)$ be a generic combination of operators which fulfills Properties $\mathcal{P}1$ and $\mathcal{P}2$, for instance

$$g_{m-2}^1 g_{m-2}^2 g_{m-2}^1 \cdots g_{m-2}^1 g_{m-2}^3 \cdots g_{m-2}^3. \tag{III.77}$$

By convention, we push all the g_{m-2}^3 operators to the very right; being c-numbers this operation is neutral. In (III.77) we have omitted the dependence on k and on z; we will take it into account when necessary. Each operator g_{m-2}^2 is estimated in companion with its neighbor g_{m-2}^1 on the right; e.g., for

$$g_{m-2}^2 = \Upsilon_{q-2}^{\pm}(k_i(q')) \quad \text{and} \quad g_{m-2}^1 = R_{q-2}^{+\perp}(z_q^{(p')} - S_{l(q),l(q-1)}(k)) \tag{III.78}$$

we can easily estimate

$$\begin{aligned}&\|\Upsilon^{\pm}_{q-2}(k_i(q'))R^{+\,\perp}_{q-2}(z^{(p')}_q - S_{\bar{l}(q),l(q-1)}(k))\| \leq \\ &\leq C \cdot \frac{1}{\sqrt{2|k_i(q')|}}||R^{+\,\perp}_{q-2}(z^{(p')}_q - S_{\bar{l}(q),l(q-1)}(k))||\end{aligned} \tag{III.79}$$

where C is α-independent; analogously,

$$\begin{aligned}&\|\Gamma^{\pm,\pm}(k_i(q'),k_j(q''))R^{+\,\perp}_{q-2}(z^{(p')}_q - S_{l(q),l(q-1)}(k))|| \leq \\ &\leq C \cdot \frac{1}{\sqrt{2|k_i(q')| \cdot 2|k_j(q'')|}}||R^{+\,\perp}_{q-2}(z^{(p')}_q - S_{l(q),l(q-1)}(k))\|.\end{aligned} \tag{III.80}$$

Following the rules in Eqs. (III.79) and (III.80), and using the usual norm estimates for the remaining $\{g^1_{m-2}\}$ operators and the c-numbers $\{g^3_{m-2}\}$, we define

$$\begin{aligned}&\mathcal{E}\Big(D_{q-2}(k,z)\Big) := \\ &= \|g^1_{m-2}\|\|g^2_{m-2}g^1_{m-2}\|' \cdots \|g^1_{m-2}\||g^3_{m-2} \cdots g^3_{m-2}|(k,z)\end{aligned} \tag{III.81}$$

where, depending on g^2_{m-2}, $\|g^2_{m-2}g^1_{m-2}\|'$ is either the right hand side in Eq. (III.79) or in Eq. (III.80). We can state the upper bound

$$||D_{q-2}(k',z)^* D_{q-2}(k,z)|| \leq \mathcal{E}\Big(D_{q-2}(k',z)\Big)\mathcal{E}\Big(D_{q-2}(k,z)\Big). \tag{III.82}$$

Finally, by definition, Property $\mathcal{P}2$ ensures that, by inserting estimate (III.82), the multiple integral (III.76) turns out to be convergent and uniformly bounded in α. Therefore the same holds for the multiple integral (III.75).

Remark

It is important not to confuse the induction index q and the scale of re-expansion, $q-2$, of the expression in Eq.(III.67).

H4) The remainder obtained from the re-expansion of (III.67) is a finite, α-independent, sum of expressions of the type

$$\begin{aligned}&\prod_{p'=q'}^{a}\Big(\prod_{j=1}^{\bar{j}(p')}\int_{\Gamma_{q'}} \alpha^{p'-q'} dz^{(p'),j}_{q'}\Big)\prod_{j=1}^{\bar{j}(q'-1)}\int_{\Gamma_{q'-1}} dz^{(q'-1),j}_{q'-1} \times \\ &\times \prod_{p=q'}^{a+1}\Big(\int_{\mathcal{K}^{p-2}_{p-1}}\prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1)\Big)\widehat{D_{q'-2}}(k,z)(\phi_{q'-2})^T \otimes C^{q'-2}_a(k)\Omega^{q'-2}_a,\end{aligned} \tag{III.83}$$

with $2 \leq q' \leq q < a \leq 2N-1$, where

(i) $C^{q'-2}_a(k)$ is a finite product of an α-independent number of photon creation operators with momenta contained in the shell $\mathcal{K}^{q'-2}_a$;

(ii) $\widehat{D_{q'-2}}(k,z)$ corresponds to $D_{q'-2}(k,z)$ except that at least one of the operators $g^1_{q'-2} \in \mathcal{G}^1_{q'-2}$ contained in $D_{q'-2}(k,z)$ is replaced by $g^1_{q'-2} - \tau^N_\sharp(g^1_{q'-2})$.

Structure of the proof.
The proof consists of three main steps.
S1) Assuming *H2)*, *H3)* and *H4)* for vectors of type ψ_q, up to $q = m-1$, we prove *H3)* and *H4)* for a vector of type ψ_m, by picking an $\bar{\alpha}_{\psi_m} > 0$ with $\bar{\alpha}_{\psi_m} \leq \bar{\alpha}(m-1)$.
S2) From *H3)* for $q = m$ and *H2)* up to $q = m-1$, we derive *H1)* for $q = m$ by picking an $\bar{\alpha}(m) > 0$ small enough and such that $\bar{\alpha}(m) \leq \bar{\alpha}(m-1)$.
S3) From *H1)* and *H4)* for $q = m-1$, we get *H2)* up to $q = m$, for $0 < \alpha < \bar{\alpha}(m)$.
In the end we check the initial step of the induction.
Remark
Concerning terminology, in order to avoid ambiguities, we will always use the word *expression* for a vector with the structure displayed either in Eq. (III.52) or in Eq. (III.67), or in Eq. (III.83). As we will see, the map $\tau^N_\sharp$ produces a sum of *expressions* starting from an original one; in other words, it is helpful to think of it as a map that produces polynomials starting from monomials. For the norm estimate of a vector corresponding to an *expression*, we always assume the estimate procedure presented in Property $\mathcal{P}2$, Hypothesis *H3)*.
S1) We start from the following expression re-expanded on scale $m-2$

$$\psi_m := \prod_{p'=m}^{a} \Big(\prod_{j=1}^{\bar{j}(p')} \int_{\Gamma_m} \alpha^{p'-m} dz_m^{(p'),j} \Big) \prod_{j=1}^{\bar{j}(m-1)} \int_{\Gamma_{m-1}} dz_{m-1}^{(m-1),j} \times \tag{III.84}$$
$$\times \prod_{p=m}^{a+1} \Big(\int_{\mathcal{K}^{p-2}_{p-1}} \prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1) \Big) D_{m-2}(k,z)(\phi_{m-2})^T \otimes C_a^{m-2}(k)\Omega_a^{m-2}.$$

