Return to Equilibrium

Volker Bach [*]
FB Mathematik; Universität Mainz;
D-55099 Mainz; Germany (vbach@mathematik.uni-mainz.de)

Jürg Fröhlich
Inst. f. Theoretische Physik; ETH Hönggerberg;
CH-8093 Zürich; Switzerland (juerg@itp.phys.ethz.ch)

Israel Michael Sigal [†]
Dept. of Math.; Univ. of Toronto;
Toronto, M5S 3G3; Canada (sigal@math.toronto.edu)

December 21, 1999

Abstract

We study an atom with finitely many energy levels in contact with a heat bath consisting of photons (black body radiation) at a temperature $T > 0$. The dynamics of this system is described by a Liouville operator, or thermal Hamiltonian, which is the sum of an atomic Liouville operator, of a Liouville operator describing the dynamics of a free, massless Bose field, and a local operator describing the interactions between the atom and the heat bath. We show that an arbitrary initial state which is normal with respect to the equilibrium state of the uncoupled system at temperature T converges to an equilibrium state of the coupled system at the same temperature, as time tends to $+\infty$ (*return to equilibrium*).

Keywords: Renormalization Group, Return to Equilibrium, KMS

---

[*] Heisenberg Fellow of the DFG and supported by SFB 288 of the DFG until 9/99, also supported by the TMR-Network on "PDE and QM".

[†] Supported by NSERC Grant NA 7901

I Introduction and Summary of Main Results

In this paper, we study open quantum systems consisting of a small, compact subsystem with finitely many degrees of freedom coupled to an infinitely extended reservoir or heat bath which, asymptotically, is in thermal equilibrium corresponding to a temperature $T > 0$. By "asymptotic thermal equilibrium" we mean that, roughly speaking, the states of interest of the system are indistinguishable from thermal equilibrium states at a fixed, positive temperature $T > 0$ in a neighbourhood of spatial infinity.

Our main concern is to analyze the phenomenon of *"return to equilibrium"*: We exhibit a class of open quantum systems with the property that the time evolution drives an *arbitrary* initial state describing "asymptotic thermal equilibrium" at a temperature $T > 0$ towards an equilibrium (or KMS) state at the *same* temperature T, as time tends to ∞. In other words, the limiting state of an open system with the property of return to equilibrium, as time tends to ∞, is a time-translation invariant KMS state corresponidng to a temperature equal to that of the heat bath near spatial infinity.

A consequence of return to equilibrium is that the entropy of the state of the small subsystem tends to *increase* under the time evolution. This means that, if only the degrees of freedom of the small subsystem are observed, the dynamics is *dissipative*, dissipation arising through energy exchange with the thermal heat bath. This kind of dissipative behaviour is sometimes called *"quantum friction"*.

The phenomenon of "return to equilibrium" is similar to the phenomenon of *"approach to a ground state"*, which is observed at zero temperature: If a suitable small subsystem, such as an electron bound to a static nucleus, is coupled to a dispersive medium with infinitely many degrees of freedom, such as the quantized electromagnetic field, at *zero* temperature, then an arbitrary initial bound state of the small subsystem approaches a *groundstate of the coupled system*, as time tends to ∞. The reason is that excited bound states of the small subsystems become *unstable* when the subsystem is coupled to the dispersive medium; they decay into lower-energy bound states through emission of dispersive modes (photons) and eventually converge to a groundstate. This phenomenon is sometimes called *"dissipation through dispersion* (emission of dispersive radiation)".

The two phenomena of "return to equilibrium" ($T > 0$) and "approch to a groundstate" ($T = 0$) can be formulated as *spectral problems* for the generator of the time evolution, i.e., for the *Liouville operator*, or thermal Hamiltonian, ($T > 0$) and the *Hamiltonian* ($T = 0$), respectively: If one can show that the *point spectrum* of the Liouville operator generating the dynamics of an open quantum system in asymptotic thermal equilibrium at a temperature $T > 0$ consists of a simple eigenvalue at 0 then the property of "return to equilibrium" can be proven to hold as a general consequence of the so-called *KMS condition.* A prerequisite for establishing "approach to a groundstate" is to show that the point spectrum of the Hamiltonian generating the dynamics of the system at *zero* temperature consists of a *single eigenvalue*, the groundstate energy, of finite multiplicity. In particular, one must show that all excited bound states of the small subsystem (e.g., an atom) are turned into *resonances* of finite life time when the latter is coupled to the dispersive medium. However, this kind of information on the energy spectrum of the Hamiltonian does *not*, by itself, suffice to prove the property of "approach to a groundstate". (In addition, one needs to establish some properties of *scattering* related to asymptotic completeness, and this tends to be a very hard analytical problem.)

The idea that initial excited bound states of an atom approach a groundstate through emission of photons, as time tends to ∞, first appeared in Bohr's theory of the hydrogen atom coupled to the quantized radiation field, at zero temperature, and remained a guiding idea in later *perturbative* analyses of the quantum theory of atoms coupled to the electromagnetic field by some of the founding fathers of quantum mechanics. See, e.g., [14] for a review of results. Mathematically rigorous, non-perturbative results on the quantum theory of charged particles interacting with the quantized radiation -(or the phonon) field at zero temperature started to appear surprisingly recently; see, e.g., [11, 19, 20, 30, 5, 6, 7, 8, 9, 10, 26, 23, 24, 39, 38, 17, 36]

First traces of the idea of "return to equilibrium" at positive temperature appear in work of Planck, in Einstein's 1917 derivation of the law of blackbody radiation, and in much subsequent work on radiation theory; see, e.g., [31, 32]. Mathematically, precise results were first obtained within various approximate treatments, such as the van Hove limit; see, e.g., [16, 29] and references given there. A complete proof of the return to equilibrium for a simple infinite quantum system, the so-called XY chain, was first presented in [35]. A reformulation of return to equilibrium as a *spectral problem* for Liouville operators was proposed by Jakšić and Pillet in [27, 28], drawing on previous fundamental work due to Araki and Woods [3], Haag, Hugenholz and Winnink [22], and Araki [1]; see also [12, 13, 21]. In the present paper, we follow the general ideas of the spectral approach to the problem of return to equilibrium due to Jakšić and Pillet [27, 28]. Due to some confusion in the literature, we find it necessary, however, to carefully review the general formalism of the quantum theory of finite and infinite systems in or close to thermal equilibrium, as developed in [3, 22, 1] (see also [12, 13, 21]), in a form convenient for applications to concrete models, and to introduce several novel technical devices within the spectral approach. Furthermore, we intend to present the general formalism in a fashion that will make future applications to more complicated problems, e.g., to the problem of *stationary states* of infinite quantum systems, or to some problems of transport theory, feasible. Unfortunately, some of these purposes of our work make the present paper quite long.

Next, we describe the class of open quantum systms considered in this paper. The *small, bounded subsystem* consists of a confined atom or molecule. In this paper, an idealized description of the small subsystem as a quantum-mechanical system with a *finite-dimensional state space*,

$$\mathcal{H}_{el} \;\cong\; \mathbb{C}^N \,, \qquad N \;<\; \infty \,, \tag{I.1}$$

is chosen, i.e., we consider an atom or molecule with only finitely many energy levels. When the coupling between the small subsystem and the reservoir, or heat bath, is turned off, the dynamics is given by a Hamiltonian, H_{el}, which is a selfadjoint operator on $\mathcal{H}_{el}$. The spectrum of H_{el} consists of eigenvalues $E_0, E_1, \ldots, E_{N-1}$. For simplicity, we assume that these eigenvalues are non-degenerate,

$$E_0 \;<\; E_1 \;<\; \cdots \;<\; E_{N-1} \,. \tag{I.2}$$

Every eigenvalue E_j corresponds to an eigenvector φ_j of $\mathcal{H}_{el}$, i.e.,

$$H_{el}\varphi_j \;=\; E_j\varphi_j \,. \tag{I.3}$$

These eigenvectors form a complete orthonormal system in $\mathcal{H}_{el}$. In particular, in the natural scalar product, $\langle\,\cdot\,|\,\cdot\,\rangle$, on $\mathcal{H}_{el}$,

$$\langle\varphi_i|\varphi_j\rangle \;=\; \delta_{ij} \,, \tag{I.4}$$

for all $i, j = 0, 1, \ldots, N_1$.

The *reservoir* is chosen to consist of the quantized electromagnetic field or of the quantized vibrations, or phonons, of an infinitely extended, harmonic material medium. The modes of the reservoir are indexed by wave vectors $\vec{k} \in \mathbb{R}^3$ and, for photons, a helicity $\lambda = \pm 1$. The Hilbert space, h, describing pure states of a *single* phonon or photon is given by

$$h \;=\; \begin{cases} L^2(\mathbb{R}^3, dk) \,, & \text{for phonons,} \\ L^2(\mathbb{R}^3 \times \mathbb{Z}_2, dk) \,, & \text{for photons,} \end{cases} \tag{I.5}$$

where

$$\int f(k)\, dk \;:=\; \begin{cases} \int_{\mathbb{R}^3} f(\vec{k})\, d^3k \,, & \text{for phonons,} \\ \sum_{\lambda=\pm 1} \int_{\mathbb{R}^3} f(\vec{k})\, d^3k \,, & \text{for photons.} \end{cases} \tag{I.6}$$

At zero temperature, the Hilbert space of pure state vectors of the reservoir is chosen to be the *Fock space*

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} h^{(n)} \,, \tag{I.7}$$

where $h^{(0)} := \mathbb{C}$, and $h^{(n)} := h^{\otimes_s n}$, $n \geq 1$, denotes the symmetric tensor product appropriate for the description of quantum-mechanical particals with Bose-Einstein statistics. A vector $\Psi \in \mathcal{F}$ is a sequence,

$$\Psi \;=\; \{\psi_n\}_{n=0}^{\infty} \,, \tag{I.8}$$

of wave functions, $\psi_n(k_1, \dots , k_n) \in h^{(n)}$, with $k_j = \vec{k}_j \in \mathbb{R}^3$, for phonons, and $k_j = (\vec{k}_j, \lambda_j) \in \mathbb{R}^3 \times \mathbb{Z}_2$, for photons. These wave functions are completely symmetric in their arguments. The scalar product for two vectors, Ψ and Φ, in $\mathcal{F}$ is given by

$$\langle \Psi | \Phi \rangle \;:=\; \sum_{n=0}^{\infty} \int dk_1 \cdots \int dk_n \, \overline{\psi_n(k_1, \dots , k_n)} \, \varphi_n(k_1, \dots , k_n) \,. \tag{I.9}$$

For $f \in h$, we define an *annihilation operator*, $a(f)$, on $\mathcal{F}$ by setting

$$a(f)\Psi \;:=\; \{\varphi_{n,f}(k_1, \dots , k_n)\}_{n=0}^{\infty} \,, \tag{I.10}$$

where

$$\varphi_{n,f}(k_1, \dots , k_n) \;:=\; \sqrt{n+1} \int dk_{n+1} \, f(k_{n+1}) \, \psi_{n+1}(k_1, \dots , k_n, k_{n+1}) \,. \tag{I.11}$$

(Note that $f \mapsto a(f)$ is linear in f, rather than anti-linear.) For every $f \in h$, $a(f)$ extends to an unbounded, densely defined, closed operator on $\mathcal{F}$. For $f \in h$, we define τf to be the complex conjugate of f, $(\tau f)(k) := \overline{f(k)}$. We define the *creation operator*, $a^*(f)$, to be the unbounded, densely defined, closed operator on $\mathcal{F}$ given by the adjoint of $a(\tau f)$, i.e.,

$$a^*(f) \;:=\; \big(a(\tau f) \big)^* \,. \tag{I.12}$$

Annihilation- and creation operators obey the canonical commutation relations

$$\big[a^*(f), a^*(g)\big] \;=\; \big[a(f), a(g)\big] \;=\; 0 \,, \tag{I.13}$$

$$\big[a(\tau f), a^*(g)\big] \;=\; \langle f | g \rangle_h \, \mathbf{1} \,, \tag{I.14}$$

where $\langle \cdot | \cdot \rangle_h$ denotes the scalar product on h. We note that the *vacuum vector*, $\Omega = (1, 0, 0, \cdots)$, spans $h^{(0)}$ and has the property that

$$a(f)\, \Omega \; = \; 0 \, , \qquad \text{for all } f \in h. \tag{I.15}$$

The *dynamics* of the reservoir at *zero temperature* is determined by the Hamiltonian

$$H_f \; = \; \int dk \, a^*(k) \, \omega(k) \, a(k) \, . \tag{I.16}$$

We choose $\omega(k)$ to be given by

$$\omega(k) \; = \; |\vec{k}| \, , \tag{I.17}$$

corresponding to *massless*, relativistic photons or phonons. The operator H_f defined by (I.16) extends to an unbounded, selfadjoint, positive operator on $\mathcal{F}$. It has a simple eigenvalue at 0, corresponding to the eigenvector Ω. The rest of the spectrum of H_f is purely absolutely continuous. See [34, 33] for a more complete summary of the theory of free, quantized fields.

The Hilbert space of the combinded system, consisting of the idealized atom and the reservoir, at *zero temperature*, is given by

$$\mathcal{H} \; := \; \mathcal{H}_{el} \otimes \mathcal{F} \, . \tag{I.18}$$

When the coupling between the atom and the reservoir is turned off, the dynamics is given by the Hamiltonian

$$H_0 \; = \; H_{el} \otimes \mathbf{1} \; + \; \mathbf{1} \otimes H_f \, , \tag{I.19}$$

with H_{el} as in (I.2), (I.3) and H_f as in (I.16). The coupling between the atom and the reservoir is described by an interaction gI, where $g \in \mathbb{R}$ is a coupling constant, and

$$I \; := \; \int dk \big\{ G(k) \otimes a^*(k) \; + \; G(k)^* \otimes a(k) \big\} \, , \tag{I.20}$$

where, for each $k \in \mathbb{R}^3 (\times \mathbb{Z}_2)$, $G(k)$ is an operator on $\mathcal{H}_{el}$, i.e., an $N \times N$ complex matrix. One could add to I terms quadratic in a^* and a, (or even of higher than second order). But, for the sake of a clear exposition of the key ideas of our analysis, let us not do that. The Hamiltonian, H_g, of the combined, coupled system at *zero temperature* corresponds to the formal sum

$$H_g \; = \; H_0 \, + \, gI \, . \tag{I.21}$$

For notational simplicity, we assume without loss of generality that $g \geq 0$. We have studied the zero-temperature dynamics of similar systems in [5, 7, 8, 9]. The purpose of this paper is to characterize the space of states of the class of open systems introduced in (I.1)–(I.3) and (I.7), (I.16), (I.18)–(I.21) describing "asymptotic thermal equilibrium" at a *positive* temperature $T > 0$, and to study properties of the dynamics of such states, as *formally* generated by the Hamiltonian H of Eq. (I.21). Under appropriate conditions on H_{el} and on the coupling matrix $G(k)$ appearing in (I.20), we establish *"return to equilibrium"* for inital states describing asymptotic equilibrium. Somewhat surprisingly, it appears that the proper mathematical formalism enabling us to formulate these problems precisely and then solve them is not widely known, although it has been developed in the sixties and early seventies. For this reason, a self-contained summary is presented in Sects. II, III, and IV.1.

Before we give a survey of the contents of this paper, we now state the conditions on H_{el} and on the coupling matrix $G(k)$ on which our analysis is based. In later sections, we refer to these conditions whereever we formulate precise results. Our first condition is as follows.

Hypothesis H- 1. *The spectrum of H_{el} consists of simple eigenvalues*

$$E_0 \; < \; E_1 \; < \; \cdots \; < \; E_{N-1} \tag{I.22}$$

corresponding to a complete, orthonormal system $\{\varphi_j\}_{j=0}^{N-1} \subseteq \mathcal{H}_{el}$ of eigenvectors.

Next, let

$$G_{i,j}(k) \; := \; \langle \varphi_i | \, G(k) \, \varphi_j \rangle \tag{I.23}$$

denote the matrix elements of the coupling matrix, $G(k)$, see Eq. (I.20), in the basis of eigenvectors of H_{el}. These matrix elements are assumed to have the following properties.

Hypothesis H- 2. *For $\theta \in \mathbb{R}$, consider the functions $G_{i,j}(e^{-\theta} k)$. For each wave vector $k \in \mathbb{R}^3$ (and helicity $\lambda = \pm 1$) and all $i, j = 1, \ldots, N$, $G_{i,j}(e^{-\theta} k)$ extends to a function of θ, also denoted by $G_{i,j}(e^{-\theta} k)$,* analytic *on a domain in $\mathbb{C}$ containing the strip*

$$\Sigma_{\vartheta_0} \; := \; \big\{ \theta \; \big| \; |\mathrm{Im}\, \theta| < \vartheta_0 \big\} \, , \tag{I.24}$$

for some $\vartheta_0 > 0$ independent *of $\vec{k} \in \mathbb{R}^3$, ($\lambda = \pm 1$) and i and j. The same property then holds for the functions*

$$G^*_{i,j}(e^{-\theta} k) \; := \; \overline{\big\langle \varphi_j \; \big| \; G(e^{-\bar{\theta}} k) \, \varphi_i \big\rangle} \, . \tag{I.25}$$

We also require a variety of *boundedness conditions* on the coupling functions $\{G_{i,j}(e^{-\theta}k)\}$.

Hypothesis H- 3. *We assume that there exists* positive *constants* $\mu > 0$ *and* $M < \infty$*, such that, for all* $\theta \in \Sigma_{\vartheta_0}$ *and* $\vec{k} \in \mathbb{R}^3$ *(and* $\lambda \pm 1$*),*

$$\sum_{i,j=1}^{N} \big| G_{i,j}(e^{-\theta}k) \big| \; \leq \; e^{M\,|\mathrm{Re}\,\theta|}\, \omega(k)^{\mu} \, , \tag{I.26}$$

where $\omega(k) = |\vec{k}|$*.*

In concrete physical models, based on the dipole approximation for the coupling of an atom with finitely many energy levels to the quantized electromagnetic field, one finds that (I.26) holds for $\mu = 1/2$.

Our next requirement is a condition on the choice of an *ultraviolet cutoff* in the interaction I which can be stated in the form of decay properties of the coupling functions $G_{i,j}(e^{-\theta}k)$, as $|\vec{k}| \to \infty$.

Hypothesis H- 4. *There exists a constant* $0 < \Lambda < \infty$ *such that, for all* $\theta \in \Sigma_{\vartheta_0}$*,*

$$\sum_{i,j=1}^{N} \int \big| G_{i,j}(e^{-\theta}k) \big|^2 \Big[\omega(k) + \omega(k)^{-3} \Big] \, dk \; \leq \; e^{2M\,|\mathrm{Re}\,\theta|}\, \Lambda^2 \, , \tag{I.27}$$

where $M < \infty$ *is the same constant as in Hypothesis H-3.*

This condition will play a crucial role in our analysis of spectral properties of the Liouville operator or thermal Hamiltonian. Among such properties the most crucial one is that *all but one eigenvalues* of the Liouville operator of the uncoupled system ($g = 0$) consisting of the (finite-level) atom and the reservoir *dissolve* in (absolutely) continuous spectrum when the interaction between the atom and the reservoir is turned on. We will show that the Liouville operator of the interacting system ($g > 0$) has a simple eigenvalue at 0 corresponding to its unique equilibrium (KMS) state, the rest of the spectrum of the Liouville operator being purely *absolutely continuous.* This is quite remarkable, because the Liouville operator of the uncoupled system ($g = 0$) has eigenvalues at $\{E_i - E_j \mid i, j = 1, \ldots, N, \; i \neq j\}$ and an N-fold degenerate eigenvalue at 0. Our proof that $N^2 - 1$ of these N^2 eigenvalues dissolve in continuous spectrum when the interaction is turned on is based on a mathematically rigorous variant of *Fermi's Golden Rule.* To make this method work, we require the following condition.

Hypothesis H- 5. *Let* $\tau_{min} := \min\big\{|E_i - E_j| \big| i \neq j\big\}$ *and* $\tau_{max} := \max\big\{|E_i - E_j|\big\}$. *For any* $i \neq j$ *and any* $\tau_{min} \leq \tau \leq 2\tau_{max}$,

$$\int dk\, \delta\big[\omega(\vec{k}) - \tau\big]\, \big|G_{i,j}(k)\big|^2 \;>\; 0\,. \tag{I.28}$$

Actually, condition (H-5) can be weakened considerably at the price of rendering the computational aspects of our analysis more complicated, see Appendix B.

Our proof of the result that the spectrum of the Liouville operator of the interacting system is purely absolutely continuous, away from 0, will involve a combination of the method of *complex spectral deformations*, more precisely a novel variant of dilatation analyticity, with a mathematically precise form of Fermi's Golden Rule based on the so-called *Feshbach map* of [7, 9]. The appearance of the complex parameter $e^{-\theta}$, $\theta \in \mathbb{C}$, in conditions (H-2) through (H-4), above, can be traced to our use of dilatation analyticity.

We now state our main results in the form of a theorem.

Theorem I.1. *Consider an open quantum system with dynamics corresponding to the formal Hamiltonian* H_g *defined in (I.21), where* H_0 *is given in (I.19) and* I *in (I.20). We assume that* H_0 *and* I *satisfy conditions (H-1) through (H-5) described above. Let*

$$L_g \;=\; L_g^{(\beta)}\,, \qquad \beta = (k_B T)^{-1}\,, \tag{I.29}$$

denote the Liouville operator acting on a Hilbert space, $\widehat{\mathcal{H}}_\beta$, *of states of the system describing asymptotic thermal equilibrium at a temperature* $T > 0$. *The operator* $L_g^{(\beta)}$ *generates the dynamics of the states in* $\widehat{\mathcal{H}}_\beta$.

Then we have that, for an arbitrary *temperature* $T > 0$, *the following hold:*

(i) $L_g^{(\beta)}$ *is essentially selfadjoint on a natural domain dense in* $\widehat{\mathcal{H}}_\beta$.

(ii) *If* $0 \leq g < g^*$, *for some* $g^* > 0$ independent *of* T, *then* $L_g^{(\beta)}$ *has a simple eigenvalue at 0 corresponding to the unique equilibrium (KMS) state of the system, and the rest of the spectrum of* $L_g^{(\beta)}$ *covers the entire real axis and is absolutely continuous.*

(iii) *Under the same assumptions, the states in* $\widehat{\mathcal{H}}_\beta$ *have the property of "return to equilibrium", in the sense described above.*

(iv) *Under certain more stringent assumptions, see Sect. V.3, (in particular,* $\mu = \frac{1}{2}$ *in (I.26)), there exists a natural linear subspace,* $\mathcal{D}_0$, *of states* *dense in $\widehat{\mathcal{H}}_\beta$ with the property that every vector in $\mathcal{D}_0$ converges to the unique equilibrium state of the system at temperture $T = (k_B\beta)^{-1}$* exponentially fast in time.

Remark I.2. *Under the hypotheses of part (iv) of Theorem I.1 stated above, one can establish precise links between our methods and those in [28], on one hand, and various heuristic treatments of "return to equilibrium" involving quantum master equations, on the other hand.*

We conclude this introduction with a brief summary of contents of the various sections of this paper.

In Sect. II, we review the general theory of pure and mixed states and of their dynamics (in the "Schrödinger picture") for quantum mechanical systems confined to bounded regions of physical space. We characterize their thermal equilibrium states and derive the Kubo-Martin-Schwinger (KMS) condition. We derive explicit expressions for the Liouville operator (or thermal Hamiltonian) in terms of the Hamiltonian and for the "modular operator" and the "modular conjugation". We describe perturbation methods for the construction of equilibrium states of interacting systems.

In Sect. III, we extend the results of Sect. II to the thermodynamic limit, following [22] and [1]. In particlar, we clarify what we mean by the notion of states in "asymptotic thermal equilibrium" at temperature $T = (k_B\beta)^{-1}$; see Sect. III.2 and III.3. We introduce the Liouville operators generating the dynamics on states which are in "asymptotic thermal equilibrium"; see Sect. III.2 and III.3. In Sect. III.4, we derive the property of "return to equilibrium" from a spectral property of the Liouville operator of a system and the KMS condition characterizing its equilibrium states. In Sect. III.5, we review the perturbation theory for equilibrium states in the thermodynamic limit.

In Sect. IV, we first review the Araki-Woods representation of equilibrium states of the quantized, free electromagnetic field. We then introduce a class of open quantum systems describing an idealized, confined atom coupled to the quantized electromagnetic field in "asymptotic thermal equilibrium" at a temperature $T > 0$. We establish selfadjointness of the Liouville operators of such systems and of related operators needed in the perturbation theory of equilibrium states. We prove that, at an arbitrary temperature $T > 0$, the systems studied in this paper have an equilibrium state which can be constructed from the equilibrium state of the quantized electromagnetic field with the help of convergent perturbation theory; see Sect. IV.3. We also establish some simple technical estimates important for our analysis.

Our *main results* (see Theorem I.1 stated above) are proven in Sect. V. In Sect. V.1, we describe these results and sketch the basic analytical methods, a novel form of *dilatation analyticity* for the Liouville operators encountered in the analysis of our class of systems and the Feshbach map of [7, 8], on which our proofs are based. All key elements of our proofs of the relevant spectral properties of the Liouville operators are explained quite carefully. In Sect. V.2, we compare and combine our approach with the one proposed in [28]. We exhibit a dense set of states in "asymptotic thermal equilibrium" which converge to a unique equilibrium state exponentially fast in time.

The remaining subsections of Sect. V and two appendices are devoted to a variety of (partly rather tedious) technical considerations. We recommend especially Sect. V.7 (a renormalization group analysis of the spectrum of Liouville operators) and Appendix B (concerning Fermi's Golden Rule) to the attention of the reader.

II Thermal Equilibrium States of Finite Systems

In this section, we recapitulate some results of [22]; see also [21, 12, 13].

II.1 Pure and Mixed States of Quantum-Mechanical Systems

We consider a quantum-mechanical system confined to a bounded region of physical space. The pure states of the system correspond to unit rays in a separable Hilbert space $\mathcal{H}$, with scalar product denoted by $(\cdot \,|\, \cdot)$. The algebra of observables of the system is a C^*-algebra, $\mathcal{A}$, contained in or equal to the algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. We assume that $\mathcal{A}$ contains the identity operator $\mathbf{1}$. The dynamics of the system is determined by a Hamiltonian, H, which is a semibounded, selfadjoint operator on $\mathcal{H}$ with discrete spectrum.

Let $\mathrm{Tr}[\cdot]$ denote the usual trace on $B(\mathcal{H})$. We assume that $\exp(-\beta H)$ is trace-class, i.e.,

$$\mathrm{Tr}\big[e^{-\beta H}\big] \; < \; \infty \, , \tag{II.1}$$

for arbitrary $\beta > 0$.

We are interested in describing general mixed states of the system, including its thermal equilibrium states, for arbitrary inverse temperature $\beta = (k_B T)^{-1}$, where k_B is Boltzmann's constant and T denotes the absolute temperature. Furthermore, we wish to study the time evolution of general mixed states, as determined by the Hamiltonian H.

According to Landau and von Neumann, a mixed state of the system corresponds to a density matrix, i.e., to a positive, selfadjoint operator on $\mathcal{H}$ of unit trace.

The two-sided ideal of trace-class operators in $B(\mathcal{H})$ is denoted by $\mathcal{L}^1(\mathcal{H})$, the two-sided ideal of Hilbert-Schmitt operators in $B(\mathcal{H})$ by $\mathcal{L}^2(\mathcal{H})$. A density matrix ρ is a positive element of $\mathcal{L}^1(\mathcal{H})$ of unit trace. Then

$$\kappa \;:=\; \rho^{1/2} \tag{II.2}$$

is Hilbert-Schmitt, with

$$\mathrm{Tr}\big[\kappa^2\big] \;=\; \mathrm{Tr}\big[\kappa^*\kappa\big] \;=\; \mathrm{Tr}[\rho] \;=\; 1\,. \tag{II.3}$$

As a *linear space*, $\mathcal{L}^2(\mathcal{H})$ is a Hilbert space with scalar product given by

$$\langle\cdot|\cdot\rangle \;:\; \mathcal{L}^2(\mathcal{H})\times\mathcal{L}^2(\mathcal{H})\to\mathbb{C}\,,\quad (\kappa,\sigma)\mapsto\langle\kappa|\sigma\rangle \;:=\; \mathrm{Tr}[\kappa^*\sigma]\,. \tag{II.4}$$

For brevity, we denote $\mathcal{L}^2(\mathcal{H})$ by $\mathcal{K}$. This Hilbert space is isomorphic to $\mathcal{H}\otimes\mathcal{H}$. It carries a representation ℓ of the algebra $\mathcal{A}$ given by

$$\ell[a]\,\kappa \;:=\; a\,\kappa \;\in\; \mathcal{K}\,, \tag{II.5}$$

for arbitrary $\kappa\in\mathcal{K}$, $a\in\mathcal{A}$. To every element $\kappa\in\mathcal{K}$, we can associate a state of the system given by the density matrix

$$\rho \;:=\; \langle\kappa|\kappa\rangle^{-1}\,\kappa\,\kappa^*\,. \tag{II.6}$$

The expectation value of an observable $a\in\mathcal{A}$ in the state ρ is given by

$$\begin{aligned}\langle a\rangle_\rho \;&:=\; \mathrm{Tr}[\rho\,a] \;=\; \langle\kappa|\kappa\rangle^{-1}\,\mathrm{Tr}\big[\kappa\kappa^*\,a\big]\\ &=\; \langle\kappa|\kappa\rangle^{-1}\,\mathrm{Tr}\big[\kappa^*\,a\,\kappa\big] && \text{(II.7)}\\ & && \text{(II.8)}\\ &=\; \langle\kappa|\kappa\rangle^{-1}\,\langle\kappa|\,\ell[a]\,\kappa\rangle\,,\end{aligned}$$

where we have used the cyclicity of the trace. For $\kappa,\sigma\in\mathcal{K}$, with $\langle\kappa|\kappa\rangle = \langle\sigma|\sigma\rangle = 1$, and $a\in\mathcal{A}$, we may define the *transition amplitudes*

$$\langle\kappa|\,\ell[a]\,\sigma\rangle\,. \tag{II.9}$$

Pure states of the system correspond to orthogonal projections $P\in\mathcal{K}$ of rank 1, i.e.,

$$P \;=\; \psi\,(\psi|\cdot) \;=\; |\psi\rangle\langle\psi|\,,\quad \psi\in\mathcal{H}\,, \tag{II.10}$$

in Dirac's bra-ket notation. Then

$$\begin{aligned} \langle P | \ell[a] P \rangle \; &= \; \mathrm{Tr}[P\, a\, P] \; = \; \mathrm{Tr}[P\, a] \\ &= \; (\psi |\, a\, \psi)\, . \end{aligned} \tag{II.11}$$

As an algebra, $\mathcal{K} = \mathcal{L}^2(\mathcal{H})$ is what is called a *Hilbert algebra*, i.e., it is a $*$-algebra, and, as a linear space, it is a Hilbert space, with the property that

$$\langle \kappa_1\, \kappa_2 |\, \kappa_3 \rangle \; = \; \langle \kappa_2 |\, \kappa_1^* \, \kappa_3 \rangle \quad \text{and} \quad \langle \kappa_1 |\, \kappa_2 \rangle \; = \; \langle \kappa_2^* |\, \kappa_1^* \rangle \, , \tag{II.12}$$

which follows from (II.4), see [22, 18].

The time evolution of an observable $a \in \mathcal{A}$ in the *Heisenberg picture* is defined, as usual, by

$$\alpha_t(a) \; := \; e^{itH}\, a\, e^{-itH} \, . \tag{II.13}$$

Then, for $\kappa, \sigma \in \mathcal{K}$,

$$\begin{aligned} \langle \kappa |\, \ell[\alpha_t(a)]\, \sigma \rangle \; &= \; \mathrm{Tr}\big[\kappa^* \, \alpha_t(a)\, \sigma\big] \\ &= \; \mathrm{Tr}\big[\kappa^* \, e^{itH}\, a\, e^{-itH}\, \sigma\big] \end{aligned} \tag{II.14}$$

$$= \; \mathrm{Tr}\big[(e^{-itH}\, \kappa\, e^{itH})^* \, a\, (e^{-itH}\, \sigma\, e^{itH})\big] \tag{II.15}$$

$$= \; \big\langle \alpha_{-t}(\kappa) \big|\, \ell[a]\, \alpha_{-t}(\sigma) \big\rangle \, , \tag{II.16}$$

by cyclicity of the trace. Thus, it is useful to define the time evolution of an element $\kappa \in \mathcal{K}$ in the *Schrödinger picture* by

$$\kappa \; \mapsto \; \kappa_t \; := \; \alpha_{-t}(\kappa) \; := \; e^{-itH}\, \kappa\, e^{itH} \, , \tag{II.17}$$

for $t \in \mathbb{R}$. We define a selfadjoint linear operator $\mathcal{L}$, the *Liouvillian*, on $\mathcal{K}$ by setting

$$\mathcal{L}\, \kappa \; := \; [\, H \, , \, \kappa \,] \, , \tag{II.18}$$

where $[\cdot, \cdot]$ denotes the commutator. Under our hypotheses on H, the operator $\mathcal{L}$ is essentially selfadjoint on the following core $\mathcal{D}$ dense in $\mathcal{K}$,

$$\mathcal{D} \; := \; \mathrm{span}\big\{ \, |\psi_i\rangle\langle\psi_j| \; \big| \; i, j \; = \; 0, 1, 2, \ldots \big\} \, , \tag{II.19}$$

where $\{\psi_i\}_{i=0}^{\infty}$ is a complete orthonormal system of eigenvectors of H, i.e.,

$$H\psi_i \; = \; E_i \psi_i \, , \quad i \; = \; 0, 1, 2, \ldots \, , \tag{II.20}$$

with

$$E_0 \;\le\; E_1 \;\le\; E_2 \;\le\; \cdots \,. \tag{II.21}$$

It is instructive to verify that $\mathcal{L}$ is a symmetric operator on $\mathcal{D} \ni \kappa, \sigma$ by computation,

$$\begin{aligned} \big\langle \mathcal{L}\kappa \big| \, \sigma \big\rangle \;&=\; \mathrm{Tr}\big[(\mathcal{L}\kappa)^* \, \sigma\big] \;=\; \mathrm{Tr}\big([\, H \,,\, \kappa\,]^* \, \sigma\big) \\ &=\; -\mathrm{Tr}\big([\, H \,,\, \kappa^* \,] \, \sigma\big) \;=\; \mathrm{Tr}\big(\kappa^* [\, H \,,\, \sigma\,]\big) \\ &=\; \big\langle \kappa \big| \, \mathcal{L}\sigma \big\rangle \,, \end{aligned} \tag{II.22}$$

using the cyclicity of the trace. Eq. (II.17) can now be rewritten as

$$\kappa_t \;=\; e^{-it\mathcal{L}} \, \kappa \,, \tag{II.23}$$

and one easily verifies that

$$e^{it\mathcal{L}} \, \ell[a] \, e^{-it\mathcal{L}} \;=\; \ell(\alpha_t(a)) \,. \tag{II.24}$$

II.2 Equilibrium States of Bounded Systems

Since we are interested in studying systems in thermal equilibrium, we must identify those vectors in $\mathcal{K}$ which describe equilibrium states at an arbitrary inverse temperature β; see [22, 18, 12, 13] for more details. Let $\mathcal{A}'$ denote the von Neumann algebra of all bounded operators on $\mathcal{H}$ which commute with all operators in $\mathcal{A}$, the *commutant* of $\mathcal{A}$. A selfadjoint operator Q on $\mathcal{H}$ is said to be affiliated with $\mathcal{A}'$ iff all spectral projections of Q belong to $\mathcal{A}'$. We say that Q commutes with H iff all spectral projections of Q commute with all spectral projections of H.

According to Gibbs, Landau, and von Neumann, every equilibrium state of the system at inverse temperature β is given by a density matrix

$$\rho_{\beta,Q} \;:=\; Z_{\beta,Q}^{-1} \, \exp[-\beta(H - Q)] \,, \tag{II.25}$$

where Q is an arbitrary selfadjoint operator on $\mathcal{H}$ affiliated with $\mathcal{A}'$, commuting with H, and such that

$$Z_{\beta,Q}^{-1} \;:=\; \mathrm{Tr}\big(\exp[-\beta(H - Q)]\big) \;<\; \infty \,. \tag{II.26}$$

The physical interpretation of Q is that of a *conserved charge* of the system. Since Q is affiliated with $\mathcal{A}'$, observables (i.e., elements of $\mathcal{A}$) are *neutral* with respect to Q.

An element κ of $\mathcal{K}$ describes a thermal equilibrium state of the system iff

$$\kappa \;\equiv\; \kappa_{\beta,Q} \;=\; Z_{\beta,Q}^{-1/2}\, \exp[-\beta(H-Q)/2]\, U\,, \tag{II.27}$$

for an arbitrary unitary operator U on $\mathcal{H}$. For,

$$\begin{aligned} \langle \kappa_{\beta,Q} |\, \ell[a]\, \kappa_{\beta,Q} \rangle \;&=\; \mathrm{Tr}\big[\kappa_{\beta,Q}^{*}\, a\, \kappa_{\beta,Q}\big] \\ &=\; Z_{\beta,Q}^{-1}\, \mathrm{Tr}\big[\kappa_{\beta,Q}\, \kappa_{\beta,Q}^{*}\, a\big] \\ &=\; Z_{\beta,Q}^{-1}\, \mathrm{Tr}\big(\exp[-\beta(H-Q)]\, a\big) \;=\; \langle a \rangle_{\beta,Q}\,. \end{aligned} \tag{II.28}$$

Next, we recall some general properties of equilibrium states (see [22, 21, 12, 13]).

First, since $\rho_{\beta,Q}$ is strictly positive, for $\beta < \infty$, we have that, for arbitrary $a \in \mathcal{A}$,

$$\langle\, a^*\, a\, \rangle_{\beta,Q} \;=\; 0 \quad\Longrightarrow\quad a \;=\; 0\,. \tag{II.29}$$

Equivalently,

$$\ell[a]\, \kappa_{\beta,Q} \;=\; 0 \quad\Longrightarrow\quad a \;=\; 0\,. \tag{II.30}$$

Thus, the vector $\kappa_{\beta,Q} \in \mathcal{K}$ is *separating* for the algebra $\ell[\mathcal{A}]$.

Second, the state $\rho_{\beta,Q}$ satisfies the *Kubo-Martin-Schwinger (KMS) condition*:

$$\begin{aligned} \langle\, a\, \alpha_t(b)\, \rangle_{\beta,Q} \;&=\; Z_{\beta,Q}^{-1}\, \mathrm{Tr}\big[\exp[-\beta(H-Q)]\, a\, e^{itH}\, b\, e^{-itH}\big] \\ &=\; Z_{\beta,Q}^{-1}\, \mathrm{Tr}\big[e^{\beta Q}\, e^{itH}\, b\, e^{-(\beta+it)H}\, a\big] \\ &=\; Z_{\beta,Q}^{-1}\, \mathrm{Tr}\big[e^{-\beta(H-Q)}\, e^{(\beta+it)H}\, b\, e^{-(\beta+it)H}\, a\big] \\ &=\; \langle\, \alpha_{-i\beta+t}(b)\, a\, \rangle_{\beta,Q}\,. \end{aligned} \tag{II.31}$$

In the second equation, we have used the cyclicity of the trace and the fact that H and b commute with $e^{\beta Q}$, in the strong sense specified above. Defining

$$F_{ab}(t) \;:=\; \langle\, a\, \alpha_t(b)\, \rangle_{\beta,Q}\,, \tag{II.32}$$

$$G_{ab}(t) \;:=\; \langle\, \alpha_t(b)\, a\, \rangle_{\beta,Q}\,, \tag{II.33}$$

the KMS condition says that the function $G_{ab}(t)$ is the boundary value of a function $G_{ab}(z)$ analytic in z in the strip

$$S_{-\beta} := \{\, \zeta \;|\; -\beta < \mathrm{Im}\zeta < 0\,\}\,, \tag{II.34}$$

and

$$\lim_{\eta \nearrow \beta} G_{ab}(t - i\eta) \;=\; F_{ab}(t)\,. \tag{II.35}$$

Equivalently, $F_{ab}(t)$ is the boundary value of a function $F_{ab}(z)$ analytic in z in the strip S_β, with

$$\lim_{\eta \nearrow \beta} F_{ab}(t + i\eta) \;=\; G_{ab}(t)\,. \tag{II.36}$$

Third, we have that

$$\langle\, a\, \alpha_t(b)\,\rangle_{\beta,Q} \;=\; \langle\, \alpha_{-t}(a)\, b\,\rangle_{\beta,Q}\,, \tag{II.37}$$

as follows from a trivial calculation similar to that in (II.31) or directly from the KMS condition. In particular,

$$\langle\, \alpha_t(b)\,\rangle_{\beta,Q} \;=\; \langle\, b\,\rangle_{\beta,Q}\,, \tag{II.38}$$

i.e., the state $\rho_{\beta,Q}$ is *time-translation invariant*; (set $a = \mathbf{1}$ in (II.37)).

We also note that the time-translation invariance (II.38) and the KMS condition imply that

$$\begin{aligned} \big\langle a^*\, b \big\rangle_{\beta,Q} \;&=\; \big\langle \alpha_{i\beta/2}(a^*\, b) \big\rangle_{\beta,Q} \;=\; \big\langle \alpha_{i\beta/2}(a^*)\, \alpha_{i\beta/2}(b) \big\rangle_{\beta,Q} \\ &=\; \big\langle \alpha_{-i\beta/2}(b)\, \alpha_{i\beta/2}(a^*) \big\rangle_{\beta,Q} \;=\; \big\langle \alpha_{-i\beta/2}(b)\, \big(\alpha_{-i\beta/2}(a)\big)^* \big\rangle_{\beta,Q}\,. \end{aligned} \tag{II.39}$$

We have used (II.38) in the first and the KMS condition (II.31) in the third equation.

We have now summarized all important elements of the quantum mechanics of finite systems in or close to thermal equilibrium. However, we shall shortly pass to the study of infinite systems which may be viewed as thermodynamic limits of finite systems. We shall analyze their properties in or close to thermal equilibrium and their behaviour under small perturbations of their dynamics by coupling them to finite subsystems. In order to prepare the ground for our analysis, we must elaborate on several aspects of the theory of finite systems.

II.3 The Commutant of the Representation ℓ of $\mathcal{A}$ on $\mathcal{K}$.

First, we note that the Hilbert space $\mathcal{K}$ of Hilbert-Schmitt operators carries a second, *anti-linear* representation, r, of the observable algebra $\mathcal{A}$ which commutes with the representation ℓ introduced in (II.5). It is defined by

$$r[a]\,\kappa \;:=\; \kappa\, a^*\,, \tag{II.40}$$

for $\kappa \in \mathcal{K}$ and $a \in \mathcal{A}$. It is now clear that ℓ stands for *left-representation* and r for *right-representation.* Obviously

$$r[za] \;=\; \bar{z}\, r[a]\,, \quad z \in \mathbb{C}\,, \tag{II.41}$$

and

$$r[a\, b] \;=\; r[a]\, r[b]\,. \tag{II.42}$$

It is not hard to show that if $\mathcal{A}$ is weakly dense in $B(\mathcal{H})$, i.e.,

$$\overline{\mathcal{A}}^{\,w} \;=\; (\mathcal{A}')' \;\equiv\; \mathcal{A}'' \;=\; B(\mathcal{H})\,, \tag{II.43}$$

then

$$\ell[\mathcal{A}]'' \;=\; r[\mathcal{A}]'\,. \tag{II.44}$$

It is instructive to try to understand where Eq. (II.44) comes from. Let C be an anti-unitary involution on $\mathcal{H}$, i.e.,

$$C^2 \;=\; \mathbf{1} \quad \text{and} \quad (C\psi | C\varphi) \;=\; (\varphi | \psi)\,, \tag{II.45}$$

for arbitrary $\varphi, \psi \in \mathcal{H}$. (In a suitable orthonormal basis of $\mathcal{H}$, C acts on $\psi \in \mathcal{H}$ as ± complex conjugation of the components of ψ in that basis.) Given C, we construct an isomorphism,

$$I_C \;:\; \mathcal{K} \;\to\; \mathcal{H} \otimes \mathcal{H}\,. \tag{II.46}$$

If $\kappa = (\psi_2 | \cdot\,)\, \psi_1 \in \mathcal{K}$, i.e.,

$$\kappa \;=\; |\psi_1\rangle\langle\psi_2|\,, \tag{II.47}$$

in Dirac's bra-ket notation, then

$$I_C \kappa \;:=\; \psi_1 \otimes C\psi_2 \;\in\; \mathcal{H} \otimes \mathcal{H}\,. \tag{II.48}$$

Next, we note that, for κ as in (II.47),

$$I_C \ell[a]\kappa \;=\; I_C(a\kappa) \;=\; I_C(|a\psi_1\rangle\langle\psi_2|) \;=\; a\psi_1 \otimes C\psi_2 \;=\; (a \otimes \mathbf{1}) I_C \kappa\,, \tag{II.49}$$

and

$$I_C r[a]\kappa \;=\; I_C(\kappa a^*) \;=\; I_C(|\psi_1\rangle\langle a\psi_2|) \;=\; \psi_1 \otimes Ca\psi_2 \;=\; (\mathbf{1} \otimes CaC) I_C \kappa\,, \tag{II.50}$$

where we have used (II.40) in the first, (II.47) in the second, (II.48) in the third, and (II.45) ($C^2 = \mathbf{1}$) in the last equation.

