Nodal sets, multiplicity and superconductivity in non simply connected domains.[1] [2]

Bernard Helffer[1], Maria Hoffmann-Ostenhof[2], Thomas Hoffmann-Ostenhof[3,4], Mark Owen[5]

[1]Département de Mathématiques: UMR CNRS 8628, Université Paris-Sud
[2]Institut für Mathematik, Unversität Wien
[3]Institut für Theoritische Chemie, Universität Wien
[4]International Erwin Schrödinger Institut for Mathematical Physics
[5]Institute of Statistics, Probability Theory and Actuarial Mathematics, Technische Universität Wien

September 29, 1999

Abstract

This is a survey on [HHOO] and further developments of the theory [He4]. We explain in detail the origin of the problem in superconductivity as first presented in [BeRu], recall the results of [HHOO] and explain the extension to the Dirichlet case. As illustration of the theory, we detail some semi-classical aspects and give examples where our estimates are sharp.

1 Zero sets in superconductivity

Following the paper by Berger-Rubinstein [BeRu], who refer to the to the Little Parks experiment [LiPa] we would like to understand the minima (or more generally the extrema) of the following Ginzburg-Landau functional which plays an important role in superconductivity. We are working in a bounded, connected, regular, open set $\Omega \subset \mathbb{R}^2$ and, for any $\lambda > 0$ and $\kappa > 0$, we consider $G_{\lambda,\kappa}$ defined, for $u \in H^1(\Omega;\mathbb{C})$ and $A \in H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ such that $\operatorname{curl} A \in L^2$, by

$$\begin{aligned} G_{\lambda,\kappa}(u,A) \quad &= \int_\Omega (\lambda(-|u|^2 + \tfrac{1}{2}|u|^4) + |(\nabla - iA)u|^2) dx_1 \cdot dx_2 \\ &+\kappa^2\lambda^{-1} \int_{\mathbb{R}^2} |\operatorname{curl} A - H_e|^2 dx_1 \cdot dx_2 \ . \end{aligned} \tag{1.1}$$

We note here that u (which is called in this theory the order parameter) is defined only in Ω, but that the magnetic potential A is defined in $\mathbb{R}^2$. Here H_e is a C_0^∞ function on $\mathbb{R}^2$ or more generally some function in $L^2(\mathbb{R}^2)$. Physically H_e represents the exterior magnetic field.

We shall sometimes use the identification between vector fields A and 1-forms ω_A.

[1]Funded by the European Union TMR grant FMRX-CT 96-0001
[2]Supported by the Austrian Science Foundation grantnumber P-12864-MAT

When analyzing the extrema of the GL-functional, it is natural to first analyze the corresponding Euler-Lagrange equations. This is a system of two equations (with a boundary equation) :

$$\begin{array}{ll} (GL)_1 & -(\nabla - iA)^2 u + \lambda u(|u|^2 - 1) = 0 \ , \ \forall x \in \Omega \ , \\ (GL)_2 & \operatorname{curl}^*(\operatorname{curl} A - H_e) = \lambda \kappa^{-2} \Im[\bar{u} \cdot (\nabla - iA)u)] \cdot 1_\Omega \\ (GL)_3 & (\nabla - iA)u \cdot \nu = 0 \ , \ \forall x \in \partial\Omega \ . \end{array} \tag{1.2}$$

Here ν is a unit exterior normal to $\partial\Omega$ and 1_Ω is the chracteristic function of Ω. The operator curl^* is defined by $\operatorname{curl}^* f := (\partial_{x_2} f, -\partial_{x_1} f)$.
Moreover, without loss of generality in our problem, we may add the condition

$$(GL)_4 \qquad \operatorname{div} A = 0 \ \text{ in } \Omega. \tag{1.3}$$

One can also assume if necessary that the vector potential satisfies

$$A \cdot \vec{\nu} = 0 \ , \tag{1.4}$$

on the boundary of Ω.
Let A_e be a solution of

$$\begin{array}{l} \operatorname{curl} A_e = H_e \\ \operatorname{div} A_e = 0 \ . \end{array} \tag{1.5}$$

Such a vector potential always exists in $H^1_{loc}(I\!R^2)$.
The first remark is that the corresponding pair $(0, A_e)$ is a solution of the Ginzburg-Landau system. This solution is called the normal solution.

Remark 1.1 .
Note also that the normalization of the functional leads to the property that

$$G_{\lambda,\kappa}(0, A_e) = 0 \ . \tag{1.6}$$

Following for example [DGP], one can show that, if Ω is bounded, then the functional $G_{\lambda,\kappa}$ admits a global minimizer which is a solution of the equation. If (u, A) is such a minimizer (or more generally a local minimizer) and if it is different from the normal solution, it is a natural question to ask for the structure of the nodal sets of u. This problem was analyzed in [ElMaQi] in the case of a simply connected domain Ω and with different boundary conditions. The authors show that the solutions have as zero sets points or lines which necessarily end at the boundary.
Inspired by [BeRu], our aim is to analyze the non simply-connected situation. In the next section we shall show how to reduce this non-linear problem to a linear problem. Instead of following closely [BeRu], we will show how to implement some ideas of [HHOO] for an alternative presentation of their results and extend it to the multiple holes case. We refer to [DuHe] (or [He4]) for a more complete analysis of the bifurcation (stability) (in connection with [Du]).

2 Bifurcation from normal solutions. Preliminaries.

Starting from one normal solution, a natural way for finding new solutions is to increasing λ from 0 and to see if one can bifurcate for a specific value of λ. One can show by a priori estimates that it is impossible for small λ, if the lowest eigenvalue $\lambda^{(1)}$ of the Neumann realization of $-\Delta_{A_e}$ in Ω is strictly positive :

$$\lambda^{(1)} > 0 \; . \tag{2.1}$$

Here, we denote by $-\Delta_A$, for a magnetic potential A, the magnetic Laplacian :

$$-\Delta_{A_e} = \sum_{j=1}^{2} (\frac{1}{i}\partial_{x_j} - A_j(x))^2 \; . \tag{2.2}$$

It is rather standard that a necessary condition for having a bifurcation is actually that λ becomes an eigenvalue of $-\Delta_{A_e}$ ([BeRu], [DuHe]). We shall consider only what is going on around $\lambda^{(1)}$. Note here that there is an intrinsic degeneracy to the problem related to the existence of an S^1 action. We observe indeed the property :

$$\text{If } (u, A) \text{ is a solution of the GL-system, then } (\exp i\theta\, u, A) \text{ is a solution,} \tag{2.3}$$

for any $\theta \in I\!R$.
In order to go further, we add the assumption

$$\lambda^{(1)} \text{ is a simple eigenvalue.} \tag{2.4}$$

We will come back later to the properties of the spectrum of $-\Delta_A$ and in particular to the strict positivity (2.1) and to the multiplicity of the lowest eigenvalue. Let only say here that (2.4) is rather generic.
Now, one can try to apply the general bifurcation theory due to Crandall-Rabinovitz [CrRa]. Note that, although, the eigenvalue is assumed to be simple, it is not exactly a simple eigenvalue in the sense of Crandall-Rabinowitz who are working with real Banach spaces. Actually, this is only simple modulo this S^1-action which was observed in (2.3). In this context, the theory developped for example [GoSc], (Chapter VII, §3 and Chapter VIII (Proposition 2.2)) is relevant, but special cases involving Schrödinger operators with magnetic field are treated in [Od], [BaPhTa], [Ta] and [Du]. There are actually various ways to remove the degeneracy and we shall present one which seems quite adapted to our peculiar situation by introducing a notion of real solution in Section 5.
All the considered operators are (relatively to the wave function or order parameter) suitable realizations of operators of the type

$$u \mapsto -\Delta_A u - \lambda f(|u|^2)u \; ,$$

with $f(0) = 1$. For our model (see $(GL)_1$), we have indeed $f(t) = 1 - t$.
In the case when $||u||_{C^0(\bar{\Omega})}$ is sufficiently small (which will be for example a

consequence of $||u||_{H^2(\bar{\Omega})}$ small), we observe that if u is a non zero solution of

$$-\Delta_A u - \lambda f(|u|^2)u = 0 \ , \tag{2.5}$$

then

$$v := \sqrt{f(|u|^2)}u \tag{2.6}$$

is a non zero solution of the linear equation :

$$-(\sqrt{f(|u|^2)})^{-\frac{1}{2}}\Delta_A(\sqrt{f(|u|^2)})^{-\frac{1}{2}}v - \lambda v = 0 \ , \tag{2.7}$$

and has same nodal set as u. Moreover u and v satisfy the same boundary condition.
The last remark is here that, if $\lambda - \lambda^{(1)}$ and $||u||_{C^0(\bar{\Omega})}$ are small enough and (2.4) is satisfied, then λ is the lowest eigenvalue (with multiplicity 1) of the Neumann-realization of the linear operator $\mathcal{L}_{u,A} := -\sqrt{f(|u|^2)}^{-\frac{1}{2}}\Delta_A\sqrt{f(|u|^2)}^{-\frac{1}{2}}$ and we have to analyze the properties of the groundstate v of this operator. Note that the assumptions that $\lambda - \lambda^{(1)}$ and $||u||_{C^0(\bar{\Omega})}$ are small enough will be satisfied when we consider bifurcation starting from the normal solution. The conclusion is that we can apply the linear theory developed for the Schrödinger operator with magnetic field Δ_A modulo a small (but easy) extension to the operators $\mathcal{L}_{u,A}$. So before to go further it seems useful to recall some basics on the Schrödinger operator with magnetic field.