Let $D_{m-2}(k,z)$ be, for example, the combination of operators considered before in Hypothesis *H3*:

$$g^1_{m-2} g^2_{m-2} g^1_{m-2} \cdots g^1_{m-2} g^3_{m-2} \cdots g^3_{m-2} \tag{III.85}$$

which fulfills Properties $\mathcal{P}1$, $\mathcal{P}2$, by hypothesis.
We replace each operator in $D_{m-2}(k,z)$ and belonging to $\mathcal{G}^1_{m-2}$ by the corresponding one under the map $\tau^N_\sharp$ which is well defined thanks to Hypothesis *H2)*, assuming that $0 < \alpha < \bar{\alpha}(m-1)$:

$$g^1_{m-2} g^2_{m-2} g^1_{m-2} \cdots g^1_{m-2} g^3_{m-2} \cdot \cdot g^3_{m-2} \mapsto \tag{III.86}$$
$$\tau^N_\sharp(g^1_{m-2}) g^2_{m-2} \tau^N_\sharp(g^1_{m-2}) \cdot \cdot \tau^N_\sharp(g^1_{m-2}) g^3_{m-2} \cdot \cdot g^3_{m-2}. \tag{III.87}$$

Exploiting Lemmata III.1, III.2 and Hypothesis *H2)*, the operator norm of the difference between (III.86) and (III.87) is controlled as follows

$$\begin{aligned}&\|g^1_{m-2}g^2_{m-2}g^1_{m-2}\cdots g^1_{m-2}g^3_{m-2}\cdots g^3_{m-2}\\ &\qquad -\tau^N_\sharp(g^1_{m-2})g^2_{m-2}\tau^N_\sharp(g^1_{m-2})\cdots\tau^N_\sharp(g^1_{m-2})g^3_{m-2}\cdots g^3_{m-2}\|\leq\\ &\leq\ C\alpha^{N+\frac{1}{2}}\cdot\mathcal{E}\Big(g^1_{m-2}g^2_{m-2}g^1_{m-2}\cdot\cdot g^1_{m-2}g^3_{m-2}\cdots g^3_{m-2}\Big)\end{aligned}\tag{III.88}$$

where C is α-independent.
Then we split the vertex operators $\{\Upsilon^{\pm}_{m-2}(k_i(m'))\}$, $m'\geq m-2$, into two pieces:

$$\Upsilon^+_{m-3}(k_i(m'))\quad;\quad[a(\vec{G}^{m-1}_m)+a^*(\vec{G}^{m-3}_{m-2})]\cdot\vec{\varepsilon}(k_i(m'))\frac{e^{-i\alpha\vec{k}_i(m'))\cdot\vec{x}}}{\sqrt{2|k_i(m')|}}\tag{III.89}$$

$$\Upsilon^-_{m-3}(k_i(m'))\quad;\quad[a(\vec{G}^{m-3}_{m-2})+a^*(\vec{G}^{m-3}_{m-2})]\cdot\vec{\varepsilon}(k_i(m'))^*\frac{e^{+i\alpha\vec{k}_i(m'))\cdot\vec{x}}}{\sqrt{2|k_i(m')|}}\,,\tag{III.90}$$

while the other vertex operators, $\Gamma^{\pm}_{m-2}$, remain unchanged. We use the notation δg^2_{m-2} for the general element in the new family of vertex operators. Hence we can rewrite (III.86) as an α-independent sum of the following products

$$\tau^N_\sharp(g^1_{m-2})\delta g^2_{m-2}\tau^N_\sharp(g^1_{m-2})\cdot\cdot\tau^N_\sharp(g^1_{m-2})g^3_{m-2}\cdots g^3_{m-2}.\tag{III.91}$$

Next, in Expression (III.84), we rewrite $(\phi_{m-2})^T$ as $\tau^N(P_{m-2})(\tilde{\phi}_{m-3})^T$; see the definitions in Eqs. (II.11), (II.13) and (III.22).
We denote by $\widetilde{D}_{m-3}(k,z)$ the general term, i.e., product of operators, that we obtain from $D_{m-2}(k,z)P_{m-2}$ under the action of the map $\tau^N_\sharp$ and after breaking the vertex operators according to (III.89) and (III.90). In $\widetilde{D}_{m-3}(k,z)$ we distinguish three different groups of elementary objects:

$\widetilde{\mathcal{G}}^1_{m-3}$

$$\widetilde{P}_{m-3}\tag{III.92}$$

$$\widetilde{R}^+_{m-3}(z^{(m')}_{m-2})\widetilde{P}^\perp_{m-3}\tag{III.93}$$

$$\widetilde{R}^+_{m-3}(z^{(m')}_{m-1}-S_{l(m-1)}(k))\widetilde{P}^\perp_{m-3}\tag{III.94}$$

where $m'\geq m-1$, $0\leq l(m-1)\leq\bar{l}(m-1)$.

$\widetilde{\mathcal{G}}^2_{m-3}$

$$\Upsilon^{\pm}_{m-3}(k_i(q')) \quad ; \quad \Gamma^{\pm,\pm}(k_i(q'),k_j(q'')) \tag{III.95}$$

$$\Delta\Upsilon^{+}_{m-3}(k_i(q')) := [a(\vec{G}^{m-3}_{m-2}) + a^*(\vec{G}^{m-3}_{m-2})]\cdot\vec{\varepsilon}(k_i(m'))\frac{e^{-i\alpha\vec{k}_i(q'))\cdot\vec{x}}}{\sqrt{2|k_i(q')|}} \tag{III.96}$$

$$\Delta\Upsilon^{-}_{m-3}(k_i(q')) := [a(\vec{G}^{m-3}_{m-2}) + a^*(\vec{G}^{m-3}_{m-2})]\cdot\vec{\varepsilon}(k_i(m'))^*\frac{e^{+i\alpha\vec{k}_i(q'))\cdot\vec{x}}}{\sqrt{2|k_i(q')|}}$$

$$W^{m-3}_{m-2} - \langle W^{m-3}_{m-2}\rangle_{\Omega_{m-2}} \tag{III.97}$$

where $m-2 \le q', q'' \le a$, $1 \le i, j \le \bar{i}(p-1)$ with $m \le p \le a+1$.