Thus I_C intertwines the linear representation ℓ of $\mathcal{A}$ on $\mathcal{H} \otimes \mathcal{H}$ given by

$$\ell[a] \;=\; a \otimes \mathbf{1}\,, \quad a \in \mathcal{A}\,, \tag{II.51}$$

with the linear representation ℓ of $\mathcal{A}$ on $\mathcal{K}$ and the anti-linear representation r of $\mathcal{A}$ on $\mathcal{H} \otimes \mathcal{H}$ given by

$$r[a] \;=\; \mathbf{1} \otimes C\,a\,C\,, \quad a \in \mathcal{A}\,, \tag{II.52}$$

with the anti-linear representation r of $\mathcal{A}$ on $\mathcal{K}$.

If $\mathcal{A}$ is weakly dense in $B(\mathcal{H})$ then

$$\ell[\mathcal{A}]'' \;=\; B(\mathcal{H}) \otimes \mathbf{1}\,, \quad r[\mathcal{A}]'' \;=\; \mathbf{1} \otimes B(\mathcal{H})\,, \tag{II.53}$$

where we use that the weak closure of a $*$-algebra of operators on a separable Hilbert space is equal to its double commutant. Clearly,

$$\big(\mathbf{1} \otimes B(\mathcal{H})\big)' \;=\; B(\mathcal{H}) \otimes \mathbf{1}\,, \tag{II.54}$$

and, using that the commutant of a $*$-algebra of operators on a separable Hilbert space is the same as the commutant of its weak closure, (II.44) follows.

II.4 The modular operators $\mathcal{S}$ and $\mathcal{J}$

There is a distinguished linear operator E acting on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, defined by

$$E(\varphi \otimes \psi) \;:=\; \psi \otimes \varphi\,. \tag{II.55}$$

The operator E is called *exchange operator*. In terms of E and the anti-unitary involution C, we may define what is called the *modular conjugation* J by setting

$$J \;:=\; E\,(C \otimes C) \;=\; (C \otimes C)\,E\,. \tag{II.56}$$

Clearly, J is an anti-unitary involution, and, remarkably (though trivially),

$$J\,\ell\,J \;=\; r\,. \tag{II.57}$$

For

$$\begin{aligned} J\,\ell[a]\,J\,(\psi \otimes \varphi) \;&=\; J\,\ell[a]\,C\varphi \otimes C\psi \;=\; J\,(aC\varphi \otimes C\psi) \;=\; \psi \otimes CaC\varphi \\ &=\; (\mathbf{1} \otimes C\,a\,C)\,(\psi \otimes \varphi) \;=\; r[a]\,(\psi \otimes \varphi)\,. \end{aligned} \tag{II.58}$$

It is easy to check that

$$I_C^{-1}\, J\, I_C\, \kappa \;=\; \kappa^* \,, \quad \text{for } \kappa \in \mathcal{K}. \tag{II.59}$$

The operator $I_C^{-1}\, J\, I_C$ is denoted by $\mathcal{J}$.

Let $\mathcal{A}$ be weakly dense in $B(\mathcal{H})$, and let ρ be a *stricly positive* density matrix on $\mathcal{H}$. Let $\kappa_\rho := \rho^{1/2}$. As in (II.29)–(II.30), one notes that κ_ρ is separating for $\ell[\mathcal{H}]$ and for $r[\mathcal{H}]$. The separating property of κ_ρ and Eq. (II.44) then imply that κ_ρ is *cyclic* for $r[\mathcal{H}]$ and for $\ell[\mathcal{H}]$, i.e., the subspaces

$$\ell[\mathcal{A}]\kappa_\rho \quad \text{and} \quad r[\mathcal{A}]\kappa_\rho \tag{II.60}$$

are *dense* in $\mathcal{K}$. We may therefore define an (in general unbounded) anti-linear operator $\mathcal{S}$, the *modular operator*, by setting

$$\mathcal{S}\big(\ell[a]\,\kappa_\rho\big) \;:=\; \ell[a^*]\,\kappa_\rho \,. \tag{II.61}$$

Thus

$$\begin{aligned} \mathcal{S}\big(\ell[a]\,\kappa_\rho\big) \;&=\; a^*\,\kappa_\rho \;=\; \kappa_\rho^{-1}\,\kappa_\rho\, a^*\,\kappa_\rho \\ &=\; \kappa_\rho^{-1}\,(r[a]\,\kappa_\rho)\,\kappa_\rho \;=\; \kappa_\rho^{-1}\,\big(\mathcal{J}(\ell[a]\,\kappa_\rho)\big)\,\kappa_\rho \,, \end{aligned} \tag{II.62}$$

by (II.57)–(II.58), and because $\kappa_\rho^* = \kappa_\rho$. Since ρ is strictly positive, the operator

$$H \;:=\; -\frac{1}{\beta}\ln\rho \tag{II.63}$$

is a semibounded, selfadjoint operator on $\mathcal{H}$: Eq. (II.62) can then be rewritten as

$$\mathcal{S}\big(\ell[a]\,\kappa_\rho\big) \;=\; e^{\beta H/2}\,\big(\mathcal{J}(\ell[a]\,\kappa_\rho)\big)\,e^{-\beta H/2} \,. \tag{II.64}$$

If $\mathcal{L}$ denotes the Liouvillian associated with the *Hamiltonian* H, i.e., if

$$\mathcal{L}\kappa \;=\; [\,H\,,\,\kappa\,] \;=\; \big(\ell[H] - r[H]\big)\,\kappa \,, \tag{II.65}$$

then (II.64) boils down to

$$\mathcal{S}\big(\ell[a]\,\kappa_\rho\big) \;=\; e^{\beta\mathcal{L}/2}\,\mathcal{J}(\ell[a]\,\kappa_\rho) \,, \tag{II.66}$$

or, by (II.60),

$$\mathcal{S} \;=\; e^{\beta\mathcal{L}/2}\,\mathcal{J} \,, \tag{II.67}$$

i.e.,

$$e^{\beta\mathcal{L}/2} \;=\; |\mathcal{S}|\,, \quad \mathcal{J} \;=\; |\mathcal{S}|^{-1}\,\mathcal{S}\,. \tag{II.68}$$

In other words, (II.67) describes the *polar decomposition* of $\mathcal{S}$. Eq. (II.61) is the starting point of a theory which works in much greater generality: *Tomita-Takesaki theory* (see, e.g., [41, 12]). Eqs. (II.68) then *define* $\exp[\beta\mathcal{L}/2]$ and $\mathcal{J}$, respectively (after a painful proof that the operator $\mathcal{S}$ defined by (II.67) is closed).

If we prefer to work on the Hilbert space $\mathcal{H}\otimes\mathcal{H}$, instead of $\mathcal{K}$, the operators $\mathcal{S}$ and $\mathcal{L}$ are replaced by

$$S \;=\; I_C\,\mathcal{S}\,I_C^{-1}\,, \quad L \;=\; I_C\,\mathcal{L}\,I_C^{-1}\,, \tag{II.69}$$

and $J \;=\; I_C\,\mathcal{J}\,I_C^{-1}$ is given by (II.56).

It is instructive to determine the explicit forms of S and L. By (II.51) and (II.69),

$$\begin{aligned} S\,\ell[a]\,I_C\,\kappa_\rho \;&=\; S\,I_C\,\ell[a]\,\kappa_\rho \;=\; I_C\,\mathcal{S}\,I_C^{-1}\,I_C\,\ell[a]\,\kappa_\rho \\ &=\; I_C\,\ell[a^*]\,\kappa_\rho \;=\; \ell[a^*]\,I_C\,\kappa_\rho\,. \end{aligned} \tag{II.70}$$

Next, for $\kappa = |\psi\rangle\langle\varphi|$, with $\psi, \varphi \in \mathcal{H}$,

$$\mathcal{L}\,\kappa \;=\; |H\psi\rangle\langle\varphi| \;-\; |\psi\rangle\langle H\varphi|\,, \tag{II.71}$$

by (II.18). Thus

$$\begin{aligned} I_C\,\mathcal{L}\,\kappa \;&=\; (H\psi)\otimes(C\varphi) \;-\; \psi\otimes(CH\varphi) \\ &=\; (H\psi)\otimes(C\varphi) \;-\; \psi\otimes\big((CHC)C\varphi\big) \\ &=\; (H\otimes\mathbf{1} \;-\; \mathbf{1}\otimes CHC)\,(\psi\otimes C\varphi) \\ &=\; (H\otimes\mathbf{1} \;-\; \mathbf{1}\otimes CHC)\,I_C\,\kappa\,. \end{aligned} \tag{II.72}$$

We conclude that

$$L \;=\; H\otimes\mathbf{1} \;-\; \mathbf{1}\otimes CHC\,. \tag{II.73}$$

In their important paper [3] on the equilibrium states of non-interacting bosons, Araki and Woods make a special choice for C: They choose C to be given by the *time-reversal operator*, T, which, according to a result of Wigner, is indeed an anti-unitary involution on $\mathcal{H}$. For this choice,

$$T\,H\,T \;=\; H\,, \tag{II.74}$$

and hence, for the Araki-Woods isomorphism $I_T : \mathcal{K} \to \mathcal{H}\otimes\mathcal{H}$,

$$L \;=\; H\otimes\mathbf{1} \;-\; \mathbf{1}\otimes H\,. \tag{II.75}$$

II.5 Perturbation Theory for Equilibrium States

Next, we consider a finite system with dynamics determined by a Hamiltonian H of the form

$$H \;=\; H_0 \,+\, I\,, \tag{II.76}$$

where H_0 is the Hamiltonian of an (unperturbed) *reference* system, and I is a perturbation. In this section, we assume that I is a *bounded* selfadjoint operator, (because we want to avoid obscuring the general theory with operator domain problems). The equilibrium state of the system is given by the density matrix

$$\rho_{\beta,Q} \;=\; Z_{\beta,Q}^{-1}\,\exp[-\beta(H-Q)] \tag{II.77}$$

which corresponds to the vector

$$\kappa_{\beta,Q} \;=\; \rho_{\beta,Q}^{1/2} \;\in\; \mathcal{K}\,. \tag{II.78}$$

We assume that Q is affiliated with $\mathcal{A}'$ and that it commutes with H_0 and with I and thus with H. Then

$$\kappa_{\beta,Q} \;=\; e^{\beta Q/2}\,Z_{\beta,Q}^{-1/2}\,e^{-\beta H/2}\,. \tag{II.79}$$

The equilibrium state of the reference system is given by

$$\kappa_{\beta,Q}^{0} \;=\; \big(Z_{\beta,Q}^{0}\big)^{-1/2}\,e^{\beta Q/2}\,e^{-\beta H_0/2} \tag{II.80}$$

corresponding to the density matrix

$$\rho_{\beta,Q}^{0} \;=\; \big(\kappa_{\beta,Q}^{0}\big)^{2}\,. \tag{II.81}$$

Note that

$$\begin{aligned} Z_{\beta,Q} \;&=\; \mathrm{Tr}\big[e^{-\beta(H-Q)}\big] \;=\; \mathrm{Tr}\big[e^{-\beta(H_0-Q+I)}\big] \;\le\; \mathrm{Tr}\big[e^{-\beta(H_0-Q)}\,e^{-\beta I}\big] \\ &\le\; e^{\beta\|I\|}\,\mathrm{Tr}\big[e^{-\beta(H_0-Q)}\big] \;=\; e^{\beta\|I\|}\,Z_{\beta,Q}^{0}\,, \end{aligned} \tag{II.82}$$

where the first inequality is the so-called Golden-Thompson inequality (which follows from the Trotter product formula and the Hölder inequality, see, e.g., [13, 37]), and the second inequality follows from $\|e^{-\beta I}\| \le e^{\beta\|I\|}$, for $\beta > 0$. Thus, when I is bounded, $Z_{\beta,Q}$ is finite iff $Z_{\beta,Q}^{0}$ is.

The Liouvillian of the reference system and the Liouvillian of the interacting system are given by

$$\mathcal{L}_0 \;:=\; \ell[H_0] \,-\, r[H_0]\,, \tag{II.83}$$

$$\mathcal{L} \;:=\; \ell[H] \,-\, r[H] \;=\; \mathcal{L}_0 \,+\, \ell[I] \,-\, r[I]\,, \tag{II.84}$$

respectively. We also define two *Radon-Nikodym operators*, $\mathcal{L}_\ell$ and $\mathcal{L}_r$, by setting

$$\mathcal{L}_\ell \;:=\; \mathcal{L}_0 + \ell[I]\,, \quad \mathcal{L}_r \;:=\; \mathcal{L}_0 - r[I]\,. \tag{II.85}$$

Note that

$$e^{z\ell[a]} \;=\; \ell\big[e^{z\,a}\big]\,, \quad e^{zr[a]} \;=\; r\big[e^{\bar{z}\,a}\big]\,, \tag{II.86}$$

for $a \in \mathcal{A}$ and $z \in \mathbb{C}$, as follows from the fact that ℓ is a linear homomorphism, while r is an anti-linear homomorphism. By (II.85) and (II.86),

$$\begin{aligned} e^{-\beta\mathcal{L}_\ell/2}\,\kappa^0_{\beta,Q} \;&=\; e^{-\frac{\beta}{2}\big(\ell[H_0]+\ell[I]-r[H_0]\big)}\,\kappa^0_{\beta,Q} \;=\; e^{-\frac{\beta}{2}(H_0+I)}\,\kappa^0_{\beta,Q}\,e^{\frac{\beta}{2}H_0} \\ &=\; \big(Z^0_{\beta,Q}\big)^{-1/2}\,e^{-\frac{\beta}{2}H}\,e^{\frac{\beta}{2}Q}\,e^{-\frac{\beta}{2}H_0}\,e^{\frac{\beta}{2}H_0} \\ &=\; \big(Z_{\beta,Q}/Z^0_{\beta,Q}\big)^{1/2}\,\big(Z_{\beta,Q}\big)^{-1/2}\,e^{-\frac{\beta}{2}(H-Q)} \\ &=\; \big(Z_{\beta,Q}/Z^0_{\beta,Q}\big)^{1/2}\,\kappa_{\beta,Q}\,, \end{aligned} \tag{II.87}$$

and a similar calculation yields

$$e^{\beta\mathcal{L}_r/2}\,\kappa^0_{\beta,Q} \;=\; \big(Z_{\beta,Q}/Z^0_{\beta,Q}\big)^{-1/2}\,\kappa_{\beta,Q}\,. \tag{II.88}$$

It follows from (II.82) and (II.87), (II.88) that $\kappa^0_{\beta,Q}$ is in the domain of definition of the (generally *unbounded*) operators $\exp[-\beta\mathcal{L}_\ell/2]$ and $\exp[\beta\mathcal{L}_r/2]$, and

$$\kappa_{\beta,Q} \;=\; \big(Z^0_{\beta,Q}/Z_{\beta,Q}\big)^{1/2}\,e^{-\frac{\beta}{2}\mathcal{L}_\ell}\,\kappa^0_{\beta,Q} \;=\; \big(Z^0_{\beta,Q}/Z_{\beta,Q}\big)^{1/2}\,e^{\frac{\beta}{2}\mathcal{L}_r}\,\kappa^0_{\beta,Q}\,. \tag{II.89}$$

Formula (II.89) is a non-commutative version of the Radon-Nikodym derivative in measure theory; see [1, 2].

Under the isomorphism I_C, $\mathcal{L}$, $\mathcal{L}_\ell$, and $\mathcal{L}_r$ are mapped to

$$L \;:=\; I_C\,\mathcal{L}\,I_C^{-1} \;=\; L_0 + I\otimes\mathbf{1} - \mathbf{1}\otimes CIC \;=:\; L_0 + W \tag{II.90}$$

$$L_\ell \;:=\; I_C\,\mathcal{L}_\ell\,I_C^{-1} \;=\; L_0 + I\otimes\mathbf{1}\,, \tag{II.91}$$

$$L_r \;:=\; I_C\,\mathcal{L}_r\,I_C^{-1} \;=\; L_0 - \mathbf{1}\otimes CIC\,, \tag{II.92}$$

with

$$L_0 \;:=\; H_0\otimes\mathbf{1} - \mathbf{1}\otimes CH_0C\,, \tag{II.93}$$

$$W \;:=\; I\otimes\mathbf{1} - \mathbf{1}\otimes CIC\,. \tag{II.94}$$

These formulae will turn out to be very useful in our analysis of concrete systems.

III Equilibrium States in the Thermodynamic Limit

III.1 Thermodynamic Limit

In this section, we recapitulate the general theory of infinite systems, i.e., systems in the thermodynamic limit, in or close to thermal equilibrium, and we discuss spectral properties of their time evolution that guarantee *return to equilibrium* of states which are local perturbations of equilibrium states.

Finite systems can be indexed by regions, Λ, in physical space. The thermodynamic limit is reached, as Λ increases to all of physical space. It is reasonable to start from the following assumptions.

III.1.1 Observable Algebras

Let $\mathcal{A}_\Lambda$ denote the observable algebra of a system confined to Λ. Then, for $\Lambda_1 \subseteq \Lambda_2$,

$$\mathcal{A}_{\Lambda_1} \subseteq \mathcal{A}_{\Lambda_2} \subseteq \mathcal{A} , \tag{III.1}$$

where $\mathcal{A}$ is a C^*-algebra describing the *observables* of the system in the thermodynamic limit. It is usually assumed that if $\{\Lambda_\imath\}_{\imath \in \mathbb{N}}$ is a family of regions increasing to all of physical space then

$$\mathcal{A} = \overline{\bigvee_{\imath \in \mathbb{N}} \mathcal{A}_{\Lambda_\imath}}^{n} , \tag{III.2}$$

where the closure is taken in the operator norm.

III.1.2 Time Evolution

As described in Sect. II, the algebras $\mathcal{A}_\Lambda$ are assumed to be contained in $B(\mathcal{H}_\Lambda)$, where $\mathcal{H}_\Lambda$ is a separable Hilbert space whenever Λ is a bounded subset of physical space. The dynamics is determined by a semibounded, selfadjoint Hamiltonian H_Λ on $\mathcal{H}_\Lambda$ with the property that $\exp[-\beta H_\Lambda]$ is trace class, i.e.,

$$\exp[-\beta H_\Lambda] \in \mathcal{L}^1(\mathcal{H}_\Lambda) , \tag{III.3}$$

for arbitrary $\beta > 0$, and that

$$\alpha_t^\Lambda(a) = e^{itH_\Lambda} \, a \, e^{-itH_\Lambda} , \quad \text{for } a \in \mathcal{A}_\Lambda, \tag{III.4}$$

defines a $*$-automorphism group of $\mathcal{A}_\Lambda$, i.e., $\alpha_t^\Lambda(a)\,\alpha_t^\Lambda(b) = \alpha_t^\Lambda(ab)$, $\alpha_t^\Lambda(a)^* = \alpha_t^\Lambda(a^*)$, and $\alpha_t^\Lambda(a) \in \mathcal{A}_\Lambda$, for arbitrary $a \in \mathcal{A}_\Lambda$, $t \in \mathbb{R}$. By (III.1), $\alpha_t^\Lambda(a)$ is well-defined and belongs to $\mathcal{A}_\Lambda$, for all $a \in \mathcal{A}_{\Lambda'}$, with $\Lambda' \subseteq \Lambda$. We denote $\bigvee_{\imath\in\mathbb{N}} \mathcal{A}_{\Lambda_\imath}$ by $\mathcal{A}_\infty$. For any $a \in \mathcal{A}_\infty$, $\alpha_t^{\Lambda_\imath}(a)$ is thus well-defined, for $\imath$ sufficiently large. In a general theory of thermal equilibrium states, one will assume that

$$\lim_{\imath} \alpha_t^{\Lambda_\imath}(a) \;=:\; \alpha_t(a) \tag{III.5}$$

exists, for all $a \in \mathcal{A}_\infty$ and all $t \in \mathbb{R}$, and that $\{\alpha_t\}_{t\in\mathbb{R}}$ is a one-parameter $*$-automorphism group of the algebra $\mathcal{A}$. It describes the time evolution of the *observables* of the infinite system in the Heisenberg picture.

III.1.3 Equilibrium States [40]

As discussed in Subsect. III.1.2, Assumptions (III.3) and (III.4) guarantee that a finite system confined to a region Λ has equilibrium states $\langle\,\cdot\,\rangle_{\beta,Q}^{\Lambda}$, see Eq. (II.28), which satisfy the KMS condition, Eqs. (II.31), (II.35), (II.36).

For $a \in \mathcal{A}_\infty$, we may consider the sequence of expectation values $\langle a\rangle_{\beta,Q}^{\Lambda_\imath}$, which are well defined if $\imath$ is large enough, depending on a. Let $\omega_\beta(\,\cdot\,)$ denote a limit of a (suitable subsequence of) the sequence $\langle\,\cdot\,\rangle_{\beta,Q}^{\Lambda_\imath}$ of states on $\mathcal{A}_\infty$ Then ω_β is a *time-translation invariant* KMS *state* on $\mathcal{A}_\infty$, i.e.,

$$\omega_\beta\big(\alpha_t(a)\big) \;=\; \omega_\beta(a)\,, \tag{III.6}$$

and, defining

$$F_{ab}(t) \;:=\; \omega_\beta\big(a\,\alpha_t(b)\big)\,, \tag{III.7}$$

$$G_{ab}(t) \;:=\; \omega_\beta\big(\alpha_t(b)\,a\big)\,, \tag{III.8}$$

we have the KMS condition

$$G_{ab}(t - i\beta) \;=\; F_{ab}(t)\,, \tag{III.9}$$

$$F_{ab}(t + i\beta) \;=\; G_{ab}(t)\,, \tag{III.10}$$

for $a, b \in \mathcal{A}_\infty$; see Eqs. (II.38), (II.35), (II.36). By continuity, these equations continue to hold for arbitrary $a, b \in \mathcal{A}$.

We define a $*$-algebra $\overset{\circ}{\mathcal{A}}$ by

$$\overset{\circ}{\mathcal{A}} := \left\{ a_f \;=\; \int dt\, f(t)\,\alpha_t(a) \;\middle|\; a \in \mathcal{A}\;,\; \hat{f} \in C_0^\infty(\mathbb{R}) \right\}, \tag{III.11}$$

where $\hat{f}$ denotes the Fourier transform of f. Since α_t has been assumed to be a $*$-automorphism group of $\mathcal{A}$, $\overset{\circ}{\mathcal{A}}$ is a $*$-subalgebra of $\mathcal{A}$, and α_t leaves

$\overset{\circ}{\mathcal{A}}$ invariant. For any $a \in \overset{\circ}{\mathcal{A}}$, $\alpha_t(a)$ is the boundary value of an $\overset{\circ}{\mathcal{A}}$-valued entire function $\alpha_z(a)$, $z \in \mathbb{C}$. For $a \in \mathcal{A}$ and $b \in \overset{\circ}{\mathcal{A}}$, we can rewrite the KMS condition as the equation

$$\omega_\beta\big(a\,\alpha_t(b)\big) \;=\; \omega_\beta\big(\alpha_{-i\beta+t}(b)\,a\big)\,. \tag{III.12}$$

Note that

$$\alpha_z(a)^* \;=\; \alpha_{\bar{z}}(a^*)\,, \tag{III.13}$$

for $z \in \mathbb{C}$, $a \in \overset{\circ}{\mathcal{A}}$. The invariance (III.6), the KMS condition (III.12), and (III.13) then imply that, for $a, b \in \overset{\circ}{\mathcal{A}}$,

$$\omega_\beta\big(a^*\,b\big) \;=\; \omega_\beta\big(\alpha_{i\beta/2}(a^*\,b)\big) \;=\; \omega_\beta\big(\alpha_{-i\beta/2}(b)\,[\alpha_{-i\beta/2}(a)]^*\big)\,. \tag{III.14}$$

This equation has a noteworthy consequence: If $\mathcal{A}$ is a *simple* C^*-algebra (i.e., $\mathcal{A}$ does not contain any two-sided $*$-ideals other than $\{0\}$ and $\mathcal{A}$) then Eq. (III.14) implies that, for any $a \in \mathcal{A}$,

$$\omega_\beta\big(a^*\,a\big) \;=\; 0 \quad\Longrightarrow\quad a \;=\; 0\,. \tag{III.15}$$

To see this, we show that $\mathcal{N} := \{a \in \mathcal{A} \mid \omega_\beta(a^*a) = 0\,\}$ is a two-sided $*$-ideal. Clearly, if $\omega_\beta(a^*a) = 0$ and $b \in \mathcal{A}$ then

$$\omega_\beta\big(a^*\,b^*\,b\,a\big) \;\le\; \omega_\beta\big(a^*\,b^*\,b\,b^*\,b\,a\big)^{1/2}\,\omega_\beta\big(a^*\,a\big)^{1/2} \;=\; 0\,, \tag{III.16}$$

by the Schwarz inequality. Hence $\mathcal{A}\mathcal{N} \subseteq \mathcal{N}$. Furthermore, if $\omega_\beta(a^*a) = 0$ and $b \in \overset{\circ}{\mathcal{A}}$ then

$$\begin{aligned} \omega_\beta\big(b^*\,a^*\,a\,b\big) \;&=\; \omega_\beta\big((b^*\,a^*\,a)\,b\big) \;=\; \omega_\beta\big(\alpha_{-i\beta/2}(b)\,\alpha_{-i\beta/2}(b^*a^*a)\big) \\ &=\; \omega_\beta\big(\alpha_{-i\beta}(b)\,b^*\,a^*\,a\big) \;=\; 0\,. \end{aligned} \tag{III.17}$$

In the second equation, we have used (III.14), in the third one invariance, i.e., (III.6), and in the last one again the Schwarz inequality, i.e.,

$$\big|\omega_\beta\big(b^*\,a\big)\big|^2 \;\le\; \omega_\beta\big(b^*\,b\big)\,\omega_\beta\big(a^*\,a\big)\,. \tag{III.18}$$

Thus, $\mathcal{A}\mathcal{N}, \mathcal{N}\mathcal{A} \subseteq \mathcal{N}$, and $\mathcal{N}$ is a two-sided $*$-ideal.

III.2 The GNS Construction

For the purpose of mathematical precision, it is useful to assume that there exists a *denumerable* subspace $\widetilde{\mathcal{A}} \subseteq \mathcal{A}$ such that, for every $a \in \mathcal{A}$, there exists a sequence $\{a_\lambda\}_{\lambda \in \mathbb{N}} \subseteq \widetilde{\mathcal{A}}$ with the property that

$$\lim_{\lambda \to \infty} \omega_\beta\big((a - a_\lambda)^* \, (a - a_\lambda)\big) \ = \ 0 \, , \tag{III.19}$$

and that

$$\omega_\beta\big(a \, \alpha_t(b)\big) \ = \ \omega_\beta\big(\alpha_{-t}(a) \, b\big) \tag{III.20}$$

is *continuous* in t, for arbitrary $a, b \in \widetilde{\mathcal{A}}$.

Next, we recapitulate the GNS *construction* in a situation where (III.6), (III.12), (III.19), and (III.20) hold: To the data $(\mathcal{A}, \alpha_t, \omega_\beta)$, as specified above, one can associate a separable Hilbert space $\mathcal{H}_\beta$, a representation ℓ of $\mathcal{A}$ on $\mathcal{H}_\beta$, a vector $\Omega_\beta \in \mathcal{H}_\beta$ which is cyclic for $\ell[\mathcal{A}]$, and a continuous one-parameter group of unitary operators $\{e^{-it\mathcal{L}}\}_{t \in \mathbb{R}}$, where $\mathcal{L}$ is a selfadjoint operator on $\mathcal{H}_\beta$, such that, for all $a \in \mathcal{A}$,

$$\begin{aligned} \omega_\beta(a) \ &= \ \langle \Omega_\beta \, | \, \ell[a] \, \Omega_\beta \rangle \, , && \text{(III.21)} \\ \ell\big[\alpha_t(a)\big] \ &= \ e^{it\mathcal{L}} \, \ell[a] \, e^{-it\mathcal{L}} \, , && \text{(III.22)} \\ \mathcal{L} \, \Omega_\beta \ &= \ 0 \, . && \text{(III.23)} \end{aligned}$$

To construct $\mathcal{H}_\beta$, Ω_β, ℓ, and $\mathcal{L}$, let $\mathcal{N} := \{ a \in \mathcal{A} \mid \omega_\beta(a^* a) = 0 \}$. As noted above, $\mathcal{N}$ is trivial if $\mathcal{A}$ is simple, by the KMS condition. We set $[a] := a$ mod $\mathcal{N}$, for all $a \in \mathcal{A}$. Clearly, $\mathcal{D} := \{[a] \mid a \in \mathcal{A}\}$ is a linear space. It is equipped with a scalar product

$$\langle \, [a] \, | \, [b] \, \rangle \ := \ \omega_\beta\big(a^* \, b\big) \, . \tag{III.24}$$

The Hilbert space $\mathcal{H}_\beta$ is then the closure of $\mathcal{D}$ in the norm induced by $\langle \cdot | \cdot \rangle$. By (III.19), $\mathcal{H}_\beta$ is separable. We set $\Omega_\beta := [\mathbf{1}]$ and define the representation $\ell : \mathcal{A} \to B(\mathcal{H}_\beta)$ by

$$\ell[a] \, [b] \ := \ [ab] \, , \tag{III.25}$$

which extends continuously from $\mathcal{D}$ to $\mathcal{H}_\beta$. Finally, we define a one-parameter unitary group $\{e^{-it\mathcal{L}}\}_{t \in \mathbb{R}}$ on $\mathcal{H}_\beta$ by

$$e^{-it\mathcal{L}} \, [a] \ := \ \big[\alpha_t(a)\big] \, . \tag{III.26}$$

Unitarity follows from the invariance of ω_β under α_t. By (III.20), $e^{-it\mathcal{L}}$ is strongly continuous on the separable Hilbert space $\mathcal{H}_\beta$, and hence it is generated by a selfadjoint operator $\mathcal{L}$, the *Liouvillian*; (Stone's theorem).

III.3 Modular Operator and Modular Conjugation

Whether $\mathcal{A}$ is simple or not, we shall henceforth always assume that Ω_β is separating for $\ell[\mathcal{A}]$, i.e., Eq. (III.15), or, equivalently, that $\mathcal{N} = 0$. Then, by (III.20), Ω_β is also cyclic and separating for $\ell[\mathring{\mathcal{A}}]$, with $\mathring{\mathcal{A}}$ as in (III.11). Under these assumptions, we can define an (unbounded) anti-linear operator $\mathcal{S}$ on $\mathcal{H}_\beta$ by

$$\mathcal{S}\big(\ell[a]\,\Omega_\beta\big) \;:=\; \ell[a]^*\,\Omega_\beta\,, \tag{III.27}$$

for $a \in \mathcal{A}$ (or $a \in \mathring{\mathcal{A}}$). Since Ω_β is separating for $\ell[\mathcal{A}]$, $\mathcal{S}$ is well-defined, and, since Ω_β is cyclic for $\ell[\mathcal{A}]$, it is densely defined.

For $a \in \mathring{\mathcal{A}}$, we define an operator $\mathcal{J}$ by

$$\mathcal{J}\big(\ell[a]\,\Omega_\beta\big) \;:=\; \mathcal{S}\,\ell\big[\alpha_{-i\beta/2}(a)\big]\,\Omega_\beta \;=\; \ell\big[\alpha_{i\beta/2}(a^*)\big]\,\Omega_\beta\,, \tag{III.28}$$

by (III.27) and (III.13). Then, for $a, b \in \mathring{\mathcal{A}}$,

$$\begin{aligned}\big\langle \mathcal{J}\,\ell[a]\,\Omega_\beta \,\big|\, \mathcal{J}\,\ell[b]\,\Omega_\beta\big\rangle \;&=\; \big\langle \ell\big[\alpha_{i\beta/2}(a^*)\big]\,\Omega_\beta \,\big|\, \ell\big[\alpha_{i\beta/2}(b^*)\big]\,\Omega_\beta\big\rangle \\ =\; \big\langle \Omega_\beta \,\big|\, \ell\big[\alpha_{-i\beta/2}(a)\,\alpha_{i\beta/2}(b^*)\big]\,\Omega_\beta\big\rangle \;&=\; \omega_\beta\big(\,\alpha_{-i\beta/2}(a)\,\alpha_{i\beta/2}(b^*)\,\big) \\ =\; \omega_\beta\big(\,b^*\,a\,\big) \;=\; \big\langle \ell[b]\,\Omega_\beta \,\big|\, \ell[a]\,\Omega_\beta\big\rangle \;&=\; \overline{\big\langle \ell[a]\,\Omega_\beta \,\big|\, \ell[b]\,\Omega_\beta\big\rangle}\,,\end{aligned} \tag{III.29}$$

where we have used Eq. (III.14) in the fourth equation. It follows from (III.29) that $\mathcal{J}$ extends to an *anti-unitary operator* on $\mathcal{H}_\beta$, which is called *modular conjugation.* Note that, by (III.26) and (III.28),

$$\begin{aligned}\mathcal{J} \;&=\; \mathcal{S}\,e^{\beta\mathcal{L}/2} \;=\; e^{-\beta\mathcal{L}/2}\,\mathcal{S}\,, \quad \text{i.e.,} \\ \mathcal{S} \;&=\; \mathcal{J}\,e^{-\beta\mathcal{L}/2} \;=\; e^{\beta\mathcal{L}/2}\,\mathcal{J}\,,\end{aligned} \tag{III.30}$$

which describes the polar decomposition of $\mathcal{S}$. *Tomita-Takesaki theory* (see, e.g., [41, 12]) is a far-reaching generalization of these considerations [22], which *starts* from the definition (III.27), then shows that $\mathcal{S}$ is *closable*, and, finally, constructs $\mathcal{J}$ and $\exp[\pm\beta\mathcal{L}/2]$ by polar decomposition of $\mathcal{S}$. But we shall not have any occasion to make use of this theory.

Next, we establish some remarkable properties of the modular conjugation $\mathcal{J}$. Using $\mathcal{J}$, we may define an *anti-linear* representation, r, of $\mathcal{A}$ on $\mathcal{H}_\beta$:

$$r[a] \;:=\; \mathcal{J}\,\ell[a]\,\mathcal{J}\,, \quad \text{for } a \in \mathcal{A}. \tag{III.31}$$

We claim that

$$r[\mathcal{A}] \;\subseteq\; \ell[\mathcal{A}]'\,, \tag{III.32}$$

i.e., $r[a]$ commutes with $\ell[b]$, for arbitrary $a, b \in \mathcal{A}$. It is instructive to verify (III.32): Since Ω_β is cyclic for $\ell[\overset{\circ}{\mathcal{A}}]$, it suffices to show that

$$r[a]\,\ell[b]\,\ell[c]\,\Omega_\beta \;=\; \ell[b]\,r[a]\,\ell[c]\,\Omega_\beta\,, \tag{III.33}$$

for arbitrary $a, b, c \in \overset{\circ}{\mathcal{A}}$. Eq. (III.33) follows from the definition (III.28) of $\mathcal{J}$ by a little algebra,

$$\begin{aligned} r[a]\,\ell[b]\,\ell[c]\,\Omega_\beta \;&=\; \mathcal{J}\,\ell[a]\,\ell\big[\alpha_{i\beta/2}(c^* b^*)\big]\,\Omega_\beta \\ &=\; \mathcal{J}\,\ell\big[a\,\alpha_{i\beta/2}(c^* b^*)\big]\,\Omega_\beta \;=\; \ell\big[\alpha_{i\beta/2}\big(\alpha_{-i\beta/2}(bc)\,a^*\big)\big]\,\Omega_\beta \\ &=\; \ell[b]\,\ell\big[\alpha_{i\beta/2}\big([a\,\alpha_{i\beta/2}(c^*)]^*\big)\big]\,\Omega_\beta \;=\; \ell[b]\,\mathcal{J}\,\ell\big[a\,\alpha_{i\beta/2}(c^*)\big]\,\Omega_\beta \\ &=\; \ell[b]\,\mathcal{J}\,\ell[a]\,\mathcal{J}\,\ell[c]\,\Omega_\beta \;=\; \ell[b]\,r[a]\,\ell[c]\,\Omega_\beta\,, \end{aligned} \tag{III.34}$$

which proves (III.33). One can show (see [22]) that, under our assumptions,

$$\ell[\mathcal{A}]'' \;=\; r[\mathcal{A}]'\,. \tag{III.35}$$

Next, we show that $\mathcal{J}$ commutes with the time evolution, i.e.,

$$\mathcal{J}\,e^{-it\mathcal{L}} \;=\; e^{-it\mathcal{L}}\,\mathcal{J}\,. \tag{III.36}$$

For $a \in \overset{\circ}{\mathcal{A}}$, Eqs. (III.26) and (III.28) yield

$$\begin{aligned} \mathcal{J}\,e^{-it\mathcal{L}}\,\ell[a]\,\Omega_\beta \;&=\; \mathcal{J}\,\ell\big[\alpha_{-t}(a)\big]\,\Omega_\beta \;=\; \ell\big[\alpha_{i\beta/2}\big(\alpha_{-t}(a^*)\big)\big]\,\Omega_\beta \\ &=\; \ell\big[\alpha_{-t}\big(\alpha_{i\beta/2}(a^*)\big)\big]\,\Omega_\beta \;=\; e^{-it\mathcal{L}}\,\ell\big[\alpha_{i\beta/2}(a^*)\big]\,\Omega_\beta \\ &=\; e^{-it\mathcal{L}}\,\mathcal{J}\,\ell[a]\,\Omega_\beta\,, \end{aligned} \tag{III.37}$$

and we have used (III.13). As a corollary of (III.36), (III.31), and (III.26), we have that

$$e^{it\mathcal{L}}\,\ell[a]\,e^{-it\mathcal{L}} \;=\; \ell\big[\alpha_t(a)\big]\,, \tag{III.38}$$

$$e^{it\mathcal{L}}\,r[a]\,e^{-it\mathcal{L}} \;=\; r\big[\alpha_t(a)\big]\,. \tag{III.39}$$

Eq. (III.36) implies that $\mathcal{J}\,i\,\mathcal{L} = i\,\mathcal{L}\,\mathcal{J}$, where i is multiplication by $\sqrt{-1}$. Since $\mathcal{J}$ is anti-unitary, this is equivalent to

$$\mathcal{J}\,\mathcal{L} \;=\; -\,\mathcal{L}\,\mathcal{J}\,, \tag{III.40}$$

which is consistent with the last equation in (III.30). This equation has an interesting consequence: If ψ is an eigenvector of $\mathcal{L}$ corresponding to an eigenvalue λ, and

$$\mathcal{J}\,\psi \;=\; \psi \tag{III.41}$$

then $\lambda = 0$. This is seen as follows.

$$\begin{aligned} \lambda \psi \ &= \ \mathcal{L} \psi \ = \ \mathcal{L} \mathcal{J} \psi \ = \ -\mathcal{J} \mathcal{L} \psi \\ &= \ -\mathcal{J} \lambda \psi \ = \ -\bar{\lambda} \mathcal{J} \psi \ = \ -\bar{\lambda} \psi \, , \end{aligned} \tag{III.42}$$

and hence

$$\lambda + \bar{\lambda} \ = \ 0 \, . \tag{III.43}$$

Since $\mathcal{L}$ is selfadjoint, λ is real, and hence (III.43) implies that $\lambda = 0$. (A slight generalization of this fact will be used in Sect. V.)

III.4 Return to Equilibrium

A state, ρ, on the C^*-algebra $\mathcal{A}$ is *normal* with respect to the representation ℓ (and the representation r) iff $0 \leq \rho \in \mathcal{L}^1(\mathcal{H}_\beta)$, i.e., ρ is of the form

$$\rho(a) \ = \ \sum_{n=1}^{\infty} p_n \, \langle \psi_n \mid \ell[a] \, \psi_n \rangle \, , \tag{III.44}$$

where $\psi_n \in \mathcal{H}_\beta$, with $\langle \psi_n | \psi_n \rangle = 1$, and $p_n \geq 0$, for all $n \in \mathbb{N}$; $\sum_{n=1}^{\infty} p_n = 1$. Every vector ψ_n can be approximated in norm by vectors of the form $\ell[a_n^m]\Omega_\beta$, with $a_n^m \in \mathcal{A}$, $m \in \mathbb{N}$, by the cyclicity of Ω_β.

The time evolution, ρ_t, $t \in \mathbb{R}$, of a normal state ρ is defined by

$$\rho_t(a) \ := \ \rho\big(\alpha_t(a)\big) \, . \tag{III.45}$$

We are interested in understanding under which conditions

$$\rho_t(a) \ \to \ \omega_\beta(a) \, , \quad \text{as } t \to \infty, \tag{III.46}$$

in a sense to be made precise. Eq. (III.46) expresses the property of *return to equilibrium.* We give sufficient conditions involving spectral properties of $\mathcal{L}$ for return to equilibrium.

Lemma III.1. *Assume that* 0 *is a simple eigenvalue of* $\mathcal{L}$ *corresponding to the eigenvector* Ω_β *and that the rest of the spectrum of* $\mathcal{L}$ *is continuous. Let* $a \in \mathcal{A}$ *and* ρ *be a normal state. Then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_{\pm t}(a) \, dt \ = \ \omega_\beta(a) \, . \tag{III.47}$$

If $\sigma(\mathcal{L}) \setminus \{0\}$ *is absolutely continuous then*

$$\lim_{t \to \pm\infty} \rho_t(a) \ = \ \omega_\beta(a) \, . \tag{III.48}$$

Proof. First, the continuity of the spectrum away from 0 implies that

$$w - \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{\pm it\mathcal{L}}\, dt \;=\; |\Omega_\beta\rangle\langle\Omega_\beta| \,. \tag{III.49}$$

If $\sigma[\mathcal{L}] \setminus \{0\}$ is absolutely continuous then we even have that

$$\mathrm{w} - \lim_{t\to\pm\infty} e^{\pm it\mathcal{L}} \;=\; |\Omega_\beta\rangle\langle\Omega_\beta| \,. \tag{III.50}$$

To derive (III.47) from (III.49), it is enough to show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big\langle \ell[b]\Omega_\beta \;\big|\; \ell[\alpha_{\pm t}(a)]\, \ell[c]\Omega_\beta \big\rangle\, dt \;=\; \omega_\beta(b^* c)\, \omega_\beta(a) \,, \tag{III.51}$$

for arbitrary $a, b, c \in \overset{\circ}{\mathcal{A}}$, by (III.44) and the remarks thereafter. Using the KMS condition (III.12), the integrand on the left side of (III.51) is seen to equal

$$\begin{aligned} \omega_\beta\big(b^*\, \alpha_{\pm t}(a)\, c\big) \;&=\; \omega_\beta\big(\alpha_{-i\beta}(c)\, b^*\, \alpha_{\pm t}(a)\big) \\ &=\; \big\langle \ell[b]\, \ell[\alpha_{i\beta}(c^*)]\, \Omega_\beta \;\big|\; \ell[\alpha_{\pm t}(a)]\, \Omega_\beta \big\rangle \\ &=\; \big\langle \ell[b]\, \ell[\alpha_{i\beta}(c^*)]\, \Omega_\beta \;\big|\; e^{\pm it\mathcal{L}}\, \ell[a]\, \Omega_\beta \big\rangle \,. \end{aligned} \tag{III.52}$$

Since $a, b, c \in \overset{\circ}{\mathcal{A}}$,

$$\ell[b]\, \ell[\alpha_{i\beta}(c^*)]\, \Omega_\beta \,, \quad \ell[a]\, \Omega_\beta \;\in\; \mathcal{H}_\beta \,. \tag{III.53}$$

Thus, using (III.49) and (III.52), we find that

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_0^T \omega_\beta\big(b^*\, \alpha_{\pm t}(a)\, c\big)\, dt \\ &=\; \big\langle \ell[b]\, \ell[\alpha_{i\beta}(c^*)]\, \Omega_\beta \;\big|\; \Omega_\beta \big\rangle \big\langle \Omega_\beta \;\big|\; \ell[a]\, \Omega_\beta \big\rangle \\ &=\; \omega_\beta\big(\alpha_{-i\beta}(c)\, b^*\big)\, \omega_\beta(a) \;=\; \omega_\beta(b^* c)\, \omega_\beta(a) \,, \end{aligned} \tag{III.54}$$

and we have used the KMS condition once more. Thus (III.51), and hence (III.47) are proven. The proof that (III.50) implies (III.48) is similar.