3 The linear Schrödinger operator with magnetic field.

Our starting point is the Laplace operator with magnetic field in a C^∞, relatively compact domain $\Omega \subset I\!R^2$. Here we discuss also the more general Schrödinger operators of the form $-\Delta_A + V$ where V is a C^∞ potential. This was not needed for the application to superconductivity but we want to emphasize how general is our approach. In particular there would be no use of analyticity properties like for example in [ElMaQi]. More generaly, we could also look at operators in the form $-g(x)^{-1}\Delta_A g(x)^{-1} + V(x)$ in order to treat the examples appearing previously but we do not do this here for simplicity. So we are considering the operator

$$u \mapsto P_{A,V}u := -(\nabla - iA)^2 u + Vu \ . \tag{3.1}$$

We are interested in the Neumann-realization $P^N_{A,V}$ or in the Dirichlet realization $P^D_{A,V}$. The basic properties of these operators are discussed in [AHS] (See also [He2]). It is well known that if we denote by $\lambda^{(1),N}(A)$ (resp. $\lambda^{(1),D}(A)$) the lowest eigenvalue of $P^N_{A,V}$ (resp. $P^D_{A,V}$) , then we always have the property

$$\begin{aligned} \lambda^{(1),N}(A) &\geq \lambda^{(1),N}(0) \ ; \\ \lambda^{(1),D}(A) &\geq \lambda^{(1),D}(0) \ . \end{aligned} \tag{3.2}$$

The natural question is then to decide when we have equality. The first necessary condition is

$$B = \operatorname{curl} A \equiv 0\ . \tag{3.3}$$

This condition is also sufficient in the case when Ω is simply connected. When Ω is not simply connected, we have also to consider, for any closed path γ in Ω, the quantity $\Phi(\gamma) := \frac{1}{2\pi}\int_\gamma (A_1 dx_1 + A_2 dx_2)$ which will be called the (normalized) circulation of A along γ. Of course, when $B = 0$ and Ω is simply connected, this quantity is equal to 0 but it is no more the case when we have holes. In the general context, it can be proved that $\lambda^{(1),N}(A) = \lambda^{(1),N}(0)$ if and only if $B = 0$ and, for any closed path γ in Ω

$$\Phi(\gamma) \in \mathbb{Z}\ . \tag{3.4}$$

A similar result holds for the Dirichlet case. This was proved in the case of the Dirichlet realization (See Lavine, O'Caroll [LaO], [He1], [He2]) and a similar result is also true in the case of Neumann [HHOO]. We observe that (3.3) and (3.4) are just the conditions under which there exists a gauge transform such that the operator is unitary equivalent to the case without magnetic potential. In this last case, the lowest eigenvalue is simple and the corresponding ground-state has no zero.

We will discuss later the natural question of determining for which potentials A, such that $\operatorname{curl} A = 0$ in Ω, the eigenvalues $\lambda^{(1),N}(A)$ or $\lambda^{(1),D}(A)$ are maximal. Note also that, in the case when $V \equiv 0$ and for the Neumann realization, the condition (2.1) is a consequence of the non validity of (3.3) or (3.4). We have indeed $\lambda^{(1),N}(0) = 0$ and strict inequality in the first line of (3.2).

We close this section by discussing briefly the question of multiplicity. It has been observed [AHS] that the property that the lowest eigenvalue is simple when $A = 0$ (which is a consequence of the Krein-Rutman theorem, cf for example [ReSi]) is no more true in general. As we shall discuss later, there is a strong connection between the nodal sets and the multiplicity.

4 The operator K

We now consider the case with zero magnetic field (Assumption (3.3)). We consider the non simply connected situation. The connected components of the complementary set of Ω in $\mathbb{R}^2$ is the union of k holes O_i and of an unbounded connected component. Let us denote by σ_i a closed path[3] which parametrizes the boundary Σ_i of the i-th hole and turns once in an anti-clockwise direction. Let us consider the circulations

$$\Phi_i = \frac{1}{2\pi}\oint_{\sigma_i} A \cdot d\mathbf{x}, \tag{4.1}$$

[3] A piecewise smooth mapping $\gamma : [0,1] \to X$ is called a path in X. The point $\gamma(0)$ is called the initial point and $\gamma(1)$ is called the final point. The image $\Gamma = \gamma([0,1])$ of the path is called a curve.

of A round the i-th hole ($i = 1, \dots, k$).
Let us assume that

$$2\Phi_i = 2\ell + 1 \ , \quad \text{with } \ell \in \mathbb{Z} \ . \tag{4.2}$$

This case is quite specific in the sense that, in this case,

$$P_{A,V} \text{ and } P_{-A,V} \text{ are gauge equivalent.} \tag{4.3}$$

One can indeed find a multivalued C^∞ function[4] ϕ in Ω such that $\exp i\phi$ is univalued and such that

$$d\phi = -2(A_1 \ dx_1 + A_2 \ dx_2) \ . \tag{4.4}$$

We then have

$$\exp -i\phi \ P_{A,V} \ \exp i\phi = P_{-A,V} \ . \tag{4.5}$$

We also observe that, if Γ is the conjugation operator

$$u \mapsto \Gamma u = \bar{u} \ , \tag{4.6}$$

then

$$\Gamma \Delta_A = \Delta_{-A} \Gamma \ , \tag{4.7}$$

and consequently

$$\Gamma P_{A,V} = P_{-A,V} \Gamma \ , \tag{4.8}$$

Combining (4.5) and (4.8), we obtain, for the operator

$$K := \exp -i\phi \ \Gamma \ , \tag{4.9}$$

which satisfies

$$K^2 = Id \ , \tag{4.10}$$

the following commutation relation

$$K \ \Delta_A = \Delta_A \ K \ . \tag{4.11}$$

Let us also observe that the Neumann conditions and the Dirichlet conditions are respected by K.
As a corollary, we get

Lemma 4.1 .
If u is an eigenvector of Δ_A^N (resp. Δ_A^D), then Ku has the same property.

Let us now explain the possible role of the operator K. Similarly to what is done with the Laplacian and the operator Γ, we observe that one can find an orthonormal basis of "real" eigenvectors u_j, i. e. they satisfy the condition

$$Ku_j = u_j \ . \tag{4.12}$$

In particular, if u is a simple eigenvector, then we can multiply it by a constant so that it becomes a "real" eigenvector.

[4] Along a path γ of index 1 around one of the holes contained in Ω, the variation of ϕ is $2n\pi$ ($n \in \mathbb{Z}$).

Remark 4.2 .
Here we would like to say that all the properties of $P_{A,V}$ described in this section are also true for the operators $-g(x)^{-1}\Delta_A g(x)^{-1} + V(x)$. Note also that if $Ku = u$, then the function v defined by (2.6) will also satisfy the same condition.