In group $\widetilde{\mathcal{G}}^2_{m-3}$, we distinguish the operators in Eqs. (III.95) and (III.96), which come from a vertex operator in $\mathcal{G}^2_{m-2}$ and that we have already denoted by δg^2_{m-2}, from the operators in Eq. (III.97) which we denote by $\mathcal{W}^{m-3}_{m-2}$. This notation is useful in the proof of next Lemma III.5.

$\widetilde{\mathcal{G}}^3_{m-3}$

$$\frac{1}{S_{l(m-2)}(k) - z^{(m-2)}_{m-2}}\mathbf{1}_{m-2} \tag{III.98}$$

$$\frac{1}{S_{l(p'),l(p'-1)}(k) - \alpha^{p'-m-1}z^{(p')}_{m-1}}\mathbf{1}_{m-2}\,;\ \frac{1}{S_{l(p')}(k) - \alpha^{p'-m-1}z^{(p')}_{m-1}}\mathbf{1}_{m-2} \tag{III.99}$$

$$S_{l(m)}(k)\mathbf{1}_{m-2}, (z^{(p'')}_{m} - z^{(p'')}_{m-1})\mathbf{1}_{m-2}, \tag{III.100}$$
$$\langle W^{m-3}_{m-2}\rangle_{\Omega_{m-2}}\mathbf{1}_{m-2}; (E^{m-3}_{m-2})^T_\sharp\mathbf{1}_{m-2}; \{\xi^{m-2}_{m-1}\mathbf{1}_{m-2}\}. \dots . \{\xi^{a-1}_{a}\mathbf{1}_{m-2}\}.$$

with $m-1 \le p' \le a$, $m \le p'' \le a$.

With the definitions above, $\widetilde{D}_{m-3}(k,z)$ has the following form

$$\prod_{j=1}^{\bar{j}}\int_{\Gamma_{m-2}} dz^{(m-2),j}_{m-2}\ \underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}\times \tag{III.101}$$
$$\times\underbrace{\tilde{g}^2_{m-3}}\cdot\cdot\underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}\,\underbrace{\tilde{g}^3_{m-3}\cdot\cdot\tilde{g}^3_{m-3}}(k,z,z^{(m-2)}_{m-2})$$

where:

(i) The arguments k and z are the momenta (and helicities) integration variables and the contour integration variables in expression (III.84), respectively; the argument $z^{(m-2)}_{m-2}$ stands for all the integration variables – $\{z^{(m-2),j}_{m-2} : j = 1, \cdot\cdot, \bar{j}\}$, where $\bar{j}$ is α-independent – in the contour integrals which are related to the expansion of P_{m-2} and $P^{\perp}_{m-2}$;

(ii) The first collection of operators corresponds, up to multiplicative c-numbers (see below, point iv)), to one of the terms in $\tau_\sharp^N(g_{m-2}^1)$, g_{m-2}^1 being the first operator in (III.86);

(iii) The second collection of operators consists of one operator which corresponds to one of type δg_{m-2}^2, g_{m-2}^2 being the second operator in (III.86);

(iv) The other collections of operators have an analogous origin from the combination of operators in (III.86), except the last one which is the collection of all c-numbers, the original c-number operators in (III.85) and the ones which come from the terms in $\tau_\sharp^N(g_{m-2}^1)$ and that have been pushed to the very right.

Hypothesis *H2)* together with the results in Lemmata III.1, III.2 and Eq. (I.54) imply

$$\begin{aligned}\|\widetilde{D}_{m-3}(k',z)^*\widetilde{D}_{m-3}(k,z)\| &\le \|\widetilde{D}_{m-3}(k',z)^*\|\|\widetilde{D}_{m-3}(k,z)\| \\ &\le C\cdot\mathcal{E}(D_{m-2}(k',z))\mathcal{E}(D_{m-2}(k,z)) \qquad \text{(III.102)}\end{aligned}$$

where C is an α-independent constant.

Then, starting from the vector in Eq. (III.84), under the action of the map $\tau_\sharp^N$, and after the partial re-expansion of $(\phi_{m-2})^T$ $(=\tau^N(P_{m-2})(\tilde{\phi}_{m-3})^T)$ we get an α-independent sum of expressions of the type

$$\begin{aligned}&\prod_{p'=m}^{a}\Big(\int_{\Gamma_m}\alpha^{p'-m}\prod_{j=1}^{\bar{j}(p')}dz_m^{(p'),j}\Big)\int_{\Gamma_{m-1}}\prod_{j=1}^{\bar{j}(m-1)}dz_{m-1}^{(m-1),j}\times \qquad \text{(III.103)}\\ &\times\prod_{p=m}^{a+1}\Big(\int_{\mathcal{K}_{p-1}^{p-2}}\prod_{i=1}^{\bar{i}(p-1)}dk_i(p-1)\Big)\widetilde{D}_{m-3}(k,z)(\tilde{\phi}_{m-3})^T\otimes C_a^{m-2}(k)\Omega_a^{m-2}.\end{aligned}$$

Because of Property $\mathcal{P}2$ which holds for $D_{m-2}(k,z)$, by hypothesis, and because of inequality (III.102), the norm of each Expression (III.103) is, at most, of order $\mathcal{O}(1)$ in α. In addition, because of Inequality (III.88), the sum of the new expressions of type (III.103) differs from the original one (III.84) by a remainder of order, at most, $o(\alpha^N)$ in norm.