□

III.5 Perturbation Theory

This section amounts to a transcription of Sect. II.5 to the thermodynamic limit. Let $I \in \mathcal{A}$. To the pair $(\{\alpha_t\}_{t\in\mathbb{R}}, I)$ we can associate a perturbed Heisenberg picture time evolution as follows: Let

$$I(t) \;:=\; \alpha_t(I) \,. \tag{III.55}$$

Then the equation

$$\begin{aligned}\alpha_t^{(I)}(a) \ &:= \ \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \\ &\cdots \int_0^{t_{n-1}} dt_n \Big[I(t_n) \, , \, \cdots \big[I(t_2) \, , \, [I(t_1) \, , \, \alpha_t(a)] \big] \cdots \Big]\end{aligned} \tag{III.56}$$

defines a perturbed one-parameter group $\{\alpha_t^{(I)}\}_{t\in\mathbb{R}}$ on the algebra $\mathcal{A}$. In the GNS representation ℓ of $\mathcal{A}$ corresponding to an α_t-KMS state ω_β on $\mathcal{A}$, $\alpha_t^{(I)}$ can be implemented unitarily,

$$\ell\big[\alpha_t^{(I)}(a)\big] \ = \ \exp\big(it\mathcal{L}_I^{(\ell)}\big) \, \ell[a] \, \exp\big(-it\mathcal{L}_I^{(\ell)}\big) \, . \tag{III.57}$$

Using (III.38) and applying the representation ℓ to (III.56), we readily find that

$$\mathcal{L}_I^{(\ell)} \ = \ \mathcal{L} \, + \, \ell[I] \, + \, \mathcal{R} \, , \tag{III.58}$$

where $\mathcal{R}$ is an arbitrary operator in $\ell[\mathcal{A}]' = r[\mathcal{A}]''$. Of course, $\alpha_t^{(I)}$ can also be implemented unitarily on $\mathcal{H}_\beta$ in the anti-linear representation r of $\mathcal{A}$,

$$\begin{aligned} r\big[\alpha_t^{(I)}(a)\big] \ &= \ \mathcal{J} \, \ell\big[\alpha_t^{(I)}(a)\big] \, \mathcal{J} \ = \ \mathcal{J} \, \exp\big(it\mathcal{L}_I^{(\ell)}\big) \, \ell[a] \, \exp\big(-it\mathcal{L}_I^{(\ell)}\big) \, \mathcal{J} \\ &= \ \exp\big(it\mathcal{L}_I^{(r)}\big) \, r[a] \, \exp\big(-it\mathcal{L}_I^{(r)}\big) \, , \end{aligned} \tag{III.59}$$

where

$$\exp\big(it\mathcal{L}_I^{(r)}\big) \ = \ \mathcal{J} \, \exp\big(it\mathcal{L}_I^{(\ell)}\big) \, \mathcal{J} \, , \tag{III.60}$$

i.e., because $\mathcal{J}$ is anti-linear,

$$\mathcal{L}_I^{(r)} \ = \ -\mathcal{J} \, \mathcal{L}_I^{(\ell)} \, \mathcal{J} \ = \ \mathcal{L} \, - \, r[I] \, - \, \mathcal{J}\mathcal{R}\mathcal{J} \, , \tag{III.61}$$

where we have used (III.58), (III.40), and (III.31). It would be pleasant to have $\mathcal{L}_I^{(\ell)} = \mathcal{L}_I^{(r)}$. This equation has the unique solution

$$\mathcal{L}_I^{(\ell)} \ = \ \mathcal{L}_I^{(r)} \ =: \ \mathcal{L}_I \ = \ \mathcal{L} \, + \, \ell[I] \, - \, r[I] \, + \, \mathcal{Z} \, , \tag{III.62}$$

where $\mathcal{Z}$ is in the center of the von Neumann algebra $\ell[\mathcal{A}]''$, which is given by $\ell[\mathcal{A}]'' \cap r[\mathcal{A}]''$. Without loss of generality, we set $\mathcal{Z} = 0$. Then

$$\mathcal{L}_I \ = \ -\mathcal{J} \, \mathcal{L}_I \, \mathcal{J} \, . \tag{III.63}$$

In our applications of the general theory, we shall use the following notation:

$$\mathcal{L}_I \;:=\; \mathcal{L} + W \;:=\; \mathcal{L} + \ell[I] \,-\, r[I]\,, \tag{III.64}$$

with $W := \ell[I] - r[I]$. It is natural to ask whether we can construct a perturbed KMS state for the dynamics described by $\{\alpha_t^{(I)}\}_{t\in\mathbb{R}}$ starting from the data $(\mathcal{H}_\beta, \ell[\mathcal{A}], \mathcal{J}, \mathcal{L}_I)$. The considerations presented in Sect. II.5 suggest that the answer is affirmative. As in Eq. (II.85), we introduce the *Radon-Nikodym operators*

$$\mathcal{L}_{I,\ell} \;:=\; \mathcal{L} + \ell[I]\,, \qquad \mathcal{L}_{I,r} \;:=\; \mathcal{L} - r[I]\,. \tag{III.65}$$

We note that, by (III.31) and (III.40),

$$\mathcal{J}\,\mathcal{L}_{I,\ell}\,\mathcal{J} \;=\; -\mathcal{L}_{I,r}\,. \tag{III.66}$$

We claim that the vector Ω_β is in the domain of the unbounded operators $\exp[-\beta\mathcal{L}_{I,\ell}/2]$ and $\exp[\beta\mathcal{L}_{I,r}/2]$, and that the vector

$$\Omega_{\beta,I} \;:=\; Z_{\beta,I}^{-1}\, e^{-\beta\mathcal{L}_{I,\ell}/2}\,\Omega_\beta \;=\; Z_{\beta,I}^{-1}\, e^{\beta\mathcal{L}_{I,r}/2}\,\Omega_\beta \tag{III.67}$$

defines a KMS state, $\omega_{\beta,I}$, on $\mathcal{A}$, for the time evolution given by $\{\alpha_t^{(I)}\}_{t\in\mathbb{R}}$. In (III.67), $Z_{\beta,I}$ is a *positive, finite* normalization factor for which we shall give an explicit formula. The equality between the two definitions of $\Omega_{\beta,I}$ implies that

$$\begin{aligned} \mathcal{J}\,\Omega_{\beta,I} \;&=\; Z_{\beta,I}^{-1}\,\mathcal{J}\,e^{-\beta\mathcal{L}_{I,\ell}/2}\,\Omega_\beta \;=\; Z_{\beta,I}^{-1}\,e^{\beta\mathcal{L}_{I,r}/2}\,\mathcal{J}\,\Omega_\beta \\ &=\; Z_{\beta,I}^{-1}\,e^{\beta\mathcal{L}_{I,r}/2}\,\Omega_\beta \;=\; \Omega_{\beta,I}\,, \end{aligned} \tag{III.68}$$

by (III.66); i.e., $\Omega_{\beta,I}$ is invariant under $\mathcal{J}$. The state $\omega_{\beta,I}$ is defined by

$$\omega_{\beta,I}(a) \;:=\; \langle \Omega_{\beta,I} \mid \ell[a]\,\Omega_{\beta,I}\rangle\,, \tag{III.69}$$

for $a \in \mathcal{A}$. Araki [1, 2] has proven that $\omega_{\beta,I}$ is indeed a KMS state for $\{\alpha_t^{(I)}\}_{t\in\mathbb{R}}$ and hence is invariant under $\{\alpha_t^{(I)}\}_{t\in\mathbb{R}}$. (The time-translation invariance of a KMS state is a simple consequence of Liouville's theorem, which says that a bounded entire function on $\mathbb{C}$ is constant.) From the fact that $\exp(-it\mathcal{L}_I)$ implements $\alpha_t^{(I)}$ unitarily on $\mathcal{H}_\beta$ and Eqs. (III.63) and (III.68) it follows that

$$\mathcal{L}_I\,\Omega_{\beta,I} \;=\; 0\,, \tag{III.70}$$

i.e., 0 is an eigenvalue of $\mathcal{L}_I$ with corresponding eigenvector $\Omega_{\beta,I}$.

The construction of $\Omega_{\beta,I}$ can thus also be viewed as a *spectral problem* for $\mathcal{L}_I$: If we can show that 0 is a simple eigenvalue of $\mathcal{L}_I$ then $\Omega_{\beta,I}$ can be defined as the corresponding eigenvector. The results summarized in this subsection, mostly due to Araki [1, 2], are plausible extensions of those of Sect. II.5. For KMS states obtained as thermodynamic limits of equilibrium states of bounded systems, they could be derived from the results in Sect. II.5 by limiting arguments. However, like *all other results* in Sect. III, they can be proven *directly*, by using the KMS condition for ω_β, the Lie-Schwinger series (III.56) and the Dyson series for $\exp[-\beta\mathcal{L}_{I,\ell}/2]\cdot\exp[\beta\mathcal{L}/2]$ in moderately clever (and somewhat tedious) ways. Reviewing the details goes beyond the scope of this paper; but see [1, 2, 12, 13]. But we present the most essential tools and explicit formulae for $\Omega_{\beta,I}$ and $Z_{\beta,I}$.

For $a \in \overset{\circ}{\mathcal{A}}$, we may define

$$a(\tau + it) \;:=\; \alpha_{i(\tau \mod \beta)+t}\,(a)\,. \tag{III.71}$$

Temperature (imaginary-time) *ordering*, T, of a product of operators $a(\tau+it)$ is defined by

$$\mathrm{T}\big[a_1(\tau_1+it_1)\cdots a_n(\tau_n+it_n)\big] \;:=\; a_{\pi(1)}(\tau_{\pi(1)}+it_{\pi(1)})\cdots a_{\pi(n)}(\tau_{\pi(n)}+it_{\pi(n)})\,, \tag{III.72}$$

where π is the permutation of $\{1,2,\dots,n\}$ with the property that, for $\tau_i \neq \tau_j$, $i \neq j$,

$$\tau_{\pi(1)} \;<\; \tau_{\pi(2)} \;<\; \cdots \;<\; \tau_{\pi(n)}\,, \tag{III.73}$$

for arbitrary $a_1, a_2, \dots, a_n \in \overset{\circ}{\mathcal{A}}$. Then the KMS condition for ω_β implies that, for arbitrary $n \in \mathbb{N}$, the *temperature-ordered Green functions*

$$\omega_\beta\big(\,\mathrm{T}\big[a_1(\zeta_1)\cdots a_n(\zeta_n)\big]\,\big) \tag{III.74}$$

are analytic in $\zeta_1, \zeta_2, \dots, \zeta_n$ on the domains

$$\mathcal{T}_n^\pi \;:=\; \big\{\zeta_1,\dots,\zeta_n \;\big|\; 0 < \mathrm{Re}\zeta_{\pi(1)} < \cdots < \mathrm{Re}\zeta_{\pi(n)} < \beta\,\big\}\,, \tag{III.75}$$

with

$$\big|\,\omega_\beta\big(\,\mathrm{T}\big[a_1(\zeta_1)\cdots a_n(\zeta_n)\big]\,\big)\,\big| \;\leq\; \prod_{j=1}^n \|a_j\|\,, \tag{III.76}$$

where $\|a\|$ is the C^*-norm of a; see [1, 2]. Furthermore,

$$\omega_\beta\big(\,\mathrm{T}\big[a_1(\zeta_1)\cdots a_n(\zeta_n)\big]\,\big) \;=\; \omega_\beta\big(\,\mathrm{T}\big[a_1(\zeta_1+\tau)\cdots a_n(\zeta_n+\tau)\big]\,\big)\,, \tag{III.77}$$

for any real τ. By (III.76), (III.74), Eqs. (III.75) and (III.77) hold for arbitrary $a_1, \dots, a_n \in \mathcal{A}$, $n \in \mathbb{N}$.

The GNS construction implies that

$$\begin{aligned} \omega_\beta\big(\mathrm{T}\big[a_1(\zeta_1)\cdots a_n(\zeta_n)\big] \big) &= \Big\langle \Omega_\beta \,\Big|\, \prod_{j=1}^{n} \ell[a_{\pi(j)}]\, e^{(\zeta_{\pi(j)}-\zeta_{\pi(j+1)})\mathcal{L}}\, \Omega_\beta \Big\rangle & \text{(III.78)} \\ &= \Big\langle \Omega_\beta \,\Big|\, \prod_{j=1}^{n} r[a_{\pi(j)}]\, e^{-(\bar\zeta_{\pi(j)}-\bar\zeta_{\pi(j+1)})\mathcal{L}}\, \Omega_\beta \Big\rangle \,. \end{aligned}$$

The second equation follows from the first one by conjugating with $\mathcal{J}$ and using that $\mathcal{J}\Omega_\beta = \Omega_\beta$.

Applying the *Duhamel* (or *Dyson*) *series* for the expressions for $\Omega_{\beta,I}$ in (III.67), we find that

$$\begin{aligned} \Omega_{\beta,I} &= Z_{\beta,I}^{-1}\, e^{-\beta\mathcal{L}_{I,\ell}/2}\Omega_\beta & \text{(III.79)} \\ &= Z_{\beta,I}^{-1} \sum_{n=0}^{\infty} \int_0^{\beta/2} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n\, \ell[I(\tau_n)]\cdots\ell[I(\tau_1)]\,\Omega_\beta \,. \end{aligned}$$

Normalizing Ω_β and $\Omega_{\beta,I}$ to have norm 1 and using (III.78), we find that

$$Z_{\beta,I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\beta} d\tau_1 \cdots \int_0^{\beta} d\tau_n\, \omega_\beta\big(\mathrm{T}\big[I(\tau_1)\cdots I(\tau_n)\big] \big) \,. \qquad \text{(III.80)}$$

With (III.76) we find that, for $0 < \beta < \infty$,

$$0 \le Z_{\beta,I} \le \exp\big(\beta\,\|I\|\big) \,. \qquad \text{(III.81)}$$

Similar formulae hold when one replaces $\mathcal{L}_{I,\ell}$ by $\mathcal{L}_{I,r}$. The KMS condition for $\omega_{\beta,I}$ (see (III.69)) with respect to $\alpha_t^{(I)}$ can be derived from (III.56), (III.77) and (III.79) by straightforward, albeit somewhat tedious, calculations.

Formulae (III.79) and (III.80) are very useful in the analysis of concrete models; see Sect. V.

IV KMS States and Liouvillians for Idealized Atoms coupled to the Quantized Electromagnetic Field

IV.1 KMS States for the Quantized Free Electromagnetic Field

In this subsection, we illustrate the general theory developed in Sects. II and III on the example of the equilibrium (KMS) states of the free electromagnetic field in the thermodynamic limit, as described by Araki and Woods in [3]. Similar results can be derived for gases of free fermions at positive density and temperature; see [4].

It is convenient to describe the free electromagnetic field in terms of creation -and annihilation operators $a^*_\lambda(\vec{k})$, $a_\lambda(\vec{k})$ satisfying the canonical commutation relations,

$$[a^*_\lambda(\vec{k})\,,\,a^*_{\lambda'}(\vec{k}')] \;=\; [a_\lambda(\vec{k})\,,\,a_{\lambda'}(\vec{k}')] \;=\; 0\,, \tag{IV.1}$$

$$[a_\lambda(\vec{k})\,,\,a^*_{\lambda'}(\vec{k}')] \;=\; \delta_{\lambda,\lambda'}\,\delta(\vec{k}-\vec{k}')\,, \tag{IV.2}$$

as described in the introduction. We thus consider observable algebras which are $*$-algebras of unbounded operators, instead of the C^*-algebras appearing in the general theory of Sects. II and III. By using the bounded Weyl operators we could, however, translate our results into C^*-algebra language. But in the analysis of concrete models, $*$-algebras are more convenient.

For notational convenience, we set $k := (\vec{k},\lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$, where $\vec{k} \in \mathbb{R}^3$ is the wave vector and $\lambda = \pm 1$ is the polarization index, and we denote $\int dk := \sum_{\lambda=\pm 1} \int d^3k$. We define

$$\delta(k-k') \;:=\; \delta_{\lambda,\lambda'}\,\delta(\vec{k}-\vec{k}')\,, \tag{IV.3}$$

and

$$a^*(k) \;:=\; a^*_\lambda(\vec{k})\,, \qquad a(k) \;:=\; a_\lambda(\vec{k})\,. \tag{IV.4}$$

Let $\mathcal{S}_0(\mathbb{R}^3)$ denote the Schwartz space test functions vanishing at the origin of $\mathbb{R}^3$. For $f = (f_+, f_-) \in \mathcal{S}_0(\mathbb{R}^3)^2$, we define

$$a^*(f) \;:=\; \sum_{\lambda=\pm} \int d^3k\, f_\lambda(\vec{k})\, a^*_\lambda(\vec{k})\,, \quad a(f) \;:=\; \sum_{\lambda=\pm} \int d^3k\, f_\lambda(\vec{k})\, a_\lambda(\vec{k})\,, \tag{IV.5}$$

and the complex conjugation

$$(\tau f)[k] \;\equiv\; \big((\tau f)_+(\vec{k})\,,\,(\tau f)_-(\vec{k})\big) \;:=\; \big(\overline{f_+(\vec{k})}\,,\,\overline{f_-(\vec{k})}\big)\,. \tag{IV.6}$$

We set

$$(f,g) \;:=\; \sum_{\lambda=\pm} \int d^3k\, \overline{f_\lambda(\vec{k})}\, g_\lambda(\vec{k})\,. \tag{IV.7}$$

Then (IV.1) and (IV.2) can be written as

$$[a^*(f)\,,\,a^*(g)] \;=\; [a(f)\,,\,a(g)] \;=\; 0\,, \tag{IV.8}$$

$$[a(\tau f)\,,\,a^*(g)] \;=\; (f,g)\,\mathbf{1}\,. \tag{IV.9}$$

We shall, however, continue to also use the operator-valued distributions $a(k)$, $a^*(k)$.

Let $\mathcal{P}$ denote the polynomial algebra generated by

$$\big\{ a(f), a^*(g) \;\big|\; f, g \in \mathcal{S}_0(\mathbb{R}^3)^2 \big\} \,. \tag{IV.10}$$

It is a $*$-algebra for the $*$-operation defined by

$$\big(a(f)\big)^* \;:=\; a^*(\tau f)\,. \tag{IV.11}$$

The time evolution of operators in $\mathcal{P}$ is described by a one-parameter group of $*$-automorphisms, $\{\alpha_t^f\}_{t\in\mathbb{R}}$, of $\mathcal{P}$ determined by

$$\alpha_t^f\big(a^*(k)\big) \;:=\; e^{it\omega(k)} a^*(k)\,, \qquad \alpha_t^f\big(a(k)\big) \;:=\; e^{-it\omega(k)} a(k)\,, \tag{IV.12}$$

where $\omega(k) := |\vec{k}|$ is the energy of a photon of wave vector $\vec{k}$; (we set $\hbar = 1$).

A *quasi-free* state ρ on the $*$-algebra $\mathcal{P}$ defined in (IV.11) is a state with the property that the connected, or "truncated", expectations

$$\rho\big(a^{\#_1}(f_1)\cdots a^{\#_n}(f_n)\big)^c \tag{IV.13}$$

all vanish, except for $n = 1$ and 2. It is not hard to show that there is a *unique* state ω_β^f on $\mathcal{P}$ which is a KMS state for the time evolution $\{\alpha_t^f\}_{t\in\mathbb{R}}$ at inverse temperature β. The state ω_β^f is quasi-free and hence completely determined by the equations

$$\omega_\beta^f\big(a^*(k)\big) \;=\; \omega_\beta^f\big(a(k)\big) \;=\; \omega_\beta^f\big(a^*(k)a^*(k')\big) \;=\; \omega_\beta^f\big(a(k)a(k')\big) \;=\; 0\,, \tag{IV.14}$$

and

$$\omega_\beta^f\big(a(k)a^*(k')\big) \;=\; \frac{\delta(k-k')}{e^{\beta\omega(k)} - 1}\,. \tag{IV.15}$$

Expectation values of products of more than two creation- and annihilation operators are given by sums of products of expectation values of $a^*(k)a(k')$, as given by (IV.15), according to *Wick's theorem*, which holds for quasi-free states. The KMS condition for ω_β^f is a direct consequence of applying (IV.2) and (IV.12) to (IV.15):

$$\begin{aligned} \omega_\beta^f\big(a(k)a^*(k')\big) \;&=\; \delta(k-k') \,+\, \omega_\beta^f\big(a^*(k')a(k)\big) \qquad\qquad \text{(IV.16)} \\ &=\; e^{\beta\omega(k)}\frac{\delta(k-k')}{e^{\beta\omega(k)} - 1} \;=\; \omega_\beta^f\big(\alpha_{-i\beta}^f\big(a^*(k')\big)\,a(k)\big)\,, \end{aligned}$$

where, in the first equation, we have used (IV.2), in the second (IV.15), and in the last one (IV.12).

The GNS construction (in a form originally due to Borchers and Wightman) tells us that the data $(\mathcal{P}, \alpha_t^f, \omega_\beta^f)$ determine a Hilbert space $\mathcal{H}_\beta^f$, a representation ℓ of $\mathcal{P}$ on $\mathcal{H}_\beta^f$, a vector $\Omega_\beta^f \in \mathcal{H}_\beta^f$ cyclic for $\ell[\mathcal{P}]$, and a unitary one-parameter group $\{e^{-it\mathcal{L}_f}\}_{t\in\mathbb{R}}$ such that

$$\omega_\beta^f(a) \;=\; \langle \Omega_\beta^f \mid \ell[a]\,\Omega_\beta^f \rangle\,, \tag{IV.17}$$

$$\ell\big[\alpha_t^f(a)\big] \;=\; e^{it\mathcal{L}_f}\,\ell[a]\,e^{-it\mathcal{L}_f}\,, \tag{IV.18}$$

for arbitrary $a \in \mathcal{P}$. Furthermore, one easily shows that Ω_β^f is separating for $\ell[\mathcal{P}]$ (which follows from the KMS condition for ω_β^f and the faithfulness of the representation ℓ; see Sect. III), and that there is a modular conjugation $\mathcal{J}$ such that the anti-linear representation r of $\mathcal{P}$ on $\mathcal{H}_\beta^f$, given by

$$r[a] \;:=\; \mathcal{J}\,\ell[a]\,\mathcal{J}\,, \quad \text{for } a \in \mathcal{P}, \tag{IV.19}$$

commutes with $\ell[b]$, for all $b \in \mathcal{P}$, and

$$\mathcal{J}\,\Omega_\beta^f \;=\; \Omega_\beta^f\,, \quad \mathcal{J}\,\mathcal{L}_f\,\mathcal{J} \;=\; -\mathcal{L}_f\,. \tag{IV.20}$$

These are immediate corollaries of the general theory outlined in Sect. III.

Following [3], we now present an explicit realization of the representations ℓ and r of $\mathcal{P}$, of the vector Ω_β^f, and of the modular conjugation $\mathcal{J}$, which is reminiscent of the description of the quantum theory of bounded systems in thermal equilibrium presented in formulae (II.48)–(II.56) of Sect. II.

Let $\mathcal{F}$ denote *Fock space* carrying the standard Fock representation of $\mathcal{P}$. Fock space contains a distinguished vector Ω (unique up to a phase) characterized by the property that

$$a(f)\,\Omega \;=\; 0\,, \quad \text{for all } f \in \mathcal{S}_0(\mathbb{R}^3)^2, \tag{IV.21}$$

which is called the *vacuum vector.* Fock space $\mathcal{F}$ and the vacuum Ω arise by GNS construction from the *quasi-free* state ω_∞ on $\mathcal{P}$ given by letting β tend to ∞ in (IV.15) and (IV.16). In our notation, we identify $\mathcal{P}$ with its representation on $\mathcal{F}$.

We define an anti-unitary operator T on $\mathcal{F}$, the *second quantization* of τ, by setting

$$T\,\Omega \;=\; \Omega\,, \tag{IV.22}$$

$$T\,a^{\#}(f)\,T \;=\; a^{\#}(\tau f)\,, \tag{IV.23}$$

$$T \;=\; T^* \;=\; T^{-1}\,. \tag{IV.24}$$

Physically, T describes *time reversal*; (compare (IV.12) and (IV.23)).

Next, we describe an isomorphism, I_T, between $\mathcal{H}_\beta^f$ and $\mathcal{F} \otimes \mathcal{F}$ and between appropriate representations. On $\mathcal{F} \otimes \mathcal{F}$, we introduce the creation- and annihilation operators

$$\begin{aligned} a_\ell^\#(f) &:= a^\#(f) \otimes \mathbf{1}\,, && \text{(IV.25)} \\ a_r^\#(f) &:= \mathbf{1} \otimes T a^\#(f) T \;=\; \mathbf{1} \otimes a^\#(\tau f)\,. && \text{(IV.26)} \end{aligned}$$

Note that a_r, a_r^* yield an *anti-linear* representation of the canonical commutation relations (IV.8)–(IV.9). Let

$$\rho_\beta(k) \;\equiv\; \rho(k) \;:=\; \frac{1}{\exp[\beta\omega(k)] \,-\, 1} \;>\; 0\,. \qquad \text{(IV.27)}$$

The isomorphism $I_T : \mathcal{H}_\beta^f \to \widehat{\mathcal{H}} := \mathcal{F} \otimes \mathcal{F}$ is determined by the following equations:

$$\begin{aligned} I_T\, \Omega_\beta^f &= \Omega \otimes \Omega\,, && \text{(IV.28)} \\ I_T\, \ell[a(f)]\, I_T^{-1} &:= a_\ell\big(\sqrt{1+\rho} f\big) \;+\; a_r^*\big(\sqrt{\rho} f\big)\,, && \text{(IV.29)} \\ I_T\, r[a(f)]\, I_T^{-1} &:= a_\ell^*\big(\sqrt{\rho}\tau f\big) \;+\; a_r\big(\sqrt{1+\rho}\tau f\big)\,. && \text{(IV.30)} \end{aligned}$$

Note that, by (IV.25)–(IV.26), Eq. (IV.29) is linear in f, while Eq. (IV.30) is anti-linear in f, as it should be. Since Ω is cyclic for $\mathcal{P}$ in $\mathcal{F}$ and Ω_β^f is cyclic for $\ell[\mathcal{P}]$ and $r[\mathcal{P}]$ in $\mathcal{H}_\beta^f$, Eqs. (IV.28)–(IV.30) determine I_T completely. It is a trivial calculation to show that $I_T \ell[a^\#] I_T^{-1}$ and $I_T r[a^\#] I_T^{-1}$ satisfy the canonical commutation relations (IV.8)–(IV.9), because $a_\ell^\#$ and $a_r^\#$ satisfy them. It follows that $\ell[\mathcal{P}]$ is $*$-*homomorphic* to $I_T \ell[\mathcal{P}] I_T^{-1}$ and $r[\mathcal{P}]$ is $*$-*homomorphic* to $I_T r[\mathcal{P}] I_T^{-1}$. Furthermore, since $a(f)\Omega = 0$, for all $f \in \mathcal{S}_0(\mathbb{R}^3)^2$, where Ω is the vacuum in $\mathcal{F}$, we find that

$$\begin{aligned} &\big\langle \Omega \otimes \Omega \;\big|\; I_T\, \ell[a^*(k)]\, \ell[a(k')]\, I_T^{-1}\, \Omega \otimes \Omega \big\rangle \\ &\quad= \big\langle \Omega \otimes \Omega \;\big|\; \sqrt{\rho(k)}\; a_r(k)\; \sqrt{\rho(k')}\; a_r^*(k')\, \Omega \otimes \Omega \big\rangle \\ &\quad= \sqrt{\rho(k)\,\rho(k')}\; \delta(k-k') \;=\; \frac{\delta(k-k')}{\exp[\beta\omega(k)] \,-\, 1} \\ &\quad= \omega_\beta^f\big(a^*(k)\, a(k')\big) \;=\; \big\langle \Omega_\beta^f \;\big|\; \ell[a^*(k)]\, \ell[a(k')]\, \Omega_\beta^f \big\rangle\,, \qquad \text{(IV.31)} \end{aligned}$$

where the first equation follows from (IV.29) and (IV.21), the second from (IV.9) and (IV.21), and the remaining equations from (IV.27), (IV.15), and (IV.17). Likewise,

$$\begin{aligned} \big\langle \Omega \otimes \Omega \;\big|\; I_T\, r[a^*(k)]\, r[a(k')]\, I_T^{-1}\, \Omega \otimes \Omega \big\rangle &= \omega_\beta^f\big(a^*(k)\, a(k')\big) && \text{(IV.32)} \\ &= \big\langle \Omega_\beta^f \;\big|\; r[a^*(k)]\, r[a(k')]\, \Omega_\beta^f \big\rangle\,. \end{aligned}$$

It follows that $I_T : \mathcal{H}_\beta^f \to \mathcal{F} \otimes \mathcal{F}$, defined by (IV.28)–(IV.30), is an isometry.

Next, we calculate $L_f := I_T \mathcal{L}_f I_T^{-1}$. We claim that

$$L_f \;=\; \int dk\, \omega(k)\, \big[a_\ell^*(k) a_\ell(k) \;-\; a_r^*(k) a_r(k)\big]\,. \tag{IV.33}$$

Then

$$\begin{aligned} e^{itL_f}\, a_\ell(k)\, e^{-itL_f} &= e^{-it\omega(k)}\, a_\ell(k)\,, && \text{(IV.34)} \\ e^{itL_f}\, a_r(k)\, e^{-itL_f} &= e^{it\omega(k)}\, a_r(k)\,. && \text{(IV.35)} \end{aligned}$$

Thus, using (IV.29), we find that

$$e^{itL_f}\, I_T\, \ell[a(k)]\, I_T^{-1}\, e^{-itL_f} \;=\; e^{-it\omega(k)}\, I_T\, \ell[a(k)]\, I_T^{-1} \;=\; I_T\, \ell\big[\alpha_t^f\big(a(k)\big)\big]\, I_T^{-1}\,, \tag{IV.36}$$

as required. Similarly, by (IV.30) and (IV.33),

$$\begin{aligned} e^{itL_f}\, I_T\, r[a(k)]\, I_T^{-1}\, e^{-itL_f} &= e^{it\omega(k)}\, I_T\, r[a(k)]\, I_T^{-1} \;=\; I_T\, r\big[e^{-it\omega(k)} a(k)\big]\, I_T^{-1} \\ &= I_T\, r\big[\alpha_t^f\big(a(k)\big)\big]\, I_T^{-1}\,, && \text{(IV.37)} \end{aligned}$$

because r is anti-linear. This, (IV.18), and the corresponding relation for r prove (IV.33).

For $\psi \otimes \varphi \in \mathcal{F} \otimes \mathcal{F}$, we define

$$E\, \psi \otimes \varphi \;=\; \varphi \otimes \psi\,, \tag{IV.38}$$

and we set

$$J \;:=\; E\, T \otimes T\,. \tag{IV.39}$$

Then

$$J\, a_\ell^\#(f)\, J \;=\; a_r^\#(\tau f)\,, \tag{IV.40}$$

and, using that $\mathcal{J} \ell[a^\#(f)] \mathcal{J} = r[a^\#(\tau f)]$ and Eqs. (IV.29)–(IV.30), we verify that

$$J\, I_T\, \ell[a^\#(f)]\, I_T^{-1}\, J \;=\; I_T\, r[a^\#(\tau f)]\, I_T^{-1} \;=\; I_T\, J\, \ell[a^\#(f)]\, J\, I_T^{-1}\,. \tag{IV.41}$$

By (IV.39),

$$J\, \Omega \otimes \Omega \;=\; \Omega \otimes \Omega\,, \tag{IV.42}$$

and $J = J^* = J^{-1}$. It follows from (IV.41)–(IV.42) that

$$J \;=\; I_T\, \mathcal{J}\, I_T^{-1}\,, \tag{IV.43}$$

i.e., J is the modular conjugation in the Araki-Woods representation. Note that

$$J\,(\mathcal{P}\otimes \mathbf{1})\,J \;=\; \mathbf{1}\otimes T\mathcal{P}T\,. \tag{IV.44}$$

Our account summarizes all essential features of the quantum theory of the free electromagnetic field in thermal equilibrium.

IV.2 An Idealized Atom and the Quantized Free Electromagnetic Field

As a next step, we consider a system consisting of an idealized atom with finitely many levels and the electromagnetic field, coupled to each other and in thermal equilibrium.

We begin by describing an idealized atom with finitely many levels. This system is a special example of those described in Subsection II.1. We briefly recall the main objects and notions. The state space $\mathcal{H}_{el}$ has dimension $N < \infty$,

$$\mathcal{H}_{el} \;=\; \mathbb{C}^N\,, \tag{IV.45}$$

and the Hamiltonian, H_{el}, is a selfadjoint $N\times N$ matrix on $\mathcal{H}_{el}$. According to Hypothesis H-1, the eigenvalues of H_{el} are simple. We choose the standard basis in $\mathcal{H}_{el}$ to consist of eigenvectors $\{\varphi_j\}_{j=0}^{N-1}$ of H_{el} corresponding to the eigenvalues $E_0 < E_1 < \cdots < E_{N-1}$, i.e., $H_{el}\varphi_j = E_j\varphi_j$. The atomic Liouvillian acts on $\mathcal{K}_{el} = \mathcal{M}_N \ni \kappa$ by

$$L_{el}\,\kappa \;=\; \big[H_{el}\,,\,\kappa\big]\,, \tag{IV.46}$$

where $\mathcal{M}_N$ denotes the algebra of complex $N\times N$ matrices, and the atomic KMS state is given by

$$\rho_\beta^{el} \;=\; Z_{\beta,0}^{-1}\sum_{j=0}^{N-1} e^{-\beta E_j}\,|\varphi_j\rangle\langle\varphi_j|\,. \tag{IV.47}$$

Recall from (IV.17) that ω_β^f denotes the unique KMS state of the electromagnetic field at inverse temperature β. The reference state of the systems consisting of an atom and the quantized radiation field is

$$\rho_\beta^0 \;:=\; \rho_\beta^{el}\otimes\omega_\beta^f\,. \tag{IV.48}$$

Let $(\mathcal{H}_\beta, \Omega_{\beta,0}, \ell, r)$ denote the GNS Hilbert space, the cyclic vector, the left repesentation of $\mathcal{A} := \mathcal{M}_N \otimes \mathcal{P}$, and the right anti-representation of $\mathcal{A}$, respectively, associated to $(\rho_\beta^0, \mathcal{A})$. The Liouvillian of the uncoupled system is

$$\mathcal{L}_0 \ = \ \mathcal{L}_{el} \, + \, \mathcal{L}_f \, , \tag{IV.49}$$

where $\mathcal{L}_f = I_T^{-1} L_f I_T$, with L_f as in (IV.33) and I_T as in (IV.28)–(IV.30). Note that ρ_β^0 is the unique KMS state of the uncoupled system.

We define $I \in \mathcal{A}$ by

$$I \ := \ \int dk \, \big\{ G(k) \otimes a^*(k) \ + \ G(k)^* \otimes a(k) \big\} \, , \tag{IV.50}$$

where, as in Hypothesis H-3, the coupling function $G : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{M}_N$ is assumed to obey

$$\| G(k) \| \ \leq \ \omega(k)^\mu \, , \tag{IV.51}$$

for some $\mu > 0$. The Liouvillian of the interacting system is defined by

$$\mathcal{L}_g \ := \ \mathcal{L}_0 \ + \ g \big\{ \ell[I] - \pi[I] \big\} \, , \tag{IV.52}$$

and the "Radon-Nikodym" operators by

$$\mathcal{L}_{g,\ell} \ := \ \mathcal{L}_0 \ + \ g\ell[I] \, , \qquad \mathcal{L}_{g,r} \ := \ \mathcal{L}_0 \ - \ gr[I] \, . \tag{IV.53}$$

For the purpose of our analysis it is convenient to work in the Araki-Woods representation, i.e., to conjugate the above operators by the isomorphism

$$I_0 \ := \ I_C \otimes I_T \ : \ \mathcal{H}_\beta \ \to \ \widehat{\mathcal{H}} := \mathcal{H}_{el} \otimes \mathcal{H}_{el} \otimes \mathcal{F} \otimes \mathcal{F} \, , \tag{IV.54}$$

where conjugation by I_C denotes complex conjugation in $\mathcal{M}_N$ in the standard basis $\{\varphi_i \otimes \varphi_j\}_{i,j=0}^{N-1}$. We set

$$\widehat{\Omega}_{\beta,0} \ := \ I_0 \, \Omega_{\beta,0} \tag{IV.55}$$

and note that

$$\widehat{\Omega}_{\beta,0} \ = \ Z_{\beta,0}^{-1} \sum_{j=0}^{N-1} e^{-\beta E_j} \, \varphi_j \otimes \varphi_j \otimes \Omega \otimes \Omega \, . \tag{IV.56}$$

Defining

$$L_0 \ := \ I_0 \, \mathcal{L}_0 \, I_0^{-1} \ , \qquad L_g \ := \ I_0 \, \mathcal{L}_g \, I_0^{-1} \, , \tag{IV.57}$$

we find that

$$L_g \ = \ L_0 \, + \, gW \, , \tag{IV.58}$$

where

$$\begin{aligned} L_0 \ &= \ L_{el} \otimes \mathbf{1}^f \, + \, \mathbf{1}^{el} \otimes L_f \, , && \text{(IV.59)} \\ L_{el} \ &= \ H_{el} \otimes \mathbf{1}_{el} - \mathbf{1}_{el} \otimes H_{el} \, , && \text{(IV.60)} \end{aligned}$$

L_f is defined in (IV.33), and the interaction is given by

$$W \ := \ I_0 \big\{ \ell[I] - r[I] \big\} I_0^{-1} \, . \tag{IV.61}$$

By means of (IV.50), and Eqs. (IV.29), (IV.30), we readily find that

$$\begin{aligned} W \ = \ \int dk \, \Big\{ & \big(\sqrt{1+\rho(k)} \, G_\ell(k) \ - \ \sqrt{\rho(k)} \ \overline{G_r^*}(k) \big) \, a_\ell^*(k) \qquad \text{(IV.62)} \\ & \big(\sqrt{1+\rho(k)} \, G_\ell^*(k) \ - \ \sqrt{\rho(k)} \ \overline{G_r}(k) \big) \, a_\ell(k) \\ & \big(\sqrt{\rho(k)} \, G_\ell^*(k) \ - \ \sqrt{1+\rho(k)} \ \overline{G_r}(k) \big) \, a_r^*(k) \\ & \big(\sqrt{\rho(k)} \, G_\ell(k) \ - \ \sqrt{1+\rho(k)} \ \overline{G_r^*}(k) \big) \, a_r(k) \Big\} \, , \end{aligned}$$

which we may rewrite as

$$\begin{aligned} W \ = \ & a_\ell^* \big(\sqrt{1+\rho} \, G_\ell \ - \ \sqrt{\rho} \, \overline{G_r^*} \big) \ + \ a_\ell \big(\sqrt{1+\rho} \, G_\ell^* \ - \ \sqrt{\rho} \, \overline{G_r} \big) \quad \text{(IV.63)} \\ & + a_r^* \big(\sqrt{\rho} \, G_\ell^* \ - \ \sqrt{1+\rho} \, \overline{G_r} \big) \ + \ a_r \big(\sqrt{\rho} \, G_\ell \ - \ \sqrt{1+\rho} \, \overline{G_r^*} \big) \, , \end{aligned}$$

using the shorthand notation $\overline{m} := I_C m I_C^{-1}$, $m_\ell := m \otimes \mathbf{1}_{el}$, $m_r := \mathbf{1}_{el} \otimes m$, and

$$a_\sigma^*(m_\mu) \ := \ \int dk \, m_\mu(k) \, a_\sigma^*(k) \quad , \quad a_\sigma(m_\mu) \ := \ \int dk \, m_\mu(k) \, a_\sigma(k) \, , \tag{IV.64}$$

for $\sigma, \mu \in \{\ell, r\}$ and matrix-valued functions $m : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{B}(\mathcal{H}_{el})$.

Similarly, setting

$$L_{g,\ell} \ := \ I_0 \, \mathcal{L}_{g,\ell} \, I_0^{-1} \ , \qquad L_{g,r} \ := \ I_0 \, \mathcal{L}_{g,r} \, I_0^{-1} \, , \tag{IV.65}$$

we have that

$$L_{g,\ell} \ = \ L_0 \, + \, gW_\ell \ , \qquad L_{g,r} \ = \ L_0 \, + \, gW_r \, , \tag{IV.66}$$

where

$$\begin{aligned} W_\ell &= I_0\,\ell[I]\,I_0^{-1} && \text{(IV.67)} \\ &= a_\ell^*\big(\sqrt{1+\rho}\,G_\ell\big) \;+\; a_\ell\big(\sqrt{1+\rho}\,G_\ell^*\big) \;+\; a_r^*\big(\sqrt{\rho}\,G_\ell^*\big) \;+\; a_r\big(\sqrt{\rho}\,G_\ell\big)\,, \\ W_r &= -I_0\,r[I]\,I_0^{-1} && \text{(IV.68)} \\ &= -a_\ell^*\big(\sqrt{\rho}\,\overline{G_r^*}\big) \;-\; a_\ell\big(\sqrt{\rho}\,\overline{G_r}\big) \;-\; a_r^*\big(\sqrt{1+\rho}\,\overline{G_r}\big) \;-\; a_r\big(\sqrt{1+\rho}\,\overline{G_r^*}\big)\,. \end{aligned}$$

Besides L_f, the following positive operator,

$$L_{aux} \;=\; H_f \otimes \mathbf{1}_f \;+\; \mathbf{1}_f \otimes H_f\,, \tag{IV.69}$$

plays an important role in our analysis. It is straigthforward to see that L_{aux} is selfadjoint on its natural domain

$$\mathrm{Dom}[L_{aux}] \;=\; \big\{\psi \in \mathcal{F}\otimes\mathcal{F} \;\big|\; \|L_{aux}\psi\| < \infty\big\} \tag{IV.70}$$

and that L_{aux} dominates L_f, in the sense that $\mathrm{Dom}[L_f] \supseteq \mathrm{Dom}[L_{aux}]$ and

$$|L_f| \;\le\; L_{aux}\,. \tag{IV.71}$$

Moreover, we have the following standard estimates (see, e.g., [7, 8, 9]),

Lemma IV.1. *If* $f \in L^2(\mathbb{R}^3\times\mathbb{Z}_2, \mathcal{M}_N)$ *and* $\omega^{-1/2} f \in L^2(\mathbb{R}^3\times\mathbb{Z}_2, \mathcal{M}_N)$ *then*

$$\left.\begin{array}{c} \big\|a_\sigma(f_\mu)\,L_{aux}^{-1/2}\,P_\Omega^\perp\big\|\,, \\ \big\|L_{aux}^{-1/2}\,P_\Omega^\perp\,a_\sigma^*(f_\mu)\big\|\,, \\ \big\|(L_{aux}+1)^{-1/2}\,a_\sigma(f_\mu)\big\|\,, \\ \big\|a_\sigma^*(f_\mu)\,(L_{aux}+1)^{-1/2}\big\| \end{array}\right\} \;\le\; \left(\int dk\,\Big(1+\frac{1}{\omega(k)}\Big)\,\|f(k)\|^2\right)^{1/2}, \tag{IV.72}$$

for $\sigma, \mu \in \{\ell, r\}$.

From these relative bounds and Hypothesis H-4, i.e.,

$$\left(\int dk\,\big(\omega(k)+\omega(k)^{-3}\big)\,\|G(k)\|^2\right)^{1/2} \;\le\; \Lambda \;<\; \infty\,, \tag{IV.73}$$

we conclude selfadjointness of L_g by Nelson's commutator theorem.

Theorem IV.2. *The operators* L_g, $L_{g,\ell}$, *and* $L_{g,r}$ *are essentially selfadjoint on* $\mathrm{Dom}[L_{aux}]$.