5 Bifurcation for special spaces.

Let us come back to the functional $G_{\lambda,\kappa}$ introduced in (1.1). Following [BeRu], we assume that

$$H_e \equiv 0 \text{ in } \bar{\Omega} \ , \tag{5.1}$$

and that (4.2) is satisfied for $B = H_e$. We would like to show how to take advantage of the existence of the operator K introduced in Section 4. We look for solutions of the GL equation in the form (u, A_e) with $Ku = u$. Let us observe that

$$L^2_K(\Omega; \mathbb{C}) := \{u \in L^2(\Omega; \mathbb{C}) \mid Ku = u\} \ , \tag{5.2}$$

is a real Hilbert subspace of $L^2(\Omega; \mathbb{C})$.
We denote by H^m_K the corresponding Sobolev spaces :

$$H^m_K(\Omega; \mathbb{C}) = H^m(\Omega; \mathbb{C}) \cap L^2_K \ . \tag{5.3}$$

We now observe the

Lemma 5.1 .
If $u \in H^1_K$, then the current $\Im(\bar{u} \cdot (\nabla - iA_e)u) = 0$ almost everywhere.

Proof of Lemma 5.1:
Let us consider a point where $u \neq 0$. Then we have $u = \rho \exp i\theta$ with $2\theta = \phi$ modulo $2\pi\mathbb{Z}$. Remembering that $A_e = \frac{1}{2}\nabla\phi$, it is easy to get the property. ∎
Once this lemma is proved, one immediately sees that (u, A_e) (with $Ku = u$) is a solution of the GL system if and only if $u \in H^1_K$ and

$$\begin{aligned} &-\Delta_{A_e} u - \lambda u(1 - |u|^2) = 0 \ , \\ &(\nabla - A_e)u \cdot \nu = 0 \ , \quad \text{on } \partial\Omega \ . \end{aligned} \tag{5.4}$$

We shall call this new system the reduced GL-equation.
But now we can apply Theorem 2.4 by Crandall-Rabinowitz [CrRa]. By assumption (2.4), the kernel of $(-\Delta_{A_e} - \lambda^{(1)})$ is now a one-dimensional real subspace in L^2_K. Let us denote by u_1 a normalized "real" eigenvector. Note that u_1 is unique up to the multiplication by ± 1 and that $u_1 \in H^m_K$, $\forall m \in \mathbb{N}$. Therefore, we have (see [BeRu] for a weaker form in the one hole case) the

Proposition 5.2 .
Under assumptions (2.4), (3.3) and (4.2), there exists a bifurcating family of solutions $(u(\cdot;\alpha), \lambda(\alpha))$ in $H^1_K \times \mathbb{R}^+$ with $\alpha \in]-\epsilon_0, +\epsilon_0[$, for the reduced GL-equation such that

$$\begin{aligned} &u(\alpha) = \alpha u_1 + \alpha^3 u_2(\alpha) \ , \\ &< u_1, u_2(\alpha) >_{L^2} = 0 \ , \\ &||u_2(\alpha)||_{H^2(\Omega)} = \mathcal{O}(1) \ ; \end{aligned} \tag{5.5}$$

$$\lambda(\alpha) = \lambda^{(1)} + c\alpha^2 + \mathcal{O}(\alpha^4)\ , \tag{5.6}$$

with

$$c = \lambda^{(1)} \cdot \int_\Omega |u_1|^4\, dx\ . \tag{5.7}$$

Moreover

$$u(-\alpha) = -u(\alpha)\ ,\ \lambda(-\alpha) = \lambda(\alpha)\ . \tag{5.8}$$

Remark 5.3 .
Note that the property (5.8) is what remains of the S^1-invariance when one considers only "real" solutions.

Let us give here the formal computations of the main terms. If we denote by L_0 the operator $L_0 := -\Delta_{A_e} - \lambda^{(1)}$. Writing $u_2(\alpha) = u_{20} + \mathcal{O}(\alpha)$, we get modulo $\mathcal{O}(\alpha^4)$.

$$(L_0 - c\alpha^2)(\alpha u_1 + \alpha^3 u_{20}) + \lambda^{(1)}\alpha^3 u_1\, |u_1|^2 = \mathcal{O}(\alpha^4)\ .$$

Projecting on u_1, we get (5.7). Projecting on the orthogonal space u_1^{perp} and denoting by M_0 the operator equal to the inverse of L_0 on this subspace and to 0 on Ker L_0, we get

$$u_{20} := -\lambda^{(1)} M_0(u_1|u_1|^2) = -\lambda^{(1)} M_0(u_1|u_1|^2 - cu_1)\ . \tag{5.9}$$

The second result is that, once we have observed that $u(\alpha)$ is a "real" solution of the reduced GL equation, then one can show that, for small $\alpha \neq 0$, the nodal set of $u(\alpha)$ will behave like the nodal set of u_1 (see Section 8 for a precise statement). In particular, if there is only one hole, then the nodal set of $u(\alpha)$ consists exactly in one line joining the interior boundary and the exterior boundary.

6 The toy model

Before to discuss the general results of [HHOO] in Sections 7 and 8, we think it is worth to discuss a model which in some sense is a limiting case of a very thin ring like open set (see Berger-Rubinstein-Schatzman [BeRuSc]). This is the case of the Schrödinger operator on the circle which is actually a rewriting of the Hill's equation. If we consider the operator

$$P_\alpha = -(\partial_\theta - i\alpha)^2\ .$$

on $L^2(S^1)$, then the spectrum is easily seen as given by

$$\sigma(P_\alpha) = \{(n-\alpha)^2\ ,\ n \in \mathbb{Z}\}\ .$$

The minimum is obtained for the n's such that $(n-\alpha)^2$ is minimal. When α is an integer, one obtains $n = \alpha$ and the eigenvalue is simple with corresponding eigenfunction $\theta \mapsto \exp i\alpha\theta$.

This property is still the same for any $\alpha \notin \frac{1}{2}+\mathbb{Z}$. When $\alpha \in \frac{1}{2}+\mathbb{Z}$, the lowest eigenvalue is $\frac{1}{4}$ and with multiplicity 2. For example, when $\alpha = \frac{1}{2}$, the eigenvectors are generated by 1 and $\exp i\theta$ or $\exp i\theta/2 \cos\theta/2$ and $\exp i\theta/2 \sin\theta/2$. The last one vanishes on $\theta = 0$. Of course one gets a continuous family of eigenvectors with zeros by considering $\exp i\theta/2 \sin(\theta-\theta_0)/2$.
It is easy to see that this degeneracy disappears when considering

$$P_{\alpha,\epsilon v} = -(\partial_\theta - i\alpha)^2 + \epsilon v(\theta) \ ,$$

perturbatively as $\epsilon \neq 0$ is small, and if the condition $\int_0^{2\pi} \exp i\theta v(\theta) d\theta \neq 0$.
We now also from the general theory of the Hill's equation (See Eastham [East]) that for any C^∞ potential v on S^1 the lowest eigenvalue $\lambda_1(\alpha)$ of $P_{\alpha,v}$ is periodic and takes its maximum for $\alpha \in \mathbb{Z}+\frac{1}{2}$. Moreover, the multiplicity is one if this last condition is not satisfied.

Our aim is to analyze the zeros of the groundstate as a function of θ. We recall that it has been proved that a "real" groundstate vanishes when $\theta = \frac{1}{2}$ modulo $\mathbb{Z}$. This was the object (in a more general case) of [HHOO].
The case when $\alpha = 0$ modulo $\mathbb{Z}$ is well known because the operator is gauge equivalent to the Schrödinger operator with $\alpha = 0$.

There is a lot of information in this case because it was analyzed in great detail when considering the Schrödinger operator with periodic potential on $L^2(\mathbb{R})$. We shall recall some of these results. The following theorem can be found in [ReSi] (Theorem XIII.89, p. 293).

Theorem 6.1 . *Let $\lambda_1(\alpha)$ the first eigenvalue of $P_{v,\alpha}$. Then we have*

1. *$\lambda_1(\alpha)$ is simple for $\alpha \in [0, \frac{1}{2}[$.*
2. *$\alpha \mapsto \lambda_1(\alpha)$ increases strictly on $[0, \frac{1}{2}[$.*

So the only case, where $\lambda_1(\alpha)$ may be degenerate is consequently the case when $\alpha = \frac{1}{2}$, modulo $\mathbb{Z}$.
Let us look in this context at the nodal sets.

Theorem 6.2 .

