ON THE SPECTRAL PROPERTIES OF WITTEN-LAPLACIANS, THEIR RANGE PROJECTIONS AND BRASCAMP–LIEB'S INEQUALITY

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A study is made of a recent integral identity of B. Helffer and J. Sjöstrand, which for a not yet fully determined class of probability measures yields a formula for the covariance of two functions (of a stochastic variable); in comparison with the Brascamp–Lieb inequality, this formula is a more flexible and in some contexts stronger means for the analysis of correlation asymptotics in statistical mechanics. Using a fine version of the Closed Range Theorem, the identity's validity is shown to be equivalent to some explicitly given spectral properties of Witten–Laplacians on Euclidean space, and the formula is moreover deduced from the obtained abstract expression for the range projection. As a corollary, a generalised version of Brascamp–Lieb's inequality is obtained. For a certain class of measures occuring in statistical mechanics, explicit criteria for the Witten-Laplacians are found from the Persson–Agmon formula, from compactness of embeddings and from the Weyl calculus, which give results for closed range, strict positivity, essential self-adjointness and domain characterisations.

1. Introduction and Main Results

1.1. Background. In 1976, H. J. Brascamp and E. H. Lieb [BL76] proved the following inequality for an arbitrary function f in $C^1(\mathbb{R}^n)\cap L^2(\mu)$, when the given measure $d\mu = e^{-\Phi}\,dx$ has a real-valued, *strictly convex* C^2 'potential' Φ with Hessian $\Phi'' = (\partial^2_{jk}\Phi)_{j,k}$:

$$\int_{\mathbb{R}^n} |f(x)-\langle f\rangle|^2 e^{-\Phi(x)}\,dx \le \int_{\mathbb{R}^n} \big(\nabla f(x)^T\Phi''(x)^{-1}\overline{\nabla f(x)}\big)e^{-\Phi(x)}\,dx; \tag{1.1}$$

the measure μ is finite and may be normalised to $\int d\mu = 1$ without loss of generality (by adding $\log\int_{\mathbb{R}^n} d\mu$ to Φ) which is done tacitly thoughout, so $\langle f\rangle := \int fe^{-\Phi}\,dx$ equals f's mean.

Since then this inequality has been used in physics, where the strict convexity assumption on Φ in some contexts is a serious restriction; e.g. this is the case for the analysis of asymptotics of correlations in statistical mechanics.

As another technique for such problems, B. Helffer and J. Sjöstrand [HS94, Sjö96, Hel98, Hel97a] have recently introduced an exact formula for the *covariance* of two functions $g_1, g_2 \in L^2(\mu)$, i.e. for $\operatorname{cov}(g_1,g_2) := \int_{\mathbb{R}^n}(g_1-\langle g_1\rangle)\overline{(g_2-\langle g_2\rangle)}e^{-\Phi}\,dx$ (in comparison the variance of f enters in (1.1)). Denoting the inner product of both $L^2(\mathbb{R}^n,\mu)$ and $L^2(\mathbb{R}^n,\mu,\mathbb{C}^n)$ by $(\,\cdot\,|\,\cdot\,)_\mu$ for

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simplicity's sake, their identity may be written as follows:

$$(g_1 - \langle g_1\rangle \,|\, g_2 - \langle g_2\rangle)_\mu = (A_1^{-1}\nabla g_1 \,|\, \nabla g_2)_\mu. \tag{1.2}$$

This uses two elliptic differential operators $A_0 \geq 0$ and $A_1 \geq 0$ on $\mathbb{R}^n$ (although A_0 does not appear explicitly in (1.2)); these are equivalent, as observed in [Sjö96], to Witten's Laplacians [Wit82], and they have the expressions

$$A_0 = -\Delta + \Phi' \cdot \nabla, \qquad A_1 = A_0 \otimes I + \Phi''. \tag{1.3}$$

In the proofs of Helffer and Sjöstrand, Φ has had a rather specific nature, e.g. with Φ''_{jk} bounded and $x\cdot\Phi' \geq C|x|^{1+\omega}$ in [Sjö96]; the Φ treated in [Hel98] is essentially $|x|^4 - |x|^2$, see Example 7.3 below. Formula (1.2) has also been used by A. Naddaf and T. Spencer [NS97], V. Bach, T. Jecko and J. Sjöstrand [BJS98], B. Helffer [Hel97b] and others. Indirectly A_0, A_1 appeared earlier in [Sjö93, HS94, Hel95].