Proof. By (IV.71) and (IV.72), we have, for $\varphi, \psi \in \mathrm{Dom}[L_{aux}]$,

$$|\langle\psi|\, L_{g,\#}\, \varphi\rangle| \;\le\; \big(1 \,+\, 16g\beta^{-1/2}(1+\beta)^{1/2}\,\Lambda\big)\, \big\|(L_{aux}+1)^{1/2}\psi\big\|\, \big\|(L_{aux}+1)^{1/2}\varphi\big\|\,, \tag{IV.74}$$

where $L_{g,\#} = L_g$, $L_{g,\ell}$, or $L_{g,r}$. Next, we observe that L_f and L_{aux} commute and that on $\mathrm{Dom}[L_{aux}^{3/2}]$, we have

$$\big[a_\sigma(f_\mu)\,,\, L_{aux}\big] \;=\; a_\sigma(\omega\, f_\mu)\,. \tag{IV.75}$$

Hence, for $\varphi, \psi \in \mathcal{D}(L_{aux}^{3/2})$,

$$\big|\big\langle\psi\big|\,[L_{aux}+1\,,\, L_{g,\#}]\,\varphi\big\rangle\big| \;\le\; 16\, g\, \beta^{-1/2}(1+\beta)^{1/2}\,\Lambda\, \big\|(L_{aux}+1)^{1/2}\psi\big\|\, \big\|(L_{aux}+1)^{1/2}\varphi\big\|\,. \tag{IV.76}$$

Thus, L_g is essentially selfadjoint by a variant [33, Thm. X.36'] of Nelson's commutator theorem. □

Next, we recall the expression for the modular conjugation in the Araki-Woods representation,

$$J \;=\; I_0\, \mathcal{J}\, I_0^{-1} \;=\; E\,(\,C\otimes C\otimes T\otimes T\,)\,, \tag{IV.77}$$

where the exchange operator acts as

$$E\,(\varphi\otimes\psi\otimes\varphi'\otimes\psi') \;=\; (\psi\otimes\varphi\otimes\psi'\otimes\varphi')\,, \tag{IV.78}$$

for $\psi\otimes\varphi\otimes\psi'\otimes\varphi' \in \widehat{\mathcal{H}}$, where $\widehat{\mathcal{H}}$ is defined in (IV.54). Note that the invariance property $\mathcal{J}\Omega_{\beta,0} = \Omega_{\beta,0}$ translates to the invariance property

$$J\,\widehat{\Omega}_{\beta,0} \;=\; \widehat{\Omega}_{\beta,0}\,. \tag{IV.79}$$

IV.3 KMS States for an Idealized Atom coupled to the Quantized Electromagnetic Field

The selfadjointness of the interacting Liouvillean L_g guarantees the existence of the dynamics as a strongly continuous unitary group $\{\exp[-itL_g]\}_{t\in\mathbb{R}}$ on $\widehat{\mathcal{H}}$. We define the Heisenberg-picture time evolution of a bounded operator b on $\widehat{\mathcal{H}}$ by

$$\alpha_t^g\big(b\big) \;:=\; e^{itL_g}\, b\, e^{-itL_g}\,. \tag{IV.80}$$

In the following theorem we construct a perturbed KMS state for the dynamics described by $\{\alpha_t^g\}_{t\in\mathbb{R}}$, starting from the data $(\widehat{\mathcal{H}}, \ell[\mathcal{M}_N\otimes\mathcal{P}], J, L_g, I_0)$.

Theorem IV.3. *Assume that G fulfills Hypothesis H-4. Then the vector $\widehat{\Omega}_{\beta,0}$ is in the domain of the two unbounded operators $\exp[-\beta L_{g,\ell}/2]$ and $\exp[\beta L_{g,r}/2]$, and the vector*

$$\widehat{\Omega}_{\beta,g} \;:=\; Z_{\beta,0}^{-1}\, e^{-\beta L_{g,\ell}/2}\, \widehat{\Omega}_{\beta,0} \;=\; Z_{\beta,0}^{-1}\, e^{\beta L_{g,r}/2}\, \widehat{\Omega}_{\beta,0} \tag{IV.81}$$

where $Z_{\beta,g} = \big\| e^{-\beta L_{g,\ell}/2}\, \widehat{\Omega}_{\beta,0} \big\| = \big\| e^{\beta L_{g,r}/2}\, \widehat{\Omega}_{\beta,0} \big\|$, defines a KMS state, ω_β^g, on $\mathcal{M}_N \otimes \mathcal{P}$, for the time evolution given by $\{\alpha_t^g\}_{t\in\mathbb{R}}$. Moreover, the normalization factor $Z_{\beta,g}$ obeys the estimate

$$\begin{aligned} 0 \;&<\; Z_{\beta,g} \\ &\leq\; \sqrt{\mathrm{Tr}\big\{e^{-\beta H_{el}}\big\}}\; \exp\Big(g^2\,\beta^2 \int \big(1 + [\beta\omega(k)]^{-1}\big)\, \|G(k)\|^2\, dk\Big) \;<\; \infty\,. \end{aligned} \tag{IV.82}$$

In particular, $L_g \widehat{\Omega}_{\beta,0} = 0$.

Proof. We first note that due to the definition of $Z_{\beta,g}$,

$$Z_{\beta,g}^2 \;=\; \big\langle \widehat{\Omega}_{\beta,0} \big|\, e^{-\beta L_{g,\ell}}\, \widehat{\Omega}_{\beta,0} \big\rangle \;=\; \big\langle \Omega_{\beta,0} \big|\, e^{-\beta \mathcal{L}_{g,\ell}}\, \Omega_{\beta,0} \big\rangle\,. \tag{IV.83}$$

Thus, if we can prove (IV.82) then $\Omega_{\beta,0}$ is in the domain of $e^{-\frac{\beta}{2}\mathcal{L}_{g,\ell}}$ and, similarly, in the domain of $e^{\frac{\beta}{2}\mathcal{L}_{g,r}}$. To demonstrate (IV.82), we introduce $\mathcal{I}(\tau) := e^{-\tau L_0}\ell[I]e^{\tau L_0}$ and observe that

$$\begin{aligned} &\big\langle \Omega_{\beta,0} \big|\, e^{-\beta \mathcal{L}_{g,\ell}}\, \Omega_{\beta,0} \big\rangle \\ &\quad=\; \sum_{n=0}^{\infty} g^n \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n\, \big\langle \Omega_{\beta,0} \big|\, \mathcal{I}(\tau_1)\cdots\mathcal{I}(\tau_n)\, \Omega_{\beta,0} \big\rangle \\ &\quad=\; \sum_{n=0}^{\infty} g^{2n} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{2n-1}} d\tau_{2n}\, \big\langle \Omega_{\beta,0} \big|\, \mathcal{I}(\tau_1)\cdots\mathcal{I}(\tau_{2n})\, \Omega_{\beta,0} \big\rangle\,, \end{aligned} \tag{IV.84}$$

using that $\prod_{j=1}^n \Big(e^{\nu_n \mathcal{L}_0}\ell[I]\Big)\Omega_{\beta,0} \in \mathcal{H}_\beta$, for $\nu_1, \ldots, \nu_n \in \mathbb{R}_+$ with $\sum_{j=1}^n \nu_n \leq \beta$. Abbreviating $e^{-\tau\omega(k)}a^*(k) =: a^+(k,\tau)$, $e^{\tau\omega(k)}a(k) =: a^-(k,\tau)$, $G(k) =: G^+(k)$, and $G^*(k) =: G^-(k)$, and using (IV.36), with $t = i\tau$, and (IV.50), we may write

$$\mathcal{I}(\tau) \;=\; \sum_{\sigma=\pm} \int dk\, \ell\big[e^{-\tau H_{el}} G^\sigma(k) e^{\tau H_{el}} \otimes a^\sigma(k,\tau)\big]\,. \tag{IV.85}$$

Inserting this expression in (IV.84), we then obtain

$$\begin{aligned}
&\big\langle \Omega_\beta^0 \big| \, e^{-\beta \mathcal{L}_{g,\ell}} \, \Omega_\beta^0 \big\rangle \qquad \text{(IV.86)} \\
&= \sum_{n=0}^{\infty} \sum_{\underline{\sigma} \in \{+,-\}^{2n}} g^{2n} \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{2n-1}} d\tau_{2n} \int dk_1 \cdots dk_{2n} \\
&\qquad \mathrm{Tr}\big[e^{-(\beta+\tau_1-\tau_{2n})H_{el}} G^{\sigma_1}(k_1) e^{(\tau_1-\tau_2)H_{el}} \cdots e^{(\tau_{2n-1}-\tau_{2n})H_{el}} G^{\sigma_{2n}}(k_{2n}) \big] \\
&\qquad \omega_\beta^0\big(a^{\sigma_1}(k_1,\tau_1) \cdots a^{\sigma_{2n}}(k_{2n},\tau_{2n}) \big) \, .
\end{aligned}$$

Applying Hölder's inequality for the trace,

$$\mathrm{Tr}[A_1 B_1 \cdots A_n B_n] \;\le\; \prod_{j=1}^{n} \|B_j\| \, \prod_{j=1}^{n} \Big(\mathrm{Tr}\{ \, |A_j|^{p_j} \, \} \Big)^{1/p_j} \, , \qquad \text{(IV.87)}$$

where $p_j \geq 0$ and $1/p_1 + \cdots + 1/p_n = 1$, we observe that

$$\begin{aligned}
&\mathrm{Tr}\big[e^{-(\beta+\tau_1-\tau_{2n})H_{el}} G^{\sigma_1}(k_1) e^{(\tau_1-\tau_2)H_{el}} \cdots e^{(\tau_{2n-1}-\tau_{2n})H_{el}} G^{\sigma_{2n}}(k_{2n}) \big] \\
&\qquad \le \; \mathrm{Tr}\big[e^{-\beta H_{el}} \big] \prod_{j=1}^{2n} \| G(k_j) \| \, . \qquad \text{(IV.88)}
\end{aligned}$$

Next, since ω_β^0 is quasi-free, Wick's theorem implies that

$$\begin{aligned}
&\omega_\beta^0\big(a^{\sigma_1}(k_1,\tau_1) \cdots a^{\sigma_{2n}}(k_{2n},\tau_{2n}) \big) \;=\; \qquad \text{(IV.89)} \\
&\sum_{\pi \in \mathcal{P}_{2n}} \prod_{j=1}^{n} \omega_\beta^0\big(a^{\sigma_{\pi(2j-1)}}(k_{\pi(2j-1)}, \tau_{\pi(2j-1)}) \, a^{\sigma_{\pi(2j)}}(k_{\pi(2j)}, \tau_{\pi(2j)}) \big) \, ,
\end{aligned}$$

where $\mathcal{P}_{2n}$ is the set of pairings, i.e., all permutations $\pi \in \mathcal{S}_{2n}$ such that $\pi(1) < \pi(3) < \cdots < \pi(2n-1)$ and $\pi(2j-1) < \pi(2j)$. Since $\omega_\beta^0(a^+ a^+) = \omega_\beta^0(a^- a^-) = 0$, the only nonvanishing contributions in (IV.89) come from

$$\omega_\beta^0\big(a^-(k,\tau) \, a^+(k',\tau') \big) \;=\; \delta(k-k') \frac{e^{(\tau-\tau')\omega(k)}}{e^{\beta\omega(k)} - 1} \, , \qquad \text{(IV.90)}$$

$$\omega_\beta^0\big(a^+(k,\tau) \, a^-(k',\tau') \big) \;=\; \delta(k-k') \frac{e^{(\beta-\tau+\tau')\omega(k)}}{e^{\beta\omega(k)} - 1} \, . \qquad \text{(IV.91)}$$

Thus, for $0 \leq \tau' \leq \tau \leq \beta$, we have the estimate

$$\begin{aligned}
\delta(k-k') \frac{2 \, e^{\beta\omega(k)/2}}{e^{\beta\omega(k)} - 1} \;&\le\; \sum_{\sigma,\sigma'=\pm} \omega_\beta^0\big(a^\sigma(k,\tau) \, a^{\sigma'}(k',\tau') \big) \\
&\le\; \delta(k-k') \frac{e^{\beta\omega(k)} + 1}{e^{\beta\omega(k)} - 1} \, . \qquad \text{(IV.92)}
\end{aligned}$$

The number of pairings is

$$|\mathcal{P}_{2n}| \;=\; (2n-1)(2n-3)\cdots 1 \;=\; \frac{(2n)!}{2^n\, n!}\,. \tag{IV.93}$$

Hence, first summing over all $\underline{\sigma} \in \{+,-\}^{2n}$, taking (IV.89) and the upper bound in (IV.92) into account, and using (IV.93), we obtain

$$\begin{aligned}
&\big\langle \Omega_{\beta,0} \big|\, e^{-\beta \mathcal{L}_{g,\ell}}\, \Omega_{\beta,0} \big\rangle \, \big(\mathrm{Tr}\{ e^{-\beta H_{el}} \} \big)^{-1} \\
\leq\;& \sum_{n=0}^{\infty} \frac{(2n)!}{2^n\, n!} \Big(g^2 \int \coth[\beta\omega(k)/2]\; \|G(k)\|^2\, dk \Big)^n \int_0^{\beta} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{2n-1}} d\tau_{2n} \\
=\;& \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{g^2\, \beta^2}{2} \int \coth[\beta\omega(k)/2]\; \|G(k)\|^2\, dk \Big)^n \\
=\;& \exp\Big(\frac{g^2\, \beta^2}{2} \int \coth[\beta\omega(k)/2]\; \|G(k)\|^2\, dk \Big) \\
\leq\;& \exp\Big(g^2\, (1+\beta)\beta\, \Lambda \Big) \;<\; \infty\,,
\end{aligned} \tag{IV.94}$$

due to (IV.73), and using $\coth x \leq 2 + 2/x$. □

V Spectral Analysis of the interacting Liouvillian

V.1 Main Results and Outline of Proofs

In this section we present our main results on the spectrum of the interacting Liouvillian L_g introduced in Eqs. (IV.52), (IV.58). Throughout our analysis, we assume that Hypotheses (H-1) through (H-5) stated in the introduction, Sect. I, are satisfied. Our goal is to prove that L_g has *purely absolutely continuous spectrum* covering the real axis, except for a *simple* eigenvalue at 0. The eigenvector corresponding to this eigenvalue is the perturbed KMS state $\widehat{\Omega}_{\beta,g}$ constructed in Theorem IV.3.

Our method to prove this result involves *two key ingredients:* a novel variant of the technique of *complex spectral deformations* (see, e.g., [15, 33]), and the *isospectral Feshbach map* introduced in [7, 8]. We shall first qualitatively describe these ingredients and then outline the basic strategy underlying our method.

Recall that the Hilbert space of temperature states of the system, in the Araki-Woods representation, is given by

$$\widehat{\mathcal{H}} \;=\; \mathcal{H}_{el} \otimes \mathcal{H}_{el} \otimes \mathcal{F} \otimes \mathcal{F} \, , \tag{V.1}$$

see Eq. (IV.54). The wave function of a photon in momentum space is given by a pair of functions $(f_+(\vec{k}), f_-(\vec{k}))$, $\vec{k} \in \mathbb{R}^3$. On the space of one-photon wave functions we define *dilatations*, $\{u(\theta)\}_{\theta \in \mathbb{R}}$, by setting

$$u(\theta)\big(f_+(\vec{k}),\, f_-(\vec{k})\big) \;=\; e^{-3\theta/2}\left(f_+(e^{-\theta}\vec{k}),\, f_-(e^{-\theta}\vec{k})\right) . \tag{V.2}$$

Note that $u(\theta)$ is unitary in the usual L^2 scalar product. We define $U(\theta)$ to be the unitary operator on Fock space $\mathcal{F}$ obtained from $u(\theta)$ by second quantization; see, e.g., [7] and Sect. V.3. Then

$$U(\theta)\, \Omega \;=\; \Omega \, , \tag{V.3}$$

where Ω is the vacuum vector. We define a representation $\{\widehat{U}(\theta)\}_{\theta \in \mathbb{R}}$ of the group of dilatations on the Araki-Woods Hilbert space by

$$\widehat{U}(\theta) \;:=\; \mathbf{1} \otimes \mathbf{1} \otimes U(\theta) \otimes U(-\theta) \, . \tag{V.4}$$

For the purposes of our analysis of the spectrum of L_g it is *crucial* that the arguments, θ and $-\theta$, in the third and fourth factor on the R.S. of (V.4) have *opposite signs*. Our method of complex spectral deformations relies on extending θ to a complex domain, $\Sigma_{\pi/2}$, which is the strip symmetric about the real axis and of width π. It is easy to see that there is a natural dense domain $\mathcal{D} \subseteq \widehat{\mathcal{H}}$ with the property that, for every $\psi \in \mathcal{D}$, $\widehat{U}(\theta)\psi$ is an *analytic* $\widehat{\mathcal{H}}$-valued function of $\theta \in \Sigma_{\pi/2}$.

We start by considering the spectrum of $L_0 = L_{g=0}$; see (IV.59). Its eigenvalues are given by those of L_{el}, i.e., by $\{E_i - E_j \mid i, j = 0, \dots , N-1\}$; the eigenvalue 0 is thus N-fold degenerate. These eigenvalues are covered by N^2 branches of continuous spectrum which are copies of the continuous spectrum of L_f. In the example, where $N = 2$, $E_0 = 0$, $E_1 = \varepsilon_0$, the spectrum of L_0 is depicted in Fig. 1.

We define the dilated Liouvillian by

$$L_0(\theta) \;:=\; \widehat{U}(\theta)\, L_0\, \widehat{U}(-\theta) \, . \tag{V.5}$$

From formulae (IV.33), (IV.59), and (IV.69), we infer that

$$L_0(\theta) \;=\; L_{el} \;+\; \cosh(\theta) \cdot L_f \;-\; \sinh(\theta) \cdot L_{aux} \, , \tag{V.6}$$

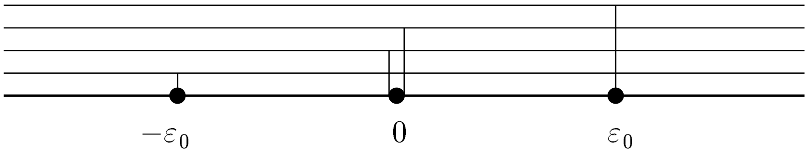

Figure 1: The spectrum of L_0.

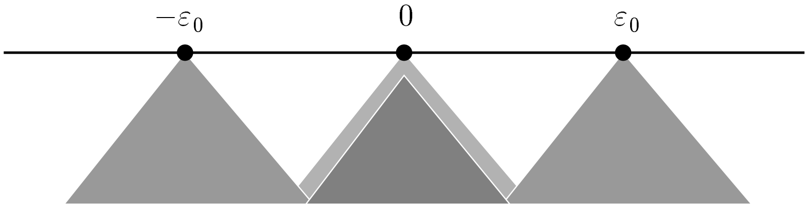

Figure 2: The spectrum of $L_0(\theta)$, for $\mathrm{Re}\,\theta = 0$, $\mathrm{Im}\,\theta = \vartheta > 0$.

where L_{aux} is the *positive* operator defined in (IV.69). The operator $L_0(\theta)$ is clearly analytic in θ on the strip $\Sigma_{\pi/2}$. If $\theta \notin \mathbb{R}$ then the spectrum of $L_0(\theta)$ intersects the real axis *only* in the eigenvalues $\{E_i - E_j \mid i, j = 0, \ldots, N-1\}$ of L_{el}. If $\mathrm{Im}\,\theta =: \vartheta > 0$ it is contained in the closed lower half plane, while if $\vartheta < 0$ it lies in the closed upper half plane. In deriving Eq. (V.6) and establishing these properties of $\sigma(L_0(\theta))$, the *relative minus sign* between the third and the fourth argument on the R.S. of Eq. (V.4) is *crucial*! In the example considered above, the spectrum of $L_0(\theta)$ for $\mathrm{Re}\,\theta = 0$ and $\mathrm{Im}\,\theta = \vartheta > 0$ is depicted in Fig. 2.

The absolutely continuous nature of the spectrum of $L_0 = L_0(\theta = 0)$ away from its eigenvalues can be inferred from the spectral properties of $L_0(\theta)$, $\theta \notin \mathbb{R}$, by considering matrix elements of the resolvent of L_0 between vectors in the dense domain $\mathcal{D}$ of dilatation-analytic vectors and using that

$$\left\langle \varphi \mid (L_0 - z)^{-1}\,\psi \right\rangle \;=\; \left\langle \widehat{U}(\bar{\theta})\varphi \mid (L_0(\theta) - z)^{-1}\,\widehat{U}(\theta)\psi \right\rangle, \tag{V.7}$$

for φ and ψ in $\mathcal{D}$. Clearly the R.S. of (V.7) is analytic in z in the complement of $\sigma(L_0(\theta))$, and this provides an analytic continuation in z of the L.S. of (V.7) to the complement of $\sigma(L_0(\theta))$.

The idea is now to study what happens to the spectrum of $L_0(\theta)$ when the perturbation

$$gW(\theta) \;:=\; g\,\widehat{U}(\theta)\,W\,\widehat{U}(\theta)^{-1} \tag{V.8}$$

is added to $L_0(\theta)$. In defining the operator $W(\theta)$ for complex values of $\theta \in \Sigma_{\vartheta_0}$, we shall make use of Hypothesis (H-2) stated in Sect. I. There are some important technical points in the construction of an analytic continuation of the operator $W(\theta)$ and of the interacting Liouvillian

$$L_g(\theta) \;=\; \widehat{U}(\theta)\, L_g\, \widehat{U}(-\theta) \;=\; L_0(\theta) \,+\, gW(\theta) \tag{V.9}$$

that require careful examination. They are dealt with in Sect. V.3 and in Appendix A. The upshot is that there exists a natural domain $\mathcal{D}_0 \subseteq \mathcal{D}$ dense in $\widehat{\mathcal{H}}$ such that, for arbitrary vectors φ and ψ in $\mathcal{D}_0$, the matrix elements

$$\big\langle \widehat{U}(\theta)\varphi \;\big|\; (L_g(\theta) - z)^{-1}\, \widehat{U}(\theta)\psi \big\rangle\,, \qquad \mathrm{Im}\, z \neq 0\,, \qquad \theta \in \mathbb{R}\,, \tag{V.10}$$

are the boundary values on the real axis of the function

$$\big\langle \widehat{U}(\bar{\theta})\varphi \;\big|\; (L_g(\theta) - z)^{-1}\, \widehat{U}(\theta)\psi \big\rangle\,, \tag{V.11}$$

which is analytic in θ on the strip Σ_{ϑ_0}, defined in Eq. (I.24), thanks to Hypothesis H-2. Since $U(\theta)^{-1} = U(\theta)^* = U(-\theta)$, for $\theta \in \mathbb{R}$, the matrix elements (V.10) are independent of θ, and hence

$$\big\langle \varphi \;\big|\; (L_g - z)^{-1}\, \psi \big\rangle \;=\; \big\langle \widehat{U}(\bar{\theta})\varphi \;\big|\; (L_g(\theta) - z)^{-1}\, \widehat{U}(\theta)\psi \big\rangle\,, \tag{V.12}$$

for $\varphi, \psi \in \mathcal{D}_0$, and $\mathrm{Im}\, z > 0$, $0 < \mathrm{Im}\, \theta < \vartheta_0$.

If we are able to find out where the spectrum, $\sigma(L_g(\theta))$, of $L_g(\theta)$ is located for, e.g., purely imaginary $\theta = i\vartheta$, with $0 < \vartheta < \vartheta_0$, then we can use Eq. (V.12) to construct an analytic continuation in z of matrix elements of $(L_g - z)^{-1}$ between vectors in $\mathcal{D}_0$ from the upper half plane to the complement of $\sigma(L_g(\theta))$ in the lower half plane.

We shall attempt to locate the spectrum of $L_g(i\vartheta)$ with the help of *perturbative methods*, using that we know $\sigma(L_0(i\vartheta))$ explicitly. The form of $\sigma(L_0(i\vartheta))$, for $\vartheta > 0$, see Fig. 2, Formula (V.6), and the bounds presented in Lemma IV.1 suggest to apply the method of the *isospectral Feshbach map* developed in [7, 8], in order to explore the properties of $\sigma(L_g(i\vartheta))$, $0 < \vartheta < \vartheta_0$. We thus recall the definition and properties of the *Feshbach map*.

Let H be a closed operator on a Hilbert space $\mathcal{H}$ and let P be a closed bounded projection operator whose range is in the domain of H. We define

$$\overline{H} \;:=\; \overline{P}\, H\, \overline{P}\,, \qquad \overline{P} := \mathbf{1} - P\,. \tag{V.13}$$

Let z belong to the *resolvent set* of $\overline{H}|_{\overline{P}\mathcal{H}}$. We assume that the operators

$$P\, H\, \overline{P}\, |\overline{H} - z|^{-1/2} \quad \text{and} \quad |\overline{H} - z|^{-1/2}\, \overline{P}\, H\, P \tag{V.14}$$

are bounded. Then we can define an operator $\mathcal{F}_P(H - z)$, the *Feshbach map* (associated to the projection P) at $H - z$, acting on the Hilbert space $P\mathcal{H}$, by setting

$$\mathcal{F}_P(H - z) \;:=\; P(H - z)P \;-\; P\, H\, \overline{P}\, (\overline{H} - z)^{-1}\, \overline{P}\, H\, P\,. \tag{V.15}$$

In [7, 8] we have proven the following theorem.

Theorem V.1. *Under the hypotheses on H, P, and z just stated, one has that*

- **(i)** *z is an eigenvalue of H iff 0 is an eigenvalue of $\mathcal{F}_P(H-z)$, and the multiplicity of $z \in \sigma_{\rm pp}(H)$ is the same as the multiplicity of $0 \in \sigma_{\rm pp}(\mathcal{F}_P(H-z))$;*
- **(ii)** *z belongs to the resolvent set of H iff 0 belongs to the resolvent set of $\mathcal{F}_P(H-z)$;*
- **(iii)** *For $\varphi, \psi \in P\mathcal{H}$ and $z \notin \sigma(H)$,*

$$\big\langle \varphi \, \big| \, (H-z)^{-1} \, \psi \big\rangle \; = \; \big\langle \varphi \, \big| \, \mathcal{F}_P(H-z)^{-1} \, \psi \big\rangle \, . \tag{V.16}$$

Our strategy, in this section, is based on applying Theorem V.1 to the concrete situation studied in this paper, with the following identifications:

$$\mathcal{H} \; := \; \widehat{\mathcal{H}} \, , \quad H \; = \; L_g(i\vartheta) \, , \tag{V.17}$$

for some $0 < \vartheta < \vartheta_0$, to be chosen optimally. Furthermore, the projection P is given by

$$P \; := \; P_\eta^{el} \otimes P_\rho^{aux} \tag{V.18}$$

where η is an eigenvalue of L_{el} (i.e., $\eta = E_i - E_j$, with E_i, E_j eigenvalues of H_{el}, $i, j = 0, \dots, N-1$), and P_η^{el} is the orthogonal projection onto the eigenspace of L_{el} corresponding to the eigenvalue η. Moreover, P_ρ^{aux} is a spectral projection of the operator L_{aux} introduced in (IV.69); more precisely

$$P_\rho^{aux} \;\; := \;\; \chi[L_{aux} < \rho] \, , \quad\; \rho > 0 \, , \tag{V.19}$$

where $\chi[x < \rho]$ is the characteristic function of $(-\infty, \rho)$. The positive number ρ is later chosen to depend on the coupling constant g; ($\rho \sim g^{2-0(\varepsilon)}$, or $\rho \sim g^{2+0(\varepsilon)}$, for a small $\varepsilon > 0$).

Next, we define a family of subsets, $\mathcal{S}_>$, $\mathcal{S}_{0,>}$, $\mathcal{S}_{0,<}$, and $\mathcal{S}_{i,j}$, $1 \le i, j \le N$, $i \ne j$, of the complex plane. Our choice of a projection P, as in Eqs. (V.18), (V.19), in the definition of the Feshbach map, $\mathcal{F}_P$, applied to the operator $L_g(i\vartheta) - z\mathbf{1}$, i.e. ,

$$\mathcal{F}_P\big(L_g(i\vartheta) - z\big) \, , \tag{V.20}$$

will depend on which of these subsets, $\mathcal{S}_{(\,\cdot\,)}$, the variable z belongs to. For the definition of $\mathcal{S}_{(\,\cdot\,)}$, we pick $0 < \varepsilon < 2/3$ and set

$$\rho_0 \; := \; g^{2-\varepsilon} \quad , \qquad \rho_1 \; := \; g^{2+\varepsilon/2} \quad , \tag{V.21}$$

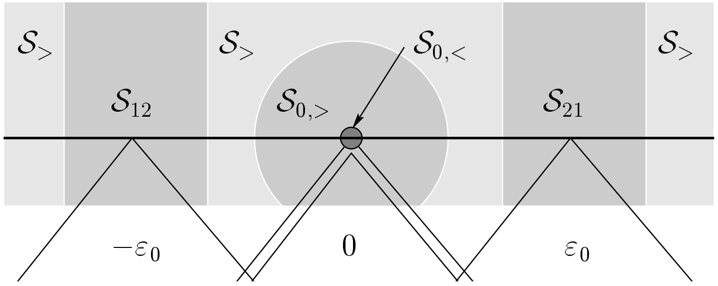

Figure 3: The subsets $\mathcal{S}_>$, $\mathcal{S}_{0,>}$, $\mathcal{S}_{0,<}$, $\mathcal{S}_{i,j}$.

and furthermore

$$\mathcal{S} \; := \; \Big\{ z \in \mathbb{C} \; \Big| \; \mathrm{Im}\, z > -\frac{\sin\vartheta}{4}\, \rho_0 \Big\} \, . \tag{V.22}$$

Using ρ_0, ρ_1, and $i, j \in \{0, \ldots, N-1\}$, $i \neq j$, the sets $\mathcal{S}_{(\,\cdot\,)}$ are defined to be the following subsets of $\mathcal{S}$,

$$\mathcal{S}_> \; := \; \Big\{ z \in \mathcal{S} \; \Big| \; \mathrm{dist}\big(\mathrm{Re}\, z \, , \; \sigma[L_{el}]\big) \; \geq \; \frac{\rho_0}{2} \Big\} \, , \tag{V.23}$$

$$\mathcal{S}_{i,j} \; := \; \Big\{ z \in \mathcal{S} \; \Big| \; \big|\mathrm{Re}\, z \, - \, E_i + E_j\big| \; \leq \; \frac{\rho_0}{2} \Big\} \, , \tag{V.24}$$

$$\mathcal{S}_{0,>} \; := \; \Big\{ z \in \mathcal{S} \; \Big| \; \frac{\sin(\vartheta)}{2}\, \rho_1 \leq |z| \leq \frac{\rho_0}{2} \Big\} \, , \tag{V.25}$$

$$\mathcal{S}_{0,<} \; := \; \Big\{ z \in \mathcal{S} \; \Big| \; |z| < \frac{\sin(\vartheta)}{2}\, \rho_1 \Big\} \, . \tag{V.26}$$

In the example, where $N = 2$, $E_0 = 0$ and $E_1 = \varepsilon_0$, these subsets are shown in Fig. 3.

We note that (V.23)–(V.26) define a covering of $\mathcal{S}$,

$$\mathcal{S} \; \subseteq \; \mathcal{S}_> \, \cup \, \Big(\bigcup_{i \neq j} \mathcal{S}_{i,j} \Big) \, \cup \, \mathcal{S}_{0,>} \, \cup \, \mathcal{S}_{0,<} \, , \tag{V.27}$$

as one easily checks.

Next, we describe, qualitatively, how one analyzes the intersection of the spectrum of $L_g(i\vartheta)$ with any one of the sets $\mathcal{S}_>, \mathcal{S}_{0,>}, \mathcal{S}_{0,<}$, and $\mathcal{S}_{i,j}$, $i \neq j$. The easiest problem is the determination of

V.1.1 $\sigma[L_g(i\vartheta)] \cap \mathcal{S}_>$.

We show that

$$\sigma[L_g(i\vartheta)] \cap \mathcal{S}_> \; = \; \emptyset \, , \quad \text{for } g > 0 \text{ sufficiently small.} \tag{V.28}$$

To this end, we show that, for $z \in \mathcal{S}_>$, $(L_g(i\vartheta) - z)^{-1}$ is a bounded operator. This can be done by expanding $(L_g(i\vartheta) - z)^{-1}$ in a *Neumann series* in the perturbation $gW(\theta)$ and, after using that $z \in \mathcal{S}_>$ and applying the bounds of Lemma IV.1, proving that this Neumann series *converges in norm*, for $g > 0$ small enough. Details are presented in Sect. V.4

The second easiest problem is the study of

V.1.2 $\sigma[L_g(i\vartheta)] \cap \mathcal{S}_{i,j}$,

for $g > 0$ sufficiently small. Here we make use of the Feshbach map associated to the projection

$$P \;=\; P_{i,j} \;:=\; P^{el}_{E_{i,j}} \otimes P^{aux}_{\rho_0} \,, \tag{V.29}$$

where $\rho_0 = g^{2-2\varepsilon}$, for a suitable $\varepsilon > 0$, and $E_{i,j} := E_i - E_j$. Without loss of generality, we assume that the coupling constant g is so small that

$$|E_{i,j} \;-\; E_{k,\ell}| \;>\; 2\,\rho_0 \tag{V.30}$$

if $E_{i,j} \neq E_{k,\ell}$. For simplicity, we also assume that $E_{i,j}$ is a simple eigenvalue of L_{el}, but this assumption is only made, in order to explain the key ideas without technical complications.

We now note that if $z \in \mathcal{S}_{i,j}$, as defined in (V.24), then

$$\Big[\big(\overline{L}_g(i\vartheta) - z \big) \Big|_{\overline{P}_{i,j}\widehat{\mathcal{H}}} \Big]^{-1} , \tag{V.31}$$

where $\overline{L}_g(i\vartheta) = \overline{P}_{i,j}\, L_g(i\vartheta)\, \overline{P}_{i,j}$, see (V.13), is a bounded operator on $\overline{P}_{i,j}\widehat{\mathcal{H}}$. This is seen by expanding the resolvent (V.31) in a *Neumann series* in $g\overline{W}(i\vartheta)$, where $\overline{W}(i\vartheta) \;=\; \overline{P}_{i,j}\, W(\theta)\, \overline{P}_{i,j}$. Using (V.30) and the definition of $P^{aux}_{\rho_0}$, Eq. (V.19), one proves norm-convergence of this Neumann series, for sufficiently small $g > 0$, with the help of the bounds of Lemma IV.1. Similarly, one proves that

$$P_{i,j}\, W(i\vartheta)\, \overline{P}_{i,j} \left| \overline{L}_g(i\vartheta) - z \right|^{-1/2} , \qquad \left| \overline{L}_g(i\vartheta) - z \right|^{-1/2} \overline{P}_{i,j}\, W(i\vartheta)\, P_{i,j} \tag{V.32}$$

are bounded operators; note that

$$P_{i,j}\, g\, W(i\vartheta)\, \overline{P}_{i,j} \;=\; P_{i,j}\, L_g(i\vartheta)\, \overline{P}_{i,j} \,, \tag{V.33}$$

because $P_{i,j}$ commutes with $L_0(i\vartheta)$, (see (V.6)).

Thus the Feshbach map on $L_g(i\vartheta) - z\mathbf{1}$ is well defined. It is given by

$$\begin{aligned} \mathcal{F}_{P_{i,j}}\big(L_g(i\vartheta) - z\big) \;&=\; P_{i,j}\, \big(L_g(i\vartheta) - z\big)\, P_{i,j} \\ &\quad - g^2\, P_{i,j}\, W(i\vartheta)\, \overline{P}_{i,j}\, \big(\overline{L}_g(i\vartheta) - z\big)^{-1}\, \overline{P}_{i,j}\, W(i\vartheta)\, P_{i,j} \,. \end{aligned} \tag{V.34}$$

The operator $\mathcal{F}_{P_{i,j}}\big(L_g(i\vartheta) - z\big)$ acts on the space $P_{i,j}\widehat{\mathcal{H}}$ and is bounded. The *leading contribution* to the first term on the R.S. of Eq. (V.34) is given by

$$\Big[E_{i,j}\,\mathbf{1} \,+\, L_f(i\vartheta)\big|_{\chi[L_{aux}<\rho_0]}\Big]\Big|_{P_{i,j}\widehat{\mathcal{H}}}\,, \tag{V.35}$$

up to corrections of $o(g^2)$; (see (V.19)).

Since, for $z \in \mathcal{S}_{i,j}$, the resolvent (V.31) has a norm-convergent Neumann series in $g\overline{W}(i\vartheta)$, the leading contribution to the second term on the R.S. of (V.34) is seen to be given by

$$-g^2\,P_{i,j}\,W(i\vartheta)\,\overline{P}_{i,j}\big(\overline{L}_0(i\vartheta) - E_{i,j}\big)^{-1}\,\overline{P}_{i,j}\,W(i\vartheta)\,P_{i,j} \tag{V.36}$$

up to corrections of $o(g^2)$.

In Eq. (V.36), one may replace the projections $P_{i,j}$ on the left and the right by $P^{el}_{E_{i,j}} \otimes P_{\Omega\otimes\Omega}$, at the price of an error term of $o(g^2)$. The resulting operator is then *independent* of ϑ, for $0 \leq \vartheta < \vartheta_0$, by analyticity. We decompose it into "real" and "imaginary" part, i.e., into a selfadjoint and an anti-selfadjoint operator. The real (selfadjoint) part is denoted by $\Delta E_{i,j}(g) \otimes P_{\Omega\otimes\Omega}$, while the imaginary part is written as $ig^2\Gamma^{(i,j)} \otimes P_{\Omega\otimes\Omega}$ where $\Gamma^{(i,j)}$ is a selfadjoint operator. Since we temporarily assumed $E_{i,j}$ to be a simple eigenvalue of L_{el}, the rank of $P^{el}_{E_{i,j}}$ is one, and $\Delta E_{i,j}(g) = g^2\,\Delta e_{i,j}\,P^{el}_{E_{i,j}}$ and $\Gamma_{i,j} = \gamma_{i,j}\,P^{el}_{E_{i,j}}$ are determined by two numbers, $\Delta e_{i,j}$ and

$$\gamma^{(i,j)} \;:=\; \big\langle \overline{P}_{i,j}\,W\,\psi_{i,j} \,\big|\, \delta(\overline{L}_0 - E_i + E_j)\,\overline{P}\,W\,\psi_{i,j}\big\rangle\,, \tag{V.37}$$

with $\psi_{i,j} := \varphi_i \otimes \varphi_j \otimes \Omega \otimes \Omega$, and φ_i is the eigenvector of H_{el} corresponding to the eigenvalue E_i; see Sect. IV.2, after Eq. (IV.45).

Expression (V.37) and Hypothesis (H-5) on $G(k)$ stated in Sect. I guarantee that $\gamma^{(i,j)}$ is *strictly positive*,

$$\gamma^{(i,j)} \;>\; 0\,. \tag{V.38}$$

An explicit estimate of $\gamma^{(i,j)}$ can be found in Appendix B.

Putting everything together, Eqs. (V.34) through (V.38), and recalling that the Feshbach map $\mathcal{F}_{P_{i,j}}$ is *isospectral*, more specifically, applying Theorem V.1, (ii), we conclude that

$$\mathrm{dist}\big(\sigma[L_g(i\vartheta)] \cap \mathcal{S}_{i,j}\,,\,\mathbb{R}\big) \;>\; \frac{1}{2}\,g^2\,\gamma^{(i,j)}\,, \tag{V.39}$$

for $g > 0$ sufficiently small. For more (but standard) details see Sect. V.5.

We turn to the study of

V.1.3 $\sigma[L_g(i\vartheta)] \cap \mathcal{S}_{0,>}$,

for $g > 0$ small enough, to which we now turn our attention. For this purpose we consider the Feshbach map, $\mathcal{F}_{P_0}$, applied to the operator $L_g(i\vartheta) - z$,

$$\mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big) \, , \qquad z \in \mathcal{S}_{0,>} \, , \tag{V.40}$$

where

$$P_0 \ := \ P_0^{el} \otimes P_{\rho_0}^{aux} \, . \tag{V.41}$$

In (V.41), P_0^{el} is the orthogonal projection onto the N-dimensional subspace, $\mathcal{H}_{el}^{(0)}$, of $\mathcal{H}_{el} \otimes \mathcal{H}_{el}$ given by

$$\mathrm{span}\big\{\varphi_0 \otimes \varphi_0, \, \varphi_1 \otimes \varphi_1, \, \dots \, , \, \varphi_{N-1} \otimes \varphi_{N-1}\big\} \, , \tag{V.42}$$

which is the kernel of L_{el}.

In a first step, we proceed as in Subsect. V.1.2, above. The Feshbach map

$$L_g(i\vartheta) - z\mathbf{1} \ \longmapsto \ \mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big) \, , \tag{V.43}$$

with $z \in \mathcal{S}_{0,>}$, is well defined, by the same arguments as in Subsect. V.1.2, and $\mathcal{F}_{P_0}(L_g(i\vartheta) - z)$ is given by formula (V.34), with $P_{i,j}$, $\overline{P}_{i,j}$, replaced by P_0, $\overline{P}_0$, respectively. Thus

$$\begin{aligned} \mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big) \ &= \ P_0 \, \big(L_g(i\vartheta) - z\big) \, P_0 \\ &\quad - g^2 \, P_0 \, W(i\vartheta) \, \overline{P}_0 \, \big(\overline{L}_g(i\vartheta) - z\big)^{-1} \, \overline{P}_0 \, W(i\vartheta) \, P_0 \, , \end{aligned} \tag{V.44}$$

with $0 < \vartheta < \vartheta_0$. This is a bounded operator on $P_0\widehat{\mathcal{H}}$. The first term on the R.S. of (V.44) is given by

$$L_f(i\vartheta)\Big|_{P_0\widehat{\mathcal{H}}} \ + \ o(g^2) \, , \tag{V.45}$$

which is shown in the same way as (V.35). Up to errors of order g^ε, the second term on the R.S. of (V.44) is given by

$$-ig^2\Big(\Gamma^{(0)}(g,z) \otimes P_{\rho_0}^{aux}\Big) \, , \tag{V.46}$$

where the operator $\Gamma^{(0)} \equiv \Gamma^{(0)}(g,z) = P_0^{el} \, \Gamma^{(0)}(g,z) \, P_0^{el}$ is given by

$$\Gamma^{(0)} \otimes P_\Omega \ := \ P_0^{el} \otimes P_\Omega \, W \, \delta(L_0) \, W \, P_0^{el} \otimes P_\Omega \, . \tag{V.47}$$

More explicitly, the matrix elements $\Gamma^{(0)}_{i,j} := \langle \varphi_i \otimes \varphi_i |\, \Gamma^{(0)}\, \varphi_j \otimes \varphi_j \rangle$ of $\Gamma^{(0)}$ in the orthonormal basis $\{\varphi_i \otimes \varphi_i\}_{i=0}^{N-1}$ of $\operatorname{Ran} P^{el}_0$ are given by

$$\Gamma^{(0)}_{i,j} \;=\; \big\langle W\, \varphi_i \otimes \varphi_i \otimes \Omega \otimes \Omega \,\big|\, \delta(L_0)\, W\, \varphi_j \otimes \varphi_j \otimes \Omega \otimes \Omega \big\rangle\,. \tag{V.48}$$

It follows from arguments similar to those in (III.41)–(III.43) that, for $\theta = i\vartheta$, $0 < \vartheta < \vartheta_0$, and for z purely imaginary, the spectrum of $\Gamma^{(0)}(g,z)$ is symmetric about the real axis, and the coefficient, $\Gamma^{(0)}$, of the leading contribution to $\Gamma^{(0)}(g,z)$ is an $N \times N$ matrix acting on $\mathcal{H}^0_{el}$ with *real* spectrum. From Eq. (V.48) it is obvious that

$$\Gamma^{(0)} \;\geq\; 0\,. \tag{V.49}$$

The matrix $\Gamma^{(0)}$ is studied in Appendix B. One result of the calculations in Appendix B is that $\Gamma^{(0)}$ satisfies a *detailed-balance equation*

$$\Gamma^{(0)}_{i,i}\, e^{-\frac{\beta}{2}\, E_i} \;=\; -\sum_{k \neq i} \Gamma^{(0)}_{i,k}\, e^{-\frac{\beta}{2}\, E_k}\,, \tag{V.50}$$

and that

$$\Gamma^{(0)}_{i,i} > 0\,, \qquad \Gamma^{(0)}_{i,k} \leq 0\,, \quad \text{for } i \neq k. \tag{V.51}$$

From Eq. (V.50) it follows that

$$\kappa_\beta \;:=\; \sum_{i=0}^{N-1} e^{-\frac{\beta}{2} E_i}\, \varphi_i \otimes \varphi_i \tag{V.52}$$

is an eigenvector of $\Gamma^{(0)}$ corresponding to the eigenvalue 0. Eq. (V.51) then implies that 0 is an eigenvalue of $\Gamma^{(0)}$ of *multiplicity* 1. The last claim follows from (V.51) with the help of a standard *Perron-Frobenius* argument. Note that κ_β is the *unperturbed Gibbs state* of the atom (in the Araki-Woods representation (II.46)–(II.51)). It follows that there is a positive constant $\widehat{\gamma}_0 > 0$ such that

$$\big\langle \psi \,\big|\, \Gamma^{(0)}\, \psi \big\rangle \;\geq\; \widehat{\gamma}_0 \;>\; 0\,, \tag{V.53}$$

for all normalized vectors $\psi \in \mathcal{H}^{(0)}_{el}$ which are orthogonal to κ_β. In Appendix B, we give a lower bound on the value of $\widehat{\gamma}_0 > 0$.