1. *If $\alpha \notin \frac{1}{2}+\mathbb{Z}$, the groundstate does not vanish.*
2. *If $\alpha \in \frac{1}{2}+\mathbb{Z}$, the multiplicity may be 1 or 2 and the "real" groundstates have a single zero in S^1.*

Proof.
We observe that $\theta \mapsto v_1(\theta,\alpha) = \exp -i\alpha\theta \ u_1(\theta,\alpha)$ is a solution in $C^\infty(\mathbb{R})$ of $-d^2u/d\theta^2 + v(\theta)u - \lambda_1(\alpha)u = 0$ on $\mathbb{R}$. Here v_1 is considered as a function of θ (here α is fixed and $\alpha \notin \mathbb{Z}$) and has the same zeros $u_1(\theta,\alpha)$. Let us assume the existence in $[0,2\pi[$ of a zero of u_1 and consequently of v_1. Let us consider the decomposition : $v_1 = f + ig$, with f and g real.
Two cases have to be considered.

- f and g are linearly dependent. In this case, possibly changing of v_1 we can assume that v_1 is real. But the Floquet condition will give $v_1(\theta+2\pi,\alpha) = \exp -i2\pi\alpha\, v_1(\theta,\alpha)$ and this implies $\sin(2\pi\alpha)=0$. This corresponds to the case when $\alpha \in \mathbb{Z}/2$. We are in the second case and get the announced result.

- We now consider the case when f and g are linearly independent. Here we get two linearly independent solutions of the same second order equation vanishing at the same point. By the Cauchy uniqueness theorem they have to be proportional. This gives the contradiction in the first case. So u_1 can not vanish as initially assumed.

7 Variation of the ground state energy

Let us consider the non-simply connected situation. We observe that, if the circulations $\Phi = (\Phi_1, \ldots, \Phi_k)$ of two distinct vector potentials A and A' are equal modulo $\mathbb{Z}^k$ then the corresponding operators $P_{A,V}$ and $P_{A',V}$ are unitarily equivalent under a gauge transformation. The following theorem was given in [HHOO].

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^2$ be a region with smooth boundary, which is homeomorphic to a disk with k holes. For a given smooth potential V, the first eigenvalue $\lambda^{(1)}$ of the magnetic Schrödinger operator $P_{A,V}$, where A satisfies (3.3), depends only on the circulations $\Phi = (\Phi_1, \ldots, \Phi_k)$ of A:*

$$\lambda^{(1)} = \lambda_1(\Phi)\ .$$

The function $\lambda_1(\Phi)$ has the following properties (in which $l \in \mathbb{Z}^k$ is arbitrary):

$$\lambda_1(\Phi + l) = \lambda_1(\Phi)\ . \tag{7.1}$$

$$\lambda_1(l/2 + \Phi) = \lambda_1(l/2 - \Phi)\ . \tag{7.2}$$

$$\lambda_1(\Phi) > \lambda_1(0, \ldots, 0) \quad \textit{for } \Phi \notin \mathbb{Z}^k\ . \tag{7.3}$$

For the case $k = 1$, we have in addition to inequality (7.3) that

$$\lambda_1(\Phi) < \lambda_1(1/2) \tag{7.4}$$

for $\Phi \notin 1/2 + \mathbb{Z}$.

Equations (7.1), (7.2) and inequality (7.3) are straightforward. In this context we should also mention the recent very interesting results [HN] by Herbst and Nakamura concerning large magnetic fields. The Neumann boundary conditions on $P_{A,V}$ are motivated by superconductivity but our results are also valid for the case of Dirichlet boundary condition. Dirichlet boundary conditions are related to the Aharonov-Bohm effect for bound states. See [LaO, He1, He2, He3]. This will be discussed in Section 11.

Remark 7.2 *The proof of (7.4) uses a connection between the maximality of the first eigenvalue for circulation* $1/2$ *and the structure of the nodal set of groundstates. In a recent paper [HHON] it was shown that for* $\Phi \in [0, 1/2)$ *(7.4) can be strenghtened to* $d/d\Phi \lambda_1(\Phi) \geq 0$.

Using semiclassical arguments, one can show that in general the first eigenvalue is not necessarily maximised for circulation $(1/2, \ldots, 1/2)$, this will be discussed in Section 11.

8 Nodal sets

Let us now analyze the structure of the nodal set of the groundstate of $P_{A,V}$ under the assumptions (3.3) and (4.2). For the toy model, one has to think that α was the constant magnetic potential on the circle, and the condition (4.2) corresponds exactly to the second case in Theorem 6.2. We have seen in Section 4, that, without any assumption on the multiplicity, it is enough to analyze the zero sets of "real" groundstates (See (4.12)). But let us start by a precise definition of this set.

Definition 8.1 *The nodal set* $\mathcal{N}(u)$ *of an eigenfunction* u *of a magnetic Schrödinger operator on a manifold* Ω *with smooth boundary is defined in* $\overline{\Omega}$ *by*

$$\mathcal{N}(u) := \overline{\{x \in \Omega : u(x) = 0\}}. \tag{8.1}$$

Useful informations on nodal sets of real valued eigenfunctions of non-magnetic Schrödinger equations in two dimensions are given in [Bers, Che, HOMN] and in Proposition 4.1 in [HHOO]. In particular such nodal sets consist of the finite union of smoothly immersed circles and lines. "Generically", the nodal set of every complex eigenfunction of a magnetic Schrödinger operator consists of isolated points of intersection of the lines of zeros of the real and imaginary parts of the function. See [ElMaQi].

The local properties of the nodal sets of eigenfunctions of the operator $P_{A,V}$ are the same as the local properties of complex solutions of non-magnetic Schrödinger equations. More precisely, since we may find at every point a local gauge $e^{i\phi}$ satisfying : $\nabla\phi = A$, we may multiply any eigenfunction of $P_{A,V}$ by a local gauge so that the product solves a non-magnetic Schrödinger equation. The nodal set is invariant under local gauge transformations.

We shall see in what follows that although the local properties of nodal sets of eigenfunctions of our magnetic Schrödinger operator are the same as the properties of a non-magnetic Schrödinger operator, the global properties differ in the case where $\Phi = (1/2, \ldots, 1/2)$. In particular, in the non-magnetic case we see that (since a real eigenfunction must change sign at the nodal set) an even number of nodal lines (or perhaps no nodal lines) of an eigenfunction emerges from each boundary component of the region. In Theorem 8.3 we show that for $\Phi = (1/2, \ldots, 1/2)$, an odd number of nodal lines of the groundstate emerge from each component. The following definition was introduced in [HHOO].

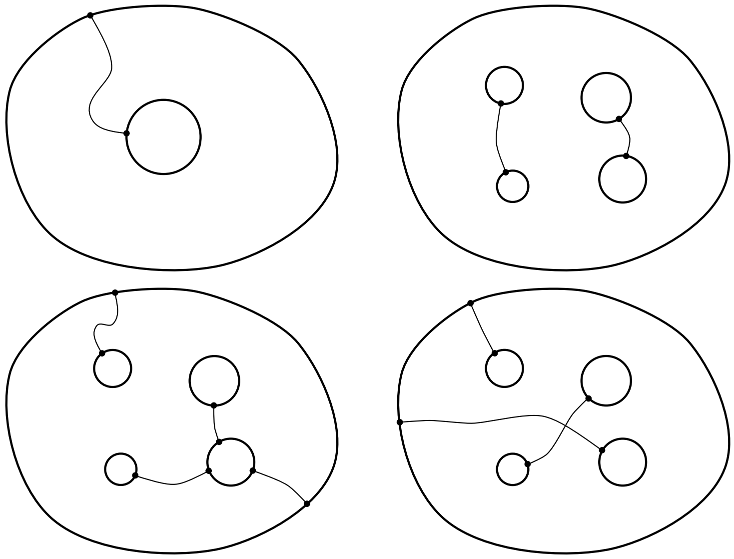

Figure 1: Examples of some sets which slit $\overline{\Omega}$

Definition 8.2 *We say that a (nodal) set $\mathcal{N}$* ***slits*** $\overline{\Omega}$ *if it is the union of a collection of piecewise smooth, immersed lines such that*

1. *each line starts and finishes at the boundary $\partial\Omega$ and leaves the boundary transversally;*
2. *internal intersections between lines are transversal;*
3. *the complement $\Omega \setminus \mathcal{N}$ is connected;*
4. *an odd number of nodal lines leaves each interior boundary component.*

We shall say that a collection of paths slits $\overline{\Omega}$ if the union of the images of the paths slits $\overline{\Omega}$.