Concerning formula (1.2), it should be noted that g_j only enters in the covariance through $Pg_j := g_j - \int g_j\, d\mu$, which is the orthogonal projection onto $L^2(\mu)\ominus\mathbb{C}$, i.e. onto the complement of the constant functions. Because the gradient provides another means to remove the part of g_j lying in $\mathbb{C}$, it is natural to have ∇g_1 and ∇g_2 on the right hand side of (1.2) and to have an inverse of a second order differential operator, like A_1^{-1}, to counteract the gradients.

Viewed thus, (1.2) may seem plausible, and this article presents a study of it, resulting first of all in more general sufficient conditions for the formula; secondly, conditions that are equivalent to (1.2) are given at an abstract level (although these two improvements are formally substantial, the consequences for statistical mechanics are to be investigated). Thirdly a more systematic and streamlined approach to (1.2) is presented.

Remark 1.1. For the general importance of formula (1.2), recall that for a stochastic variable $X\colon \Omega \to \mathbb{R}^n$ with distribution μ on $\mathbb{R}^n$, the left hand side of (1.2) equals $\operatorname{cov}(g_1(X), g_2(X))$. So when μ is of the type treated here, then (1.2) provides a formula also for such covariances. However, this is clear, and hence this consequence shall not be treated below.

1.2. **Summary.** In this paper various — abstract and explicit — conditions are deduced for (1.2). To give an application of these, (1.1) is derived in the general strictly convex case from (1.2) and moreover extended to the case $f \in H^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^2(\mu)$; this supplements an explanation of B. Helffer of how (1.2) implies the validity of (1.1) for certain f when Φ is uniformly strictly convex [Hel98].

In the applications estimates like $A_1 \geq c_0 > 0$ would clearly give a bounded inverse A_1^{-1}, so that (1.2) would yield control of the covariance on the left hand side there. However, it turns out that already (1.2) *itself* follows from such a strict positivity.

This fact is observed here (while further properties of the A_j entered in [Sjö96, Hel98]) and proved for rather more general Φ than those considered hitherto. Actually, by means of Hilbert space methods (the Closed Range Theorem), even weaker sufficient conditions on A_0 or A_1 (such as closed range of A_0) are proved to be equivalent to (1.2); this analysis is furthermore valid for arbitrary probability measures μ on $\mathbb{R}^n$.

These techniques are also applied to the associated d-complex in μ-weighted spaces and to the associated A_k on forms of higher degrees, see (1.8) and (1.7) below. This is done because these objects may be of interest in statistical mechanics, and because the cases $k = 0$ and 1 have to be treated anyway in order to settle the relations between the various conditions found for A_0 and A_1.

Explicit criteria in terms of Φ are given partly by means of compact embeddings or bounds of essential spectra; partly by a pseudo-differential treatment of the Witten–Laplacians on functions and 1-forms on $\mathbb{R}^n$, i.e. of

$$\Delta_\Phi^{(0)} = -\Delta + \tfrac{1}{4}|\Phi'|^2 - \tfrac{1}{2}\,\Delta\,\Phi, \qquad \Delta_\Phi^{(1)} = \Delta_\Phi^{(0)} \otimes I + \Phi''; \tag{1.4}$$

these are unitarily equivalent to A_0 and A_1. This approach consists in a study of the $\Delta_\Phi^{(k)}$ by means of the Weyl calculus of Hörmander [Hör85, 18.4–6], and it circumvents the difficulty that pseudo-differential techniques do not play well together with the *weighted* Sobolev spaces in which the A_k act. In this analysis, sufficient conditions for closed range of A_0, strict positivity of A_1 and essential self-adjointness as well as characterisations of the maximal domains of A_0 and A_1 are established (through similar results for the $\Delta_\Phi^{(k)}$).

When combined, the ps.d. and Hilbert space analyses show formula (1.2) for a class of Φ which contains e.g. all polynomials of even degree $r \geq 2$, which have positive definite part of degree r (a change of the constant term will renormalise to $\int d\mu = 1$); in comparison the polynomials contained in the class of [Sjö96] have $r = 2$ (a rather simpler case because the term Φ'' in (1.4) is bounded on $L^2(\mathbb{R}^n, \mathbb{C}^n)$), while [Hel98] covered cases with $r = 4$.