Let $P^{el}_{\kappa_\beta}$ denote the orthogonal projection onto κ_β, and let $P^{el}_{0,\perp} := P^{el}_0 - P^{el}_{\kappa_\beta}$. Our analysis shows that, for$z \in \mathcal{S}_{>,0}$,

$$\mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big) \;=\; \mathcal{F}_1 \,+\, \mathcal{F}_2 \,+\, O\big(g^{2+\varepsilon}\big)\,, \tag{V.54}$$

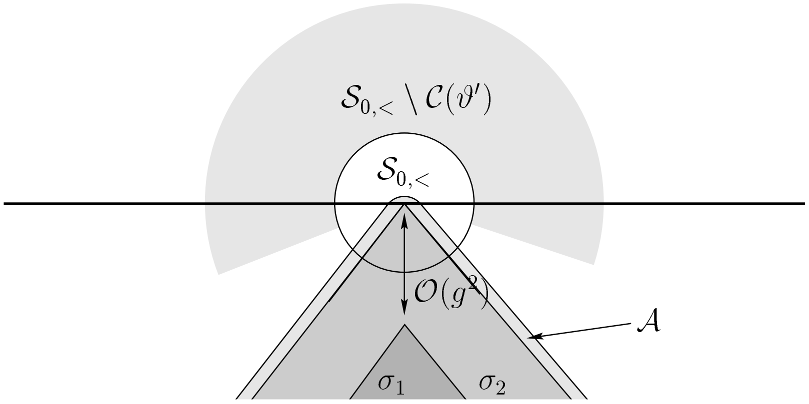

Figure 4: $\sigma_1 = \sigma[\mathcal{F}_1]$, $\sigma_2 = \sigma[\mathcal{F}_2]$, $\sigma[\mathcal{F}_{P_0}(L_g(i\vartheta))] \subseteq \sigma_1 \cup \sigma_2 \cup \mathcal{A}$, $z = 0$. The region $\mathcal{A}$ is of width $\mathcal{O}(g^{2+\varepsilon})$, $\vartheta' < \vartheta$.

where

$$\mathcal{F}_1 \; := \; P^{el}_{\kappa_\beta} \otimes P^{aux}_{\rho_0} \big(L_f(i\vartheta) - z \big) \, , \tag{V.55}$$

and

$$\mathcal{F}_2 \; := \; P^{el}_{0,\perp} \otimes P^{aux}_{\rho_0} \big[L_f(i\vartheta) - z - ig^2 \Gamma^{(0)} \big] \, . \tag{V.56}$$

It follows from these formulae that (for $z = 0$!) the spectra of $\mathcal{F}_{P_0}(L_g(i\vartheta) - z)$, $\mathcal{F}_1$, and $\mathcal{F}_2$ are contained in the shaded regions sketched in Fig. 4. In Sect. V.6 we then use this information to prove the invertibility of $\mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big)$, for $z \in \mathcal{S}_{0,>} \setminus \mathcal{C}(\vartheta')$, $0 < \vartheta' < \vartheta$, and $g > 0$ sufficiently small, where $\mathcal{C}(\alpha) \subseteq \mathbb{C}$ is the cone (see Fig. 4)

$$\mathcal{C}(\alpha) \; := \; \Big\{ z \in \mathbb{C} \, \Big| \, |\arg(z) - 3\pi/2| \geq (\pi/2) - \alpha \Big\} \, . \tag{V.57}$$

By far the *hardest analytical problems*, and the physically most interesting phenomena, appear in the study of

V.1.4 $\sigma[L_g(i\vartheta)] \cap \mathcal{S}_{0,<}$.

Formulae (V.54)–(V.56) and Fig. 4 suggest to apply a second Feshbach map to the operator $\mathcal{F}_{P_0}(L_g(i\vartheta) - z)$, requiring now that z belongs to the set $\mathcal{S}_{0,<}$ defined in Eq. (V.26). For this purpose, we define an orthogonal projection, $P_{0,<}$, by setting

$$P_{0,<} \; := \; P^{el}_{\kappa_\beta} \otimes P^{aux}_{\rho_1} \, , \tag{V.58}$$

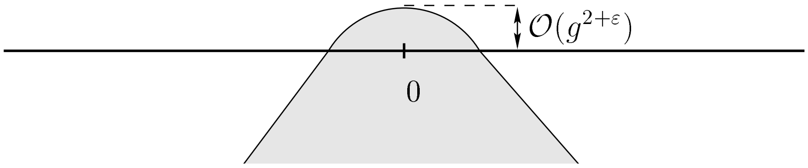

Figure 5: Set containing $\sigma[K^{(0)}(g,z)]$.

recalling from (V.21) that $\rho_1 = g^{2+\varepsilon/2}$. With the help of formula (V.54), (V.55), (V.56) and Lemma IV.1 one verifies without difficulties that the Feshbach map

$$\mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big) \ \mapsto \ K^{(0)}(g,z) \ := \ \mathcal{F}_{P_{0,<}}\Big(\mathcal{F}_{P_0}\big(L_g(i\vartheta) - z\big)\Big) \tag{V.59}$$

is well defined, for $z \in \mathcal{S}_{0,<}$, and that the spectrum of $K^{(0)}(g,z)$ is contained in the shaded region shown in Fig. 5.

The operator $K^{(0)}(g,z)$ is now chosen as the initial condition for a *renormalization operator* (-map), $\mathcal{R}$, very similar to the one introduced in [7, 8]. The effect of the renormalization operator is to *lower the spectral scale* corresponding to L_{aux} by a factor ρ, $0 < \rho < 1$, to be chosen appropriately. It is defined as the *composition* of a *Feshbach map* with a dilatation, $\rho^{-1}\, U\big(\ln(1/\rho)\big) \otimes U\big(\ln(1/\rho)\big)$; (note that the signs in the two arguments are now *equal*.) The Feshbach map involved in the definition of $\mathcal{R}$ maps operators on the range of

$$P^{(n-1)} \ := \ P^{el}_{\kappa_\beta} \otimes P^{aux}_{\rho_1 \rho^{n-1}} \tag{V.60}$$

to operators on the range of $P^{(n)}$, for arbitrary $n = 1, 2, \dots$, while simultaneously locating the "spectral parameter" z in *ever smaller disks* around a point $E_{(\infty)} \in \mathbb{C}$ (depending on the initial condition). For the initial condition $K^{(0)}(g,z)$ it follows from Theorem V.1, (i), and the fact that L_g has an eigenvalue at 0, proven in Theorem IV.3, that $E_{(\infty)} = 0$. Using Hypothesis (H-3), Sect. I, on the interaction I, one sees that iterated application of the renormalizatin map $\mathcal{R}$ to $K^{(0)}(g,z)$ drives this operator towards a *trivial fixed point*, which is given by the operator $L_f(i\vartheta)\, P^{(0)}$. This is the phenomenon of *infrared asymptotic freedom*, which one encounters in all the models studied in [5, 7, 8] as well as in the model studied in this paper. With a little experience, the details of this process of infrared renormalization can be carried out by inspection. They are studied in more detail in Sects. V.6 and V.7.

The conclusion of the discussion presented in Subsects. V.1.1–V.1.4, above, is that, for $0 < \vartheta < \vartheta_0$, and for $g > 0$ sufficiently small, there is an angle

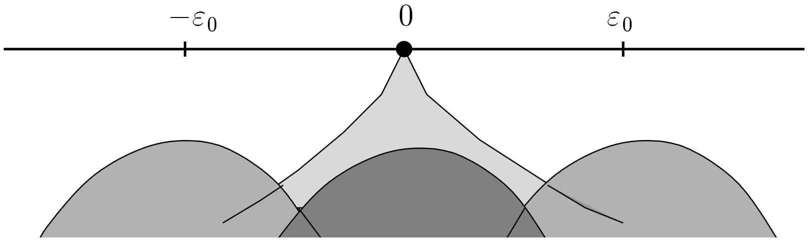

Figure 6: Approximate location of $\sigma[L_g(i\vartheta)]$.

$0 < \vartheta'' < \vartheta$ such that

$$\begin{aligned} &\sigma[L_g(i\vartheta)] \cap \left\{ z \in \mathbb{C} \,\middle|\, \operatorname{Im} z \geq -g^2 \right\} \qquad \text{(V.61)} \\ &\quad \subseteq \; \mathcal{C}(\vartheta'') \; = \; \left\{ z \in \mathbb{C} \,\middle|\, \operatorname{Im} z \leq 0 \,, \;\; |\operatorname{Re} z| \leq \cot(\vartheta'') \, |\operatorname{Im} z| \right\} , \end{aligned}$$

Furthermore, L_g has a *simple* eigenvalue at 0. In the example, where $N = 2$, $E_0 = 0$, $E_1 = \varepsilon_0$, the spectrum of $L_g(i\vartheta)$, $0 < \vartheta < \vartheta_0$, for $g > 0$ small enough, is contained in the shaded region, shown in Fig. 6.

Using Eq. (V.12) we see that our analysis proves that, away from the *simple* eigenvalue 0 of L_g, the spectrum of L_g is *purely absolutely continuous.* The general results of Sect. III.4 then imply that the model studied in this paper has the property of *"return to equilibrium"*.

V.2 A Comment on Exponentially Fast Return to Equilibrium

The results on the spectrum of $L_g(i\vartheta)$ presented in the last subsection are not sufficient to prove *exponentially fast* return to equilibrium of dilatation-analytic initial states. In fact, there is no compelling reason to expect that, for the general class of interactions between the atom and the quantized radiation field considered in this paper (see Hypotheses (H-1)–(H-5) of Sect. I), arbitrary dilatation-analytic initial states of the system return to the unique equilibrium state constructed in Theorem IV.3 exponentially fast. However, for a rather *special* class of interactions introduced by Jakšić and Pillet [27, 28], one can prove exponentially fast return to equilibrium of initial states belonging to a certain fairly natural dense subset of $\widehat{\mathcal{H}}$ by combining our methods with some of the ideas developed in [27, 28]. The key arguments are outlined in this subsection.

To begin with, it is useful to use polar coordinates in momentum space,

$$\mathbb{R}^3 \ni \vec{k} = \omega \, \vec{e} \, , \tag{V.62}$$

where $\omega = |\vec{k}|$ and $\vec{e}$ is a unit vector in $\mathbb{R}^3$. Then

$$d^3k \rightarrow \omega^2 \, d\omega \, d\Omega_{\vec{e}} \, , \tag{V.63}$$

where $d\Omega_{\vec{e}}$ is the uniform measure on the unit sphere in $\mathbb{R}^3$. In the following, we shall extend the range of values of the variable ω from the positive half-axis to the *entire real line.*

It is convenient to introduce new creation- and annihilation operators, α^* and α, by setting

$$\alpha^{\#}(\omega, \varepsilon) := \begin{cases} \omega \, a^{\#}_{\ell,\lambda}(\omega\vec{e}) \, , & \omega > 0 \, , \\ \omega \, a^{\#}_{r,\lambda}(-\omega\vec{e}) \, , & \omega < 0 \, , \end{cases} \tag{V.64}$$

where $\lambda = \pm 1$ is the polarization index and $\varepsilon = (\vec{e}, \lambda)$. We also define

$$d\Omega_\varepsilon := \sum_{\lambda = \pm 1} d\Omega_e \, , \tag{V.65}$$

and

$$\tilde{\delta}(\varepsilon - \varepsilon') := \delta_{\lambda\lambda'} \, \delta(\vec{e} - \vec{e}) \, . \tag{V.66}$$

Then $\alpha(\omega, \varepsilon)$ and $\alpha^*(\omega, \varepsilon)$ satisfy the canonical commutation relations

$$\big[\alpha(\omega, \varepsilon) \, , \, \alpha(\omega', \varepsilon')\big] = \big[\alpha^*(\omega, \varepsilon) \, , \, \alpha^*(\omega', \varepsilon')\big] = 0 \, , \tag{V.67}$$

and

$$\big[\alpha(\omega, \varepsilon) \, , \, \alpha^*(\omega', \varepsilon')\big] = \delta(\omega - \omega') \, \tilde{\delta}(\varepsilon - \varepsilon') \, . \tag{V.68}$$

We set

$$\phi := \Omega \otimes \Omega \tag{V.69}$$

and note that

$$\alpha(\omega, \varepsilon) \, \phi = 0 \, , \quad \text{for all } \omega, \, \varepsilon. \tag{V.70}$$

The Liouvillian, L_f, of the radiation field is then given by

$$L_f = \int_{\mathbb{R}} d\omega \int d\Omega_\varepsilon \, \alpha^*(\omega, \varepsilon) \, \omega \, \alpha(\omega, \varepsilon) \, ; \tag{V.71}$$

see [27], while L_{aux} becomes

$$L_{aux} \;=\; \int_{\mathbb{R}} d\omega \int d\Omega_\varepsilon \; \alpha^*(\omega,\varepsilon)\,|\omega|\,\alpha(\omega,\varepsilon)\,. \tag{V.72}$$

Let $G(k)$ denote the form factor of the interaction, I, between atom and quantized radiation field, as defined in (I.20) and (IV.50); $G(k)$ is an operator on $\mathcal{H}_{el}$, i.e., it is an $N \times N$ matrix. We define matrix-valued functions, $F_\ell(\omega,\varepsilon)$ and $F_r(\omega,\varepsilon)$, by

$$F_\ell(\omega,\varepsilon) \;:=\; \begin{cases} \omega^{-1/2}\, G_\ell(\omega\vec{e},\lambda)\,, & \omega > 0 \\ -(-\omega)^{-1/2}\, G_\ell^*(-\omega\vec{e},\lambda)\,, & \omega < 0 \end{cases}\,, \tag{V.73}$$

and

$$F_r(\omega,\varepsilon) \;:=\; \begin{cases} \omega^{-1/2}\, C G_r^*(\omega\vec{e},\lambda) C\,, & \omega > 0 \\ -(-\omega)^{-1/2}\, C G_r(-\omega\vec{e},\lambda) C\,, & \omega < 0 \end{cases}\,, \tag{V.74}$$

where C is the conjugation introduced in Sect. II.3.

We now assume that $F_\ell(\omega,\varepsilon)$ and $F_r(\omega,\varepsilon)$ are the restrictions to the real axis of matrix-valued functions, also denoted by $F_\ell(\omega,\varepsilon)$ and $F_r(\omega,\varepsilon)$, which are *analytic* in ω on the strip

$$\Sigma_\tau \;:=\; \big\{\omega \;\big|\; |\mathrm{Im}\,\omega| < \tau\big\}\,, \tag{V.75}$$

for some positive $\tau \le \infty$. We also assume that the L^2-norm (w.r. to $d\xi$) of

$$\big(|\xi|^{3/2} + |\xi|^{-1/2}\big)\, \big\|F_\#(\xi + i\eta,\varepsilon)\big\|_{B(\mathcal{H}_{el})} \tag{V.76}$$

is *bounded uniformly* in η, if $|\eta| \le \delta\tau$, for an arbitrary $\delta < 1$.

It is not hard to construct form factors $G(k)$ for which these assumptions hold; see [27, 28] for some simple examples. But we emphasize that if

$$\|G(k)\|_{B(\mathcal{H}_{el})} \;\sim\; |\vec{k}|^\mu\,, \quad \text{for } |\vec{k}| \to 0, \tag{V.77}$$

for some $\mu > 0$ (as required in this paper), then the assumptions described in (V.73)–(V.76) *only* hold if

$$\mu \;=\; \frac{1}{2},\, \frac{3}{2},\, \dots\,, \tag{V.78}$$

while the techniques described in Sect. V.1 are applicable for *arbitrary* $\mu > 0$!

Given an inverse temperature $\beta > 0$, we define

$$F_\ell^{(\beta)}(\omega,\varepsilon) \;:=\; \sqrt{\omega\,(1 - e^{-\beta\omega})^{-1}}\; F_\ell(\omega,\varepsilon)\,, \tag{V.79}$$

and

$$F_r^{(\beta)}(\omega,\varepsilon) \;:=\; \sqrt{-\omega\,(1-e^{\beta\omega})^{-1}}\,F_r(\omega,\varepsilon)\,. \tag{V.80}$$

It is important to observe that the functions

$$\sqrt{\pm\omega\,(1-e^{\pm\beta\omega})^{-1}} \tag{V.81}$$

are analytic in ω in the strip $\Sigma_{2\pi\beta^{-1}}$; (the function under the square root has simple poles at the points $\pm 2\pi i\beta^{-1}n$, $n=1,2,3,\dots$). Thus $F_\ell^{(\beta)}$ and $F_r^{(\beta)}$ are analytic on the strip Σ_κ, with $\kappa=\min\{\tau,2\pi\beta^{-1}\}$.

The interacting Liouvillian, L_g, can be expressed in terms of α, α^*, $F_\ell^{(\beta)}$, and $F_r^{(\beta)}$ by

$$L_g \;=\; L_{el} \,+\, L_f \,+\, g(W_\ell - W_r)\,, \tag{V.82}$$

where L_f is given by (V.71), and

$$W_\# \;=\; \int d\omega \int d\Omega_\varepsilon\,\left[\alpha^*(\omega,\varepsilon)\,F_\#^{(\beta)}(\omega,\varepsilon) \,+\, \alpha(\omega,\varepsilon)\,F_\#^{(\beta)}(\omega,\varepsilon)^*\right], \tag{V.83}$$

where $\#=\ell$ or r. Formula (V.83) follows directly from (V.79), (V.80), and Eqs. (IV.67), (IV.68).

Given an $N\times N$ matrix $M(\omega)=(M_{i,j}(\omega))$ expressed in the basis $\{\varphi_i\}_{i=0}^{N-1}$ of eigenvectors of H_{el}, we define

$$\overline{M}_{i,j}(\omega) \;:=\; \overline{M_{i,j}(\bar\omega)}\,, \tag{V.84}$$

and

$$M^*_{i,j}(\omega) \;:=\; \overline{M_{j,i}(\bar\omega)}\,. \tag{V.85}$$

We introduce the generator, T, of *translations along the ω-axis*

$$T \;:=\; \int d\omega \int d\Omega_\varepsilon\;\alpha^*(\omega,\varepsilon)\,\Big(-i\frac{\partial}{\partial\omega}\,\alpha\Big)(\omega,\varepsilon)\,. \tag{V.86}$$

Then

$$L_g(\sigma) \;:=\; e^{-i\sigma T}\,L_g\,e^{i\sigma T} \;=\; L_{el} \,+\, L_f \,-\, \sigma N \,+\, g(W_\ell(\sigma) - W_r(\sigma)) \tag{V.87}$$

where $W_\#(\sigma)$ is obtained from $W_\#$ by replacing $F_\#^{(\beta)}(\omega,\varepsilon)$ in (V.83) by $F_\#^{(\beta)}(\omega+\sigma,\varepsilon)$, with $\#=\ell$ or r, and

$$N \;=\; \int d\omega \int d\Omega_\varepsilon\;\alpha^*(\omega,\varepsilon)\,\alpha(\omega,\varepsilon) \tag{V.88}$$

is the *number operator*. The assumptions on the form factor $G(k)$ described above and Eqs. (V.84), (V.85) ensure that the operator $L_g(\sigma)$ extends to a family of operators *analytic* on the strip

$$\Sigma_{\min\{\tau, 2\pi\beta^{-1}\}} \ := \ \big\{ \sigma \; \big| \; |\mathrm{Im}\, \sigma| < \min\{\tau, 2\pi\beta^{-1}\} \big\} \, . \tag{V.89}$$

The proof is similar to the one presented in Appendix A. Clearly, there are natural dense sets of translation-analytic vectors in $\widehat{\mathcal{H}}$. Thus, we can apply the techniques of complex spectral deformation theory. In fact, the thing to do is to *combine complex translations* with the *complex dilatations* used in Sect. V.1, i.e.,

$$\omega \ \longmapsto \ \omega + \sigma \ \longmapsto \ e^{-\theta}(\omega + \sigma) \, . \tag{V.90}$$

Note that translations and dilatations do not commute. Hence, it is important that we *first* translate and *then* dilate L_g. Reversing the order of these operations does not yield an analytic family since, e.g., L_{aux} is not translation analytic. So, after translating and then dilating L_g, we obtain a family of operators

$$\big\{ L_g(\sigma, \theta) \big\} \tag{V.91}$$

analytic in σ and θ on a domain

$$\Lambda_{\eta_0, \vartheta_0} \ := \ \big\{ (\sigma, \tau) \in \mathbb{C}^2 \; \big| \; |\mathrm{Im}\, \sigma| < \eta_0 \, , \ \ |\mathrm{Im}\, \theta| < \vartheta_0 \big\} \tag{V.92}$$

in $\mathbb{C}^2$, for some positive constants

$$\eta_0 \ = \ O(\beta^{-1}) \quad \text{and} \quad \vartheta_0 \ = \ O(1) \, . \tag{V.93}$$

There is a natural *dense* domain $\mathcal{D}_A$, of vectors in $\widehat{\mathcal{H}}$ which are contained in $D(L_{aux}) \cap D(N)$ and are translation-and dilatation-analytic on the domain $\Lambda_{\eta_0, \vartheta_0}$. Assuming that the conditions on the form factor $G(k)$ described in Sects. I and V.1 and in (V.73)–(V.76) hold, we are then able to construct an analytic continuation in z of the matrix elements

$$\big\langle \psi \; \big| \; (L_g - z)^{-1} \, \varphi \big\rangle \, , \qquad \psi, \varphi \in \mathcal{D}_A \, , \tag{V.94}$$

to the complement of

$$\bigcap_{(\sigma,\theta) \in \Lambda_{\eta_0, \vartheta_0}} \sigma[L_g(\sigma, \theta)] \, . \tag{V.95}$$

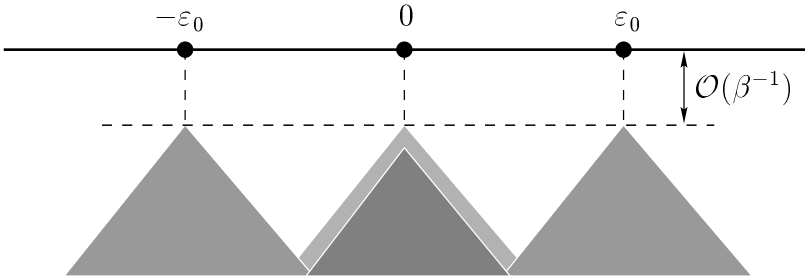

Figure 7: Sketch of $\sigma[L_0(i\eta, i\vartheta)]$.

In order to see what can be accomplished with these methods, we set

$$\sigma \;=\; i\eta\,, \quad \theta \;=\; i\vartheta\,, \tag{V.96}$$

with $0 < \eta < \eta_0$, $0 < \vartheta < \vartheta_0$, and study the spectrum of $L_g(i\eta, i\vartheta)$ with the help of the techniques described in the last section. In the example where $N = 2$, $E_0 = 0$, and $E_1 = \varepsilon_0$, the spectrum of $L_0(i\eta, i\vartheta)$ has the form sketched in Fig. 7.

We define subsets $\mathcal{S}_>$, $\mathcal{S}_{i,j}$, $\mathcal{S}_{0,>}$, and $\mathcal{S}_{0,<}$ of $\mathbb{C}$ in a way very similar to the one in Sect. V.1. The analysis of the spectrum of $L_g(i\eta, i\vartheta)$ on the subsets $\mathcal{S}_>$, $\mathcal{S}_{i,j}$, and $\mathcal{S}_{0,>}$ is virtually *identical* to the one of $\sigma[L_g(i\vartheta)]$ outlined in Sect. V.1 (and completed in Sects. V.3–V.7). It is only in the analysis of

$$\sigma[L_g(i\eta, i\vartheta)] \;\cap\; \mathcal{S}_{0,<} \tag{V.97}$$

where the usefulness of complex translations becomes manifest: Applying the renormalization operator $\mathcal{R}$ mentioned in Sect. V.1 (see [7, 8] and Sect. V.7) to the operator

$$L_{(0)}(g, z) \;:=\; \Gamma(\rho_1)\, \mathcal{F}_{P_{0,<}}\Big(\mathcal{F}_{P_0}\big(L_g(i\eta, i\vartheta) - z\big)\Big)\, \Gamma(\rho_1)^*\,, \tag{V.98}$$

where we use the notation introduced in Sect. V.1, and $\Gamma(\rho_1) := \mathbf{1} \otimes \mathbf{1} \otimes U(-\ln\rho_1) \otimes U(-\ln\rho_1)$ is the unitary dilatation that maps L_{aux} into $\rho_1\, L_{aux}$, one encounters the phenomenon that, for $\eta > 0$, $\vartheta > 0$, the continuous spectrum of $L_{(0)}(g, z)$ is pushed farther and farther into the lower half-plane. If the renormalization operator $\mathcal{R}$ lowers the scale of the L_{aux}-spectrum by a factor $\rho < 1$, then the distance between the continuous spectrum of $\widetilde{L}_{(0)}(g, z)$, $|z| \sim O(\rho^n)$, and the real axis, after n applications of the renormalization operator $\mathcal{R}$, is $O(\rho^{-n})$. This follows from the fact that

$$\begin{aligned} \mathcal{R}\big(L_f(i\eta, i\vartheta)\big) \;&=\; \mathcal{R}\big(\cos(\vartheta) L_f \,-\, i\sin(\vartheta) L_{aux} \,-\, i\eta N\big) \\ &=\; \cos(\vartheta) L_f \,-\, i\sin(\vartheta) L_{aux} \,-\, i\eta\rho^{-1} N \\ &=\; L_f\big(i\eta\rho^{-1}, i\vartheta\big)\,, \end{aligned} \tag{V.99}$$

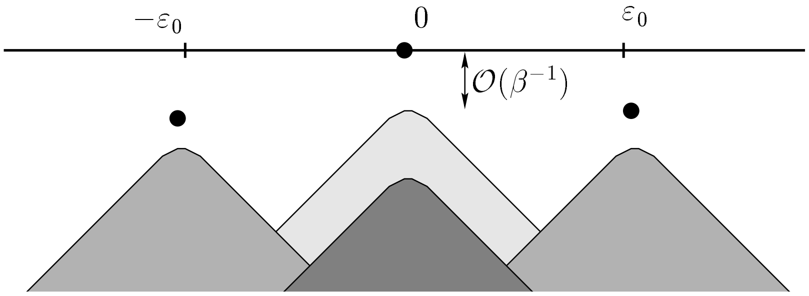

Figure 8: Sketch of $\sigma[L_g(i\eta, i\vartheta)]$.

which follows from Eqs. (V.6), (V.87) and from the definition of $\mathcal{R}$; see (V.59)–(V.60) and Sect. V.7. Thanks to *infrared asymptotic freedom*, it follows that, for $z \in \mathcal{S}_{0,<}$,

$$\begin{aligned} L_{(n)}(g,z) \; &:= \; \mathcal{R}^n\big(L_{(0)}(g,z)\big) && \text{(V.100)} \\ &= \; L_f(i\eta\rho^{-n}, i\vartheta) P^{el}_{\kappa_\beta} \otimes P^{aux}_1 \; + \; O\big(|z|\rho^{-n}\big) \; + \; O\big(g\rho^{\alpha n}\big) \, , \end{aligned}$$

for some $\alpha > 0$ (which depends on the behaviour of $G(k)$ near $|\vec{k}| = 0$).

It follows from the isospectrality of the Feshbach map, Theorem V.1, (i), and Theorem IV.3 (existence of a perturbed KMS state) that $L_{(n)}(g,0)$ has an eigenvalue at 0. This fact and Eq. (V.100) then imply that 0 is a *simple* eigenvalue (analytic perturbation theory!) and that for $0 < |z| < \mathcal{O}(1)$, $L_{(n)}(g,z)$ is *invertible* on the range of $P^{el}_{\kappa_\beta} \otimes P^{aux}_1$.

By Theorem V.1, we conclude that, in a small disk around 0, the spectrum of $L_g(i\eta, i\vartheta)$ is empty, except for a simple eigenvalue at 0.

In the example where $N = 2$, $E_0 = 0$, and $E_1 = \varepsilon_0$, the location of $\sigma[L_g(i\eta, i\vartheta)]$ is sketched in Fig. 8, for $g > 0$ small enough.

It follows from these results by arguments due to Hunziker [25, 9], that an initial state $\psi \in \mathcal{D}_A \subset \widehat{\mathcal{H}}$ *returns to equilibrium exponentially fast*, with a *rate* of $\mathcal{O}(\beta^{-1})$, for $g > 0$ sufficiently small.

In the remaining subsections and in two appendices, we present some analytical details required to render the analysis presented in this and the last subsection mathematically rigorous.

V.3 Complex Dilatation of the Liouvillian

In this section, we discuss the dilatation analyticity of the Liouvillian L_g. The technical details of this discussion are given in Appendix A.

Recall from (V.2)–(V.4) the definition of the unitary dilatation operator $\widehat{U}(\theta)$, for $\theta \in \mathbb{R}$. We define the dilated Liouvillian by

$$L_g(\theta) \ := \ \widehat{U}(\theta)\, L_g\, \widehat{U}(\theta)^{-1} \, . \tag{V.101}$$

We find that

$$L_g(\theta) \ = \ L_{el} \otimes \mathbf{1}^f \ + \ \mathbf{1}^{el} \otimes L_f(\theta) \ + \ gW(\theta)\, , \tag{V.102}$$

where

$$L_f(\theta) \ = \ e^{-\theta}(H_f \otimes \mathbf{1}_f) \ - \ e^{\theta}(\mathbf{1}_f \otimes H_f)\, , \tag{V.103}$$

and

$$\begin{aligned} W(\theta) \ = \ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(V.104)} \\ & e^{-3\theta/2} \int dk \, \Big\{ \big(\sqrt{1+\rho(e^{-\theta}k)}\, G_\ell(e^{-\theta}k) \ - \ \sqrt{\rho(e^{-\theta}k)}\, \overline{G_r^*}(e^{-\theta}k) \big)\, a_\ell^*(k) \\ & \qquad\qquad + \big(\sqrt{1+\rho(e^{-\theta}k)}\, G_\ell^*(e^{-\theta}k) \ - \ \sqrt{\rho(e^{-\theta}k)}\, \overline{G_r}(e^{-\theta}k) \big)\, a_\ell(k) \Big\} \\ & + e^{3\theta/2} \int dk \, \Big\{ \big(\sqrt{\rho(e^{\theta}k)}\, G_\ell^*(e^{\theta}k) \ - \ \sqrt{1+\rho(e^{\theta}k)}\, \overline{G_r}(e^{\theta}k) \big)\, a_r^*(k) \\ & \qquad\qquad + \big(\sqrt{\rho(e^{\theta}k)}\, G_\ell(e^{\theta}k) \ - \ \sqrt{1+\rho(e^{\theta}k)}\, \overline{G_r^*}(e^{\theta}k) \big)\, a_r(k) \Big\} \end{aligned}$$

and $e^{-\theta}k := (e^{-\theta}\vec{k}, \lambda)$.

In order to obtain an analytic continuation of the dilated interaction $W(\theta)$ in Eq. (V.104) from real to complex $\theta \in \Sigma_{\vartheta_0}$, we recall that Hypothesis H-2 insures the dilatation analyticity of $G(e^{-\theta}k)$, for $\theta \in \Sigma_{\vartheta_0}$, and thus also insures the dilatation analyticity of $e^{\theta}\omega(k)^{-1/2}G(e^{-\theta}k)$. We follow the convention that, for a matrix-valued function $M(z)$, $z \in \mathbb{C}^3$,

$$\big\langle \varphi_i \big| \, \overline{M}(z)\, \varphi_j \big\rangle \ := \ \overline{\big\langle \varphi_i \big| \, M(\,\overline{z}\,)\, \varphi_j \big\rangle} \tag{V.105}$$

$$\big\langle \varphi_i \big| \, M^*(z)\, \varphi_j \big\rangle \ := \ \overline{\big\langle \varphi_j \big| \, M(\,\overline{z}\,)\, \varphi_i \big\rangle}\, . \tag{V.106}$$

Furthermore, for $\theta \in \mathbb{R}$, we set

$$\mu_\theta(k) \ := \ \sqrt{\omega(k)\, \rho(e^{-\theta}k)} \ = \ \sqrt{\omega(k) \big(\exp[e^{-\theta}\beta\omega(k)] \ - \ 1 \big)^{-1}}\, , \tag{V.107}$$

$$\nu_\theta(k) \ := \ \sqrt{\omega(k) \big(1 \ + \ \rho(e^{-\theta}k) \big)} \ = \ \exp[e^{-\theta}\beta\omega(k)/2]\, \mu_\theta(k)\, , \tag{V.108}$$

and similar to the discussion of the function in (V.81), we extend $\theta \mapsto \mu_\theta(k), \nu_\theta(k)$ to the strip $\Sigma_{\pi/2} = \{ |\mathrm{Im}\, z| < \pi/2 \}$ about $\mathbb{R}$ by analytic continuation.

With Eqs. (V.105)–(V.108), we see that the interaction is dilatation analytic.

Lemma V.2. *Assume Hypotheses H-1, H-2, and H-4. Then the map*

$$W \; : \; \Sigma_{\vartheta_0} \to \mathcal{B}\big[\mathrm{Dom}[L_{aux}], \mathcal{H}\big] \; , \quad \theta \mapsto W(\theta) \tag{V.109}$$

is analytic.

Proof. We first write $\alpha := \mathrm{Re}\,\theta$, $\vartheta := \mathrm{Im}\,\theta$ and observe that

$$\begin{aligned} & \big|\exp[e^{-\theta}\beta\omega(k)] \; - \; 1\big| \\ = \; & \big|\exp[e^{-\alpha}\cos(\vartheta)\beta\omega(k)]\exp[ie^{-\alpha}\sin(\vartheta)\beta\omega(k)] \; - \; 1\big| \\ \geq \; & \big|\exp[e^{-\alpha}\cos(\vartheta)\beta\omega(k)] \; - \; 1\big| \; \geq \; e^{-\alpha}\cos(\vartheta)\,\beta\,\omega(k) \, , \end{aligned} \tag{V.110}$$

and hence

$$|\mu_\theta(k)| \; , \; |\nu_\theta(k)| \;\; \leq \;\; \sqrt{\omega(k)} \left(\frac{e^{\alpha/2}}{\sqrt{\cos(\vartheta)\,\beta\,\omega(k)}} \; + \; 1 \right) . \tag{V.111}$$

So, defining

$$w^{(\theta)}_{+,\ell}(k) \;\; := \;\; e^{-\theta}\,\omega(k)^{-1/2}\,\big(\nu_\theta(k)\,G_\ell(e^{-\theta}k) \; - \; \mu_\theta(k)\,\overline{G^*_r}(e^{-\theta}k)\big) \, , \tag{V.112}$$

$$w^{(\theta)}_{-,\ell}(k) \;\; := \;\; e^{-\theta}\,\omega(k)^{-1/2}\,\big(\nu_\theta(k)\,G^*_\ell(e^{-\theta}k) \; - \; \mu_\theta(k)\,\overline{G_r}(e^{-\theta}k)\big) \, , \tag{V.113}$$

$$w^{(\theta)}_{+,r}(k) \;\; := \;\; e^{\theta}\,\omega(k)^{-1/2}\,\big(\mu_{-\theta}(k)\,G^*_\ell(e^{\theta}k) \; - \; \nu_{-\theta}(k)\,\overline{G_r}(e^{\theta}k)\big) \, , \tag{V.114}$$

$$w^{(\theta)}_{-,r}(k) \;\; := \;\; e^{\theta}\,\omega(k)^{-1/2}\,\big(\mu_{-\theta}(k)\,G_\ell(e^{\theta}k) \; - \; \nu_{-\theta}(k)\,\overline{G^*_r}(e^{\theta}k)\big) \, , \tag{V.115}$$

Hypotheses H-3 and H-4 insure the following estimate,

$$\big\| w^{(\theta)}_{\sigma,\tau}(k) \big\| \;\; \leq \;\; \frac{2e^{(M+1)|\mathrm{Re}\,\theta|}}{\sqrt{\cos(\vartheta)}} \left(1 \; + \; \frac{1}{\beta\omega(k)} \right)^{1/2} |\vec{k}|^{\mu}\,\kappa(k) \, , \tag{V.116}$$

for any $\theta in \Sigma_{\vartheta_0}$, with $\vartheta := \mathrm{Im}\,\theta$ $M < \infty$ as in Hypotheses H-3 and 4, and $\kappa \leq 1$ is a function such that

$$\int \Big(\omega(k) + \omega(k)^{-3} \Big) \, |\vec{k}|^{2\mu} \, |\kappa(k)|^2 \, dk \;\; \leq \;\; \Lambda^2 \, . \tag{V.117}$$

Furthermore, the matrix-valued functions $\theta \mapsto w^{(\theta)}_{\sigma,\tau}(k)$ are analytic in Σ_{ϑ_0}. Hence the standard bounds given in Lemma IV.1 insure that

$$W(\theta) \;\; = \;\; a^*_\ell(w^{(\theta)}_{+,\ell}) \; + \; a_\ell(w^{(\theta)}_{-,\ell}) \; + \; a^*_r(w^{(\theta)}_{+,r}) \; + \; a_r(w^{(\theta)}_{-,r}) \;\; = \;\; \sum_{\sigma=\pm} \sum_{\tau=\ell,r} a^\sigma_\tau(w^{(\theta)}_{\sigma,\tau}) \, , \tag{V.118}$$

with $a^+_\# := a^*_\#$ and $a^-_\# := a_\#$, is dilatation analytic, and the following norm bound holds, for some constant $C < \infty$,

$$\big\| W(\theta)(L_{aux} + 1)^{-1} \big\| \;\le\; \frac{C\, e^{(M+1)|\mathrm{Re}\,\theta|}\,(\beta + 1)}{\beta \cos(\vartheta)}\, \Lambda\,. \qquad \square \tag{V.119}$$

Our next goal is to establish the analyticity of the resolvent

$$R_g(\theta, z) \;:=\; \big(L_g(\theta) \;-\; z \big)^{-1}\,, \tag{V.120}$$

as an operator-valued function of θ and z. This is not immediate from a direct application of standard techniques in dilatation analyticity because $\theta \mapsto L_g(\theta)$ is *not* a family of type A, for $\theta \in \Sigma_{\vartheta_0}$. Indeed, as we point out in Appendix A, $\theta \mapsto L_g(\theta)$ is not even an analytic family on $\mathcal{H}$ in the sense of Kato (see, e.g., [33]).

To make a precise statement, we introduce a dense set of vectors,

$$\mathcal{D} \;:=\; \mathcal{D}_1 \cap \mathcal{D}_2 \;\subseteq\; \mathcal{H}\,, \tag{V.121}$$

where

$$\mathcal{D}_1 \;:=\; \bigcap_{|\theta| \le \theta_0} \mathrm{Dom}\big[(L_{aux} + 1)\widehat{U}(\theta) \big]\,, \tag{V.122}$$

and $\mathcal{D}_2$ consists of all vectors which are analytic w.r.t. $\theta \in \Sigma_{\vartheta_0}$. We see that $\mathcal{D} \subseteq \mathcal{H}$ is dense by observing that

$$\mathcal{D} \;\supseteq\; \mathrm{span}\Big\{ \varphi_\ell \otimes \varphi_r \otimes \psi_\ell \otimes \psi_r \;\Big|\; \varphi_{\ell/r} \in \mathcal{H}_{el}\,, \quad \psi_{\ell/r} \in \mathcal{D}_{Gauss} \Big\}\,, \tag{V.123}$$

where $\mathcal{D}_{Gauss}$ consists of all translates and dilates of Gaussians,

$$\begin{aligned} \mathcal{D}_{Gauss} \;:=\; \Big\{ a^*(f_1) \cdots a^*(f_n)\Omega \;\Big|\; & f_j(\vec{k}, \lambda) = \exp[-(\vec{k} - \vec{k}_j)^2/\sigma_j]\, \delta_{\lambda_j, \lambda}\,, \\ & n \in \mathbb{N}_0\,,\; \vec{k}_j \in \mathbb{R}^3\,,\; \sigma_j > 0\,,\; \lambda_j \in \{\pm 1\} \Big\}\,. \end{aligned} \tag{V.124}$$

For $\varphi, \psi \in \mathcal{D}$, we establish the existence of the desired analytic continuation of

$$(\theta, z) \;\mapsto\; F_{\varphi,\psi}(\theta, z) \;:=\; \big\langle \widehat{U}(\bar{\theta})\varphi \big|\, \big(L_g(\theta) \;-\; z \big)^{-1}\, \widehat{U}(\theta)\psi \big\rangle \tag{V.125}$$

in the following theorem, which is an immediate consequence of Theorem A.4 proven in Appendix A.

Theorem V.3. *Let* $0 < \vartheta_0' < \vartheta_0 < \pi/2$, *and assume that* $\mathrm{Im}\, z > 0$, $0 < g \ll \vartheta_0'(\pi/2 - \vartheta_0)$, *and* $\theta \in \Sigma_{\vartheta_0}$, $\mathrm{Im}\, \theta \geq \vartheta_0'$. *Then*

(i) *For all* $\varphi, \psi \in \mathcal{D}$, *the function* $z \mapsto \langle \varphi | (L_g - z)^{-1} \psi \rangle$ *has an analytic continuation from the upper half-plane into* $\mathcal{G}'$ *given by*

$$\big\langle \varphi \big| \big(L_g \, - \, z \big)^{-1} \psi \big\rangle \; = \; \big\langle \widehat{U}(\bar{\theta}) \varphi \big| \big(L_g(\theta) \, - \, z \big)^{-1} \widehat{U}(\theta) \psi \big\rangle \, . \qquad \text{(V.126)}$$

(ii) *On the rectangular domain* $R := \big\{ \theta \in \mathbb{C} \; \big| \; |\mathrm{Re}\, \theta| \leq \alpha \, , \; \vartheta_0' \leq \mathrm{Im}\, \theta \leq \vartheta_0 \big\}$, *where* $\alpha := -\frac{1}{2} \ln \cos(2\vartheta_0') > 0$, *the map*

$$R \ni \theta \;\; \mapsto \;\; L_g(\theta) \, \in \, \mathcal{B}\big[\mathrm{Dom}[L_{aux}] \, , \, \mathcal{H} \big] \qquad \text{(V.127)}$$

is an analytic family of type A.

(iii) $\sigma[L_g(\theta)] \subseteq \overline{\mathbb{C}^-} \setminus \mathcal{G}'$, *where* $\mathcal{G}'$ *is the connected component of* $\{ z \in \mathbb{C} | \; \|R_g(\theta, z)\| < \infty \}$ *containing* $\mathbb{C}^+$.

V.4 Invertibility of $L_g(\theta) - z$ on $\mathcal{S}_>$

In this subsection we study the spectral properties of $L_g(\theta)$ on $\mathcal{S}_>$. In the previous subsection, we established the analytic continuation of matrix elements of $(L_g(\theta) - z)^{-1}$ in θ and z. This allows us now to choose purely imaginary values of θ, and we shall henceforth assume that $0 < \vartheta_0' < \vartheta_0 < \pi/2$,

$$\theta \; = \; i\vartheta \quad \text{and} \quad \vartheta_0' \; \leq \; \vartheta \; \leq \; \vartheta_0 \, . \qquad \text{(V.128)}$$

We first demonstrate that $\mathcal{S}_> \subseteq \rho[L_g(\theta)]$, the resolvent set of $L_g(\theta)$, simply by expanding $(L_g(\theta) - z)^{-1}$ in a norm-convergent Neumann series.

Theorem V.4. *For* $g > 0$ *sufficiently small,* $\mathcal{S}_> \subseteq \rho[L_g(\theta)]$, *and*

$$\big\| (L_g(\theta) - z)^{-1} \big\| \; \leq \; \mathcal{O}\big(\rho_0^{-1} \big) \, , \qquad \text{(V.129)}$$

for any $z \in \mathcal{S}_>$.