See Figure 1 for some examples of regions which are slit.
Note that part 3 of the above definition is the reason why a nodal set which slits $\overline{\Omega}$ contains no immersed circles, and also implies that each line of a slitting set links together a unique pair $\{\Sigma_i, \Sigma_j\}$ of distinct (i.e. $i \neq j$) boundary components. Note also that for the single hole case, a set which slits $\overline{\Omega}$ consists of one line which joins the outer boundary of Ω to the inner boundary.
The main theorem is then

Theorem 8.3 . *Let Ω be a region with smooth boundary, which is homeomorphic to a disk with k holes. Let V be a smooth potential and let A be a smooth magnetic vector potential satisfying equation (3.3), such that the value of the circulations around each hole lie in $1/2 + \mathbb{Z}$ (that is $\Phi = (1/2, \ldots, 1/2)$, modulo $\mathbb{Z}^k$).*

1. *If the first eigenvalue of $P_{A,V}$ is simple then the nodal set of the corresponding eigenfunction slits $\overline{\Omega}$. Otherwise there exists an orthonormal "real" basis $\{u_1, \ldots, u_m\}$ of the groundstate eigenspace such that the nodal set of any non-zero combination $\sum_{i=1}^m a_i u_i$, with $a_i \overline{a}_j \in I\!R$ for each $1 \leq i, j \leq m$, slits $\overline{\Omega}$.*

2. *For $k = 1, 2$ with groundstate multiplicity two, the nodal sets of two linearly independent groundstates do not intersect. It follows that the nodal set of a combination $a_1 u_1 + a_2 u_2$ is empty whenever $a_1 \overline{a}_2 \notin I\!R$.*

Remark 8.4 .

1. *For the cases $k \geq 3$, we expect that there could be intersection of nodal sets of two independent groundstates, and correspondingly that the nodal set of a combination $a_1 u_1 + a_2 u_2$ will not in general be empty when $a_1 \overline{a}_2 \notin I\!R$.*

2. *If we assume that $P_{A,V}$ has Dirichlet boundary conditions then Theorem 8.3 hold with suitable changes to the proofs.*

The proof given in [HHOO] is inspired by one step in the proof of [BeRu] who treat only the one hole case. But it appears more convenient to lift the problem on some two-fold covering manifold where one can gauge away the magnetic potential and analyze real solutions of a real operator with some additional anti-symmetry. This will be explained in Section 9.

Let us compare with the point of view developed in [BeRu].
In the one hole case, we can consider for any slitting path γ, the selfadjoint realization of the Schrödinger operator $-\Delta + V$ in $\Omega(\gamma) := \Omega \setminus \{\gamma\}$ with Neumann boundary condition on $\partial\Omega$ and Dirichlet condition on γ. We denote the lowest eigenvalue of this operator by $\mu(\gamma)$ and the corresponding strictly positive, normalized, eigenvector by ψ_γ. Let $\chi = \chi_\gamma$ be a solution of

$$d\chi = A_1\, dx_1 + A_2\, dx_2 \ , \quad \text{in } \Omega(\gamma) \ .$$

Such a solution exists because $\Omega(\gamma)$ is simply connected, without any assumption on the circulation of A. Considering the function $u_\gamma = \exp i\chi\, \psi_\gamma$ (considered by extension as defined on $L^2(\Omega)$) (which belongs to $H^1(\Omega)$), we get by applying the minimax lemma,

$$\lambda^N_{A,V} \leq \mu(\gamma). \tag{8.2}$$

We consequently have proved, without any special assumption on the multiplicity and without any restrictive condition on the circulation, the first part of the

Proposition 8.5 . *Let V be C^∞ and A a C^∞ potential on $\bar{\Omega}$ such that $B = 0$ in Ω. Then*

$$\lambda^N_{A,V} \leq \inf_{\gamma \in \mathcal{L}^1(\Omega)} \mu(\gamma) \ , \tag{8.3}$$

where $\mathcal{L}^1(\Omega)$ is the set of all piecewise C^1 curves γ such that $\Omega(\gamma)$ is simply connected.
If (4.2) is satisfied, then we have actually equality in (8.3)

$$\lambda^N_{A,V} = \inf_{\gamma \in \mathcal{L}^1(\Omega)} \mu(\gamma) \ , \tag{8.4}$$

Remark 8.6 .
The quantity appearing in the r. h. s of (8.3) was introduced in [BeRu] in the case $V = 0$. The condition of non degeneracy which was introduced by these authors and used in their argument was actually the condition:
(C) : The infimum $\inf_\gamma \mu(\gamma)$ is realized for a unique slitting path γ.
We emphasize that we shall not use this assumption. Moreover we can prove, as a result of our analysis that either this condition is satisfied (and the lowest eigenvalue is then simple) or there is an infinite family of paths realizing the minimum.

We assume now condition (4.2) and would like to find some γ such that $\mu(\gamma) \leq \lambda^N_{A,V}$. This would give as a consequence the equality (8.4). A solution γ which gives the solution is actually the nodal line of one "real" ground state.

One can interpret all this theory as the analysis of an excited eigenvalue of a Schrödinger operator defined on the two-fold covering of Ω. This will be detailed in Section 9.

On the other hand, according to the arguments developed in Sections 3 and 5, we obtain the following theorem

Theorem 8.7 . *Under the assumptions of Proposition 5.2, there exists $\epsilon_1 > 0$ such that, for all α such that $0 < |\alpha| \leq \epsilon_1$, the zero set of the "real" bifurcating state $u(\alpha)$ slits $\bar{\Omega}$.*

Although we have not verified the mathematical details there are strong reasons to think that this nodal set tends as $|\alpha| \to 0$ to the nodal set of u_1.

9 A twofold Riemannian covering manifold

In this section we keep the assumption of Section 8 and explain some elements of the proof of the characterization of the nodal sets. The proofs of the results of [HHOO] use a twofold Riemannian covering manifold $\tilde{\Omega}$ of the domain Ω. For the case of more than one hole, there exists more than one twofold Riemannian covering manifold of Ω. We shall take a particular choice of covering manifold on which the circulation of the lifted magnetic potential $\tilde{A}$ along any closed curve is an integer. Before the precise definition, we introduce some basic notation. For further details see for example [GHL].

Definition 9.1 *Let $\tilde{\Omega}$ be a covering manifold of Ω, and let Π be the associated covering map. We denote the lifts of various quantities as follows:*

For a set $\mathcal{N}$ define $\tilde{\mathcal{N}} = \{x \in \tilde{\Omega} : \Pi(x) \in \mathcal{N}\}$. For a function $f : \Omega \to \mathbb{C}$, define $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ by $\tilde{f} = f \circ \Pi$. For a path $\sigma : [0,1] \to \Omega$ and a point $x \in \tilde{\Omega}$ such that $\Pi(x) = \sigma(0)$ let $\tilde{\sigma} : [0,1] \to \tilde{\Omega}$ denote the unique lifted path such that $\tilde{\sigma}(0) = x$ and $\Pi \circ \tilde{\sigma} = \sigma$.

We endow the covering manifold with the metric obtained by lifting the flat Euclidean metric of Ω to $\tilde{\Omega}$. This is the unique metric which makes Π a local isometry, and therefore a Riemannian covering map. Let $\tilde{\Delta}$ denote the Laplace-Beltrami operator on $L^2(\tilde{\Omega})$ induced by the lifted metric on Ω, and let $\omega_{\tilde{A}}$ be the 1-form on $\tilde{\Omega}$ obtained by lifting the 1-form ω_A associated with the smooth vector potential A defined on Ω.

Let $\tilde{\Omega}_\infty$ be the universal covering manifold of Ω and let Π_∞ be the associated covering map. Note that due to (4.2) if two points $x_\infty, y_\infty \in \tilde{\Omega}_\infty$ satisfy $\Pi_\infty(x_\infty) = \Pi_\infty(y_\infty)$ then for any path σ joining x_∞ to y_∞, the integral

$$\frac{1}{2\pi} \int_{\Pi_\infty \circ \sigma} A \cdot dx \tag{9.1}$$

lies either in $1/2+\mathbb{Z}$ or in $\mathbb{Z}$ and is independent of the path σ because $\operatorname{curl} A = 0$ and because the universal covering manifold is simply connected. We therefore construct the twofold covering manifold (as a quotient of the universal covering manifold) as follows:

Definition 9.2 .