1.3. **The main results.** In formula (1.2) the probability measure μ will be arbitrary to begin with, while g_j will be considered either in the weighted Sobolev space $H^1(\mu)$ equal to $\{\, u \in L^2(\mu) \mid \forall j = 1, \dots, n\colon \partial_j u \in L^2(\mu)\,\}$, whereby $L^2(\mu) := L^2(\mathbb{R}^n, \mu, \mathbb{C})$, or in $L^2(\mu)$ when this is justified. When the measure μ is such that $d\mu = e^{-\Phi(x)}\,dx$, then it will throughout be assumed that

$$\int_{\mathbb{R}^n} e^{-\Phi(x)}\,dx = 1 \quad \text{and} \quad \Phi \in C^2(\mathbb{R}^n, \mathbb{R}). \tag{1.5}$$

To simplify notation (cf. (1.7) ff. below), differentials will hereafter be used instead of gradients, so A_1 will act on suitable 1-forms in $L^2(\mathbb{R}^n, \mu, \wedge^1\mathbb{C}^n)$. To explicate the operators that appear as d_0 and d_0^* in (1.8) below and onwards, note that when 1-forms are identified with vector functions

v, then $d_0 f$, $d_0^* v$ identify with ∇f and $(\Phi' - \nabla)\cdot v$, respectively. Also for simplicity, $(\,\cdot\,|\,\cdot\,)_\mu$ will for any k refer to the scalar product in $L^2(\mathbb{R}^n, \mu, \wedge^k\mathbb{C}^n)$, i.e. the space of k-forms with coefficients in $L^2(\mu)$; on this space the following norm is used:

$$\| \sum_{j_1<\dots<j_k} f_{j_1\dots j_k} dz^{j_1} \wedge \dots \wedge dz^{j_k} \|_\mu = (\int_{\mathbb{R}^n} (\sum |f_{j_1\dots j_k}|^2)\, d\mu)^{1/2}. \tag{1.6}$$

It is of course a central question how the operators A_0 and A_1 are defined precisely, but it will be equally important to make sense of the inverse A_1^{-1}. For this reason, it is worthwhile to define A_0 and A_1 as 'Hodge Laplacians' to begin with; this is not only a most general approach (it works for arbitrary probability measures μ), but it also leads in a very natural way to a fruitful discussion, by simple operator theoretical methods, of the invertibility of A_1. This will be explained in the following, before the theorems are presented.

But first of all it should be emphasised that the present article focuses on the following problem for the identity (1.2):

> Which probability measures μ have the property that formula (1.2) holds for all functions f, g in $H^1(\mu)$?

Of course other problems would be equally meaningful (such as fixing two functions f and g and then search for the μ for which (1.2) would be true); but for simplicity's sake the discussion will here be restricted to the above-mentioned problem.

Secondly, as the point of departure it is useful to adopt the following definitions of A_0 and A_1 as the Hodge Laplacians[1]

$$A_k^{(\mathrm{H})} = d_k^* d_k + d_{k-1} d_{k-1}^*, \qquad D(A_k^{(\mathrm{H})}) = D(d_k^* d_k) \cap D(d_{k-1} d_{k-1}^*), \tag{1.7}$$

associated to the complex (where $d_{-1} \equiv 0$)

$$(0) \longrightarrow L^2(\mu) \xrightarrow{d_0} L^2(\mu, \wedge^1\mathbb{C}^n) \xrightarrow{d_1} \dots \xrightarrow{d_{k-1}} L^2(\mu, \wedge^k\mathbb{C}^n) \xrightarrow{d_k} \dots. \tag{1.8}$$

Here d_k denotes the exterior differential of k-forms; this is in general a first order differential operator acting in the distribution sense, but in the context above, d_k is equipped with its maximal domain as an unbounded, closed, densely defined operator from $L^2(\mu, \wedge^k\mathbb{C}^n)$ to $L^2(\mu, \wedge^{k+1}\mathbb{C}^n)$; clearly $D(d_0) = H^1(\mu)$. This is an example of a Hilbert complex in the sense of J. Brüning and M. Lesch [BL92], where such complexes are described in a clear way (by comparison the present article focuses on the conditions implying that (1.8) is a Fredholm complex rather than the conclusions that would follow from this property).