Proof. We expand the inverse of $L_g(\theta) - z$ in a Neumann series,

$$\big(L_g(\theta) - z \big)^{-1} \; = \; \sum_{n=0}^{\infty} \big(L_0(\theta) - z \big)^{-1} \Big\{ -gW(\theta) \, \big(L_0(\theta) - z \big)^{-1} \Big\}^n \, , \qquad \text{(V.130)}$$

which is easily seen to be norm-convergent since, by Lemma IV.1, we have

$$\big\| \, (L_{aux} + \rho_0)^{-1/2} \, gW(\theta) \, (L_{aux} + \rho_0)^{-1/2} \, \big\| \; = \; \mathcal{O}\big(g \, \rho_0^{-1/2} \big) \; = \; \mathcal{O}\big(g^{\varepsilon} \big) \, , \qquad \text{(V.131)}$$

and

$$\left\| \frac{L_{aux} + \rho_0}{L_0(\theta) - z} \right\| \;=\; \sup_{\substack{0 \le |r_f| \le r_{aux} \\ \lambda \in \sigma[L_{el}]}} \left| \frac{r_{aux} + \rho_0}{\lambda + \cos(\vartheta) r_f - i \sin(\vartheta) r_{aux} - z} \right| \;\le\; C(\vartheta) \, , \tag{V.132}$$

where $C(\vartheta) < \infty$ is a ϑ-dependent constant, and r_{aux}, r_f are points in the joint spectrum of L_{aux}, L_f, respectively.

To check (V.132), we distinguish the cases $r_{aux} < \tau \rho_0$ and $r_{aux} \ge \tau \rho_0$, with $\tau \ge 0$ to be picked later. For $r_{aux} < \tau \rho_0$, we have

$$\begin{aligned} |\lambda + \cos(\vartheta) r_f - i \sin(\vartheta) r_{aux} - z| \;&\ge\; |\lambda + \cos(\vartheta) r_f - \mathrm{Re}(z)| \\ \ge\; \Big(\frac{\rho_0}{2} - \cos(\vartheta) \tau \rho_0 \Big)_+ \;&\ge\; \frac{(1 - 2\tau \cos\vartheta)_+}{2(1+\tau)} \, (r_{aux} + \rho_0) \, , \end{aligned} \tag{V.133}$$

and, for $r_{aux} \ge \tau \rho_0$,

$$\begin{aligned} |\lambda + \cos(\vartheta) r_f - i \sin(\vartheta) r_{aux} - z| \;&\ge\; |\sin(\vartheta) r_{aux} + \mathrm{Im}(z)| \\ \ge\; \Big(\sin(\vartheta) - \frac{\tan\vartheta}{4\tau} \Big)_+ r_{aux} \;&\ge\; \frac{(\tau \sin\vartheta - \tan\vartheta)_+}{4(1+\tau)} \, (r_{aux} + \rho_0) \, . \end{aligned} \tag{V.134}$$

Choosing $\tau := (\sin\vartheta + 2\cos\vartheta)(4\cos\vartheta)^{-1}(\sin\vartheta + \cos\vartheta)^{-1}$, we obtain $C(\vartheta) = (\sin\vartheta/2)(\sin\vartheta + \cos\vartheta)^{-1}$. □

V.5 Invertibility of $L_g(\theta) - z$ in the Vicinity of Atomic Eigenvalues away from Zero

In this subsection we investigate the invertibility of $L_g(\theta) - z$ in $\mathcal{S}_{i,j}$, for any $i \ne j$. In Theorem V.7 below, we show the existence of a positive constant $\gamma_{\ne 0} > 0$ such that

$$\mathcal{S}_{i,j} \,\cap\, \{ z \in \mathbb{C} : \; \mathrm{Im}\, z > -g^2 \gamma_{\ne 0} \} \;\subseteq\; \rho[L_g(\theta)] \, , \tag{V.135}$$

provided $g > 0$ is sufficiently small, and $\varepsilon > 0$ is as in Eq. (V.21). Theorem V.7 and Theorem V.4 imply, in particular, that the spectrum of $L_g(\theta)$ in $(-\infty \, , -\rho_0] \cup [\rho_0 \, , \infty)$ is absolutely continuous.

Let $z \in \mathcal{S}_{i,j}$. We recall from the definition (V.24) of $\mathcal{S}_{i,j}$ that

$$|\mathrm{Re}(z) - \eta| \;\le\; \rho_0/2 \quad \text{and} \quad \mathrm{Im}\, z \;\ge\; -\frac{\sin(\vartheta)}{4} \, \rho_0 \, , \tag{V.136}$$

where $\eta := E_{i,j} \neq 0$ is a nonzero eigenvalue of L_{el}. The proximity of z to η implies that z is uniformly in g away from any other eigenvalue of L_{el},

$$\mathrm{dist}\big(z\,,\ \sigma[L_{el}] \setminus \{\eta\} \big) \ \geq\ \frac{1}{2} \min_{\substack{\mu,\nu\in\sigma[L_{el}] \\ \mu\neq\nu}} |\mu - \nu| \ >\ 0\,. \tag{V.137}$$

As in [7, Sect. IV] and in [9, Sect. III], we establish the invertibility of $L_g(\theta) - z$ by means of the *Feshbach map*, $\mathcal{F}_P(L_g(\theta) - z)$, corresponding to the projection

$$P_\eta \ :=\ P_\eta^{el} \otimes P_{\rho_0}^{aux} \quad , \quad \overline{P}_\eta \ :=\ \mathbf{1} - P_\eta\,, \tag{V.138}$$

where $P_\eta^{el} := \chi_{\{\eta\}}[L_{el}]$ is the projection of L_{el} onto η, and $P_{\rho_0}^{aux} := \chi[L_{aux} < \rho_0]$ is the projection onto spectral values of L_{aux} strictly less than ρ_0. To prove existence of $\mathcal{F}_{P_\eta}(L_g(\theta) - z)$, we require the following preparatory lemma.

Lemma V.5. *Assume (V.128) and (V.136). Then, for $g > 0$ sufficiently small, $\overline{P}_\eta\big(L_g(\theta) - z\big)\overline{P}_\eta$ is invertible on* $\mathrm{Ran}\,\overline{P}_\eta$.

Proof. We construct the inverse of $\overline{P}_\eta L_g(\theta)\overline{P}_\eta - z$ on $\mathrm{Ran}\,\overline{P}_\eta$ by an expansion in a Neumann series,

$$\begin{aligned}
\big(\overline{P}_\eta L_g(\theta)\overline{P}_\eta - z\big)^{-1}\overline{P}_\eta \ &=\ \sum_{n=0}^{\infty} \frac{\overline{P}_\eta}{L_0(\theta) - z}\left(-gW(\theta)\,\frac{\overline{P}_\eta}{L_0(\theta) - z}\right)^n \\
=\ \frac{(L_{aux} + \rho_0)^{1/2}\overline{P}_\eta}{L_0(\theta) - z} &\sum_{n=0}^{\infty} \Bigg\{ (L_{aux} + \rho_0)^{-1/2}\,\big(-gW(\theta)\big)\,(L_{aux} + \rho_0)^{-1/2} \\
&\left(\frac{(L_{aux} + \rho_0)\overline{P}_\eta}{L_0(\theta) - z}\right)\Bigg\}^n\ (L_{aux} + \rho_0)^{-1/2}\,.
\end{aligned} \tag{V.139}$$

By Lemma IV.1, we have

$$\big\|\,(L_{aux} + \rho_0)^{-1/2}\,gW(\theta)\,(L_{aux} + \rho_0)^{-1/2}\big\| \ =\ \mathcal{O}\big(g\,\rho_0^{-1/2}\big)\,. \tag{V.140}$$

We make use of

$$\overline{P}_\eta \ =\ \overline{P}_\eta^{(1)} + \overline{P}_\eta^{(2)}\,, \tag{V.141}$$

$$\overline{P}_\eta^{(1)} \ :=\ \overline{P_\eta^{el}} \otimes \mathbf{1}_f\,, \quad \overline{P}_\eta^{(2)} \ :=\ P_\eta^{el} \otimes \chi[L_{aux} \geq \rho_0]\,, \tag{V.142}$$

where $\overline{P_\eta^{el}} := \chi_{\mathbb{R}\setminus\{\eta\}}[L_{el}]$ is the projection onto the eigenvalues of L_{el} different from η. Since both $\overline{P}_\eta^{(1)}$ and $\overline{P}_\eta^{(2)}$ commute with L_{aux} and with $L_0(\theta)$, we have that

$$\left\|\frac{(L_{aux} + \rho_0)\overline{P}_\eta}{L_0(\theta) - z}\right\| \ =\ \max_{j=1,2}\left\|\frac{(L_{aux} + \rho_0)\overline{P}_\eta^{(j)}}{L_0(\theta) - z}\right\|\,. \tag{V.143}$$

On $\operatorname{Ran} \overline{P}_\eta^{(1)}$, we observe that

$$\left\| \frac{(L_{aux} + \rho_0)\overline{P}_\eta^{(1)}}{L_0(\theta) - z} \right\| \;=\; \sup\left\{ \left| \frac{r + \rho_0}{\mu + \cos(\vartheta) r_f - i \sin(\vartheta) r_{aux} - z} \right| \right\}, \tag{V.144}$$

where the supremum is taken over $\mu \in \sigma[L_{el}] \setminus \{\eta\}$ and $0 \le r_f \le r_{aux}$. Since $\sin\vartheta \ge \sin\vartheta_0' > 0$ and $|\mu - \operatorname{Re} z| \ge c$, for some constant $c > 0$, we have

$$\begin{aligned} & |\mu + \cos(\vartheta) r_f - i \sin(\vartheta) r_{aux} - z| \\ & \quad \ge \; \frac{1}{2}\big(c - \cos(\vartheta) r_f\big)_+ \;+\; \frac{1}{4}\big(r_{aux} - \sin(\vartheta)\rho_0\big)_+ \\ & \quad \ge \; \frac{1}{2}\big(2\rho_0 - r_{aux}\big)_+ \;+\; \frac{1}{4}\big(r_{aux} - \rho_0\big)_+ \;\ge\; \frac{1}{8}\big(\rho_0 + r\big) \,, \end{aligned} \tag{V.145}$$

and hence

$$\left\| \frac{(L_{aux} + \rho_0)\overline{P}_\eta^{(1)}}{L_0(\theta) - z} \right\| \;\le\; 8\,. \tag{V.146}$$

On $\operatorname{Ran} \overline{P}_\eta^{(2)}$, we estimate

$$\left\| \frac{(L_{aux} + \rho_0)\overline{P}_\eta^{(2)}}{L_0(\theta) - z} \right\| \;\le\; \sup_{r \ge \rho_0}\left\{ \frac{r + \rho_0}{|r \sin\vartheta + \operatorname{Im} z|} \right\} \le \sup_{r \ge \rho_0}\left\{ \frac{2(r + \rho_0)}{r - \rho_0/2} \right\} \;=\; 8\,, \tag{V.147}$$

and putting together (V.146) and (V.147), we obtain

$$\left\| \frac{(L_{aux} + \rho_0)\overline{P}_\eta}{L_0(\theta) - z} \right\| \;\le\; 8\,. \tag{V.148}$$

Inserting (V.148) and (V.140), the Neumann series (V.139) is seen to converge in norm since $\mathcal{O}(g\,\rho_0^{-1/2}) = \mathcal{O}(g^\varepsilon) \ll 1$, for $g > 0$ sufficiently small. □

Lemma V.5 establishes the existence of the following Feshbach operator,

$$\begin{aligned} \mathcal{F}_{P_\eta} \;&:=\; \mathcal{F}_{P_\eta}\big(L_g(\theta) - z\big) \\ &:=\; \big(L_0(\theta) - z\big) P_\eta \;+\; g P_\eta W(\theta) P_\eta \\ &\quad - g^2 P_\eta W(\theta) \overline{P}_\eta \big(\overline{P}_\eta L_g(\theta) \overline{P}_\eta - z\big)^{-1} \overline{P}_\eta W(\theta) P_\eta . \end{aligned} \tag{V.149}$$

The importance of $\mathcal{F}_{P_\eta}$ lies in the following identity (see [7, 8, 9]),

$$\begin{aligned}\big(L_g(\theta)-z\big)^{-1} \;=\;& \Big[P_\eta \,-\, g\big(\overline{P}_\eta L_g(\theta)\overline{P}_\eta - z\big)^{-1}\overline{P}_\eta W(\theta) P_\eta\Big] \\ & \mathcal{F}_{P_\eta}^{-1}\Big[P_\eta \,-\, g P_\eta W(\theta)\overline{P}_\eta\big(\overline{P}_\eta L_g(\theta)\overline{P}_\eta - z\big)^{-1}\Big] \\ & + \overline{P}_\eta\big(\overline{P}_\eta L_g(\theta)\overline{P}_\eta - z\big)^{-1}\overline{P}_\eta\,, \end{aligned} \tag{V.150}$$

which manifestly shows that $L_g(\theta) - z$ is invertible iff $\mathcal{F}_{P_\eta}(L_g(\theta) - z)$ is invertible on $\operatorname{Ran} P_\eta$. Indeed,

$$\big\|\,\big(L_g(\theta)-z\big)^{-1}\,\big\| \;\le\; \Big(1 \,+\, \mathcal{O}\big(g\,\rho_0^{-1/2}\big)\Big)\,\big\|\,\mathcal{F}_{P_\eta}\big(L_g(\theta)-z\big)\,\big\| \;+\; \mathcal{O}\big(\rho_0^{-1/2}\big)\,, \tag{V.151}$$

by Lemma V.5 and the estimates in its proof.

To apply the Feshbach map, we introduce the level-shift operator,

$$Q_E \otimes P_\Omega \;:=\; \lim_{\varepsilon\searrow 0}\big\{\, P_\Omega\, W\, \overline{P}_\Omega\,(L_0 - E - i\varepsilon)^{-1}\,\overline{P}_\Omega\, W P_\Omega\,\big\}\,, \tag{V.152}$$

where $E \in \mathbb{R}$ and $P_\Omega := |\Omega\otimes\Omega\rangle\langle\Omega\otimes\Omega|$ is the projection onto the vacuum $\Omega\otimes\Omega$ in $\mathcal{F}\otimes\mathcal{F}$. An explicit computation shows that Q_E is bounded and has a nonnegative imaginary part

$$\Gamma_E \otimes P_\Omega \;:=\; \operatorname{Im} Q_E \otimes P_\Omega \;=\; P_\Omega\, W\,\overline{P}_\Omega\,\delta(L_0 - E)\,\overline{P}_\Omega\, W P_\Omega \;\ge\; 0\,. \tag{V.153}$$

It is convenient to omit the trivial tensor factor $\otimes P_\Omega$ in our notation, i.e., to identify Q_E with $Q_E \otimes P_\Omega$ and Γ_E with $\Gamma_E \otimes P_\Omega$.

In Theorem B.1 of Appendix B, we show that if $\eta = E_i - E_j \neq 0$ then

$$P_\eta\,\Gamma_\eta\,P_\eta \;\ge\; \gamma^{(i,j)}\,P_\eta\,, \tag{V.154}$$

for some strictly positive constant $\gamma^{(i,j)} > 0$.

Theorem V.6. *Assume (V.128). For any* $0 < \varepsilon < 2/3$,

$$\big\|\mathcal{F}_{P_\eta} \,-\, P_\eta\big(L_0(\theta) - z - g^2 Q_\eta\big)P_\eta\big\| \;=\; \mathcal{O}\big(g^{2+\varepsilon}\big)\,. \tag{V.155}$$

Proof. Denoting $\chi_r(\omega) := \chi[\omega < r]$, we first observe that, by Lemma IV.1,

$$\begin{aligned} \big\|a_\ell(w_{\sigma,\nu}^{(\theta)})\,P_\eta\big\| \;&=\; \big\|a_\ell(\chi_{\rho_0}\,w_{\sigma,\nu}^{(\theta)})\,P_\eta\big\| \\ \le\;& \mathcal{O}\big(\rho_0^{1/2}\big)\,\big\|a_\ell(\chi_{\rho_0}\,w_{\sigma,\nu}^{(\theta)})\,(L_{aux} + \rho_0)^{-1/2}\,P_\eta\big\| \\ \le\;& \mathcal{O}\big(\rho_0^{1/2}\big)\left(\int_{\omega(k)<\rho_0} dk\,\Big(1 + \frac{1}{\omega(k)}\Big)\,\big\|w_{\sigma,\nu}^{(\theta)}(k)\big\|^2\right)^{1/2} \\ \le\;& \mathcal{O}\big(\rho_0^{(1+\tau)/2}\,\beta^{-1/2}(\beta+1)^{1/2}\big)\left(\int_{\omega(k)<\rho_0} dk\,\Big(1 + \frac{1}{\omega(k)^{2+\tau}}\Big)\,\frac{4\,e^{2\pi}\,\kappa(k)}{\cos^2\vartheta}\right)^{1/2} \\ =\;& \mathcal{O}\big(\vartheta^{-4}\,\rho_0^{(1+\tau)/2}\,\beta^{-1/2}(\beta+1)^{1/2}\big)\,, \end{aligned} \tag{V.156}$$

for any $0 < \tau < 1$. Using similar estimates for the other terms, we obtain

$$\big\| \, g \, P_\eta \, W(\theta) \, P_\eta \, \big\| \;=\; \mathcal{O}\big(g \, \rho_0^{(1+\tau)/2}\big) \;=\; \mathcal{O}\big(g^{1+(1-\varepsilon)(1+\tau)}\big) \, , \tag{V.157}$$

Compare to [7, (IV.101)]. Next, the second resolvent equation and Lemma V.5 yield

$$\begin{aligned}
&\Big\| g^2 P_\eta W(\theta) \overline{P}_\eta \big(\overline{P}_\eta L_g(\theta) \overline{P}_\eta - z \big)^{-1} \overline{P}_\eta W(\theta) P_\eta \\
&\qquad\qquad - \, g^2 P_\eta W(\theta) \overline{P}_\eta \big(\overline{P}_\eta L_0(\theta) \overline{P}_\eta - z \big)^{-1} \overline{P}_\eta W(\theta) P_\eta \Big\| \\
&\quad = \; \mathcal{O}\big(g^3 \, \rho_0^{-1/2}\big) \;=\; \mathcal{O}\big(g^{2+\varepsilon}\big) \, ,
\end{aligned} \tag{V.158}$$

Compare to [7, (IV.101)]. Third, we define $\overline{P}_\eta(\omega) := \mathbf{1} - \chi_{\{\eta\}}[L_{el}] \otimes \chi[L_f + \omega < \rho_0]$ and

$$\begin{aligned}
M(\theta, z) \;\; := \;\; & \int dk \Big\{ w^{(\theta)}_{-,\ell}(k) \Big(\frac{\overline{P}_\eta(\omega(k))}{L_0(\theta) + e^{-i\vartheta}\omega(k) - z} \Big) w^{(\theta)}_{+,\ell}(k) \\
& \qquad - \, w^{(\theta)}_{-,r}(k) \Big(\frac{\overline{P}_\eta(\omega(k))}{L_0(\theta) + e^{i\vartheta}\omega(k) - z} \Big) w^{(\theta)}_{+,r}(k) \Big\} \, ,
\end{aligned} \tag{V.159}$$

similar to Q in [7, (IV.67)]. A normal-ordering procedure as in [7, (IV.66)–(IV.76)] then gives, for any $0 < \tau' < 1$,

$$\begin{aligned}
&\Big\| g^2 P_\eta W(\theta) \overline{P}_\eta \big(\overline{P}_\eta L_0(\theta) \overline{P}_\eta - z \big)^{-1} \overline{P}_\eta W(\theta) P_\eta \; - \; g^2 P_\eta M(\theta, z) P_\eta \Big\| \\
&\qquad\qquad = \; \mathcal{O}\big(g^2 \, \rho_0^{\tau'}\big) \;=\; \mathcal{O}\big(g^{2+2\tau'(1-\varepsilon)}\big) \, .
\end{aligned} \tag{V.160}$$

Fourth, using the first resolvent equation, we obtain

$$\begin{aligned}
&\Big\| g^2 P_\eta \big(M(\theta, z) \; - \; M(\theta, \eta) \big) P_\eta \Big\| \\
&\quad \le \; g^2 \, |\mathrm{Im} z| \int dk \, \frac{8 \, e^{2\pi}}{\cos^2 \vartheta} \Big(1 + \frac{1}{\beta \omega(k)} \Big) \Big(\frac{\kappa(k)}{\rho_0 + \omega(k)} \Big)^2 \\
&\quad = \; \mathcal{O}\big(g^2 \, \rho_0^{1-\tau''}\big) \;=\; \mathcal{O}\big(g^{2+(1-\varepsilon)(1-\tau'')}\big) \, ,
\end{aligned} \tag{V.161}$$

for any $0 < \tau'' < 1$. Fifth, a similar estimate as (V.161) and an analytic continuation $\vartheta \to 0$ give

$$\big\| g^2 \, P_\eta \, M(\theta, \eta) \, P_\eta \; - \; g^2 \, P_\eta \, Q_\eta \, P_\eta \big\| \;=\; \mathcal{O}\big(g^2 \, \rho_0^{1-\tau'''}\big) \;=\; \mathcal{O}\big(g^{2+(1-\varepsilon)(1-\tau''')}\big) \, , \tag{V.162}$$

for any $0 < \tau''' < 1$. Estimates (V.161) and (V.162) are similar to [7, Lemmata IV.11, IV.12]. Choosing $\tau''' := \tau'' := 1-\tau' := 1-\tau$ and observing that $1+2\tau(1-\varepsilon) \le \min\{1+(1-\varepsilon)(1+\tau)\,,\ 2+2\tau(1-\varepsilon)\}$, we arrive at

$$\left\| \mathcal{F}_{P_\eta} \;-\; P_\eta\big(L_0(\theta) - z - g^2 Q_\eta\big)P_\eta \right\| \;=\; \mathcal{O}\big(g^{2+\varepsilon} \;+\; g^{1+2\tau(1-\varepsilon)}\big)\,, \tag{V.163}$$

from which (V.155) follows upon choosing $\tau := 2\varepsilon(1-\varepsilon)^{-1}$. □

Next, thanks to (V.154), we have

$$\begin{aligned} \left\| \big(L_0(\theta) - z - g^2 Q_\eta\big)^{-1} P_\eta \right\| \;&\le\; \mathrm{dist}\big\{\sigma[L_0(\theta) - g^2 Q_\eta]\,,\, z\big\} \\ &\le\; 2\left(\gamma^{(i,j)}\right)^{-1} g^{-2}\,, \end{aligned} \tag{V.164}$$

whenever $|\mathrm{Im} z| \le \gamma^{(i,j)} g^2/2$. Combining this with (V.155), we obtain a convergent Neumann series expansion for $\mathcal{F}_{P_\eta}^{-1}$,

$$\left\| \mathcal{F}_{P_\eta}^{-1} \right\| \;\le\; C\, g^{-2} \sum_{n=0}^{\infty} \left(\frac{C\, g^{2+\varepsilon}}{g^2} \right)^n \;=\; \mathcal{O}\big(g^{-2}\big)\,. \tag{V.165}$$

Defining

$$\gamma_{\neq 0} \;:=\; \min\big\{\gamma^{(i,j)} \;\big|\; i \neq j\ \big\} \;>\; 0\,, \tag{V.166}$$

we thus arrive at the following theorem.

Theorem V.7. *Assume (V.128), $z \in \mathcal{S}_{i,j}$, $i \neq j$, and $\mathrm{Im}(z) \ge -\gamma_{\neq 0}\, g^2/2$. For sufficiently small $g > 0$, the dilated Liouvillian $L_g(\theta) - z$ is bounded invertible,*

$$\sigma[L_g(\theta)] \,\cap\, \big\{z \in \mathbb{C} \;\big|\; \mathrm{Re}\, z \ge \rho_0/2\,,\ \mathrm{Im}\, z \ge -\gamma_{\neq 0}\, g^2/2\big\} \;=\; \emptyset\,. \tag{V.167}$$

V.6 Invertibility of $L_g(\theta) - z$ in $\mathcal{S}_{0,>}$ outside the cone $\mathcal{C}(\vartheta') = \big\{|\mathrm{Re}\, z| \le -\cot(\vartheta')\mathrm{Im}\, z\big\}$, for $\vartheta' < \vartheta$

The purpose of this subsection is to study the invertibility of $L_g(\theta) - z$ in $\mathcal{S}_{0,>}$. Thus we henceforth assume (V.128) and $|z| \le \rho_0/2$. We introduce the projections

$$P_0 \;:=\; P_0^{el} \otimes \chi[L_{aux} < \rho_0]\,, \tag{V.168}$$

$$\overline{P}_0 \;:=\; \mathbf{1} - P_0 \;=\; \overline{P}_0^{(1)} + \overline{P}_0^{(2)}\,, \tag{V.169}$$

$$\overline{P}_0^{(1)} \;:=\; \overline{P_0^{el}} \otimes \mathbf{1} \quad,\quad \overline{P}_0^{(2)} \;:=\; P_0^{el} \otimes \chi[L_{aux} \ge \rho_0]\,, \tag{V.170}$$

where P_0^{el} is the projection of rank N onto $\mathrm{Ker}\, L_{el}$,

$$P_0^{el} \;=\; \chi_{\{0\}}[L_{el}] \;=\; \sum_{n=0}^{N-1} |\varphi_n \otimes \varphi_n\rangle \, \langle \varphi_n \otimes \varphi_n| \,, \tag{V.171}$$

and $\chi[L_{aux} < \rho_0]$ is the projection onto spectral values of L_{aux} strictly less than ρ_0. Again, we first establish the applicability of the Feshbach map.

Lemma V.8. *Assume (V.128) and* $|z| \leq \rho_0/2$. *Then, for* $g > 0$ *sufficiently small,* $\overline{P}_0 \big(L_g(\theta) - z \big) \overline{P}_0$ *is invertible on* $\mathrm{Ran}\, \overline{P}_0$.

Proof. Analogous to Lemma V.5. □

By Lemma V.8, $L_g(\theta) - z$ is invertible (on $\mathcal{H}$) iff $\mathcal{F}_{P_0} := \mathcal{F}_{P_0}(L_g(\theta) - z)$ is invertible on $\mathrm{Ran}\, P_0$. As in the previous subsection, the level shift operator Q_E, introduced in (V.152), plays an important role. Note that $J(\varphi_n \otimes \varphi_n \otimes \Omega \otimes \Omega) = \varphi_n \otimes \varphi_n \otimes \Omega \otimes \Omega$, and hence

$$P_0^{el} \otimes P_\Omega \;=\; J \, P_0^{el} \otimes P_\Omega \;=\; P_0^{el} \otimes P_\Omega \, J \,, \tag{V.172}$$

where J is defined in (IV.77). Since furthermore, $J \, L_0 \, J \;=\; -L_0$, and $J \, W \, J \;=\; -W$, we have that

$$\begin{aligned} & P_0^{el} \otimes P_\Omega \, W \, \overline{P}_\Omega \, (L_0 - i\varepsilon)^{-1} \, \overline{P}_\Omega \, W \, P_0^{el} \otimes P_\Omega \\ & \quad = \; P_0^{el} \otimes P_\Omega \, J \, W \, \overline{P}_\Omega \, (L_0 - i\varepsilon)^{-1} \, \overline{P}_\Omega \, W \, J \, P_0^{el} \otimes P_\Omega \\ & \quad = \; - \, P_0^{el} \otimes P_\Omega \, W \, \overline{P}_\Omega \, (L_0 + i\varepsilon)^{-1} \, \overline{P}_\Omega \, W \, P_0^{el} \otimes P_\Omega \,, \end{aligned} \tag{V.173}$$

and therefore

$$Q_0 \, P_0 \;=\; i \, \Gamma_0 \, P_0 \tag{V.174}$$

is purely imaginary.

Lemma V.9. *Assume (V.128) and* $|z| \leq \rho_0/2$. *Then, for* $g > 0$ *sufficiently small and any* $0 < \varepsilon < 2/3$,

$$\big\| \mathcal{F}_{P_0} \;-\; P_0 \big(L_0(\theta) - z - i g^2 \Gamma_0 \big) P_0 \big\| \;=\; \mathcal{O}\big(g^{2+\varepsilon} \big) \,. \tag{V.175}$$

Proof. Analogous to Theorem V.6, taking into account (V.174). □

We now distinguish between spectral parameters, z, very close to zero and those which are at least of magnitude $\mathcal{O}(g^{2+\varepsilon})$. The latter can be dealt with by a standard Neumann series expansion, provided they are outside the cone $\mathcal{C}(\vartheta')$, $\vartheta' < \vartheta$, while for the spectral parameters close to zero we apply the renormalization group arguments developed in [7, 8]. This is done in Subsection V.7.

We turn to proving the invertibility of the resolvent of $L_g(\theta) - z$ for z outside of $\mathcal{C}(\vartheta')$ and of magnitude between $\mathcal{O}(g^{2+\varepsilon})$ and $\rho_0/2$, see Fig. 4.

Theorem V.10. *Assume (V.128),* $0 < \vartheta' < \vartheta$, *and* $0 < \varepsilon < 1/3$. *Suppose that* $C_0\, g^{2+\varepsilon} \leq |z| \leq \rho_0/2$ *and* $z \notin \mathcal{C}(\vartheta')$*, where* $C_0 < \infty$ *is sufficiently large. Then, for* $g > 0$ *sufficiently small,* $L_g(\theta) - z$ *is bounded invertible.*

Proof. We first observe that $\big(L_0(\theta) - ig^2\Gamma_0\big)P_0$ is a normal operator. Since $\Gamma_0 \geq 0$, we have that $\sigma[L_f(\theta) - ig^2\Gamma_0] \subseteq \sigma[L_f(\theta)] = \mathcal{C}(\vartheta)$. Hence we obtain

$$\Big\| \big(L_0(\theta) - z - ig^2\Gamma_0\big)^{-1} P_0 \Big\| \;\leq\; \mathrm{dist}\big\{z\,,\,\mathcal{C}(\vartheta)\big\}^{-1} \;\leq\; \big(C_0\,\sin(\vartheta/2)\,g^{2+\varepsilon}\big)^{-1}\,. \tag{V.176}$$

Inserting this estimate into a Neumann series expansion and using (the analogue of) (V.155), we arrive at the assertion,

$$\|\mathcal{F}_{P_0}\| \;\leq\; \mathcal{O}\big(g^{-2-\varepsilon}\big) \sum_{n=0}^{\infty} \left(\frac{\mathrm{const}}{C_0\,\sin(\vartheta/2)} \right)^n \;=\; \mathcal{O}\big(g^{-2-\varepsilon}\big)\,, \tag{V.177}$$

provided C_0 is chosen sufficiently large. □

V.7 Renormalization Group Study of the Spectrum of $L_g(\theta)$ in $\mathcal{S}_{0,<}$

Having dealt with the spectral parameters of magnitude larger than $C_0\, g^{2+\varepsilon}$, we shall henceforth assume that $z \in \mathcal{S}_{0,<}$, i.e., that $|z| \leq \sin(\vartheta)\, g^{2+\varepsilon/2}/2$. The analysis of the spectrum of $L_g(\theta)$ in $\mathcal{S}_{0,<}$ is the most involved part of our analysis, as it requires an application of the renormalization transformation developed in [7, 8].

To apply the renormalization group map, it is necessary to convert $\mathcal{F}_{P_0} \equiv \mathcal{F}_{P_0}(L_g(\theta) - z)$ into a normal-ordered form. More precisely, we expand $\mathcal{F}_{P_0}$ in a Neumann series,

$$\mathcal{F}_{P_0} \;=\; (L_0(\theta) - z)P_0 \;+\; \sum_{\nu=1}^{\infty} (-1)^{\nu-1} g^{\nu}\, P_0\, W(\theta) \left(\frac{\overline{P}_0}{L_0(\theta) - z}\, W(\theta)\, P_0 \right)^{\nu-1}\,, \tag{V.178}$$

which is norm-convergent, as we have seen in the previous subsection.

To convert $\mathcal{F}_{P_0}$ into its normal-ordered form. it is convenient to adopt the following notation. We henceforth denote

$$\begin{aligned} k &:= (\vec{k},\lambda,\tau) \;\in\; \mathbb{R}^3 \times \{1,2\} \times \{\ell, r\}\,, && \text{(V.179)} \\ \int dk\, f(k) &:= \sum_{\lambda=1,2} \sum_{\tau=\ell,r} \int_{\mathbb{R}^3} d^3\vec{k}\, f(\vec{k},\lambda,\tau)\,, && \text{(V.180)} \end{aligned}$$

and

$$\begin{aligned} a^+(\vec{k},\lambda,\tau) := a^*_\tau(\vec{k},\lambda) \;\;,\;\; a^-(\vec{k},\lambda,\tau) &:= a_\tau(\vec{k},\lambda)\;\;, && \text{(V.181)} \\ w^{(\theta)}_\pm(\vec{k},\lambda,\tau) &:= w^{(\theta)}_{\pm,\tau}(\vec{k},\lambda)\,. && \text{(V.182)} \end{aligned}$$

Furthermore, we write $\omega(k) := |\vec{k}|$, etc. In this new notation the operators to deal with appear in a more compact form as

$$\begin{aligned} W(\theta) &= a^+(w^{(\theta)}_+) + a^-(w^{(\theta)}_-)\,, && \text{(V.183)} \\ L_{aux} = \int dk\, \omega(k)\, a^+(k) a^-(k) \;\;,\;\; L_f &= \int dk\, \tau\, \omega(k)\, a^+(k) a^-(k)\,. && \text{(V.184)} \end{aligned}$$

Thus the term in (V.178) of order g^ν can be written as

$$\sum_{\sigma_1,\cdots,\sigma_\nu=\pm} P_0\, a^{\sigma_1}(w^{(\theta)}_{\sigma_1}) \left(\frac{\overline{P}_0}{L_0(\theta)-z}\right) \cdots \left(\frac{\overline{P}_0}{L_0(\theta)-z}\right) a^{\sigma_\nu}(w^{(\theta)}_{\sigma_\nu})\, P_0\,. \quad \text{(V.185)}$$

For future purpose, we introduce some more notation. We collect the eigenvalues of L_{el} in a set $\{\eta_0, \eta_1, \cdots, \eta_M\} = \{E_{i,j} | 0 \le i,j \le N-1\}$, where $\eta_0 := 0$ and $M \le N(N-1)$. We then introduce

$$\begin{aligned} \chi^{(\alpha)}_{el} &:= P^{el}_{\eta_\alpha} = \chi_{\{\eta_\alpha\}}[L_{el}]\,, && \text{(V.186)} \\ \bar{\chi}^{(\alpha)}_f(\omega) &:= \begin{cases} \mathbf{1}^f & \text{for } \alpha = 1,2,\ldots,M, \\ \chi[L_{aux} + \omega < \rho_0] & \text{for } \alpha = 0, \end{cases} && \text{(V.187)} \end{aligned}$$

and we observe that

$$\overline{P}_0 = \sum_{\alpha=0}^{M} \chi^{(\alpha)}_{el} \otimes \bar{\chi}^{(\alpha)}_f(0)\,. \quad \text{(V.188)}$$

For $k_1, \dots, k_m, \tilde{k}_1, \dots, \tilde{k}_n \in \mathbb{R}^3 \times \{1,2\} \times \{\ell, r\}$, we further denote

$$
\begin{aligned}
k^{(m)} &:= (k_1, \dots, k_m)\,, \qquad \tilde{k}^{(n)} := (\tilde{k}_1, \dots, \tilde{k}_n)\,, && \text{(V.189)} \\
\underline{L} &:= (L_f, L_{aux})\,, \qquad \underline{r} := (r_f, r_{aux})\,, && \text{(V.190)} \\
a^+(k^{(m)}) &:= \prod_{j=1}^{m} a^+(k_j)\,, \qquad a^-(\tilde{k}^{(n)}) := \prod_{j=1}^{n} a^-(\tilde{k}_j)\,, && \text{(V.191)} \\
dk^{(m)} &:= \prod_{j=1}^{m} dk_j\,, \qquad d\tilde{k}^{(n)} := \prod_{j=1}^{n} d\tilde{k}_j\,, && \text{(V.192)} \\
K^{(m,n)} &:= (k^{(m)}, \tilde{k}^{(n)})\,, \qquad dK^{(m,n)} := dk^{(m)}\, d\tilde{k}^{(n)}\,, && \text{(V.193)} \\
\omega(k^{(m)}) &:= \sum_{j=1}^{m} \omega(k_j)\,, \qquad \chi_\rho[\omega] := \chi_{[0,\rho)}[\omega]\,. && \text{(V.194)}
\end{aligned}
$$

Equipped with this notation, we rewrite (V.185) as

$$
\sum_{\substack{\sigma_1, \dots, \sigma_\nu = \pm \\ \alpha_1, \dots, \alpha_{\nu-1} = 0, \dots, M}} \int dk^{(\nu)} \chi_{el}^{(0)} w_{\sigma_1}^{(\theta)}(k_1) \chi_{el}^{(\alpha_1)} \cdots \chi_{el}^{(\alpha_{\nu-1})} w_{\sigma_\nu}^{(\theta)}(k_\nu) \chi_{el}^{(0)} \otimes \qquad \text{(V.195)}
$$

$$
\chi_{\rho_0}[L_{aux}]\, a^{\sigma_1}(k_1)\, \frac{\bar{\chi}_f^{(\alpha_1)}(0)}{\eta_{\alpha_1} + L_f(\theta) - z} \cdots \frac{\bar{\chi}_f^{(\alpha_{\nu-1})}(0)}{\eta_{\alpha_{\nu-1}} + L_f(\theta) - z} a^{\sigma_\nu}(k_\nu)\, \chi_{\rho_0}[L_{aux}]\,.
$$

Now we normal-order the product of creation- and annihilation operators in the second line of (V.195). By (a two-component variant of) [8, Lemma A.3], we have, for arbitrary functions $f_1, f_2, \dots, f_{\nu-1}$,

$$
\begin{aligned}
& a^{\sigma_1}(k_1)\, f_{\alpha_1}[L_f, L_{aux}] \cdots f_{\alpha_{\nu-1}}[L_f, L_{aux}]\, a^{\sigma_\nu}(k_\nu) && \text{(V.196)} \\
&= \sum_{\mathcal{Q} \subseteq \mathcal{N}} \prod_{j \in \mathcal{Q}_+} a^+(k_j) \Big\langle \Omega \Big| \prod_{j=1}^{\nu} \Big\{ [a^{\sigma_j}(k_j)]^{\chi[j \notin \mathcal{Q}]} \\
&\qquad f_{\alpha_j} \Big[L_f + r_f + \sum_{\substack{i=1 \\ i \in \mathcal{Q}_-}}^{j} (-1)^{\tau_j} \omega(k_i) + \sum_{\substack{i=1 \\ j+1 \in \mathcal{Q}_+}}^{\nu} (-1)^{\tau_j} \omega(k_i)\ , \\
&\qquad\quad L_{aux} + r_{aux} + \sum_{\substack{i=1 \\ i \in \mathcal{Q}_-}}^{j} \omega(k_i) + \sum_{\substack{i=1 \\ j+1 \in \mathcal{Q}_+}}^{\nu} \omega(k_i) \Big] \Big\} \Omega \Big\rangle \Big|_{\underline{r} = \underline{L}} \prod_{j \in \mathcal{Q}_-} a^+(k_j)\,,
\end{aligned}
$$

where $\mathcal{N} := \{1, 2, \dots, \nu\}$, $\mathcal{Q}_\pm := \{j \in \mathcal{Q} | \sigma_j = \pm\}$, and $[a^{\sigma_j}(k_j)]^{\chi[j \notin \mathcal{Q}]} = a^{\sigma_j}(k_j)$, for $j \notin \mathcal{Q}$, and $[a^{\sigma_j}(k_j)]^{\chi[j \notin \mathcal{Q}]} = 1$, for $j \in \mathcal{Q}$. To apply this formula to the second line of (V.195), we choose

$$
f_\alpha[r_f, r_{aux}] := \frac{\bar{\chi}_f^{(\alpha)}(r_{aux})}{\eta_\alpha + \cos(\vartheta) r_f - i \sin(\vartheta) r_{aux} - z}\,. \qquad \text{(V.197)}
$$

Note that, similar to (V.143)–(V.148), we have

$$\sup_{\substack{0\leq|r_f|\leq r_{aux}\\ \alpha=0,\dots,M}} \Big\{ \big(r_{aux}+\rho_0\big)\,\big|f_\alpha[r_f,r_{aux}]\big| \Big\} \;\leq\; C\,, \tag{V.198}$$

for some $C<\infty$. Inserting formula (V.196) into (V.195) and summing up the contributions to all orders in g, we obtain an effective Liouvillian on $\mathrm{Ran}\,P_0$,

$$\begin{aligned} L_{(0)}[z]-z \;&:=\; \mathcal{F}_{P_0}(L_g(\theta)-z) \\ &=\; P_0\,\Big(E_{(0)}[z]\;-\;z\;+\;T_{(0)}[z,\underline{L}]\;+\;W_{(0)}[z]\Big)\,P_0\,, \end{aligned} \tag{V.199}$$

where

$$\begin{aligned} E_{(0)}[z] \;&:=\; \chi_{el}^{(0)}\otimes P_\Omega\,\Big(\mathcal{F}_{P_0}(L_g(\theta)-z)\;+\;z\Big)\,\chi_{el}^{(0)}\otimes P_\Omega\,, \\ &=\; -\sum_{\nu=2}^{\infty}(-g)^\nu \sum_{\substack{\sigma_1,\dots,\sigma_\nu=\pm\\ \alpha_1,\dots,\alpha_{\nu-1}=0,\dots,M}} \\ &\qquad \int dk^{(\nu)}\,\chi_{el}^{(0)}w_{\sigma_1}^{(\theta)}(k_1)\chi_{el}^{(\alpha_1)}\;\cdots\;\chi_{el}^{(\alpha_{\nu-1})}w_{\sigma_\nu}^{(\theta)}(k_\nu)\chi_{el}^{(0)} \\ &\qquad \times\Big\langle\Omega\Big|\;a^{\sigma_1}(k_1)\,f_{\alpha_1}[\underline{L}]\;\cdots\;f_{\alpha_{\nu-1}}[\underline{L}]\,a^{\sigma_\nu}(k_\nu)\;\Omega\Big\rangle\,, \end{aligned} \tag{V.200}$$

$$\begin{aligned} &T_{(0)}[z,\underline{r}] \\ &\;=\; \chi_{el}^{(0)}\otimes P_\Omega\,\Big(\mathcal{F}_{P_0}(r_f(\theta)+L_g(\theta)-z)\;-\;\mathcal{F}_{P_0}(L_g(\theta)-z)\Big)\,\chi_{el}^{(0)}\otimes P_\Omega\,, \\ &\;=\; r_f(\theta)\;-\;\sum_{\nu=2}^{\infty}(-g)^\nu \sum_{\substack{\sigma_1,\dots,\sigma_\nu=\pm\\ \alpha_1,\dots,\alpha_{\nu-1}=0,\dots,M}} \\ &\qquad \int dk^{(\nu)}\chi_{el}^{(0)}w_{\sigma_1}^{(\theta)}(k_1)\chi_{el}^{(\alpha_1)}\;\cdots\;\chi_{el}^{(\alpha_{\nu-1})}w_{\sigma_\nu}^{(\theta)}(k_\nu)\chi_{el}^{(0)} \\ &\qquad \times\Big\langle\Omega\Big|\;\Big\{a^{\sigma_1}(k_1)\,f_{\alpha_1}[\underline{L}+\underline{r}]\;\cdots\;f_{\alpha_{\nu-1}}[\underline{L}+\underline{r}]a^{\sigma_\nu}(k_\nu) \\ &\qquad\qquad -\;a^{\sigma_1}(k_1)\,f_{\alpha_1}[\underline{L}]\;\cdots\;f_{\alpha_{\nu-1}}[\underline{L}]a^{\sigma_\nu}(k_\nu)\;\Omega\Big\rangle\,, \end{aligned} \tag{V.201}$$

using $\underline{L}:=(L_f,L_{aux})$, $\underline{r}:=(r_f,r_{aux})$, and $r_f(\theta):=\cos(\vartheta)r_f-i\sin(\vartheta)r_{aux}$. Furthermore,

$$W_{(0)}[z] \;:=\; \sum_{\substack{m,n=0\\ m+n\geq1}}^{\infty} W_{m,n}^{(0)}[z]\,, \tag{V.202}$$

$$W_{m,n}^{(0)}[z] \;:=\; \int dK^{(m,n)}\;a^+(k^{(m)})\;w_{m,n}^{(0)}\big[z,\underline{L},K^{(m,n)}\big]\;a^-(\tilde{k}^{(n)})\,, \tag{V.203}$$

and we shall display the dependence of $L_{(0)}$, $E_{(0)}$, $T_{(0)}$, and $W_{(0)}$ on θ unless necessary. Note that $E_{(0)}$ is an operator on $\mathrm{Ran}\, \chi_{el}^{(0)}$, i.e., $E_{(0)} \in \mathcal{M}_N$ is an $N \times N$ matrix. Similarly, $\underline{r} \mapsto T_{(0)}[z, \underline{r}]$ is an $N \times N$ matrix-valued function, and $\underline{r} \mapsto w_{m,n}^{(0)}[z, \underline{r}, K^{(m,n)}]$ are $N \times N$ matrix-valued functions, for $m + n \geq 1$, pointwise in $K^{(m,n)}$. Eq. (V.196) yields the following explicit expressions for $w_{m,n}^{(0)}$ (compare to [8, Lemma III.6]),

$$w_{m,n}^{(0)}\big[z, \underline{L}, K^{(m,n)}\big] \; = \qquad \text{(V.204)}$$
$$-\sum_{p=0}^{\infty} \sum_{b \in \mathcal{B}_{m,n,p}} \sum_{\alpha_1, \dots, \alpha_{\nu-1}=0}^{M} (-g)^{\nu} \int dX^{(p,p)} \mathcal{S}_{m,n}\Big\{ F_b\big[X^{(p,p)}, K^{(m,n)}\big] \Big\} \, ,$$

where $\mathcal{B}_{m,n,p}$ denotes the set of partitions $b = (b_k, b_{\tilde{k}}, b_x, b_{\tilde{x}})$ of $\{1, 2, \dots, m + n + 2p\}$ such that $|b_k| = m$, $|b_{\tilde{k}}| = n$, and $|b_x| = |b_{\tilde{x}} = p$, i.e., b_k, $b_{\tilde{k}}$, b_x, $b_{\tilde{x}}$ are ordered, pairwise disjoint subsets of $\{1, 2, \dots, m + n + 2p\}$ whose union give $\{1, 2, \dots, m + n + 2p\}$.