1. *We define the twofold covering manifold $\tilde{\Omega}$ by identifying points x_∞, y_∞ in $\tilde{\Omega}_\infty$ according to the equivalence relation $x_\infty \sim y_\infty$ if and only if*

$$\Pi_\infty(x_\infty) = \Pi_\infty(y_\infty) \tag{9.2}$$

   and for each path σ in $\tilde{\Omega}_\infty$ joining x_∞ to y_∞ we have

$$\frac{1}{2\pi} \int_{\Pi_\infty \circ \sigma} A \cdot dx \in \mathbb{Z}. \tag{9.3}$$

   The covering map $\Pi : \tilde{\Omega} \to \Omega$ is defined by $\Pi(x) = \Pi_\infty(x_\infty)$, where $x = [x_\infty]$ is the equivalence class (under $\sim$) containing x_∞.

2. *On our twofold covering manifold we define the symmetry map $G : \tilde{\Omega} \to \tilde{\Omega}$ by setting Gx to be the other point in $\tilde{\Omega}$ which lies above $\Pi(x) \in \Omega$. Note that $\Pi^{-1}(\Pi(x)) = \{x, Gx\}$.*

3. *We say that a function $f : \tilde{\Omega} \to \mathbb{C}$ is symmetric (resp. antisymmetric) if $f(Gx) = f(x)$ (if $f(Gx) = -f(x)$) for all $x \in \tilde{\Omega}$.*

Note that the identity map and G form a group $\mathbf{G} = \{I, G\}$, with the composition $G^2 = I$, which acts freely on $\tilde{\Omega}$. The quotient of $\tilde{\Omega}$ by $\mathbf{G}$ is the original manifold Ω. Using equation (9.3), we have

$$\frac{1}{2\pi} \oint_{\sigma} \tilde{A} \cdot \mathrm{d}\tilde{x} = \frac{1}{2\pi} \oint_{\Pi \circ \sigma} A \cdot dx \in \mathbb{Z}, \tag{9.4}$$

for any closed path σ in $\tilde{\Omega}$. Hence there exists a smooth, multivalued function θ on $\tilde{\Omega}$ such that $\exp i\theta$ is singlevalued and

$$\nabla\theta = \tilde{A}. \tag{9.5}$$

The crucial lemma in [HHOO] is

Lemma 9.3 .
1. The operator $\mathbf{L} : L^2(\Omega) \to L^2(\tilde{\Omega})$ *defined by*

$$\mathbf{L}u = \frac{1}{\sqrt{2}} e^{i\theta} \tilde{u} \tag{9.6}$$

is a isometry onto the antisymmetric functions in $L^2(\tilde{\Omega})$.
2. $\mathbf{L}$ *maps the "real" functions (that is such that* $Ku = u$*) onto the real antisymmetric functions on* $\tilde{\Omega}$.

As a consequence, we get that $\mathbf{L}$ maps eigenfunctions of $P_{A,V}$ onto antisymmetric eigenfunctions of the Schrödinger operator

$$\tilde{P}_{0,V} = -\tilde{\Delta} + \tilde{V} \tag{9.7}$$

acting on $L^2(\tilde{\Omega})$ with Neumann boundary conditions. The "real" eigenvectors become real eigenvectors of $\tilde{P}_{0,V}$.
In particular $\lambda^{(1)}$ is the lowest eigenvalue of $\tilde{P}_{0,V}$ restricted to the antisymmetric space. Any real corresponding antisymmetric groundstate has to vanish (this is the initial argument for the proof of Theorem 8.3) and can not consequently correspond to the lowest eigenvalue of $\tilde{P}_{0,V}$. Actually, we know that the groundstate corresponds to a single eigenvalue and is symmetric.
This does not mean that $\lambda^{(1)}$ is necessarily the second eigenvalue of $\tilde{P}_{0,V}$. For any $\ell \in I\!N$ ($\ell > 0$) and in the one hole case, it is indeed possible to construct potentials V such that the first ℓ eigenvalues correspond to symmetric eigenstates [Ni]. Another consequence is for the Ginzburg-Landau functional is that

$$G_\lambda(u, A_e) = \int_{\tilde{\Omega}} \left(\lambda(-|\mathbf{L}u|^2 + |\mathbf{L}u|^4) + |\tilde{\nabla}\mathbf{L}u|^2 \right) d\tilde{x} \; . \tag{9.8}$$

In particular, to look for minima for G_λ with "real" u and $A = A_e$ leads to minimize the functional

$$v \mapsto \int_{\tilde{\Omega}} \left(\lambda(-|v|^2 + |v|^4) + |\tilde{\nabla}v|^2 \right) d\tilde{x} \; ,$$

reduced to the real, antisymmetric functions on $L^2(\tilde{\Omega})$.

10 About the multiplicity

We have seen at many places that assumptions on the multiplicity of the groundstate play an important role. As we shall see in this section, this problem of multiplicity is deeply connected with the structure of nodal sets of groundstates. It has been proven in [HHOO] the following theorem.

Theorem 10.1 . *If (4.2) is satisfied then the multiplicity m of the first eigenvalue of $P_{A,V}$ (Dirichlet or Neumann realization) satisfies*

$$m \leq \begin{array}{ll} 2 & , k = 1, 2; \\ k & , \text{ if } k \text{ is odd, } k \geq 3; \\ k-1 & , \text{ if } k \text{ even, } k \geq 4 \,. \end{array} \tag{10.1}$$

Here we make some remarks connected to the above theorem.

Remark 10.2 .

1. *The above bound on the multiplicity of the first eigenvalue is sharp in the case of one hole (see Example 5.3 in [HHOO]), two holes (see Section 11), three holes (see Section 12), but it is not expected to be sharp for a large number of holes. It would be interesting to know an asymptotic result about the growth of the maximum multiplicity with the number of holes.*

2. *The proof is obtained by taking advantage of topological obstructions to nodal sets caused by the holes. These obstructions prevent the existence of high dimensional groundstate eigenspaces. Our type of method was first discovered in [Che] and has since then been taken up and used by others, e.g. [Na, HOHON, HOMN]. See also [Col] for explicit constructions of examples with high multiplicity.*

3. *This result bears similarities to bounds on multiplicities of higher eigenvalues of non-magnetic Schrödinger operators on surfaces with boundary. Some related literature on this topic is given in [Col, Na, HOHON, HOMN].*

4. *It has been shown in [BCC] that no upper bound on the multiplicity exists when one adds a general magnetic field, even on the sphere.*

5. *In [HHON] it was shown for the single hole case but for arbitrary circulation Φ that the groundstate eigenvalue is always simple unless Φ is a half integer.*

About the proof:
Let us explain the case of one hole for the Dirichlet case. The proof is by contradiction. If there are three linearly independent ground states, then one can always be reduced to the case when there are three linearly independent states u_1, u_2 and u_3 such that

$$Ku_j = u_j \ , \ \text{ for } j = 1,\ 2,\ 3\ .$$

Then we show, that, for any x and y on $\partial\Omega$, one can find a non zero ground state u ($u = \sum_j \lambda_j u_j$ with $\lambda_j \in I\!R$) such that $Ku = u$ and $(\frac{\partial}{\partial\nu}u)(x) = (\frac{\partial}{\partial\nu}u)(y) = 0$ where $\frac{\partial}{\partial\nu}$ denotes the normal derivative. This means that at x and y nodal lines hit the boundary (see below)and if $x \neq y$ and belong to the same connected

component of $\partial\Omega$, then this is in contradiction with the property of a ground state satisfying $Ku = u$.
We have indeed, as a consequence of the Hopf boundary point lemma, the following property

Lemma 10.3 .
If $x \in \partial\Omega$, and u is a groundstate for Neumann such that $u(x) = 0$. Then $x \in \mathcal{N}(u)$. If u is a groundstate for the Dirichlet case and $x \in \partial\Omega$ such that $\partial u/\partial n = 0$ then $x \in \mathcal{N}(u)$.