Under the Wick-ordering of the photon creation- and annihilation operators in the shell $\mathcal{K}_{m-2}^{m-3}$, by decomposing the identity $\mathbf{1}_{m-3}$ into $P_{m-3}+P_{m-3}^\perp$ and using the change of variables $z_m^{(p'),j}=\alpha z_{m-1}^{(p'),j}$, each Expression (III.103) yields an α-independent sum of expressions of the type in Eq. (III.84) on scale $m-3$

$$\begin{aligned}&\prod_{p'=m-1}^{a}\Big(\prod_{j=1}^{\bar{j}(p')}\int_{\Gamma_{m-1}}\alpha^{p'-m-1}dz_{m-1}^{(p'),j}\Big)\prod_{j=1}^{\bar{j}(m-2)}\int_{\Gamma_{m-2}}dz_{m-2}^{(m-2),j}\times \qquad \text{(III.104)}\\ &\times\prod_{p=m-1}^{a+1}\Big(\int_{\mathcal{K}_{p-1}^{p-2}}\prod_{i=1}^{\bar{i}(p-1)}dk_i(p-1)\Big)D_{m-3}(k,z)(\phi_{m-3})^T\otimes C_a^{m-3}(k)\Omega_a^{m-3}.\end{aligned}$$

For the expressions of type (III.104), Properties $\mathcal{P}1$, $\mathcal{P}2$ are correctly fulfilled by the operators in the term $D_{m-3}(k,z)$. Property $\mathcal{P}1$ is trivial because of the structure of the map $\tau_{\sharp}^{N}$. Property $\mathcal{P}2$ is a corollary of the following lemma:

Lemma III.5. *In an expression of the type given in Eq.(III.103), the Wick-ordering of the photon creation- and annihilation operators in the shell $\mathcal{K}_{m-2}^{m-3}$ produces a finite, α-independent, number of expressions of the type in Eq. (III.104). The order in α of the norm estimate of each of these expressions is higher or equal to the order of the norm estimate of the original expression (III.103).*

Proof. In the Appendix.
We can now pick $\bar{\alpha}_{\psi_m} > 0$, N-dependent only, with $\bar{\alpha}_{\psi_m} \leq \bar{\alpha}(m-1)$ and $\bar{\alpha}_{\psi_m} \leq \bar{\alpha}_{\psi^i_{m-1}}$ for each i, where the vectors in $\{\psi^i_{m-1}\}$ are the finitely many – the number being α-independent – expressions of type (III.104) that have been obtained. In other words, we pick $\bar{\alpha}_{\psi_m} > 0$ small enough such that we can apply Hypothesis *H3)* up to $q = m-1$, and we can get the claim for ψ_m.
Concerning Hypothesis *H4)*, we observe that the structure of the remainder - of order, at most, $o(\alpha^N)$ - that has been neglected in the step from scale $m-2$ to $m-3$ is clearly an α-independent sum of expressions of the type

$$\prod_{p'=m}^{a}\Big(\prod_{j=1}^{\bar{j}(p')}\int_{\Gamma_m}\alpha^{p'-m}dz_m^{(p'),j}\Big)\prod_{j=1}^{\bar{j}(m-1)}\int_{\Gamma_{m-1}}dz_{m-1}^{(m-1),j}\times \tag{III.105}$$
$$\times\prod_{p=m}^{a+1}\Big(\int_{\sigma_{p-1}}^{\sigma_{p-2}}\prod_{i=1}^{\bar{i}(p-1)}dk_i(p-1)\Big)\widehat{D_{m-2}}(k,z)(\phi_{m-2})^T\otimes C_a^{m-2}(k)\Omega_a^{m-2}.$$

We collect the remainders that follow from Hypothesis *H4)* up to $q = m-1$ and we add the remainder (III.105) obtained above. Thus Hypothesis *H4)* holds true also for $q = m$.
S2) We start from the expression

$$(\phi_m)^T = \tau^N(P_m)(\tilde{\phi}_{m-1})^T \tag{III.106}$$

and we assume Hypothesis *H2)* up to $q = m-1$ and $0 < \alpha < \bar{\alpha}(m-1)$. Then we successively apply the following operations:

- The Wick-ordering of the photon operators in the shell $\mathcal{K}_m^{m-1}$;
- The map $\tau_{\sharp}^{N}$ applied to the operators $R^{+\perp}_{m-1}(z_m^{(m)} - S_{\bar{l}(m)}(k))$ and P_{m-1} which are produced after the Wick-ordering and under the decomposition of $\mathbf{1}_{m-1}$ into $P_{m-1} + P_{m-1}^{\perp}$.

We keep the only expressions of type (III.84) on scale $m-2$ and neglect a remainder $o(\alpha^N)$; here we exploit Hypothesis *H2)* for $q = m-1$ and Lemma III.1. Then we can pick $\bar{\alpha}(m) > 0$, and we can single out the vector $(\phi_m)^T_\sharp$ by the procedure in step *S1)*. In particular, by choosing $\bar{\alpha}(m)$ small enough, we require that, in the interval $(0, \bar{\alpha}(m))$, the so defined re-expanded vector is not zero and that the overlap with $(\phi_{m-1})^T_\sharp$ does not vanish. This is possible because $\|(\phi_{m-1})^T_\sharp - (\phi_{m-1})^T\| \leq o(\alpha^N)$ and $\|(\phi_m)^T_\sharp - (\phi_m)^T\| \leq o(\alpha^N)$.
S3) We first prove that

$$\|\vec{v}_{m-1}[(\phi_{m-1})^T_\sharp - (\phi_{m-1})^T]\| \leq o(\alpha^N) \tag{III.107}$$

by taking into account the structure of each expression (III.83), with $a = m$, contained in the remainder $(\phi_{m-1})^T_\sharp - (\phi_{m-1})^T$. In details, for each $q' \geq 2$ with $m > q'-1$, we split $\vec{v}_{m-1}$ into

$$\vec{v}_{m-1} = \vec{v}_{q'-2} + a(\vec{G}^{q'-2}_{m-1}) + a^*(\vec{G}^{q'-2}_{m-1}). \tag{III.108}$$

and we notice that the ratio $\|v^l_{q'-2} g^1_{q'-2}\| / \|g^1_{q'-2}\|$, $l = 1, 2, 3$, is uniformly bounded in q' and in α for any operator $g^1_{q'-2}$. As far as $\vec{v}_{q'-2}$ is concerned, this is enough to get the inequality in (III.107). Next we pass to control the two operators

$$[a(\vec{G}^{q'-2}_{m-1}), C^{q'-2}_a] \quad \text{and} \quad a^*(\vec{G}^{q'-2}_{m-1}) \tag{III.109}$$

when they are applied to an expression like in Eq. (III.83). For that we use the result in