Given $b \in \mathcal{B}_{m,n,p}$ and denoting $M := m + n + 2p$, the matrix-valued function F_b is defined by

$$F_b\big[X^{(p,p)}, K^{(m,n)}\big] \; := \qquad \text{(V.205)}$$
$$\chi_{el}^{(0)} w_b^{(\theta)}(1, X^{(p,p)}, K^{(m,n)}) \chi_{el}^{(\alpha_1)} \cdots \chi_{el}^{(\alpha_{\nu-1})} w_b^{(\theta)}(M, X^{(p,p)}, K^{(m,n)}) \chi_{el}^{(0)}$$
$$\cdot \Big\langle \Omega \Big| \, a^b(1, X^{(p,p)}, K^{(m,n)}) \, f_{\alpha_1}[\underline{L} + \underline{\mu}_1]$$
$$\cdots \; f_{\alpha_{\nu-1}}[\underline{L} + \underline{\mu}_{\nu-1}] a^b(M, X^{(p,p)}, K^{(m,n)}) \, \Omega \Big\rangle \, ,$$

where

$$w_b^{(\theta)}(j, X^{(p,p)}, K^{(m,n)}) \; := \; \begin{cases} w_+^{(\theta)}(k_l) \, , & \text{if } j \text{ is the } l^{th} \text{ member of } b_k, \\ w_-^{(\theta)}(\tilde{k}_l) \, , & \text{if } j \text{ is the } l^{th} \text{ member of } b_{\tilde{k}}, \\ w_+^{(\theta)}(x_l) \, , & \text{if } j \text{ is the } l^{th} \text{ member of } b_x, \\ w_-^{(\theta)}(\tilde{x}_l) \, , & \text{if } j \text{ is the } l^{th} \text{ member of } b_{\tilde{x}}, \end{cases} \qquad \text{(V.206)}$$

and

$$a^b(j, X^{(p,p)}, K^{(m,n)}) \; := \; \begin{cases} 1 \, , & \text{if } j \in b_k \cup b_{\tilde{k}}, \\ a^+(x_l) \, , & \text{if } j \text{ is the } l^{th} \text{ member of } b_x, \\ a^-(\tilde{x}_l) \, , & \text{if } j \text{ is the } l^{th} \text{ member of } b_{\tilde{x}}. \end{cases} \qquad \text{(V.207)}$$

Moreover, $\mathcal{S}_{m,n}$ denotes the symmetrization operator,

$$\mathcal{S}_{m,n}\{F\}[X^{(p,p)}, K^{(m,n)}] \; := \qquad \text{(V.208)}$$
$$\frac{1}{m! \, n!} \sum_{\substack{\pi \in S_m \\ \tilde{\pi} \in S_n}} F\big[X^{(p,p)}, k_{\pi(1)}, \dots, k_{\pi(m)} \, ; \, \tilde{k}_{\tilde{\pi}(1)}, \dots, \tilde{k}_{\tilde{\pi}(n)}\big] \, .$$

We have the following estimates on these coefficients.

Lemma V.11. *Assume that $z \in \mathcal{S}_{0,<}$. Then there exists a constant, $C < \infty$, such that, for $g > 0$ sufficiently small,*

$$\begin{aligned} \big\| E_{(0)}[z] \, + \, i \, g^2 \, \Gamma_0 \big\| \ &\leq \ C \, g^{2+2\varepsilon} \, , && \text{(V.209)} \\ \big\| \partial_{r_{aux}} T_{(0)}[z, \underline{r}] \, + \, i \sin \vartheta \big\| \ &\leq \ C \, g^{2\varepsilon} \, , && \text{(V.210)} \end{aligned}$$

$$\big\| w_{m,n}^{(0)}[z, \underline{r}, K^{(m,n)}] \big\| \ \leq \ C \, \rho_0 \left(\frac{C \, g}{\rho_0} \right)^{m+n} \prod_{j=1}^{m} \frac{\kappa(k_j)}{\omega(k_j)^{1/2-\mu}} \prod_{j=1}^{n} \frac{\kappa(\tilde{k}_j)}{\omega(\tilde{k}_j)^{1/2-\mu}} \, , \tag{V.211}$$

$$\begin{aligned} \int_{B_{\rho_0}^{m+n}} \big\| \partial_{r_{aux}} w_{m,n}^{(0)}[z, \underline{r}, K^{(m,n)}] \big\| \prod_{j=1}^{m} \Big(\frac{d^3 k_j}{\omega(k_j)^{3/2+\mu}} \Big) \prod_{j=1}^{n} \Big(\frac{d^3 \tilde{k}_j}{\omega(\tilde{k}_j)^{3/2+\mu}} \Big) & \\ \leq \ C \, \big(C \, g \big)^{m+n} \, , & \end{aligned} \tag{V.212}$$

where $B_r := \{ k \mid \omega(k) < r \}$.

Proof. The asserted estimates follow from adaption of [8, Section III]. For illustration, we give a proof of (V.209). We first rewrite $E_{(0)}[z]$ as

$$\begin{aligned} E_{(0)}[z] \ = \ -g^2 \, M[\theta, z] \, - \, \sum_{\nu=4}^{\infty} (-g)^{\nu} & \sum_{\substack{\sigma_1, \dots, \sigma_\nu = \pm \\ \alpha_1, \dots, \alpha_{\nu-1} = 0, \dots, M}} \int dk^{(\nu)} \\ & \chi_{el}^{(0)} w_{\sigma_1}^{(\theta)}(k_1) \chi_{el}^{(\alpha_1)} \ \cdots \ \chi_{el}^{(\alpha_{\nu-1})} w_{\sigma_\nu}^{(\theta)}(k_\nu) \chi_{el}^{(0)} \\ & \times \Big\langle \Omega \Big| \ a^{\sigma_1}(k_1) \, f_{\alpha_1}[\underline{L}] \ \cdots \ f_{\alpha_{\nu-1}}[\underline{L}] \, a^{\sigma_\nu}(k_\nu) \, \Omega \Big\rangle \, , \end{aligned} \tag{V.213}$$

where

$$M[\theta, z] \ := \ \sum_{\alpha=0}^{M} \int dk \, f_\alpha[(-1)^\tau \omega(k), \omega(k)] \, \chi_{el}^{(0)} w_-^{(\theta)}(k) \chi_{el}^{(\alpha)} w_+^{(\theta)}(k) \chi_{el}^{(0)} \, . \tag{V.214}$$

Here, $k = (\vec{k}, \lambda, \tau) \in \mathbb{R}^3 \times \{1, 2\} \times \{\ell, r\}$, and $(-1)^\tau := 1$, for $\tau = \ell$, and $(-1)^\tau := -1$, for $\tau = r$. We recall that

$$f_\alpha[(-1)^\tau \omega, \omega] \ = \ \frac{\bar{\chi}_f^{(\alpha)}[\omega]}{\eta_\alpha + e^{-i(-1)^\tau \vartheta} \omega(k) - z} \tag{V.215}$$

and that an analytic continuation from $\theta = 0$ to $\theta = i\vartheta$ yields

$$\begin{aligned}\Gamma_0 \; &= \; \sum_{\alpha=0}^{M} \int dk \, \frac{1}{\eta_\alpha + e^{-i(-1)^\tau \vartheta}\omega(k)} \, \chi_{el}^{(0)} w_-^{(\theta)}(k) \chi_{el}^{(\alpha)} w_+^{(\theta)}(k) \chi_{el}^{(0)} \, , \\ &= \; \sum_{\alpha=0}^{M} \int dk \, \frac{1}{\eta_\alpha + \omega(k) - i0} \, \chi_{el}^{(0)} w_-^{(0)}(k) \chi_{el}^{(\alpha)} w_+^{(0)}(k) \chi_{el}^{(0)} \, , \qquad \text{(V.216)}\end{aligned}$$

since Γ_0 is independent of θ. In view of the fact that

$$\left| \frac{\bar{\chi}_f^{(\alpha)}[\omega(k)]}{\eta_\alpha + e^{-i(-1)^\tau \vartheta}\omega(k) - z} \; - \; \frac{1}{\eta_\alpha + e^{-i(-1)^\tau \vartheta}\omega(k)} \right| \; = \; \mathcal{O}(|z|) \, , \qquad \text{(V.217)}$$

for $\alpha \geq 1$, and

$$\left| \frac{\bar{\chi}_f^{(0)}[\omega(k)]}{e^{-i(-1)^\tau \vartheta}\omega(k) - z} \; - \; \frac{e^{i(-1)^\tau \vartheta}}{\omega(k)} \right| \; \leq \; \frac{2\,|z|}{(\omega(k) + \rho_0)^2} \; + \; \frac{\chi_{\rho_0}[\omega(k)]}{\omega(k)} \, , \qquad \text{(V.218)}$$

we obtain that

$$\big\| K[\theta, z] \; - \; i\Gamma_0 \big\| \; = \; \mathcal{O}\big(g^2\big) \, . \qquad \text{(V.219)}$$

Secondly, we observe that

$$\begin{aligned} &\Big| \Big\langle \Omega \Big| \; a^{\sigma_1}(k_1) \, f_{\alpha_1}[\underline{L}] \; \cdots \; f_{\alpha_{\nu-1}}[\underline{L}] \, a^{\sigma_\nu}(k_\nu) \; \Omega \Big\rangle \Big| \qquad \text{(V.220)} \\ &= \; \rho_0 \, \Big| \Big\langle \Omega \Big| \; (L_{aux} + \rho_0)^{-1/2} \, a^{\sigma_1}(k_1) \, (L_{aux} + \rho_0)^{-1/2} \; (L_{aux} + \rho_0) \, f_{\alpha_1}[\underline{L}] \\ &\qquad \cdots \; (L_{aux} + \rho_0) \, f_{\alpha_{\nu-1}}[\underline{L}] \; (L_{aux} + \rho_0)^{-1/2} \, a^{\sigma_\nu}(k_1) \, (L_{aux} + \rho_0)^{-1/2} \; \Omega \Big\rangle \Big| \\ &\leq \; \rho_0 \, 8^{\nu - 1} \prod_{j=1}^{\nu} \Big\| (L_{aux} + \rho_0)^{-1/2} \, a^{\sigma_j}(k_j) \, (L_{aux} + \rho_0)^{-1/2} \Big\| \, , \end{aligned}$$

using (V.198). The standard estimate from Lemma IV.1 implies that

$$\int dk_j \, \| w_{\sigma_j}^{(\theta)}(k_j) \| \; \Big\| (L_{aux} + \rho_0)^{-1/2} \, a^{\sigma_j}(k_j) \, (L_{aux} + \rho_0)^{-1/2} \Big\| \; = \; \mathcal{O}(\rho_0^{-1/2}) \, . \qquad \text{(V.221)}$$

Inserting (V.219)–(V.221) into the sum in (V.213) and summing up the terms of order $\nu \geq 4$ as in [8, Lemma III.7], we obtain that

$$\| E_{(0)}[z] \; + \; i g^2 \Gamma_0 \| \; \leq \; \sum_{\nu=4}^{\infty} C \, \rho_0 \left(\frac{C\, g}{\rho_0^{1/2}} \right)^{\nu} \; = \; \mathcal{O}\big(g^{2+2\varepsilon}\big) \, , \qquad \text{(V.222)}$$

thus establishing (V.209). □

In Theorem B.2 in Appendix B we prove that $\operatorname{Ker}\Gamma_0 = \mathbb{C}\kappa_\beta$, where

$$\kappa_\beta \;=\; Z^{-1} \sum_{n=0}^{N-1} e^{-\beta E_n/2}\, \varphi_n \otimes \varphi_n \,, \tag{V.223}$$

$Z = \sum_{n=0}^{N-1} e^{-\beta E_n}$ is a normalization factor. Hence, denoting by

$$P^{el}_{\kappa_\beta} \;:=\; |\kappa_\beta\rangle\,\langle\kappa_\beta| \tag{V.224}$$

the orthogonal projection onto κ_β, we have that

$$\Gamma_0 \;=\; \Gamma_0\, P^{el}_{\kappa_\beta} \;\geq\; \widehat{\gamma}_0\, P^{el}_{\kappa_\beta} \,, \tag{V.225}$$

for some positive $\widehat{\gamma}_0 > 0$. Our strategy is now to apply the Feshbach map again, using the projection

$$P_1 \;:=\; P^{el}_{\kappa_\beta} \otimes \chi[L_{aux} < \rho_1] \,, \tag{V.226}$$

where

$$\rho_1 \;:=\; g^{2+\varepsilon/2} \,. \tag{V.227}$$

We have $P_1 P_0 = P_0 P_1 = P_1$ and

$$\overline{P}_1 \;:=\; \mathbf{1} - P_1 \;=\; \overline{P}_1^{(1)} + \overline{P}_1^{(2)} \,, \tag{V.228}$$

where

$$\overline{P}_1^{(1)} \;:=\; \overline{P}^{el}_{\kappa_\beta} \otimes \mathbf{1} \,, \qquad \overline{P}_1^{(2)} \;:=\; P^{el}_{\kappa_\beta} \otimes \chi[L_{aux} \geq \rho_1] \,, \tag{V.229}$$

Again, for the Feshbach map to be defined, we prove the invertibility of the operator restricted to $\operatorname{Ran}\overline{P}_1\, P_0$. We divide the proof into a series of lemmata.

Lemma V.12. *Assume* $z \in \mathcal{S}_{0,<}$. *Then*

$$\left\| \frac{(T_{(0)}[z, \underline{L}] + E_{(0)}[z] - z)\, \overline{P}_1\, P_0}{L_{aux} + \rho_1} \right\| \;\leq\; \mathcal{O}(1) \,. \tag{V.230}$$

Proof. According to (V.209), (V.210) and (V.225), we have

$$-\mathrm{Im}\{T_{(0)}[z,\underline{L}] + E_{(0)}[z] - z\} \;\geq\; \big(\sin\vartheta - \mathcal{O}(g^{2\varepsilon})\big) L_{aux} \;+\; \big(\widehat{\gamma}_0 - \mathcal{O}(g^{\varepsilon/2})\big) g^2 \tag{V.231}$$

on $\mathrm{Ran}\overline{P}_1^{(1)}$. Conversely, on $\mathrm{Ran}\overline{P}^{(2)}$, we estimate

$$\begin{aligned} & -\mathrm{Im}\{T_{(0)}[z,\underline{L}] + E_{(0)}[z] - z\} \\ \geq\; & \big(\sin\vartheta - \mathcal{O}(g^{2\varepsilon})\big) L_{aux} \;-\; \frac{\sin\vartheta}{2}\rho_1 \;-\; \mathcal{O}(g^{2+2\varepsilon}) \\ \geq\; & \big(\sin(\vartheta)/2 \;-\; \mathcal{O}(g^{3\varepsilon/2})\big) L_{aux}\,, \end{aligned} \tag{V.232}$$

since $\mathcal{O}(g^{2+\varepsilon}) = \rho_1 \mathcal{O}(g^{3\varepsilon/2}) = L_{aux}\mathcal{O}(g^{3\varepsilon/2})$. □

Lemma V.13. *Assume* $z \in \mathcal{S}_{0,<}$. *Then*

$$\begin{aligned} \big\| P_0\,(L_{aux}+\rho_1)^{-1/2}\,W_{(0)}[z]\,(L_{aux}+\rho_1)^{-1/2}\,P_0 \big\| \;&=\; \mathcal{O}\big(g\,(\rho_0/\rho_1)^{1/2}\big) \\ &=\; \mathcal{O}\big(g^{1-3\varepsilon/4}\big)\,. \end{aligned} \tag{V.233}$$

Proof. The proof is an adaption of [8, Theorem B.2], using the bounds (V.211), for all $m+n \geq 1$, and summing up all contributions. □

Putting together Lemma V.12 and Lemma V.13, we obtain the invertibility of $L_{(0)}[z] - z$ restricted to $\mathrm{Ran}\,\overline{P}_1\,P_0$ by a Neumann series expansion of $(L_{(0)}[z]-z)^{-1}$ around $(T_{(0)}[z,\underline{L}] + E_{(0)}[z] - z)^{-1}$.

Lemma V.14. *Assume (V.128) and* $|z| \leq C_0 g^{2+\varepsilon}$. *Then, for* $g > 0$ *sufficiently small,* $\overline{P}_1\,(L_{(0)} - z)\,\overline{P}_1$ *is invertible on* $\mathrm{Ran}\,\overline{P}_1\,P_0$.

Lemma V.14 justifies a second application of the Feshbach map with projection P_1. That is, the operator $\mathcal{F}_{P_1}(L_{(0)}[z]-z)$ is well-defined. To formulate this result, we define a bijection,

$$Z_{(0)} \;:\; \mathcal{S}_{0,<} \to D_{1/2}\,, \qquad Z_{(0)}(\zeta) \;:=\; \frac{i\,\zeta}{\sin(\vartheta)\,\rho_1}\,, \tag{V.234}$$

and we introduce the unitary dilatation

$$U_{(0)}\,\Omega \;:=\; \Omega\,, \qquad U_{(0)}\,a^{\sigma}(\vec{k},\lambda,\tau)\,U_{(0)}^* \;:=\; \rho_1^{-3/2}\,a^{\sigma}(\vec{k}/\rho_1,\lambda,\tau)\,, \tag{V.235}$$

noting that

$$U_{(0)}\,\underline{L}\,U_{(0)}^* \;=\; \rho_1\,\underline{L}(\theta\,, \qquad U_{(0)}\,\mathrm{Ran}\,\chi_{\rho_1}[L_{aux}] \;=\; \mathrm{Ran}\,\chi_1[L_{aux}]\,, \tag{V.236}$$

where $\underline{L} = (L_f, L_{aux})$.

Theorem V.15. *Let $z \in D_{1/2}$. Then, for $g > 0$ sufficiently small, $L_g(\theta) - z$ is isospectral (in the sense of [7, 8]) to*

$$L_{(1)}[z] - z \;:=\; \frac{i}{\sin(\vartheta)\,\rho_1}\, U_{(0)}\, \mathcal{F}_{P_1}\big(L_{(0)}[Z_{(0)}^{-1}(z)] \;-\; Z_{(0)}^{-1}(z)\big)\, U_{(0)}^* \tag{V.237}$$

defined on $\mathcal{H}_{red} := \mathrm{Ran}\, \chi_1[L_{aux}]$.

Similar to $L_{(0)}[z]$, we write $L_{(1)}[z]$ as

$$L_{(1)}[z] - z \;=\; \chi_1[L_{aux}] \Big(E_{(1)}[z] \;-\; z \;+\; T_{(1)}[z, \underline{L}] \;+\; W_{(1)}[z] \Big) \chi_1[L_{aux}]\, , \tag{V.238}$$

where

$$\begin{aligned}
E_{(1)}[z] \;&:=\; \frac{i}{\sin(\vartheta)\,\rho_1} \big\langle \Omega \big|\, \mathcal{F}_{P_1}\big(L_{(0)}[Z^{-1}(z)] \;-\; Z^{-1}(z)\big)\, \Omega \big\rangle \;+\; z\, , && \text{(V.239)}\\
T_{(1)}[z, \underline{r}] \;&:=\; \frac{i}{\sin(\vartheta)\,\rho_1} \big\langle \Omega \big|\, \mathcal{F}_{P_1} \circ \mathcal{F}_{P_0}\big(\underline{r} + L_g(\theta) - Z^{-1}(z)\big) && \text{(V.240)}\\
&\qquad\qquad - \mathcal{F}_{P_1} \circ \mathcal{F}_{P_0}\big(\underline{r} + L_g(\theta) - Z^{-1}(z)\big)\, \Omega \big\rangle\, , &&\\
W_{(1)}[z] \;&:=\; \sum_{\substack{m,n=0\\ m+n\geq 1}}^{\infty} W_{m,n}^{(1)}[z]\, , && \text{(V.241)}\\
W_{m,n}^{(1)}[z] \;&:=\; \int dK^{(m,n)}\; a^+(k^{(m)})\; w_{m,n}^{(1)}\big[z, \underline{L}, K^{(m,n)}\big]\; a^-(\tilde{k}^{(n)})\, . && \text{(V.242)}
\end{aligned}$$

From Lemma V.11 and using the techniques from [8, Section IV], we derive the following estimates on these operators.

Lemma V.16. *Assume $z \in D_{1/2}$. Then there exists a constant, $C < \infty$, such that, for $g > 0$ sufficiently small,*

$$\begin{aligned}
|E_{(1)}[z]| \;&\leq\; C\, g^{3\varepsilon/2}\, , && \text{(V.243)}\\
|\partial_{r_{aux}} T_{(1)}[z, \underline{r}] \;-\; 1| \;&\leq\; C\, g^{2\varepsilon}\, , && \text{(V.244)}
\end{aligned}$$

$$\big|w_{m,n}^{(1)}[z, \underline{r}, K^{(m,n)}]\big| \;\leq\; \big(C\, g\big)^{m+n} \prod_{j=1}^{m} \frac{\kappa(k_j)}{\omega(k_j)^{1/2-\mu}} \prod_{j=1}^{n} \frac{\kappa(\tilde{k}_j)}{\omega(\tilde{k}_j)^{1/2-\mu}}\, , \tag{V.245}$$

$$\int_{B_1^{m+n}} \big|\partial_{r_{aux}} w_{m,n}^{(1)}[z, \underline{r}, K^{(m,n)}]\big| \prod_{j=1}^{m} \bigg(\frac{d^3 k_j}{\omega(k_j)^{3/2+\mu}} \bigg) \prod_{j=1}^{n} \bigg(\frac{d^3 \tilde{k}_j}{\omega(\tilde{k}_j)^{3/2+\mu}} \bigg) \;\leq\; C\, \big(C\, g\big)^{m+n}\, , \tag{V.246}$$

where $B_r := \{k \mid \omega(k) < r\}$.

Proof. To prove (V.243), we observe that due to Lemma V.13 we have

$$\begin{aligned} & \rho_1^{-1} \big| \big\langle \Omega \big| \, W_{(0)}[\zeta] \, \overline{P}_1 \, \big(\overline{P}_1 L_{(0)}[\zeta] \overline{P}_1 - \zeta \big)^{-1} \, \overline{P}_1 \, W_{(0)}[z] \, \Omega \big\rangle \big| \\ & \quad \leq \ \mathcal{O}(1) \, \big\| P_0 \, (L_{aux} + \rho_1)^{-1/2} \, W_{(0)}[\zeta] \, (L_{aux} + \rho_1)^{-1/2} \, P_0 \big\|^2 \\ & \quad = \ \mathcal{O}\big(g^{2 - 3\varepsilon/2} \big) \, . \end{aligned} \tag{V.247}$$

Thus, using (V.222), we obtain

$$\begin{aligned} |E_{(1)}[z]| \ &= \ \frac{i}{\sin(\vartheta) \, \rho_1} P_1 \, E_{(0)}[Z^{-1}(z)] \, P_1 \ + \ \mathcal{O}\big(g^{2-3\varepsilon/2} \big) & \text{(V.248)} \\ &= \ \frac{i}{\sin(\vartheta) \, \rho_1} P_1 \, \big(E_{(0)}[Z^{-1}(z)] + i g^2 \Gamma_0 \big) \, P_1 \ + \ \mathcal{O}\big(g^{2-3\varepsilon/2} \big) \\ &= \ \mathcal{O}\big(g^{3\varepsilon/2} \big) \, . \end{aligned}$$

A similar argument yields (V.244). The proof of (V.245) and (V.246) is rather lengthy, and we shall only examine the tree level contributions to $w_{m,n}^{(1)}$, i.e., those resulting from rescaling $w_{m,n}^{(0)}$. It actually turns out that these contributions are the dominant ones. We set

$$w_{m,n}^{(1),T}[z, \underline{r}, K^{(m,n)}] \ := \ \frac{i}{\sin(\vartheta)} \, \rho_1^{\frac{3}{2}(m+n)-1} \, w_{m,n}^{(0)}[Z^{-1}(z), \rho_1 \underline{r}, \rho_1 K^{(m,n)}] \, . \tag{V.249}$$

One then easily checks that (V.211) and (V.212) imply the bounds (V.245) and (V.246), with $w_{m,n}^{(1)}$ replaced by $w_{m,n}^{(1),T}$. □

Using these bounds, it is not difficult to verify that, for a suitable choice of ρ and ξ,

$$D_{1/2} \ni z \ \mapsto \ L_{(1)}[z] \in \mathcal{W}'_\Delta \tag{V.250}$$

defines an analytic family, where $\mathcal{W}'_\Delta$ is the Banach space of operators defined in [8, (I.46)]. Hence $L_{(1)} \in \mathcal{W}_\Delta$, and Lemma V.16 implies the following theorem.

Theorem V.17. *For some constant $C < \infty$ and sufficiently small $g > 0$, $L_{(1)}$ belongs to the polydisk*

$$L_{(1)} \ \in \ \mathcal{B}\big(C \, g^{2\varepsilon} \ , \ C \, g^{3\varepsilon/2} \big) \, , \tag{V.251}$$

defined in [8, (I.64)].

In other words, $L_{(1)}$ is a proper initial operator for the renormalization group map $\mathcal{R}_\rho$ defined in [8]. We may thus invoke [8, Theorems V.7 and V.10] to obtain the following result.

Theorem V.18. *Let* $0 < \vartheta' < \vartheta$*. For sufficiently small* $g > 0$*, there exists a number,* $E_{(\infty)} \in \mathcal{S}_{0,<}$*, such that*

(i) $E_{(\infty)}$ *is a simple eigenvalue of* $L_g(\theta)$*, and*

(ii) *the spectrum of* $L_g(\theta)$ *obeys*

$$\sigma[L_g(\theta)] \,\cap\, \mathcal{S}_{0,<} \;\subseteq\; \big(E_{(\infty)} \,+\, \mathcal{C}(\vartheta')\big) \,\cap\, \mathcal{S}_{0,<} \,. \tag{V.252}$$

A simple corollary (see Fig. 6) is

Corollary V.19. *Let* $0 < \vartheta' < \vartheta$*. For sufficiently small* $g > 0$*,*

(i) 0 *is a simple eigenvalue of* $L_g(\theta)$*, and*

(ii) *the spectrum of* $L_g(\theta)$ *obeys*

$$\sigma[L_g(\theta)] \,\cap\, \mathcal{S}_{0,<} \;\subseteq\; \mathcal{C}(\vartheta') \,\cap\, \mathcal{S}_{0,<} \,. \tag{V.253}$$

Proof. We first note that $\sigma[L_g(\theta)] \subseteq \overline{\mathbb{C}_-}$, by analytic continuation and the fact that the spectrum of L_g is real. Thus $\operatorname{Im} E_{(\infty)} \leq 0$.

Secondly, 0 is an eigenvalue of $L_g(\theta)$, so $0 \in E_{(\infty)} + \mathcal{C}(\vartheta')$, which implies that $E_{(\infty)} = 0$. □

A Analytic Continuation of the Resolvent of the Liouvillian

A.1 Outline of the strategy

Our goal in this appendix is to establish the analyticity of the resolvent

$$R_g(\theta, z) \;:=\; \big(L_g(\theta) \,-\, z\big)^{-1} \,, \tag{A.1}$$

as an operator-valued function of θ and z. This does not follow from a direct application of standard techniques in dilatation analyticity, in contrast to [7, 8, 9], because $\theta \mapsto L_g(\theta)$ is not a family of type A, for $\theta \in D(0, \pi/2)$. Indeed, as we point out below, $\theta \mapsto L_g(\theta)$ is not even an analytic family on $\mathcal{H}$ in the sense of Kato (see, e.g., [33]).

Note, however, that we are not really interested in global analyticity properties of $R_g(\theta, z)$. For our spectral analysis, it suffices to have an analytic continuation of $R_g(0, \lambda + i\varepsilon)$, with $\lambda + i\varepsilon \in \mathbb{C}^+$, $0 < \varepsilon \ll 1$, in the upper half-plane, to $R_g(i\vartheta, z')$, with $z' \in \mathbb{C}^-$ in the lower half-plane and $\vartheta > 0$. Hence, it suffices to have a connected domain $\mathcal{A} \subseteq \mathbb{C}^2$, containing $(0, z)$ and $(i\vartheta, z')$, such that $\mathcal{A} \ni (\theta, z) \mapsto R_g(\theta, z)$ is analytic.

The construction of $\mathcal{A}$ or, rather, of the curve in $\mathcal{A}$ linking $(0, \lambda + i\varepsilon)$ and $(i\vartheta, z')$ is as follows.

- First, using the selfadjointness of L_g, we pass from $R_g(0, \lambda + i\varepsilon)$ to $R_g(0, \lambda + 2i)$, by usual analytic continuation in z.
- Second, for $(\theta, z) = (0, \lambda + 2i)$, we pass from $R_g(\theta, z)$ to a more regular resolvent,

  $$\widetilde{R}_g(\theta, z) \ := \ (L_{aux} + 1)^{-1} \, R_g(\theta, z) \, (L_{aux} + 1)^{-1} \, . \tag{A.2}$$

  Note that the restriction of $R_g(\theta, z)$ to $\mathrm{Dom}(L_{aux})$ can be reconstructed from $\widetilde{R}_g(\theta, z)$.
- The key step of our construction is as follows. Introducing

  $$\Sigma_r^+ \ := \ \big\{ \theta \in \mathbb{C} \ \big| \ 0 \leq \mathrm{Im}\, \theta < r \big\} \, , \tag{A.3}$$

  we prove that

  $$\Sigma_{\vartheta_0}^+ \ni \theta \ \mapsto \ \widetilde{R}_g(\theta, \lambda + 2i) \tag{A.4}$$

  defines an analytic family with

  $$\big\| \, \widetilde{R}_g(\theta, \lambda + 2i) \, \big\| \ \leq \ \mathrm{dist}\{ \mathrm{NumRan}[L_g(\theta)] \, , \, \lambda + 2i \}^{-1} \, . \tag{A.5}$$

  The main difficulty we are facing here is that the coefficient in front of the dominant operator L_{aux} is linearly vanishing, as $\theta \to 0$, since $L_g(\theta) = \cosh(\theta) L_f - \sinh(\theta) L_{aux} + g W(\theta)$. So, while all other terms in $L_g(\theta)$ are relatively bounded w.r.t. L_{aux}, their relative bounds are divergent, as $\theta \to 0$. Our main observation, however, is that we only need to control the *imaginary part* of $L_g(\theta)$ and that the imaginary part is asymptotically of the form $\mathrm{Im}\, L_g(\theta) = -i \mathrm{Im}(\theta) \big(L_{aux} + g \widetilde{W}(\theta)$, where $\widetilde{W}(\theta)$ is relatively L_{aux}-bounded with zero relative bound, as $\theta \to 0$. Hence, for sufficiently small $|\theta|$ with $\mathrm{Im}\, \theta \geq 0$, the imaginary part of $L_g(\theta)$ is negative definite. The assumption $\mathrm{Im}\, z \geq 2$ now insures the differentiabilty of $\widetilde{R}_g(\theta, \lambda + 2i)$ at $\theta = 0$.

- Fourth, continuing from $\theta = 0$ to $\theta = i\vartheta$, $0 < \vartheta_0' \leq \vartheta \leq \vartheta_0 < \pi/2$, the norm estimate (A.5) enables us to analytically continue $\widetilde{R}_g(i\vartheta, z)$ in z from $z = \lambda + 2i$ to the connected component $\mathcal{G}$ of
$$\big\{ z \in \mathbb{C} \;\big|\; \|L_g(\theta) - z\| < \infty \,\big\} \tag{A.6}$$
containing $\lambda + 2i$. Since, on $\mathrm{Dom}(L_{aux})$, we can obtain $R_g(i\vartheta, z)$ from $\widetilde{R}_g(i\vartheta, z)$, we arrive at the desired analytic continuation of matrix elements $\langle \varphi | R_g(0, z)\psi \rangle$, for φ, ψ in the dense set $\mathcal{D}$ defined in (V.123).

A.2 Key Step

To carry out the third step indicated above, we first prove some preparatory lemmata.

Lemma A.1. *Let* $0 < \vartheta_0 < \pi/2$, *and assume that* $\theta \in \Sigma_{\vartheta_0}^+$, $\mathrm{Im} z \geq 2$, *and* $0 < g \ll M_{\vartheta_0} := \vartheta_0(\pi/2 - \vartheta_0)$. *Then* $L_g(\theta) - z$ *is invertible and*
$$\big\| \big(L_g(\theta) - z \big)^{-1} \big\| \;\leq\; \mathrm{dist}\{ \mathrm{NumRan}[L_g(\theta)] \,,\, z \}^{-1} \,. \tag{A.7}$$

Proof: It suffices to prove (A.7) only for purely imaginary $\theta = i\vartheta$, $0 \leq \vartheta < \vartheta_0$, since the real part of θ gives rise to a unitary dilatation which leaves norms and numerical ranges and hence both sides of (A.7) unchanged. We observe that, by Cauchy's estimate and (V.116), we have
$$\big\| \partial_\theta w_{\sigma,\tau}^{(\theta)}(k) \big\| \;\leq\; \left(\frac{2}{\pi/2 \,-\, \vartheta_0} \right) \frac{2}{\sqrt{\cos(\vartheta)}} \left(1 \,+\, \frac{1}{\beta\omega(k)} \right)^{1/2} |\vec{k}|^{\mu}\, \kappa(k) \,. \tag{A.8}$$

This and Lemma IV.1 imply that
$$\begin{aligned} \big\| \mathrm{Im}\{W(\theta)\} \, (L_{aux} + 1)^{-1} \big\| \;&=\; \big\| \mathrm{Im}\{W(\theta) - W(0)\} \, (L_{aux} + 1)^{-1} \big\| \\ &=\; \mathcal{O}\big(\, \vartheta / M_{\vartheta_0} \big) \,, \end{aligned} \tag{A.9}$$

and similarly
$$\big\| \mathrm{Re}\{W(\theta)\} \, (L_{aux} + 1)^{-1} \big\| \;=\; \mathcal{O}\big(1/M_{\vartheta_0} \big) \,. \tag{A.10}$$

Next, we note that
$$\begin{aligned} L_g(\theta)^* \;&=\; \cos\vartheta \, L_f \;+\; i \, \sin\vartheta \, L_{aux} \;+\; g W(\theta)^* \qquad\qquad (A.11) \\ &=\; \cos\vartheta \, L_f \;+\; g \mathrm{Re}\{W(\theta)\} \;+\; i \big(\sin\vartheta \, L_{aux} \;+\; g \mathrm{Im}\{W(\theta)^*\} \big) \,. \end{aligned}$$

Thus, for any $\psi \in \mathrm{Dom}[L_{aux}] \subseteq \mathrm{Dom}[L_g(\theta)^*]$, we have

$$\begin{aligned} &\mathrm{Im}\Big\langle (L_{aux}+1)^{-1}\,\psi \;\Big|\; L_g(\theta)^*\,\psi \Big\rangle \qquad \text{(A.12)} \\ &= \frac{1}{2}\Big\langle \psi \;\Big|\; \big[(L_{aux}+1)^{-1},\, g\mathrm{Re}\{W(\theta)\}\big]\,\psi \Big\rangle \;+\; \sin\vartheta \Big\langle \psi \;\Big|\Big(\frac{L_{aux}}{L_{aux}+1}\Big)\,\psi \Big\rangle \\ &\quad + \mathrm{Re}\Big\langle (L_{aux}+1)^{-1}\,\psi \;\Big|\; g\mathrm{Im}\{W(\theta)^* - W(0)\}\,\psi \Big\rangle\,. \end{aligned}$$

As in (IV.74)–(IV.75), we obtain from Lemma IV.1 that

$$\begin{aligned} &\Big| \Big\langle \psi \;\Big|\; \big[(L_{aux}+1)^{-1},\, g\mathrm{Re}\{W(\theta)\}\big]\,\psi \Big\rangle \Big| \qquad \text{(A.13)} \\ &= g \Big| \Big\langle (L_{aux}+1)^{-1}\,\psi \;\Big|\; \big[L_{aux}\,,\, \mathrm{Re}\{W(\theta)\}\big]\,(L_{aux}+1)^{-1}\,\psi \Big\rangle \Big| \\ &\leq C\,g\,\big\langle \psi \;\big|\; (L_{aux}+1)^{-1}\,\psi \big\rangle\,, \end{aligned}$$

whereas (A.9) implies that

$$\Big| \Big\langle (L_{aux}+1)^{-1}\,\psi \;\Big|\; g\mathrm{Im}\{W(\theta)^* - W(0)\}\,\psi \Big\rangle \Big| \;=\; \mathcal{O}\big(g\,\vartheta\,M_{\vartheta_0}^{-1}\big)\,\langle \psi | \psi \rangle\,. \qquad \text{(A.14)}$$

These estimates yield

$$\begin{aligned} &\mathrm{Im}\Big\langle (L_{aux}+1)^{-1}\,\psi \;\Big|\; L_g(\theta)^*\,\psi \Big\rangle \qquad \text{(A.15)} \\ &\geq \sin\vartheta \Big\langle \psi \;\Big|\Big(\frac{L_{aux}}{L_{aux}+1}\Big)\,\psi \Big\rangle \;-\; \mathcal{O}\big(g\,\vartheta\,M_{\vartheta_0}^{-1}\big)\,\langle \psi | \psi \rangle \\ &\qquad\qquad - C\,g\,\big\langle \psi \;\big|\; (L_{aux}+1)^{-1}\,\psi \big\rangle \\ &\geq -(1+Cg)\,\big\langle \psi \;\big|\; (L_{aux}+1)^{-1}\,\psi \big\rangle \;+\; \big(\sin\vartheta \;-\; (g\,\vartheta\,M_{\vartheta_0}^{-1})\big)\,\langle \psi | \psi \rangle \\ &\geq -(1+Cg)\,\big\langle \psi \;\big|\; (L_{aux}+1)^{-1}\,\psi \big\rangle\,, \end{aligned}$$

provided that $g \ll M_{\vartheta_0}$. Note that (A.15) extends to any $\psi \in \mathrm{Dom}[L_g(\theta)^*]$, by continuity.