[1] For $d\mu = e^{-\Phi}\,dx$, an interpretation of $e^{-\Phi}$ as $\sqrt{\det(g_{ij})}$ would for any Riemannian metric (g_{ij}) lead to a Laplace–Beltrami operator which resembles A_0, A_1, ... but differs; indeed, in this way already (1.6) would be unconventional in the study of the de Rham complex; see e.g. [Jos98, 2.1].

For d_k the closure of the range is denoted by X_{k+1},

$$X_{k+1} := \overline{R(d_k)} \tag{1.9}$$

and the kernel by $Z(d_k)$; as a convention $X_0 := \mathbb{C}$. The Hilbert space adjoint, d_k^*, also enters in the Hodge Laplacians, cf. (1.7); for simplicity the superscript '(H)' is suppressed in the sequel, for the definition in (1.7) will be in effect until Section 4, unless otherwise is explicitly stated.

Thirdly, a series of small remarks will clarify the situation: using the orthogonal projection P onto $L^2(\mu) \ominus \mathbb{C}$, formula (1.2) may be written as

$$(\, Pg_1 \mid Pg_2 \,)_\mu = (\, A_1^{-1} d_0 g_1 \mid d_0 g_2 \,)_\mu. \tag{1.10}$$

Moreover, $d_0 g_1$ and $d_0 g_2$ belong to $X := X_1$, and since $d \circ d \equiv 0$ implies $X_k \subset Z(d_{k+1})$, the subspace X of $L^2(\mu, \wedge^1 \mathbb{C}^n)$ is invariant under A_1 by (1.7); in fact

$$A_1|_X = (d_0 d_0^*)|_X. \tag{1.11}$$

Hence one has the following identity of unbounded operators in X:

$$(A_1^{-1})|_X = (d_0 d_0^*|_X)^{-1}; \tag{1.12}$$

in particular, $D(A_1^{-1}) \cap X$ equals the range $R(d_0 d_0^*|_X) = d_0 d_0^*(X)$.

It is now straightforward to verify the implications

$$(1.10) \text{ holds for } g_1, g_2 \in H^1(\mu) \tag{1.13}$$

$$\iff Pf = d_0^*(d_0 d_0^*)^{-1} d_0 f \quad \text{holds for } f \in H^1(\mu). \tag{1.14}$$

Indeed, these properties are, by (1.12) and the selfadjointness of P, both equivalent to

$$(\, (d_0 d_0^*)^{-1} d_0 f \mid d_0 g \,)_\mu = (\, Pf \mid g \,)_\mu \quad \text{holds for all} \quad f, g \in H^1(\mu); \tag{1.15}$$

e.g. it is found when (1.14) holds that $(d_0 d_0^*)^{-1} d_0 f$ is an element of $D(d_0^*)$, whence (1.15) follows from the definition of the adjoint. The other implications follow in a similar manner.

Therefore the formulated problem for (1.10) has been reduced to the just given property in (1.14) for P, and this rewriting as a 'linear' problem makes the analysis more straightforward, as we shall see immediately.

Lemma 1.2. *$Pf = d_0^*(d_0 d_0^*)^{-1} d_0 f$ holds whenever $f - \langle f \rangle$ belongs to $D(d_0) \cap R(d_0^*)$.*

Proof. When $f - \langle f \rangle$ is in $H^1(\mu) \cap R(d_0^*)$, then $d_0 f = d_0(d_0^* v)$ holds for some $v \in X \cap D(d_0^*)$, because $X^\perp = Z(d_0^*)$. Hence $d_0 f$ belongs to $D(d_0 d_0^*|_X^{-1})$, and so

$$d_0^*(d_0 d_0^*|_X)^{-1} d_0 f = d_0^* \cdot I v = f - \langle f \rangle, \tag{1.16}$$

since $d_0 d_0^*|_X^{-1}$ maps into $D(d_0 d_0^*) \subset D(d_0^*)$. $\square$

The usefulness of the lemma in connection with the proof of (1.2) was independently discovered by V. Bach, T. Jecko and J. Sjöstrand [BJS98, (II.15)].