Lemma III.6. *The norm of the vector obtained by applying the operator $a(\vec{G}^{q'-2}_{m-1})$ to a vector of the type in Eq.(III.83) can be estimated from above. Its order in α is higher or equal to the order of the norm estimate of the original vector.*

Proof. In the Appendix.
Moreover, from Eq. (III.107) we also get that

$$\|\vec{v}_{m-1}(\phi_m)^T_\sharp\| \leq \mathcal{O}(1) \tag{III.110}$$

because of the inequality $\|\vec{v}_{m-1}(\phi_m)^T\| \leq \mathcal{O}(1)$ which trivially follows from the definitions in Eqs. (II.11), (II.12) and (II.13). Then Hypothesis *H2)* for $q = m$ follows from the definition in Eq. (III.62) and because of Eqs. (III.107), (III.110) and due to the fact that $|((\phi_{m-1})^T_\sharp, (\phi_m)^T_\sharp)| \geq \delta > 0$, where δ is α-independent, in the interval $0 < \alpha < \bar{\alpha}(m) < \bar{\alpha}(m-1)$, which follows from step *S2)*. This can be derived from Corollary II.3.

We now check the first step of the inductive procedure.
For α in the interval $(0, \bar{\alpha}(3))$ where $\bar{\alpha}(3) > 0$:

1) The following truncated vectors can be fully re-expanded

$$
\begin{aligned}
(\phi_1)^T \;&:=\; \tilde{\phi}_0 && \text{(III.111)}\\
&\quad -\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\Gamma_1} dz_1 \frac{1}{\widetilde{H}_0^+ - z_1}[(-W_1^0)\,\frac{1}{\widetilde{H}_0^+ - z_1}]^j\tilde{\phi}_0 \\
(\phi_2)^T \;&:=\; (\tilde{\phi}_1)^T && \text{(III.112)}\\
&\quad -\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\Gamma_2} dz_2 \frac{1}{\widetilde{H}_1^+ - z_2}[(-W_2^1)\,\frac{1}{\widetilde{H}_1^+ - z_2}]^j(\tilde{\phi}_1)^T \\
(\phi_3)^T \;&:=\; (\tilde{\phi}_2)^T && \text{(III.113)}\\
&\quad -\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\Gamma_3} dz_3 \frac{1}{\widetilde{H}_2^+ - z_3}[(-W_3^2)\,\frac{1}{\widetilde{H}_2^+ - z_3}]^j(\tilde{\phi}_2)^T;
\end{aligned}
$$

2) The truncated energy shifts can be split as

$$
(E_q^{q-1})^T = (E_q^{q-1})^T_\sharp + o(\alpha^{q+\frac{1}{2}+N}) \qquad \text{(III.114)}
$$

for $1 \leq q \leq 3$;

3) It is possible to fully re-expand, up to the order $2N$ in $\alpha^{\frac{1}{2}}$, an expression of the type

$$
\begin{aligned}
\psi_1 \;&:= \Big(\prod_{j=1}^{\bar{j}(3)}\int_{\Gamma_3} dz_3^{(3),j}\prod_{j=1}^{\bar{j}(2)}\int_{\Gamma_2} dz_2^{(2),j}\Big)\times && \text{(III.115)}\\
&\times\prod_{p=3}^{4}\Big(\int_{\mathcal{K}_{p-1}^{p-2}}\prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1)\Big)D_1(k,z)(\phi_1)^T\otimes C_3^1(k)\Omega_3^1
\end{aligned}
$$

and the remainder is given as an α-independent number of expressions of the type

$$
\begin{aligned}
&\Big(\prod_{j=1}^{\bar{j}(3)}\int_{\Gamma_3} dz_3^{(3),j}\prod_{j=1}^{\bar{j}(2)}\int_{\Gamma_2} dz_2^{(2),j}\Big)\times && \text{(III.116)}\\
&\times\prod_{p=3}^{4}\Big(\int_{\mathcal{K}_{p-1}^{p-2}}\prod_{i=1}^{\bar{i}(p-1)} dk_i(p-1)\Big)\widehat{D_1}(k,z)(\phi_1)^T\otimes C_3^2(k)\Omega_3^2.
\end{aligned}
$$

The claim on the truncated energy shift $(E_1^0)^T$ and on the truncated ground-state $(\phi_1)^T$ is obvious, by definition, in a sufficiently small interval $(0,\bar{\alpha}(1))$. Then we observe that, after Wick-ordering the photon creation- and annihilation operators in the shell $\mathcal{K}_2^1$, we can pick $\bar{\alpha}(2)$ such that the vector $(\phi_2)^T$

can be fully re-expanded in the interval $0 < \alpha < \bar{\alpha}(2) \leq \bar{\alpha}(1)$ by simply applying Lemma III.1 and using $(E_1^0)^T \equiv (E_1^0)^T_\sharp$. By following the procedure in *S3)*, we can prove the claim concerning $(E_2^1)^T$; i.e., we can compute $(E_2^1)^T_\sharp$ for $0 < \alpha < \bar{\alpha}(2)$. .
Expression (III.115) can be fully re-expanded in an interval $0 < \alpha < \bar{\alpha}_{\psi_1}$, $\bar{\alpha}_{\psi_1} \leq \bar{\alpha}(1)$, by the procedure described in *S1)* since $(E_1^0)^T$ is already explicit. The remainder is trivially of the form in Eq. (III.116).
By the procedure in step *S2)*, we can set the interval $0 < \alpha < \bar{\alpha}(3)$, with $0 < \bar{\alpha}(3) \leq \bar{\alpha}(2) \leq \bar{\alpha}(1)$, where the vector $(\phi_3)^T$ can be reduced to a finite and α-independent number of expressions of the type in Eq. (III.115) and then fully re-expanded; here we use $(E_2^1)^T_\sharp$ previously determined.
Finally, following the procedure in *S3)*, $(E_3^2)^T_\sharp$ is obtained for $\alpha \in (0, \bar{\alpha}(3))$. □

In the following corollary the final result is achieved.