Assuming now that $\psi \in \mathrm{Ker}\{L_g(\theta)^* - \bar{z}\}$, we derive from Estimate (A.15) the following inequality,

$$\begin{aligned} -\mathrm{Im}(z)\,\big\langle (L_{aux}+1)^{-1}\,\psi \;\big|\; \psi \big\rangle \;&=\; \mathrm{Im}\big\langle (L_{aux}+1)^{-1}\,\psi \;\big|\; \bar{z}\,\psi \big\rangle \qquad \text{(A.16)} \\ &=\; \mathrm{Im}\big\langle (L_{aux}+1)^{-1}\,\psi \;\big|\; L_g(\theta)^*\,\psi \big\rangle \\ &\geq\; -(1+Cg)\,\big\langle \psi \;\big|\; (L_{aux}+1)^{-1}\,\psi \big\rangle\,. \end{aligned}$$

Since $\mathrm{Im}z \geq 2$, this estimate implies that $\psi = 0$, provided $g > 0$ is sufficiently small. Hence, $\mathrm{Ran}\{L_g(\theta) - z\}$ is dense, and we may define an inverse, $\big(L_g(\theta) -$

$z\big)^{-1}$. Furthermore, the density of $\mathrm{Ran}\{L_g(\theta)-z\}$ insures the validity of the second equation in the following numerical range estimate,

$$\begin{aligned}
&\| \big(L_g(\theta)-z\big)^{-1} \| && \text{(A.17)}\\
&= \sup\Big\{ \| \big(L_g(\theta)-z\big)^{-1}\psi \| \;\Big|\; \psi \in \mathrm{Dom}[L_g(\theta)]\,,\ \|\psi\|=1 \Big\} \\
&= \sup\Big\{ \| \big(L_g(\theta)-z\big)\psi \|^{-1} \;\Big|\; \varphi \in \mathrm{Ran}\{L_g(\theta)-z\}\,,\ \|\varphi\|=1 \Big\} \\
&\le \sup\Big\{ |\langle\varphi|L_g(\theta)\varphi\rangle - z|^{-1} \;\Big|\; \varphi \in \mathrm{Ran}\{L_g(\theta)-z\}\,,\ \|\varphi\|=1 \Big\} \\
&= \mathrm{dist}\big\{\mathrm{NumRan}[L_g(\theta)]\,,\, z\big\}^{-1}\,. \qquad \square
\end{aligned}$$

Lemma A.2. *Let* $0<\vartheta_0<\pi/2$, *and assume that* $\theta\in\Sigma^+_{\vartheta_0}$, $\mathrm{Im}z\ge 2$, *and* $0<g\ll M_{\vartheta_0}:=\vartheta_0(\pi/2-\vartheta_0)$. *Then*

$$B_\pm(\theta) \;:=\; (L_{aux}+1)^{\mp 1}\,\big(L_g(\theta)-z\big)^{-1}\,(L_{aux}+1)^{\pm 1} \tag{A.18}$$

is defined on $\mathrm{Dom}[L_{aux}]$ *and extends to a bounded operator on* $\mathcal{H}$ *of norm*

$$\|B_\pm(\theta)\| \;\le\; e^{2|\mathrm{Re}\,\theta|}\,\big(1+\mathcal{O}(g/M_{\vartheta_0})\big)\,\mathrm{dist}\big\{\mathrm{NumRan}[L_g(\theta)]\,,\,z\big\}^{-1}\,. \tag{A.19}$$

Proof: We first notice that $\mathrm{Dom}[B_\pm(\theta)]=\mathrm{Dom}[B_\pm(i\mathrm{Im}\,\theta)]$ and

$$\|B_\pm(\theta)\| \;\le\; e^{2|\mathrm{Re}\,\theta|}\,\|B_\pm(i\mathrm{Im}\,\theta)\|\,. \tag{A.20}$$

Thus, it suffices to prove the assertion for $\theta=i\vartheta$, $0\le\vartheta<\vartheta_0$, which we henceforth assume. Next we observe that

$$\begin{aligned}
&\big(L_g(\theta)-z\big)^{-1}(L_{aux}+1)^{-1} \;-\; (L_{aux}+1)^{-1}\big(L_g(\theta)-z\big)^{-1} && \text{(A.21)}\\
&\quad=\; \big(L_g(\theta)-z\big)^{-1}(L_{aux}+1)^{-1}\,\big[L_{aux}\,,\,L_g(\theta)\big]\,(L_{aux}+1)^{-1}\,\big(L_g(\theta)-z\big)^{-1}\,.
\end{aligned}$$

Indeed, thanks to Lemma A.1 and $\mathrm{Dom}[L_g(\theta)]\supseteq\mathrm{Dom}[L_{aux}]$, both sides in (A.21) define a bounded operator. Note that on $\mathrm{Dom}[L_{aux}]$,

$$\big[L_{aux}\,,\,L_g(\theta)\big] \;=\; g\,\big[L_{aux}\,,\,W(\theta)\big]\,, \tag{A.22}$$

and hence, by the same argument as in (IV.73), (IV.74)–(IV.75), we even have that

$$\Big\|\,(L_{aux}+1)^{-1}\,\big[L_{aux}\,,\,L_g(\theta)\big]\,\Big\| \;=\; \mathcal{O}(g)\,. \tag{A.23}$$

On the other hand, $\mathrm{Im} z \geq 2$ and $g \ll M_{\vartheta_0}$ insures the condition that $\mathrm{dist}\{\mathrm{NumRan}[L_g(\theta)]\, , \, z\} \geq 1$. Hence $\|X\| = \mathcal{O}(g)$, where

$$X \ := \ \big(L_g(\theta) - z\big)^{-1} (L_{aux} + 1)^{-1} \left[L_{aux} \, , \, L_g(\theta)\right] , \tag{A.24}$$

and

$$(\mathbf{1} - X)\,(L_{aux} + 1)^{-1} \big(L_g(\theta) - z\big)^{-1} (L_{aux} + 1)^{-1} \ = \ \big(L_g(\theta) - z\big)^{-1} . \tag{A.25}$$

This implies that $\mathbf{1} - X$ is invertible and that

$$\big\|(\mathbf{1} - X)^{-1}\big\| \ = \ 1 \, + \, \mathcal{O}(g) \, . \tag{A.26}$$

Multiplying (A.25) by $(\mathbf{1} - X)^{-1}$ and using (A.26), we arrive at the assertion for $B_+(i\vartheta)$. The proof for $B_-(i\vartheta)$ is similar. □

Putting together Lemma A.1 and Lemma A.2, we arrive at

Theorem A.3. *Let* $0 < \vartheta_0 < \pi/2$ *and* $0 < g \ll M_{\vartheta_0} := \vartheta_0(\pi/2 - \vartheta_0)$. *Then, for all* $z \in \{z \in \mathbb{C} \, | \, \mathrm{Im} z \geq 2\}$,

$$\begin{aligned} \widetilde{R}(\,\cdot\,, z) \ \ : \ \ \Sigma_{\vartheta_0}^{+} \ & \to \ \ \mathcal{B}[\mathcal{H} \otimes \mathcal{H}] \, , \\ \theta \ & \mapsto \ \ \widetilde{R}(\theta, z) \ = \ (L_{aux} + 1)^{-1} \big(L_g(\theta) - z\big)^{-1} (L_{aux} + 1)^{-1} \end{aligned} \tag{A.27}$$

is analytic, i.e., $\theta \mapsto L_g(\theta)$ *defines an analytic family on* $\Sigma_{\vartheta_0}^{+}$ *in the sense of Kato.*

Proof: We start with the observation that

$$\partial_\theta L_g(\theta) \ = \ -\mathbf{1}^{el} \otimes \Big(e^{-\theta} H_f \otimes \mathbf{1}_f + e^{\theta} \mathbf{1}_f \otimes H_f\Big) \ + \ g \, \partial_\theta W(\theta) \, , \tag{A.28}$$

where (see (V.118)

$$\begin{aligned} \partial_\theta W(\theta) \ \ = \ \ & a_\ell^*(\partial_\theta w_{+,\ell}^{(\theta)}) \, + \, a_\ell(\partial_\theta w_{-,\ell}^{(\theta)}) \\ & + a_r^*(\partial_\theta w_{+,r}^{(\theta)}) \, + \, a_r(\partial_\theta w_{-,r}^{(\theta)}) \, . \end{aligned} \tag{A.29}$$

By Lemma IV.1 and (A.8), we clearly have

$$\Big\| \, \partial_\theta W(\theta) \, (L_{aux} + 1)^{-1} \, \Big\| \ = \ \mathcal{O}\big(e^{|\mathrm{Re}\,\theta|} \, M_{\vartheta_0}^{-1}\big) \, , \tag{A.30}$$

and hence

$$\Big\| \, \partial_\theta L_g(\theta) \, (L_{aux} + 1)^{-1} \, \Big\| \ = \ \mathcal{O}\Big(e^{2|\mathrm{Re}\,\theta|} \, \big(1 + g \, M_{\vartheta_0}^{-1}\big)\Big) \, . \tag{A.31}$$

Using Lemmata A.1, A.2, and (A.31), we thus obtain

$$\begin{aligned}
&\Big\|(L_{aux}+1)^{-1}\,\partial_\theta\big\{\big(L_g(\theta)-z\big)^{-1}\big\}\,(L_{aux}+1)^{-1}\Big\| \qquad (A.32)\\
&= \ \Big\|(L_{aux}+1)^{-1}\,\big(L_g(\theta)-z\big)^{-1}\,\partial_\theta L_g(\theta)\,\big(L_g(\theta)-z\big)^{-1}\,(L_{aux}+1)^{-1}\Big\|\\
&= \ \Big\|B_+(\theta)\,(L_{aux}+1)^{-1}\,\partial_\theta L_g(\theta)\,(L_{aux}+1)^{-1}\,B_-(\theta)\Big\|\\
&\le \ C\,\mathrm{dist}\big\{\mathrm{NumRan}[L_g(\theta)]\,,\,z\big\}^{-2} \ \le \ C' \ < \ \infty\,,
\end{aligned}$$

for some constants $C, C' = \mathcal{O}\big(e^{4|\mathrm{Re}\,\theta|}\big) < \infty$. □

Theorem A.4. *Let* $0 < \vartheta_0' < \vartheta_0 < \pi/2$, *and assume that* $\mathrm{Im} z \ge 2$, $0 < g \ll \vartheta_0'(\pi/2-\vartheta_0)$, *and* $\theta \in \Sigma^+_{\vartheta_0} \cap \{\mathrm{Im}\theta \ge \vartheta_0'\}$. *Define* $R'(\theta,z) := (L_{aux}+1)\widetilde{R}(\theta,z)(L_{aux}+1)$ *on* $\mathrm{Dom}[L_{aux}]$. *Then*

(i) $R'(\theta,z)$ *extends to a bounded operator on* $\mathcal{H}\otimes\mathcal{H}$.

(ii) $L_g(\theta)-z$ *is invertible on* $\mathcal{H}\otimes\mathcal{H}$, *and its inverse,* $R(\theta,z) := (L_g(\theta)-z)^{-1}$, *is given the extension of* $R'(\theta,z)$.

(iii) *On the rectangular domain* $R := \big\{\theta\in\mathbb{C}\,\big|\ |\mathrm{Re}\,\theta| \le \alpha\,,\ \vartheta_0' \le \mathrm{Im}\,\theta \le \vartheta_0\big\}$, *where* $\alpha := -\frac{1}{2}\ln\cos(2\vartheta_0') > 0$, *the map*

$$R \ni \theta \ \mapsto \ L_g(\theta) \ \in \ \mathcal{B}\big[\mathrm{Dom}[L_{aux}]\,,\,\mathcal{H}\otimes\mathcal{H}\big] \qquad (A.33)$$

defines an analytic family of type A.

(iv) $\sigma[L_g(\theta)] \ \subseteq \ \overline{\mathbb{C}^-} \cap (\mathbb{C}\setminus\mathcal{G}')$, *where* $\mathcal{G}'$ *is the connected component of* $\{z\in\mathbb{C}|\ \|R(\theta,z)\|<\infty\}$ *containing* $\mathbb{C}^+$.

Proof: Statements (i) and (ii) easily follow from Lemmata A.1, A.2, and Theorem A.3.
For (iii), we set $\alpha := \mathrm{Re}\,\theta$, $\vartheta := \mathrm{Im}\,\theta$, and we observe that

$$L_f(\theta) \ = \ \cosh(\theta)\,L_f \ - \ \sinh(\theta)\,L_{aux} \ = \ -\sinh(\theta)\big[L_{aux} \ - \ \coth(\theta)\,L_f\big]\,. \qquad (A.34)$$

Since $\sinh\theta = \sinh\alpha\,\cos\vartheta \,+\, i\,\cosh\alpha\,\sin\vartheta$ and $\vartheta \ge \vartheta_0' > 0$, we have that

$$|\sinh\theta| \ \ge \ |\mathrm{Im}\sinh\theta| \ \ge \ \sin\vartheta \ \ge \ \sin\vartheta_0' \ > \ 0\,. \qquad (A.35)$$

Next, we observe that, for $\psi\in\mathrm{Dom}[L_{aux}^{1/2}]$,

$$\big|\big\langle\psi\,\big|\,L_f(\theta)\,\psi\big\rangle\big| \ \le \ |\sinh\theta|\,\big(1+|\coth\theta|\big)\,\big\langle\psi\,\big|\,L_{aux}\,\psi\big\rangle\,, \qquad (A.36)$$

because $\pm L_f \le L_{aux}$. To construct a lower bound, we use

$$\begin{aligned} \big| \langle \psi \,|\, L_f(\theta)\, \psi \rangle \big| \;&\le\; |\sinh\theta|\, \big|\mathrm{Re}\langle \psi \,|\, \big(L_{aux} \,-\, \coth(\theta) L_f\big)\, \psi \rangle \\ &\le\; |\sinh\theta|\, \big(1 - \mathrm{Re}\{\coth(\theta)\}\big)\, \langle \psi \,|\, L_{aux}\, \psi \rangle \,. \end{aligned} \tag{A.37}$$

To have a nontrivial lower bound, we therefore require that

$$\mathrm{Re}\big\{\coth\theta\big\} \;=\; \frac{\sinh(2\alpha)}{\cosh(2\alpha) \,-\, \cos(2\vartheta)} \;<\; 1\,, \tag{A.38}$$

or, equivalently, $\cos(2\vartheta) < \cosh(2\alpha) - |\sinh(2\alpha)|$. By our assumption on ϑ_0', we have that $\cos(2\vartheta) < \cosh(2\alpha) - |\sinh(2\alpha)|$, and (A.38) holds. Thus, for all $\theta \in D(0, \vartheta_0) \cap \{\mathrm{Im} z \ge \vartheta_0'\}$, the quadratic form $L_f(\theta)$ is sectorial, and $\mathrm{Dom}_{\mathcal{Q}}[L_f(\theta)] = \mathrm{Dom}_{\mathcal{Q}}[L_{aux}]$. Since $W(\theta)$ is a relatively bounded form perturbation w.r.t. L_{aux}, with zero relative bound, so it is with respect $L_f(\theta)$ and hence $L_0(\theta)$. This proves (iii).

To prove (iv), we observe that, by analytic continuation, $L_g(\theta) - z$ is invertible in $\mathbb{C}^+$, since $L_0(\theta) - z$ is. $\square$

B Positivity of the Level-Shift Operator Γ_E

B.1 Definitions

Recall from Hypothesis H-1 that the spectrum of H_{el} is assumed to entirely consist of simple eigenvalues $E_m < E_{m+1}$ with corresponding orthonormal eigenvectors φ_m, $m \in \mathcal{N} := \{0, 1, \ldots, N-1\}$. Thus $\{\varphi_i | i \in \mathcal{N}\} \subseteq \mathcal{H}_{el}$ is an orthonormal basis. We also have

$$L_{el} \;:=\; H_{el} \otimes \mathbf{1}_{el} - \mathbf{1}_{el} \otimes H_f\,, \tag{B.1}$$

$$L_f \;:=\; H_f \otimes \mathbf{1}_f - \mathbf{1}_f \otimes H_f\,, \tag{B.2}$$

$$L_0 \;:=\; L_{el} \otimes \mathbf{1}^f + \mathbf{1}^{el} \otimes L_f\,, \tag{B.3}$$

where $\mathbf{1}^{el} := \mathbf{1}_{el} \otimes \mathbf{1}_{el}$ and $\mathbf{1}^f := \mathbf{1}_f \otimes \mathbf{1}_f$. Henceforth, we frequently omit trivial tensor factors, like $\otimes \mathbf{1}$, unless they clarify the exposition.

Recall from (IV.62) that the interaction in the Liouvillian at inverse temperature β in the Araki-Woods representation is given by

$$\begin{aligned} W \;=\; & a_\ell^*\big(\sqrt{1+\rho}\, G_\ell \,-\, \sqrt{\rho}\, \overline{G_r^*}\big) \,+\, a_\ell\big(\sqrt{1+\rho}\, G_\ell^* \,-\, \sqrt{\rho}\, \overline{G_r}\big) \\ & + a_r^*\big(\sqrt{\rho}\, G_\ell^* \,-\, \sqrt{1+\rho}\, \overline{G_r}\big) \,+\, a_r\big(\sqrt{\rho} G_\ell \,-\, \sqrt{1+\rho}\, \overline{G_r^*}\big)\,, \end{aligned} \tag{B.4}$$

where the coupling function $G : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{B}(\mathcal{H}_{el})$ is a matrix-valued function and

$$\rho \;\equiv\; \rho(\xi) \;=\; \big(e^{\beta\omega(\xi)} \,-\, 1\big)^{-1} . \tag{B.5}$$

We further recall from (V.152)–(V.153) that, for $E \in \mathbb{R}$, the imaginary part Γ_E of the level shift operator Q_E is given by

$$\Gamma_E \otimes P_\Omega \;:=\; P_\Omega\, W\, P_\Omega^\perp\, \delta\big[L_0 P_\Omega^\perp \,-\, E\big]\, P_\Omega^\perp\, W\, P_\Omega\,, \tag{B.6}$$

where, P_Ω is the orthogonal projection onto the vacuum vector $\Omega^f := \Omega_f \otimes \Omega_f$ in $\mathcal{F}_f \otimes \mathcal{F}_f$.

Applying the modular conjugation $J = E(C \otimes C \otimes T \otimes T)$ (see Eq. (IV.77)), we see that $J(\Gamma_E \otimes P_\Omega)J = \Gamma_{-E} \otimes P_\Omega$. Introducing the corresponding restriction, $J_{el} = E_{el}(C \otimes C)$, of J to $\mathcal{H}_{el} \otimes \mathcal{H}_{el}$, where $E_{el}(\varphi \times \psi) := \psi \otimes \varphi$ is the corresponding exchange operator, we find that Γ_E is anti-unitarily equivalent to Γ_{-E},

$$J_{el}\, \Gamma_E\, J_{el} \;=\; \Gamma_{-E}\,. \tag{B.7}$$

In particular, $J_{el}\Gamma_0 J_{el} = \Gamma_0$.

For any Borel set $\Delta \subseteq \mathbb{R}$, we denote $\chi_\Delta := \chi_\Delta(L_{el})$ and further

$$\begin{aligned}
\mathcal{A}_\Delta \;&:=\; \big\{(i,j) \in \mathcal{N}^2 \,\big|\; E_{i,j} \in \Delta\big\}\,, && \text{(B.8)}\\
\mathcal{A}_\Delta^{(1)} \;&:=\; \big\{i \in \mathcal{N} \,\big|\; \exists j \in \mathcal{N} :\; E_{i,j} \in \Delta\big\}\,, && \text{(B.9)}\\
\mathcal{A}_\Delta^{(2)} \;&:=\; \big\{j \in \mathcal{N} \,\big|\; \exists i \in \mathcal{N} :\; E_{i,j} \in \Delta\big\}\,, && \text{(B.10)}\\
[\mathcal{A}_\Delta^{(\nu)}]^c \;&:=\; \mathcal{N} \setminus \mathcal{A}_\Delta^{(\nu)}\,, \quad \nu = 1,2\,. && \text{(B.11)}
\end{aligned}$$

Note that $\mathcal{A}_\Delta^{(2)} = \mathcal{A}_{-\Delta}^{(1)}$ and that $[\mathcal{A}_\Delta^{(1)}]^c = [\mathcal{A}_\Delta^{(2)}]^c = \emptyset$ if $\Delta \ni 0$. Moreover,

$$\begin{aligned}
\Delta = \{E_{i,j}\} \subseteq (0,\infty) \quad &\Longrightarrow \quad \mathcal{A}_\Delta^{(1)} \ni i\,,\;\; [\mathcal{A}_\Delta^{(1)}]^c \ni 0\,, && \text{(B.12)}\\
\Delta = \{E_{i,j}\} \subseteq (-\infty,0) \quad &\Longrightarrow \quad \mathcal{A}_\Delta^{(1)} \ni i\,,\;\; [\mathcal{A}_\Delta^{(1)}]^c \ni N-1\,. && \text{(B.13)}
\end{aligned}$$

Furthermore, we denote the corresponding canonical projections by

$$\begin{aligned}
p^{(1)} \;&:\; \mathcal{A}_\Delta \to \mathcal{A}_\Delta^{(1)}\,, \quad (i,j) \mapsto i\,, && \text{(B.14)}\\
p^{(2)} \;&:\; \mathcal{A}_\Delta \to \mathcal{A}_\Delta^{(2)}\,, \quad (i,j) \mapsto j\,, && \text{(B.15)}
\end{aligned}$$

and we call a Borel set $\Delta \subseteq \mathbb{R}$ *nondegenerate* iff $p^{(1)}$ or $p^{(2)}$ is bijective. (Note that $p^{(1)}$ is bijective iff $p^{(2)}$ is.) The assumption of simplicity of the eigenvalues $E_m < E_{m+1}$ implies that $\{E\}$ is nondegenerate, for any $E \in \mathbb{R}$. For any nondegenerate $\Delta \subseteq \mathbb{R}$, there exists a bijection $N \equiv N_\Delta : \mathcal{A}_\Delta^{(1)} \to \mathcal{A}_\Delta^{(2)}$ such that

$$\mathcal{A}_\Delta \;=\; \big\{ (i, N_\Delta(i)) \;\big|\; i \in \mathcal{A}_\Delta^{(1)} \big\} \,. \tag{B.16}$$

The case $\Delta = \{E_{i,j}\}$ and $E = E_{i,j}$ is of special interest, and we denote

$$\Gamma^{(E_{i,j})} \;:=\; \Gamma_{i,j}\Big|_{\mathrm{Ran}\, P^{el}_{E_{i,j}}} \,, \tag{B.17}$$

recalling that $P^{el}_{i,j} = \chi_{\{E_{i,j}\}}[L_{el}]$.

We conclude this section with a computation of the matrix elements of Γ_E. To this end we abbreviate $\varphi_{i,j} := \varphi_i \otimes \varphi_j$, $E_{i,j} := E_i - E_j$, and $G_{i,j} \equiv G_{i,j}(\xi)$, $\xi := (\vec{\xi}, \lambda)$. We obtain

$$\begin{aligned}
&\langle \varphi_{i,j} |\, \Gamma_E \, \varphi_{k,\ell} \rangle \\
&\quad= \big\langle \varphi_{i,j} \otimes \Omega^f \big|\, a_\ell \big(\sqrt{1+\rho}\, G^* \otimes \mathbf{1}_{el} \;-\; \sqrt{\rho}\, \mathbf{1}_{el} \otimes \overline{G} \big)\, P_\Omega^\perp \delta\big[L_0 P_\Omega^\perp - E \big]\, P_\Omega^\perp \\
&\qquad\qquad a_\ell^* \big(\sqrt{1+\rho}\, G \otimes \mathbf{1}_{el} \;-\; \sqrt{\rho}\, \mathbf{1}_{el} \otimes \overline{G}^* \big) \, \big(\varphi_{k,\ell} \otimes \Omega^f \big) \big\rangle \\
&\qquad + \big\langle \varphi_{i,j} \otimes \Omega^f \big|\, a_r \big(\sqrt{\rho}\, G \otimes \mathbf{1}_{el} \;-\; \sqrt{1+\rho}\, \mathbf{1}_{el} \otimes \overline{G}^* \big)\, P_\Omega^\perp \delta\big[L_0 P_\Omega^\perp - E \big]\, P_\Omega^\perp \\
&\qquad\qquad a_r^* \big(\sqrt{\rho}\, G^* \otimes \mathbf{1}_{el} \;-\; \sqrt{1+\rho}\, \mathbf{1}_{el} \otimes \overline{G} \big) \, \big(\varphi_{k,\ell} \otimes \Omega^f \big) \big\rangle \\
&\quad= \int d\xi \Big\{ \big\langle \varphi_{i,j} \big|\, \big(\sqrt{1+\rho}\, G^* \otimes \mathbf{1}_{el} \;-\; \sqrt{\rho}\, \mathbf{1}_{el} \otimes \overline{G} \big)\, \delta\big[L_{el} - E + \omega(\xi) \big] \\
&\qquad\qquad \big(\sqrt{1+\rho}\, G \otimes \mathbf{1}_{el} \;-\; \sqrt{\rho}\, \mathbf{1}_{el} \otimes \overline{G}^* \big)\, \varphi_{k,\ell} \big\rangle \\
&\qquad + \big\langle \varphi_{i,j} \otimes \Omega^f \big|\, \big(\sqrt{\rho}\, G \otimes \mathbf{1}_{el} \;-\; \sqrt{1+\rho}\, \mathbf{1}_{el} \otimes \overline{G}^* \big)\, \delta\big[L_{el} - E - \omega(\xi) \big] \\
&\qquad\qquad \big(\sqrt{\rho}\, G^* \otimes \mathbf{1}_{el} \;-\; \sqrt{1+\rho}\, \mathbf{1}_{el} \otimes \overline{G} \big)\, \varphi_{k,\ell} \big\rangle \Big\} \,,
\end{aligned} \tag{B.18}$$

and hence

$$\begin{aligned}
&\langle \varphi_{i,j} |\, \Gamma_E \, \varphi_{k,\ell} \rangle \;=\; \int d\xi \sum_{m,n=0}^{N-1} \\
&\quad \Big\{ \delta\big[E_{m,n} - E + \omega(\xi) \big]\, \big\langle \varphi_{i,j} \big|\, \big(\sqrt{1+\rho}\, G^* \otimes \mathbf{1}_{el} \;-\; \sqrt{\rho}\, \mathbf{1}_{el} \otimes \overline{G} \big)\, \varphi_{m,n} \big\rangle \\
&\qquad\qquad \big\langle \varphi_{m,n} \big|\, \big(\sqrt{1+\rho}\, G \otimes \mathbf{1}_{el} \;-\; \sqrt{\rho}\, \mathbf{1}_{el} \otimes \overline{G}^* \big)\, \varphi_{m,n} \big\rangle \\
&\quad + \delta\big[E_{m,n} - E - \omega(\xi) \big]\, \big\langle \varphi_{i,j} \big|\, \big(\sqrt{\rho}\, G \otimes \mathbf{1}_{el} \;-\; \sqrt{1+\rho}\, \mathbf{1}_{el} \otimes \overline{G}^* \big)\, \varphi_{k,\ell} \big\rangle \\
&\qquad\qquad \big\langle \varphi_{m,n} \big|\, \big(\sqrt{\rho}\, G^* \otimes \mathbf{1}_{el} \;-\; \sqrt{1+\rho}\, \mathbf{1}_{el} \otimes \overline{G} \big)\, \varphi_{k,\ell} \big\rangle \Big\} \,.
\end{aligned} \tag{B.19}$$

Hence, the final result is

$$\begin{aligned} &\langle \varphi_{i,j} |\, \Gamma_E \, \varphi_{k,\ell} \rangle \qquad\qquad (B.20) \\ &= \int d\xi \sum_{m,n=0}^{N-1} \Big\{ \delta\big[E_{m,n} - E + \omega(\xi)\big] \, \big(\sqrt{1+\rho}\, \overline{G}_{m,i} \, \delta_{j,n} \; - \; \sqrt{\rho}\, \delta_{i,m} \, \overline{G}_{j,n}\big) \\ &\qquad\qquad \big(\sqrt{1+\rho}\, G_{m,k} \, \delta_{n,\ell} \; - \; \sqrt{\rho}\, \delta_{m,k} \, G_{\ell,n}\big) \\ &\quad + \delta\big[E_{m,n} - E - \omega(\xi)\big] \, \big(\sqrt{\rho}\, G_{i,m} \, \delta_{j,n} \; - \; \sqrt{1+\rho}\, \delta_{i,m} \, G_{n,j}\big) \\ &\qquad\qquad \big(\sqrt{\rho}\, \overline{G}_{k,m} \, \delta_{n,\ell} \; - \; \sqrt{1+\rho}\, \delta_{m,k} \, \overline{G}_{n,\ell}\big) \Big\} \, . \end{aligned}$$

In particular, for $E = 0$, $i = j$, and $k = \ell$, we have

$$\langle \varphi_{i,i} |\, \Gamma_0 \, \varphi_{k,k} \rangle \; = \; \delta_{i,k} \Big(\sum_{j \neq k} \eta_{k,j} \, e^{-\beta E_{k,j}/2} \Big) \; - \; (1 - \delta_{i,k}) \, \eta_{i,k} \; , \tag{B.21}$$

where

$$\eta_{i,k} \; := \; 2 \int \Big(\delta\big[\omega(\xi) - E_{i,k}\big] + \delta\big[\omega(\xi) + E_{i,k}\big] \Big) \, \sqrt{\rho(\xi)(1+\rho(\xi))} \, \big|G_{i,k}(\xi)\big|^2 \, d\xi \, . \tag{B.22}$$

Formula (B.21) is equivalent to the detailed-balance equation (V.50). To derive (B.21), we use

$$\frac{1+\rho(\xi)}{\rho(\xi)} \; = \; 1 + \frac{1}{\rho(\xi)} \; = \; e^{\beta \, \omega(\xi)/2} \, , \tag{B.23}$$

which implies the following two identities,

$$\delta\big[\omega(\xi) - E_{i,k}\big] \, \big(1 + \rho(\xi)\big) \; = \; \delta\big[\omega(\xi) - E_{i,k}\big] \, \sqrt{\rho(\xi)(1+\rho(\xi)} \; e^{\beta E_{i,k}/2} \, , \tag{B.24}$$

$$\delta\big[\omega(\xi) + E_{i,k}\big] \, \rho(\xi) \; = \; \delta\big[\omega(\xi) + E_{i,k}\big] \, \sqrt{\rho(\xi)(1+\rho(\xi)} \; e^{\beta E_{i,k}/2} \, . \tag{B.25}$$

B.2 Strict Positivity of Γ_E for $E \neq 0$

Our next task is to show that under certain assumptions $\chi_\Delta \Gamma_E \chi_\Delta \geq \alpha \chi_\Delta$, for some $\alpha > 0$ and all $E \neq 0$. Since Γ_E is anti-unitarily equivalent to Γ_{-E}, we may restrict ourselves to considering $E > 0$.

Theorem B.1. *Assume Hypotheses H-1 and H-5. Let* $E > 0$ *and* $\Delta \subseteq \mathbb{R}$ *be a nondegenerate Borel set. Then*

$$\chi_\Delta \, \Gamma_E \, \chi_\Delta \; \geq \; \gamma_E(\Delta) \, \chi_\Delta \; , \tag{B.26}$$

where the number $\gamma_E(\Delta) \geq 0$ *is defined by*

$$\begin{aligned}\gamma_E(\Delta) \;&:=\; \min_{\tilde m \in \mathcal{A}_\Delta^{(1)}} \int d\xi \sum_{m \in [\mathcal{A}_\Delta^{(1)}]^c} \Big\{ \delta\big[E_{m,N(\tilde m)} - E + \omega(\xi)\big] \, \big|G_{m,\tilde m}(\xi)\big|^2 \\ &\qquad + \delta\big[E_{\tilde m,N(m)} - E - \omega(\xi)\big] \, \big|G_{N(\tilde m),N(m)}(\xi)\big|^2 \Big\} \,. \qquad \text{(B.27)}\end{aligned}$$

Furthermore, for $E_{i,j} \neq 0$*,*

$$\Gamma^{(E_{i,j})} \;\geq\; \gamma^{(i,j)} \;:=\; \gamma_{E_{i,j}}(\{E_{i,j}\}) \;>\; 0 \,, \qquad \text{(B.28)}$$

uniformly in $\beta \to \infty$*.*

Proof. Since Δ is nondegenerate, $\{\varphi_{i,N(i)} | i \in \mathcal{A}_\Delta^{(1)}\}$ is an ONB in $\mathrm{Ran}\chi_\Delta(L_{el})$. We may therefore write any normalized vector in $\Phi \in \mathrm{Ran}\chi_\Delta(L_{el})$ as $\Phi = \sum_{i\in\mathcal{N}} c_i \, \varphi_{i,N(i)}$, where $\sum_{i\in\mathcal{N}} |c_i|^2 = 1$ and $c_i = 0$, for all $i \in [\mathcal{A}_\Delta^{(2)}]^c$. Inserting this into (B.20) yields

$$\begin{aligned}\langle\Phi|\,\Gamma_E\,\Phi\rangle \;&=\; \int d\xi \sum_{m,n=0}^{N-1} \Big\{ \qquad \text{(B.29)} \\ &\delta\big[E_{m,n} - E + \omega(\xi)\big] \, \big|\sqrt{1+\rho}\, G_{m,N^{-1}(n)} \, c_{N^{-1}(n)} \;-\; \sqrt{\rho}\, G_{N(m),n} \, c_m\big|^2 \\ &+ \delta\big[E_{m,n} - E - \omega(\xi)\big] \, \big|\sqrt{\rho}\, G_{N^{-1}(n),m} \, \bar c_{N^{-1}(n)} \;-\; \sqrt{1+\rho}\, G_{n,N(m)} \, \bar c_m\big|^2 \Big\} \,.\end{aligned}$$

Now we observe that the range $\mathcal{N}^2$ of summation contains the following two disjoint subsets:

$$\mathcal{A}_\Delta^{(1)} \times [\mathcal{A}_\Delta^{(2)}]^c \,, \qquad [\mathcal{A}_\Delta^{(1)}]^c \times \mathcal{A}_\Delta^{(2)} \,. \qquad \text{(B.30)}$$

Using this and the fact that c_i vanishes for $i \in [\mathcal{A}_\Delta^{(1)}]^c$, we obtain the following lower bound,

$$\langle\Phi|\,\Gamma_E\,\Phi\rangle \;\geq\; \gamma' \,+\, \gamma'' \,, \qquad \text{(B.31)}$$

where

$$\begin{aligned}\gamma' \;&:=\; \int d\xi \sum_{m \in \mathcal{A}_\Delta^{(1)}} |c_m|^2 \sum_{n \in [\mathcal{A}_\Delta^{(2)}]^c} \Big\{ \delta\big[E_{m,n} - E + \omega(\xi)\big] \, \big|\sqrt{\rho}\, G_{N(m),n}\big|^2 \\ &\qquad + \delta\big[E_{m,n} - E - \omega(\xi)\big] \, \big|\sqrt{1+\rho}\, G_{N(m),n}\big|^2 \Big\} \qquad \text{(B.32)} \\ &:=\; \int d\xi \sum_{\tilde m \in \mathcal{A}_\Delta^{(1)}} |c_{\tilde m}|^2 \sum_{m \in [\mathcal{A}_\Delta^{(1)}]^c} \Big\{ \delta\big[E_{\tilde m,N(m)} - E + \omega(\xi)\big] \, \rho \\ &\qquad + \delta\big[E_{\tilde m,N(m)} - E - \omega(\xi)\big] \, (1+\rho) \Big\} \, \big|G_{N(\tilde m),N(m)}\big|^2 \,, \qquad \text{(B.33)}\end{aligned}$$

and similarly

$$\gamma'' := \int d\xi \sum_{\tilde m \in \mathcal{A}_\Delta^{(1)}} |c_{\tilde m}|^2 \sum_{m \in [\mathcal{A}_\Delta^{(1)}]^c} \Big\{ \delta\big[E_{m,N(\tilde m)} - E + \omega(\xi)\big] (1+\rho) \\ + \delta\big[E_{m,N(\tilde m)} - E - \omega(\xi)\big]\, \rho \Big\} \big|G_{m,\tilde m}\big|^2 . \tag{B.34}$$

Adding γ' and γ'', we arrive at the first assertion, Eq. (B.27),

$$\gamma' + \gamma'' \geq \sum_{\tilde m \in \mathcal{A}_\Delta^{(1)}} |c_{\tilde m}|^2 \int d\xi \sum_{m \in [\mathcal{A}_\Delta^{(1)}]^c} \Big\{ \delta\big[E_{m,N(\tilde m)} - E + \omega(\xi)\big] \big|G_{m,\tilde m}\big|^2 \\ + \delta\big[E_{\tilde m,N(m)} - E - \omega(\xi)\big] \big|G_{N(\tilde m),N(m)}\big|^2 \Big\} . \tag{B.35}$$

To prove (B.28), we assume that $\Delta = \{E_{i,j}\}$ and $E = E_{i,j} > 0$. Then $i \in \mathcal{A}_\Delta^{(1)}$ and $0 \in [\mathcal{A}_\Delta^{(1)}]^c$, so retaining only the term $m = 0$ in (B.27), we find that

$$\gamma^{(i,j)} \geq \min_{\tilde m \in \mathcal{A}_\Delta^{(1)}} \int d\xi \; \delta\big[E_{0,N(\tilde m)} - E_{i,j} + \omega(\xi)\big] \big|G_{m,\tilde m}(\xi)\big|^2 . \tag{B.36}$$

Since $E_{0,N(\tilde m)} \leq 0$, we have that $\tau_{min} \leq E_{i,j} - E_{0,N(\tilde m)} \leq 2\tau_{max}$, and Hypothesis H-5 directly yields $\gamma^{(i,j)} > 0$. □

B.3 Spectral Gap above 0 for $\Gamma^{(0)}$

In this section we assume that $\Delta = \{0\}$ and that $E = 0$. Furthermore, we shall make use of Hypothesis H-3, in addition to Hypotheses H-1 and H-5. Note that since the eigenvalues of H_{el} are nondegenerate, $\{0\}$ is nondegenerate and symmetric. Moreover $\Gamma^{(0)} = \Gamma_0\big|_{\mathrm{Ran}\, P_0^{el}}$ commutes with J_{el}.

Theorem B.2. *The infimum of the spectrum of* $\Gamma^{(0)}$ *is a simple eigenvalue and equals* 0, *with eigenvector* $\kappa_\beta = \sum_{m=0}^{N-1} e^{-\beta E_m/2} \varphi_{i,i}$. *Furthermore, the gap above* 0 *can be estimated by*

$$0 < \widehat{\gamma}_0 \leq \min\Big\{ \sigma\big[\Gamma^{(0)}\big] \setminus \{0\} \Big\} \tag{B.37}$$

where

$$\widehat{\gamma}_0 := 2 \left(\sum_{m=0}^{N-1} e^{-\beta E_m} \right) \min_{0 \leq m < n \leq N-1} \big\{ \eta_{m,n}\, e^{\beta(E_m+E_n)/2} \big\} , \tag{B.38}$$

and $\eta_{m,n} > 0$ *is defined in (B.21).*

Remark B.3. *Note that*

$$\liminf_{\beta\to\infty} \widehat{\gamma}_0 \;\geq\; \min_{0\leq m<n\leq N-1} \left\{ \int d\xi \, \big|G_{m,n}(\xi)\big|^2 \, \delta\big[E_{m,n} + \omega(\xi)\big] \right\}, \tag{B.39}$$

so $\widehat{\gamma}_0 > 0$, uniformly in $\beta \to \infty$, thanks to Hypothesis H-5.

Proof. Since $\{0\}$ is nondegenerate, $\{\varphi_{i,i} | i \in \mathcal{N}\}$ is an orthonormal basis in $\mathrm{Ran}\, P_0^{el}$, and we may write any normalized vector $\kappa \in \mathrm{Ran}\, P_0^{el}$ as $\kappa = \sum_{i\in\mathcal{N}} \kappa_i \, \varphi_{i,i}$, where $\sum_{i\in\mathcal{N}} |\kappa_i|^2 = 1$. Inserting this into (B.21) and denoting $\mu_{m,n} := \eta_{m,n} \, e^{\beta(E_m+E_n)/2}$, we obtain

$$\begin{aligned}
\langle \kappa | \, \Gamma_0 \, \kappa \rangle \;&=\; \sum_{m,n=0}^{N-1} \eta_{m,n} \Big(e^{-\beta E_{m,n}/2} \, |\kappa_n|^2 \;-\; \overline{\kappa_m} \, \kappa_n \Big) \\
&=\; \sum_{m,n=0}^{N-1} \mu_{m,n} \Big(e^{-\beta E_m} \, |\kappa_n|^2 \;-\; \overline{e^{-\beta E_n/2} \, \kappa_m} \, e^{-\beta E_m/2} \, \kappa_n \Big) \\
&=\; \sum_{m,n=0}^{N-1} \mu_{m,n} \Big| e^{-\beta E_m/2} \, \kappa_n \;-\; e^{-\beta E_n/2} \, \kappa_m \Big|^2 \\
&=\; 2 \sum_{0\leq m<n}^{N-1} \mu_{m,n} \Big| e^{-\beta E_m/2} \, \kappa_n \;-\; e^{-\beta E_n/2} \, \kappa_m \Big|^2 .
\end{aligned} \tag{B.40}$$

From Eq. (B.40) it is obvious that

$$\langle \kappa_\beta | \, \Gamma_0 \, \kappa_\beta \rangle \;=\; 0 . \tag{B.41}$$

Moreover, we obtain the lower bound

$$\begin{aligned}
\langle \kappa | \, \Gamma_0 \, \kappa \rangle \;&\geq\; 2 \, G' \sum_{0\leq m<n}^{N-1} \big| e^{-\beta E_m/2} \, \kappa_n \;-\; e^{-\beta E_n/2} \, \kappa_m \big|^2 \\
&=\; G' \sum_{m,n=0}^{N-1} \big| e^{-\beta E_m/2} \, \kappa_n \;-\; e^{-\beta E_n/2} \, \kappa_m \big|^2 \\
&=\; 2 \, G' \Big(Z' \;-\; \big| \langle \kappa_\beta | \kappa \rangle \big|^2 \Big) ,
\end{aligned} \tag{B.42}$$

where $Z' := \sum_{m=0}^{N-1} e^{-\beta E_m} = \langle \kappa_\beta | \kappa_\beta \rangle$ and $G' := \min_{0\leq m<n\leq N-1}\{\mu_{m,n}\}$. Hence,

$$\langle \kappa | \, \Gamma_0 \, \kappa \rangle \;\geq\; 2 \, G' \, Z' , \tag{B.43}$$

whenever $\kappa \perp \kappa_\beta$ and $\|\kappa\| = 1$. $\square$

Acknowledgements. We thank W. Aschbacher, J.-M. Barbaroux, H. Schulz-Baldes, and T. Spencer for helpful discussions.

References

[1] H. Araki. Relative Hamiltonian for faithful normal states of a von Neumann algebra. *Pub. R.I.M.S., Kyoto Univ.*, 9:165–209, 1973.

[2] H. Araki. Positive cone, Radon-Nikodym theorems, relative Hamiltonians and the Gibbs condition in statistical mechanics. In Kastler D, editor, *C^*-Algebras and their Applications to Statistical Mechanics and Quantum Field Theory*, Bologna, 1975. Editrice Comp.

[3] H. Araki and E. Woods. Representations of the canonical commutation relations describing a non-relativistic infinite free bose gas. *J. Math. Phys.*, 4:637–662, 1963.

[4] H. Araki and W. Wyss. Representations of canonical anticommutation relations. *Helv. Phys. Acta*, 37:139–159, 1964.

[5] V. Bach, J. Fröhlich, and I. M. Sigal. Mathematical theory of non-relativistic matter and radiation. *Lett. Math. Phys.* , 34:183–201, 1995.

[6] V. Bach, J. Fröhlich, and I. M. Sigal. Mathematical theory of radiation. *Found. Phys.* , 27(2):227–237, 1997.

[7] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. *Adv. in Math.* , 137:299–395, 1998.

[8] V. Bach, J. Fröhlich, and I. M. Sigal. Renormalization group analysis of spectral problems in quantum field theory. *Adv. in Math.* , 137:205–298, 1998.

[9] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Commun. Math. Phys.*, 207:249–290, 1999.

[10] V. Bach, J. Fröhlich, I. M. Sigal, and A. Soffer. Positive commutators and spectrum of nonrelativistic QED. *Commun. Math. Phys. (in Print)*, 1999.

[11] Ph. Blanchard. Discussion mathematique du modéle de Pauli et Fierz relatif à la catastrophe infrarouge. *Commun. Math. Phys.*, 15:156–172, 1969.

[12] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1.* Text and Monographs in Physics. Springer-Verlag, Berlin, 2 edition, 1987.

[13] O. Bratteli and D. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2.* Text and Monographs in Physics. Springer-Verlag, Berlin, 2 edition, 1996.

[14] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg. *Photons and Atoms – Introduction to Quantum Electrodynamics.* John Wiley, New York, 1991.

[15] H. Cycon, R. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators.* Springer, Berlin, Heidelberg, New York, 1 edition, 1987.

[16] E.B. Davies. *Quantum Theory of Open Systems.* Academic Press, New York, 1976.

[17] J. Derezinski and C. Gérard. Asymptotic completeness in quantum field theory. massive Pauli-Fierz Hamiltonians. *Rev. Math. Phys.*, 11(4):383–450, 1999.

[18] J. Dixmier. *Von Neumann Algebras*, volume 27 of *Mathematical Library.* North-Holland, Amsterdam, 1981.

[19] J. Fröhlich. On the infrared problem in a model of scalar electrons and massless scalar bosons. *Ann. Inst. H. Poincaré*, 19:1–103, 1973.

[20] J. Fröhlich. Existence of dressed one-electron states in a class of persistent models. *Fortschr. Phys.* , 22:159–198, 1974.

[21] R. Haag. *Local Quantum Physics. Fields, Particles, Algebras.* Text and Monographs in Physics. Springer, Berlin, 1992.

[22] R. Haag, N. Hugenholz, and M. Winnink. On the equilibrium states in qauntum statistical mechanics. *Commun. Math. Phys.*, 5:215–236, 1967.

[23] M. Hübner and H. Spohn. Radiative decay: nonperturbative approaches. *Rev. Math. Phys.* , 7:363–387, 1995.

[24] M. Hübner and H. Spohn. Spectral properties of the spin-boson Hamiltonian. *Ann. Inst. H. Poincare*, 62:289–323, 1995.

[25] W. Hunziker. Resonances, metastable states and exponential decay laws in perturbation theory. *Commun. Math. Phys.* , 132:177–188, 1990.

[26] V. Jakšić and C. A. Pillet. On a model for quantum friction. I: Fermi's golden rule and dynamics at zero temperature. *Ann. Inst. H. Poincaré*, 62:47–68, 1995.

[27] V. Jakšić and C. A. Pillet. On a model for quantum friction. II: Fermi's golden rule and dynamics at positive temperature. *Commun. Math. Phys.*, 176(3):619–643, 1996.

[28] V. Jakšić and C. A. Pillet. On a model for quantum friction III: Ergodic properties of the spin-boson system. *Commun. Math. Phys.*, 178(3):627–651, 1996.

[29] Ph. Martin. *Modèles en Mécanique Statistique des Processus Irréversibles*, volume 103 of *Lecture Notes in Physics*. Springer-Verlag, 1979.

[30] T. Okamoto and K. Yajima. Complex scaling technique in non-relativistic qed. *Ann. Inst. H. Poincaré*, 42:311–327, 1985.

[31] W. Pauli. Quantum mechanics. In C.P. Enz, editor, *Pauli Lectures on Physics*, volume 3. MIT Press, Cambridge, MA, 1973.

[32] W. Pauli. Statistical mechanics. In C.P. Enz, editor, *Pauli Lectures on Physics*, volume 4. MIT Press, Cambridge, MA, 1973.

[33] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: Analysis of Operators*, volume 4. Academic Press, San Diego, 1 edition, 1978.

[34] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: II. Fourier Analysis and Self-Adjointness*, volume 2. Academic Press, San Diego, 2 edition, 1980.

[35] D.W. Robinson. Return to equilibrium. *Commun. Math. Phys.*, 31:171–189, 1973.

[36] B. Schlein. Nichtrelativistische Elektronensysteme in einem quantisierten elektromagnetischen Feld. Master's thesis, ETH-Zürich, August 1999. diploma thesis.

[37] B. Simon. *The Statistical Mechanics of Lattice Gases*, volume 1 of *Princeton Series in Physics*. Princeton University Press, Princeton, NJ, 1993.

[38] E. Skibsted. Spectral analysis of N-body systems coupled to a bosonic field. *Rev. Math. Phys.*, 10(7):989–1026, 1997.

[39] H. Spohn. Asymptotic completeness for Rayleigh scattering. *J. Math. Phys.*, 38:2281–2296, 1997.

[40] M. Takesaki. Disjointness of the KMS-states of different temperatures. *Commun. Math. Phys.*, 17:33–41, 1970.

[41] M. Takesaki. *Tomita's theory of modular Hilbert algebras and its application.* Springer-Verlag, Berlin, 1970.