11 About the maximality at circulation $\frac{1}{2}$

If we introduce a semi-classical parameter like the analysis of the Aharonov-Bohm effect in [He1], one can naturally ask the following question :
Is there some limiting curve for the nodal set as $h \to 0$, for ground states of $P_{A,\frac{V}{h^2}}$ when $\operatorname{curl} A = 0$ and (4.2) is satisfied. One suspects that the Agmon distance will play an important role in the picture but this problem remains open.
We recall that in the case of one hole, we have mentioned in Section 7 that the lowest eigenvalue of $P_{A,\frac{V}{h^2}}$ is maximal, as a function of the circulation Φ around the hole, when $\Phi = \frac{1}{2}$ modulo $\mathbb{Z}$. We consider the Dirichlet situation but one can similarly treat the Neumann situation.
We would like to show that we can not hope a similar result in the many hole case. More precisely, we shall consider the 2 holes case and construct, using semiclassical analysis, an example where $\lambda^{(1)}(\frac{1}{2},\frac{1}{2}) < \lambda^{(1)}(\frac{1}{4},\frac{1}{4})$.

The comparison of $\lambda^{(1)}(\frac{1}{2},\frac{1}{2})$ and $\lambda^{(1)}(\frac{1}{4},\frac{1}{4})$ is just an application of the semi-classical analysis of the tunneling effect presented in [He1] (see also [HeSj1, HeSj2]). But let us be more precise in the assumptions. We consider the operator $-\Delta_A + \frac{V(x)}{h^2}$ in Ω where Ω has two holes O_1 and O_2. We assume that the magnetic field $B = \operatorname{curl} A$ vanishes in Ω and keep the two circulations Φ_1 and Φ_2 as parameters, together with h. We assume also that the electric potential V has a non degenerate unique minimum at x_0 in Ω and assume for normalization that $V(x_0) = 0$.
In this case it is well established that, in the semiclasssical limit (i.e. as $h \to 0$), the lowest eigenvalue is simple and given modulo $\mathcal{O}(1)$ by the lowest eigenvalue of some harmonic oscillator attached to x_0 :

$$-\Delta_y + h^{-2}\frac{1}{2}(\operatorname{Hess} V(x_0)y, y)\ . \tag{11.1}$$

We denote by $\lambda^{(1)}_\Omega(h; \Phi_1, \Phi_2)$ this lowest eigenvalue.
Moreover the normalized groundstate u_1 is known to be decaying at least as $\mathcal{O}_\epsilon(\exp -\frac{(1-\epsilon)d_V(x,x_0)}{h})$ for any $\epsilon > 0$, where d_V is the Agmon distance associated with the Agmon metric defined as $V \cdot dx^2$ on $\bar{\Omega}$.

This result was proven in [HeSj2].

We now assume that V creates a big barrier not only near $\partial\Omega$ but also between the two holes O_1 and O_2. The point is that there is a minimal path (for the length associated to the Agmon distance) starting from x_0, turning once around $O_1 \cup O_2$ and coming back to x_0 whose length is strictly less than the length of any closed path starting from x_0 and going once around only O_1 or O_2 or two-times the distance from x_0 to $\partial\Omega$. If γ is this minimal path (supposed to be unique and non degenerate), then one can find an one-hole open set Ω' such that $\Im\gamma \subset \Omega' \subset \Omega$ such that, for some $\epsilon_0 > 0$,

$$\lambda^{(1)}_{\Omega}(h;\Phi_1,\Phi_2) = \lambda^{(1)}_{\Omega'}(h,\Phi_1+\Phi_2) + \mathcal{O}(\exp -\frac{S+\epsilon_0}{h}) ,$$

where S is the length of this minimal path. The hole O' contains O_1 and O_2 and the circulation along γ is $\Phi_1 + \Phi_2$. The analysis of $\lambda_{\tilde{\Omega}}(h,\Phi)$ was performed in [He1] (Theorem 4.1.1, with $t = h$). It is shown that, for some $\epsilon_1 > 0$ and some $\alpha_0 > 0$,

$$\lambda^{(1)}_{\Omega'}(h,\Phi) = \lambda^{(1)}_{\Omega'}(h,0) + h^{\frac{1}{2}}(1-\cos\Phi)a(h)\exp -\frac{S}{h} + \mathcal{O}(\exp -\frac{S+\epsilon_1}{h}) ,$$

with $a(h) \sim \alpha_0 > 0$.
This shows that this is when $\Phi = \Phi_1 + \Phi_2$ is near $\frac{1}{2}$ (modulo $\mathbb{Z}$) that $\lambda^{(1)}_{\Omega'}(h,\Phi)$ and consequently $\lambda^{(1)}_{\Omega}(h;\Phi_1,\Phi_2)$ is maximal in the semiclassical limit.

An indirect consequence is that in this case the nodal set is a line between the two holes. The maximality of the eigenvalue for flux $(\frac{1}{2},\frac{1}{2})$ can be proven in the case when $\Omega \setminus \mathcal{N}$ is simply connected, where $\mathcal{N}$ is the nodal set of some "real" groundstate. When discussing all the possibilities of slitting of two holes the only possibility is indeed the case when Ω is slitt by a line between two holes.

Moreover, we have a good WKB approximation of the ground state in the neighborhood of γ (see [He1]) and we know from this approximation that the nodal line when Φ is an integer can not meet γ in the semiclassical limit. This is a another way to show that there is necessarily in this case a nodal line between the two holes in the case when $(\Phi_1,\Phi_2) = (\frac{1}{2},\frac{1}{2})$.
It is easy to construct a case when a groundstate has as nodal set two lines joining the boundary of the hole to the exterior boundary. This is for example the case when the multiplicity is 2 because one can always in this case construct, by considering a real linear combination of two linearly independent groundstates u_1 and u_2 satisfying $Ku_j = u_j$ $(j = 1,2)$, a groundstate with a nodal line arriving at the external boundary (and consequently with two!).
In particular we have shown that the two possible situations of nodal sets predicted by the topology in the case of two holes are effectively observed.

Remark 11.1 *The semi-classical behavior of the nodal sets is a nice problem. In the case considered before where a potential V creates a single well, the already* *developed semi-classical analysis gives some information. If the minimal non trivial path going from x_0 to x_0 (this can be interpreted in the covering manifold (See Section 9)) corresponds to an integer flux, then as already mentioned there are no nodal lines crossing this path. On the contrary, in the half integer case, one can show that the nodal line crosses as $h \to 0$ the curve γ at the point which is at the Agmon distance $S/2$. One suspects that the nodal set will asymptotically contain a line defined (say in the case of one hole) (under suitable assumptions on V) as the projection of the set in the covering $\tilde{\Omega}$ given by*

$$d_{\tilde{V}}(x, \tilde{x}_0) = d_{\tilde{V}}(x, G\tilde{x}_0) ,$$

where $\Pi(\tilde{x}_0) = x_0$.

Remark 11.2 . *One can also present, in the case of two holes, examples with $V \equiv 0$ giving rise to the two nodal sets situation. The case of three holes will be detailed in the next section.*

12 The case of 3 holes

Here we want to construct a domain Ω with 3 holes such that the multiplicity of the first eigenvalue is 3. Thereby we assume Dirichlet boundary conditions.

Pick a disk with radius 1 centered at the origin and remove 3 small disks of radius ρ such that their centers are in polar coordinates (r, ω) given by $(r, \pi/2)$ $(r, 7\pi/6)$ $(r, 11\pi/6)$. Note that it has the symmetry of an equilateral triangle.

We denote the resulting domain by $\Omega(r, \rho)$. As usually we assume that the circulation around each hole is $1/2$. Clearly $r \leq 1 - \rho$. Define $\lambda_1(\Omega(r, \rho))$ by

$$\lambda_1(\Omega(r,\rho)) := \lambda_1(r,\rho) = \inf_{\mathcal{S}} \inf\{\int |\nabla\phi|^2 dx : \phi \in C_0^\infty, ||\phi|| = 1\} \qquad (12.1)$$

according to [HHOO] where $\mathcal{S}$ runs over all slitting sets. We will always assume that ρ is sufficiently small. In fact it is not difficult to show that we could even let ρ tend to 0 and we would get still a well defined operator since the associated quadratic form stays semibounded.

We have two extreme cases.

(i) $r \to 1 - \rho$, $(\rho << 1)$.

For $r = 1 - \rho$ the disks touch the boundary and we can gauge the field away. Furthermore

$$\lim_{\rho \to 0} \lambda_1(1 - \rho, \rho) = \lambda_1(1, 0) \qquad (12.2)$$

the lowest eigenvalue of the unit disk with Dirichlet boundary conditions. Here we used results on the dependence of Dirichlet eigenvalues with respect to domain variations [Stoll]. To be more precise we can go to the double covering $\tilde{\Omega}(r, \rho)$. The spectrum of the lifted Laplacian depends continuously on r and ρ as well as the eigenprojections (in the appropriate sense).