Corollary III.7. *For any N, for α in a fixed interval $(0, \bar{\alpha})$ where $\bar{\alpha} > 0$ is N-dependent, $E_{\mathrm{gs}} \equiv E_{\mathrm{gs}}(\alpha)$ and $\phi_{\mathrm{gs}} \equiv \phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})$ have expansions of the form*

$$E_{\mathrm{gs}}(\alpha) = E_0 + \sum_{\ell=3}^{N} \varepsilon_\ell(\alpha)\, \alpha^\ell + o(\alpha^N) \tag{III.117}$$

$$\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}}) = \phi_0 + \sum_{\ell=3}^{2N} \varphi_\ell(\alpha)\, \alpha^{\ell/2} + o(\alpha^N) \tag{III.118}$$

with

$$\lim_{\alpha \to 0} \alpha^\delta |\varepsilon_\ell(\alpha)| = 0 \qquad \forall \delta > 0 \tag{III.119}$$

$$\lim_{\alpha \to 0} \alpha^\delta \|\varphi_\ell(\alpha)\| = 0 \qquad \forall \delta > 0 \tag{III.120}$$

for arbitrary $N = 1, 2, 3, \ldots$. In expansions (III.117) and (III.118), $\varepsilon_\ell(\alpha)$ and $\varphi_\ell(\alpha)$ are computable in terms of finitely many convergent integrals, for any $3 \leq \ell < \infty$.

Proof. For any N, we can select the value $\bar{\alpha}(M = 2N-1)$ such that expansions of the type in Eqs. (III.117), (III.118) hold for $\alpha \in (0, \bar{\alpha}(M = 2N-1))$. In fact, because of Theorem III.4 and because of the results in Section II, the quantities $\frac{(\phi_{M=2N-1})^T_\sharp(\alpha^{\frac{1}{2}})}{\|(\phi_{M=2N-1})^T_\sharp(\alpha^{\frac{1}{2}})\|}$ and $E_0 - \sum_{m=1}^{m=M} (E_m^{m-1})^T_\sharp(\alpha)$ are asymptotic, up to order $2N$ in $\alpha^{\frac{1}{2}}$, as $\alpha \to 0$, to $\frac{\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})}{\|\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})\|}$ and $E_{\mathrm{gs}}(\alpha)$, respectively. The coefficients $\varepsilon_\ell(\alpha)$ and the vectors $\varphi_\ell(\alpha)$ are expressed in terms of bare quantities according to Definition III.3. □

Remark
In order to find approximate expressions for α in an N-independent interval of the type $(0, \bar{\alpha})$, up to corrections of the type $const \cdot \alpha^{N+\epsilon}$, $\epsilon > 0$, we can use regular perturbation theory. The argument is the following. From Proposition I.I, we know that, for any value of α fulfilling the assumptions,

$$\left\| \frac{\phi_{2N-1}(\alpha^{\frac{1}{2}})}{\|\phi_{2N-1}(\alpha^{\frac{1}{2}})\|} - \frac{\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})}{\|\phi_{\mathrm{gs}}(\alpha^{\frac{1}{2}})\|} \right\| \le C'' \alpha^{N+\frac{1}{2}} \tag{III.121}$$

where C'' is α- and N-independent. Now consider the Hamiltonian with infrared cut-off $\sigma = \alpha^{2N-1}$ in the interaction term, with *radiative* coupling constant β and acting on the space $\mathcal{H}_{el} \otimes \mathcal{F}_{2N-1}$. It is defined as follows

$$\begin{aligned} &\hat{H}_{2N-1}(\beta, \alpha) := \qquad\qquad\qquad\qquad (\mathrm{III.122}) \\ &= \ (-i\vec{\nabla}_x \, + \, a^*(\vec{G}_{2N-1}(\beta)) \, + \, a(\vec{G}_{2N-1}(\beta)))^2 \, - \, V(x) \, + \, \check{H}_{2N-1} \, , \end{aligned}$$

where

$$\vec{G}(k)(\beta) \ := \ \frac{\beta^3}{(2\pi)^{3/2}} \, \frac{\Lambda(k)}{\sqrt{2\,|k|}} \, e^{-i\alpha\vec{k}\cdot\vec{x}} \, \vec{\varepsilon}(k) \tag{III.124}$$

From the analysis in [1] and underlying the results in Proposition I.1, for β in an interval $(-\bar{\alpha}^{\frac{1}{2}}, \bar{\alpha}^{\frac{1}{2}})$ where $\bar{\alpha}$ is a positive N-independent number, we can construct the vector $\hat{\phi}_{2N-1}(\beta, \alpha)$, that is groundstate of the Hamiltonian $\hat{H}_{2N-1}(\beta, \alpha)$, by definition. Furthermore, we know that $\mathrm{gap}[\hat{H}_{2N-1}(\beta, \alpha)] \ge c \cdot \kappa \alpha^{2N-1}$ where $0 < c < 1$.
Consider $\bar{\alpha} \ge \bar{\alpha}(M)$. We are ultimately interested in $\hat{\phi}_{2N-1}(\beta, \alpha)|_{\beta=\alpha^{\frac{1}{2}}}$ where $\alpha \ge \bar{\alpha}(M = 2N-1)$; then we can apply regular perturbation theory and we get that, in particular, the vector $\hat{\phi}_{2N-1}(\beta, \alpha)$ is $\mathcal{C}^\infty$ in β for $\beta \in (-\bar{\alpha}^{\frac{1}{2}}, \bar{\alpha}^{\frac{1}{2}})$. Therefore we can use Taylor formula and we can write for $\bar{\alpha} > \alpha \ge \bar{\alpha}(M = 2N-1)$:

$$\phi_{2N-1}(\alpha^{\frac{1}{2}}) \equiv \hat{\phi}_{2N-1}(\alpha^{\frac{1}{2}}, \alpha) = \tag{III.125}$$

$$= \ \hat{\phi}_{2N-1}(0, \alpha) + \sum_{j=1}^{2N} \frac{\alpha^{\frac{j}{2}}}{j!} \frac{d^j \hat{\phi}_{2N-1}(\beta, \alpha)}{d\beta^j}\Big|_{\beta=0} + \tag{III.126}$$

$$+ \frac{\alpha^{\frac{2N+1}{2}}}{(2N+1)!} \frac{d^{2N+1} \hat{\phi}_{2N-1}(\beta, \alpha)}{d\beta^{2N+1}}\Big|_{\beta=\alpha'^{\frac{1}{2}}} \tag{III.127}$$

where $\alpha'^{\frac{1}{2}}$ is a point in $(0, \alpha^{\frac{1}{2}})$.
A similar expansion can be provided for $E_{2N-1}(\alpha)$. Finally, in Expansions (III.117), (III.118), we can fix the constants in the error terms $o(\alpha^N)$ uniformly in $(0, \bar{\alpha})$.

A Proofs of Lemma III.5 and Lemma III.6

Proof of Lemma III.5.
We must study the norm of each distinct vector in the sum corresponding to Wick-ordering in the shell $\mathcal{K}_{m-2}^{m-3}$ of

$$\widetilde{D}_{m-3}(k,z)(\tilde{\phi}_{m-3})^T, \tag{A.1}$$

where $\widetilde{D}_{m-3}(k,z)$ is given in Eq. (III.101).
For each of these vectors, we eventually prove that the order in α of its norm is, at most, equal to the order of the norm estimate of

$$\|\widetilde{D}_{m-3}(k,z)\| \leq \mathcal{E}(D_{m-2}(k,z)P_{m-2}^T) \tag{A.2}$$

that we get in Eq. (III.102) as a byproduct of the previous control of the truncation map (see Lemmata III.1, III.2).
Without loss of generality, we study the term $\widetilde{D}_{m-3}(k,z)$ considered in Theorem III.4

$$\prod_{j=1}^{\bar{j}} \int_{\Gamma_{m-2}} dz_{m-2}^{(m-2),j} \underbrace{\tilde{g}_{m-3}^1 \tilde{g}_{m-3}^2 \tilde{g}_{m-3}^1 \cdot \cdot \tilde{g}_{m-3}^1} \times \tag{A.3}$$
$$\times \underbrace{\tilde{g}_{m-3}^2} \cdot \cdot \underbrace{\tilde{g}_{m-3}^1 \tilde{g}_{m-3}^2 \tilde{g}_{m-3}^1 \cdot \cdot \tilde{g}_{m-3}^1} \underbrace{\tilde{g}_{m-3}^3 \cdot \cdot \tilde{g}_{m-3}^3}(k, z, z_{m-2}^{(m-2)}).$$

As explained in the theorem, the operators of type $\tilde{g}_{m-3}^2$ in the first collection are in fact of type $\mathcal{W}_{m-2}^{m-3}$ (see Eq. (III.97)); the operator $\tilde{g}_{m-3}^2$ in the second collection is of type δg_{m-2}^2, since it derives from a vertex operator in the group $\mathcal{G}_{m-2}^2$. In the same way it is easy to specify the operators in the other collections.

For deriving the estimate in Eq. (A.2), among all the possible combinations of operators $\mathcal{W}_{m-2}^{m-3}$ and operators δg_{m-2}^2 which may be present in (A.3), we select the leading order ones; i.e, in applying the results of Lemmata III.1, III.2, we implicitly assume the operators δg_{m-2}^2 be different from $\Delta\Upsilon_{m-3}^{\pm}(k_i(q'))$ and the operators $\mathcal{W}_{m-2}^{m-3}$ be either equal to $\vec{v}_{m-3} \cdot a(\vec{G}_{m-2}^{m-3})$ or to $\vec{v}_{m-3} \cdot a^*(\vec{G}_{m-2}^{m-3})$.
We also note that, for determining the estimates stated in Lemmata III.1, III.2 and Proposition I.1 and underlying Inequality (A.2), we can as well use the following power counting rules.
Power-counting rules

1) For each $\mathcal{W}_{m-2}^{m-3}$ a factor $\alpha^2 \cdot \alpha^{m-3}$ is gained.

2) For each operator of the type

$$\widetilde{R}^{+\perp}_{m-3}(z^{m-2}_{m-2}) \tag{A.4}$$

$$\widetilde{R}^{+\perp}_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k)) \tag{A.5}$$

a factor α^{m-2} is lost.

3) The operators δg^2_{m-2} and $\widetilde{P}_{m-3}$ are neutral, i.e., no power of α is gained or lost.

4) The c-number operators give either positive or negative powers of α.

5) Each contour integration in Eq. (A.3) yields a power α^{m-2}.

Now we return to the vectors in Eq. (A.1). Let ψ be any vector obtained from the Wick-ordering in Eq. (A.1). The norm of ψ is bounded by

$$\begin{aligned}
&\prod_{j=1}^{\bar{j}} |\Gamma^j_{m-2}| \sup_{z^{(m-2)}_{m-2},z,k} \Big[|[\tilde{g}^3_{m-3}](k,z,z^{(m-2)}_{m-2})| \times \\
\times\ &\Big(\underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}\underbrace{\tilde{g}^2_{m-3}}\cdot\cdot\underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}(\tilde{\phi}_{m-3})^T, \\
&,\underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}\underbrace{\tilde{g}^2_{m-3}}\cdot\cdot\underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}(\tilde{\phi}_{m-3})^T\Big)^{\frac{1}{2}}_{\pi_\psi}(k,z,z^{(m-2)}_{m-2})\Big]
\end{aligned} \tag{A.6}$$

where the symbol of scalar product labeled by π_ψ means the contribution to the square norm

$$\| \underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}\underbrace{\tilde{g}^2_{m-3}}\cdot\cdot\underbrace{\tilde{g}^1_{m-3}\tilde{g}^2_{m-3}\tilde{g}^1_{m-3}\cdot\cdot\tilde{g}^1_{m-3}}(\tilde{\phi}_{m-3})^T\|^2 \tag{A.7}$$

corresponding to the specific pairings-combination π_ψ - of the photon creation- and annihilation operators contained in the terms $\mathcal{W}^{m-3}_{m-2}$ - associated to the square norm of the given vector ψ; $[\tilde{g}^3_{m-3}](k,z,z^{(m-2)}_{m-2})$ is an abbreviated notation for $\underbrace{\tilde{g}^3_{m-3}\cdot\cdot\tilde{g}^3_{m-3}}(k,z,z^{(m-2)}_{m-2})$.