(ii) If r and ρ tend to zero then $\Omega(r,\rho) \to \Omega(0,0)$ and $\Omega(0,0)$ is just the punctured unit disk. The slitting set hence consists of a nodal arc γ connecting the boundary with the center of the disk.

Clearly

$$\lambda_1(0,0) > \lambda_1(1,0). \tag{12.3}$$

(The additional Dirichlet condition at γ raises the eigenvalue, because the capacity of γ is strictly greater zero.)

We first show that for ρ small and $r = 1 - \rho - \epsilon$, ϵ small, the slitting set must consist of 3 nodal arcs which connect the holes to the boundary of the unit disk such that we have the full symmetry of the equilateral triangle. To see this suppose that we have not the full symmetry, then also the nodal set rotated by $2\pi/3$ or reflected by one of the symmetry axis will be a minimizing slitting set. Hence we would have at least multiplicity 2 for $\lambda_1(1-\rho-\epsilon,\rho)$. But then there is a function u in the corresponding eigenspace, such that $u(0) = 0$. Hence a nodal arc must pass through the origin and hit eventually two holes or a hole and the boundary. Again this cannot happen since for $\epsilon \to 0$, we would not obtain the groundstate energy.

Now consider the case (ii), namely r small and ρ even smaller. There we must have at least multiplicity 2. To see this suppose that we have multiplicity 1. Then the slitting set $\mathcal{S}$ must have the total symmetry, in particular it must be invariant under a rotation by $2\pi/3, \quad 4\pi/3$. Again this cannot happen since for $r \to 0$ and, say, $\rho = r/10$ we would end up with a slitting set which consists of three nodal arcs connecting the origin to the boundary in such a way that the symmetry of the equilateral triangle is respected, which clearly cannot be the groundstate.

We now fix ρ sufficiently small and consider $\lambda_1(r,\rho)$ as a function of r. For r small we have $m(\lambda_1(r,\rho)) = 2$ and for $r = 1-\rho-\epsilon$, $m(\lambda_1(r,\rho)) = 1$ for ϵ small. Our previous analysis shows that we have groundstates in two orthogonal symmetry subspaces, one with the symmetry of the equilateral triangle which corresponds to multiplicity 1 and the nonsymmetric one which corresponds to multiplicity (at least) 2. These eigenvalues depend continuously on r and hence there must be for some r_0 a crossing proving our assertion.

As mentioned before it would be interesting to construct examples with high multiplicity of the groundstate eigenvalue for more holes. In particular the question arises what the asymptotics of the multiplicity is as a function of the number of holes.

References

[AHS] Y. Avron, I. Herbst, B. Simon : Schrödinger operators with magnetic fields. I- General interactions. Duke Math. J. 45, p. 847-883 (1978).

[BaPhTa] P. Bauman, D. Phillips, Q. Tang : Stable nucleation for the Ginzburg-Landau system with an applied magnetic field. IMA Preprint Series 1416 (1996).

[BeRu] J. Berger, J. Rubinstein : On the zero set of the wave function in superconductivity. Commun. in Math. Physics 202, p. 621-628 (1999).

[BeRuSc] J. Berger, J. Rubinstein, M. Schatzman : Multiply connected mesoscopic superconducting structures. Preprint 1998.

[Bers] L. Bers : Local behaviour of solution of general linear elliptic equations. Comm. Pure Appl. Math., 8 (1955), p. 473-476.

[BCC] G. Besson, B. Colbois, and G. Courtois, *Sur la multiplicité de la première valeur propre de l'opérateur de Schrödinger avec champ magnétique sur la sphère* S^2, Trans. Amer. Math. Soc. **350** (1998), 331–345.

[Che] S. Y. Cheng : Eigenfunctions and nodal sets. Commentarii Math. Helv. , 51 (1976), p. 43-55.

[Col] Y. Colin de Verdière, *Multiplicités des valeurs propres. Laplaciens discrets et laplaciens continus*, Rend. Mat. Appl. **VII** (1993), 433–460.

[CrRa] M.G. Crandall, P.H. Rabinowitz : Bifurcation from simple eigenvalues. Journal of Functional Analysis 8, p. 321-340 (1971).

[Du] M. Dutour : Thèse en préparation.

[DuHe] M. Dutour, B. Helffer : In preparation.

[DGP] Q. Du, M.D. Gunzburger and J.S. Peterson : Analysis and approximation of the Ginzburg-Landau model of superconductivity. Siam Rev. 34 (1992), p. 431-448.

[East] M. S. P. Eastham : The spectral theory of periodic differential equations. Scottish Acad. Press, Edinburgh-London (1973).

[ElMaQi] C. M. Elliott, H. Matano, T. Qi : Zeros of complex Ginzburg-Landau order parameter with applications to superconductivity, Eur. J. Appl. Math. 5, p. 431-448, 1994.

[GHL] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, 2nd ed., Universitext, Springer-Verlag, 1990.

[GoSc] M. Golubitsky, D.G. Schaeffer, Vol 1. *Singularity and groups in bifurcation theory,* Appl. Math. Sciences 51, Springer Verlag.

[He1] B. Helffer : Effet d'Aharonov-Bohm pour un état borné. Comm. in Math. Phys. 119 (1988), p. 315-329.

[He2] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Lecture notes in mathematics, vol. 1336, Springer-Verlag.

[He3] B. Helffer : On spectral theory for Schrödinger operators with magnetic fields. Advanced studies in Pure Mathematics 23, p. 113-141 (1994).

[He4] B. Helffer : Some of the results were presented in a talk in Matrei (Austria) in the meeting: Geometrical aspects of spectral thery. (July 99).

[HeSj1] B. Helffer, J. Sjöstrand : Multiple wells in the semi-classical limit I- Comm. in PDE 9(4), p. 337-408 (1984).

[HeSj2] B. Helffer, J. Sjöstrand : Effet tunnel pour l'équation de Schrödinger avec champ magnétique. Annales de l'ENS de Pise 14 (1987), p. 625-657.

[HHOO] B. Helffer, M. and T. Hoffmann-Ostenhof and M. Owen : Nodal sets for the groundstate of the Schrödinger operator with zero magnetic field in a non simply connected domain. Commun. in Math. Physics. 202, p. 629-649 (1999).

[HHON] B. Helffer, M. and T. Hoffmann-Ostenhof, N. Nadirashvili : In preparation.

[HN] I. Herbst and S. Nakamura, *Schrödinger operators with strong magnetic fields: Quasi-periodicity of spectral orbits and topology*, Amer. Math. Soc. Trans, **189**, (1999), p. 105-123.

[HOHON] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, *On the multiplicity of eigenvalues of the Laplacian on surfaces*, Ann. Global Anal. Geom.,**17**,(1999), p. 43-48

[HOMN] T. Hoffmann-Ostenhof, P. Michor, and N. Nadirashvili, *Bounds on the multiplicity of eigenvalues for fixed membranes*, Preprint 1998, to appear in Geometrical and Functional Analysis.

[LaO] R. Lavine, M. O'Caroll : Ground state properties and lower bounds on energy levels of a particle in a uniform magnetic field and external potential, J. Math. Phys. 18 (1977), p. 1908-1912.

[LiPa] W.A. Little and R.D. Parks, *Observation of quantum periodicity in the transition temperature of a superconducting cylinder*, Phys. Rev. Lett. **9** (1962), p.9-12.

[Na] N. S. Nadirashvili : *Multiple eigenvalues of the Laplace operator*, Math. USSR, Sb. **61** (1988), p. 225-238.

[Ni] Chang-Shou Lin, Wei-Ming Ni : *A Counterexample to the nodal Domain conjecture and a related semilinear equation.* Proc. Amer. Math. Soc. **102**, 1988, p. 271-277

[Od] F. Odeh : Existence and bifurcation theorems for the Ginzburg-Landau equations. J. Math. Phys. 8:12, p. 2351-2357.

[ReSi] M. Reed, B. Simon : Methods of modern mathematical Physics, IV. Analysis of operators. Academic Press. New York (1978).

[Stoll] P. Stollmann: Aconvergence theorem for Dirichlet forms with applications to boundary value problems with varying domains. Math. Z. **219** (1995), p. 275-287.

[Ta] P. Takac : Bifurcations and vortex formation in the Ginzburg-Landau equations. Preprint 1999.