We can provide an upper bound to the quantity in Eq. (A.6) as follows.
For any distinct couple of creation- and annihilation operator that will be contracted in the pairings combination π_ψ, there is a distinct resolvent, between the two photon operators, of the two possible types

$$\widetilde{R}^{+,\perp}_{m-3}(z^{(m-2)}_{m-2}) \quad , \widetilde{R}^{+,\perp}_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k)). \tag{A.8}$$

Because of the pull-through formula, after a contraction, the resolvents in Eq. (A.8) assume the form

$$R^{+}_{m-3}(z^{(m')}_{m-2} - |q|) \quad , R^{+}_{m-3}(z^{(m')}_{m-1} - S_{l(m-1)}(k) - |q|), \tag{A.9}$$

with $q \in \mathcal{K}_{m-2}^{m-3}$. In the case of a contraction involving the operators $\vec{v}_{m-3} \cdot a(\check{G}_{m-2}^{m-3})$ and $\vec{v}_{m-3} \cdot a^*(\vec{G}_{m-2}^{m-3})$, an elementary calculation yields

$$\begin{aligned} &\Big\| \alpha^3 \int_{\mathcal{K}_{m-2}} dq \frac{\vec{v}_{m-3} \cdot \vec{\varepsilon}(q)^* e^{i\alpha\vec{q}) \cdot \vec{x}}}{\sqrt{|q|}} \frac{1}{H_{m-3}^+ - z_{m-2}^{(m')} + |q|} \frac{\vec{v}_{m-3} \cdot \vec{\varepsilon}(q) e^{-i\alpha\vec{q} \cdot \vec{x}}}{\sqrt{|q|}} \Big\| \\ \leq \; & C\alpha^3 \int_{\mathcal{K}_{m-2}} dq \frac{1}{|q|} \frac{1}{|-z_{m-2}^{(m')} + |q||} \leq C'\alpha^m. \end{aligned} \tag{A.10}$$

where C, C' are α-independent constants. An analogous result holds for the second resolvent in Eq. (A.9). From the calculation above, we recover the same estimate derived by the *power counting rules*: In fact

$$\alpha^m = (\alpha^2 \cdot \alpha^{m-3})^2 (\alpha^{-m+2}), \tag{A.11}$$

and the right hand side in (A.11) corresponds exactly to the factors carried by two operator $\mathcal{W}_{m-2}^{m-3}$ and a resolvent in (A.3), according to Rules 1) and 2), respectively.

Therefore, in the integral in Eq. (A.6), we can associate to each distinct contraction a distinct resolvent in the middle - this is because of the choice of a leading order combination of operators $\mathcal{W}_{m-2}^{m-3}$ and δg_{m-2}^2 in (A.3). Then, we use estimates analogous to (A.10) for the contractions; we estimate in the usual way all the projections, all the remaining resolvents and the velocity operators next to them (see Eq. (III.79). In the end, because of the result in Eq. (A.10), we get the same order in α obtained by the power counting rules for the estimate of the entire expression (A.3).

After the Wick-ordering, we decompose the identity operator $\mathbf{1}_{m-3}$ into $P_{m-3} + P_{m-3}^{\perp}$ in front of all the resolvents. Then, according to our definitions, the vector ψ corresponds to the sum of a finite and α-independent number of expressions of the type

$$\begin{aligned} &\prod_{j=1}^{\bar{j}} \int_{\Gamma_{m-2}} dz_{m-2}^{(m-2),j} \int_{\mathcal{K}_{m-2}} \prod_{i=1}^{\bar{i}(m-2)} dk_i(m-2)\, \underbrace{g_{m-3}^1 g_{m-3}^2 g_{m-3}^1 \cdots g_{m-3}^1} \times \\ &\times \underbrace{g_{m-3}^2} \cdots \underbrace{g_{m-3}^1 g_{m-3}^2 g_{m-3}^1 \cdot \cdot g_{m-3}^1} (k, k(m-2), z, z_{m-2}^{(m-2)}) \times \\ &\times (\phi_{m-3})^T \otimes C_{m-2}^{m-3}(k(m-2)) \Omega_{m-3}^{m-2} \end{aligned} \tag{A.12}$$

where we distinguish the new momenta $k(m-2)$ in our present notation. Furthermore, notice that the bound derived in Eq. (A.10) holds separately when the resolvent is decomposed into the two orthogonal pieces

$$\frac{1}{-z_{m-2}^{(m')} + |q|} P_{m-3} \quad , \quad \frac{1}{H_{m-3}^+ - z_{m-2}^{(m')} + |q|} P_{m-3}^{\perp}. \tag{A.13}$$

Then, because of this last remark concerning the operators in Eq. (A.13), each Expression (A.12) has, at most, the same order in α of the quantity in (A.6) and Property $\mathcal{P}2$ is therefore fulfilled for the expressions in Eq. (III.104).
If we consider "subleading combinations" of the operators $\mathcal{W}_{m-2}^{m-3}$ and δg_{m-2}^{2} in (A.3), similar arguments apply. All the other choices for δg_{m-2}^{2} and $\mathcal{W}_{m-2}^{m-3}$ (different from the ones underlying the operator norm estimate of (A.2)) turn out to be subleading because the operators $a^{\natural}((G_{m-2}^{m-3})^{l})$, $l = 1, 2, 3$, carry further positive powers of α, both directly and as output of a contraction. $\square$

Proof of Lemma III.6.
In order to prove the desired result, it is sufficient to observe that, for any n-photon vector and any $h \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, we have

$$
\begin{aligned}
\|a(h)\phi\| &= \| \int dq \overline{h(q)} a(q) dq \phi \| && \text{(A.14)} \\
&\leq \int |\overline{h(q)}| \|a(q)\phi\| dq && \text{(A.15)} \\
&\leq (\int |\overline{h(q)}|^2 |dq)^{\frac{1}{2}} \cdot (\int \|a(q)\phi\|^2 dq)^{\frac{1}{2}} && \text{(A.16)} \\
&= \|h\|_{L^2} \cdot n^{\frac{1}{2}} \|\phi\| && \text{(A.17)}
\end{aligned}
$$

$\square$

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