Conversely (1.14) implies that $f-\langle f\rangle = Pf$ belongs to $D(d_0)\cap R(d_0^*)$ for any $f\in H^1(\mu)$. However, it is more useful to ask the following question: which probability mesures μ have the property that

$$\forall f\in H^1(\mu)\colon\ f-\langle f\rangle\in D(d_0)\cap R(d_0^*)\quad ? \tag{1.17}$$

Since $H^1(\mu)=D(d_0)$, it is clear that μ has this property if and only if

$$H^1(\mu)\ominus\mathbb{C}\subset R(d_0^*). \tag{1.18}$$

(Here and throughout $F\ominus\mathbb{C}$ stands for the elements of a given subspace $F\subset L^2$ which are orthogonal to the constant functions.) The inclusion (1.18) would obviously be true if μ is such that

$$R(d_0^*)=L^2(\mu)\ominus\mathbb{C}; \tag{1.19}$$

since $\overline{R(d_0^*)}=Z(d_0)^\perp=\mathbb{C}^\perp$, condition (1.19) is equivalent to closedness of the range $R(d_0^*)$.

Because (1.19) also expresses that $R(d_0^*)$ should be the maximal possible subspace of $L^2(\mu)$, one could in view of (1.18) envisage that (1.19) is sufficient but unnecessary.

However, this is not the case, for a fine version of the Closed Range Theorem yields that (1.10) is equivalent to the closedness of d_0^*'s range, hence to (1.19). This is explained for arbitrary unbounded operators in general Hilbert spaces in Section 3 below, but the consequences for the A_k are summed up in the following theorem. In particular, the problem formulated for (1.2) above has been rephrased by means of a set of equivalent conditions on A_0, A_1, d_0 and d_0^*, that one finds for $k=0$ in

Theorem 1.3. *Let k be a fixed index in the complex* (1.8)*. Then closedness of each of the spaces $R(A_k)$, $R(d_k)$ and $R(d_k^*)$ as well as strict positivity of $A_{k+1}|_{X_{k+1}}$ are equivalent properties, and if P_k is the projection from $L^2(\mu,\wedge^k\mathbb{C}^n)$ onto $R(A_k)$, these properties are also equivalent to the validity of*

$$P_k=d_k^*(A_{k+1}|_{X_{k+1}})^{-1}d_k\quad\text{on all of}\quad D(d_k). \tag{1.20}$$

In the affirmative case, $A_k|_{Z(d_k)^\perp}=d_k^*d_k$ and has closed range $R(A_k)=R(d_k^*)$ and $A_{k+1}|_{X_{k+1}}=d_kd_k^*$. Moreover, there is an equivalence

$$A_k|_{Z(d_k)^\perp}=U^*A_{k+1}|_{X_{k+1}}U \tag{1.21}$$

with U equal to the unitary $d_k(d_k^*d_k|_{X_k})^{-1/2}$ from $L^2(\mu,\wedge^k\mathbb{C}^n)\ominus Z(d_k)$ to X_{k+1}, and consequently

$$\sigma(A_k|_{Z(d_k)^\perp})=\sigma(A_{k+1}|_{X_{k+1}}) \tag{1.22}$$

with the analogous relation for the essential spectra.

The point of the proof of this result is to combine the usual estimates from below of the adjoint (here d_0^*) with the fact that T^*T is self-adjoint for any densely defined, closed operator T. In the abstract set-up, the range projection of T^*T has the form $P = T^*(TT^*|_{R(T)})^{-1}T$; when applied to A_0 and A_1, this yields (1.14) and thus (1.2) at least for u, g_1 and $g_2 \in H^1(\mu)$.

However, the closedness of $R(A_0)$, $R(d_0)$, $R(d_0^*)$ or positivity of $A_1|_X$ is not easy to analyse when the A_k are defined from (1.7) for an arbitrary probability measure μ. From Section 4 below and onwards, we shall therefore work under the assumption that μ has a density $e^{-\Phi(x)}$ with respect to Lebesgue measure for some Φ fulfilling (1.5).

Using this assumption, an alternative variational definition of the A_k (i.e. by means of sesqui-linear forms, or Lax–Milgram's lemma) is introduced in Section 4 below; thereafter it is seen that this yields the Friedrichs extension from $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$, and it is then shown, for the variationally defined operators, that (1.7) holds both in the distribution sense and as a formula for unbounded operators. (Concerning essential self-adjointness of the A_k, see the remarks in Section 8 below.)

Using the definition by Lax–Milgram's lemma, it is proved in Section 5 below that the regularity assumption on the g_1, g_2 and f above may be relaxed from $H^1(\mu)$ to $L^2(\mu)$:

Theorem 1.4. *Let Φ satisfy* (1.5)*, and suppose that A_0 as an unbounded operator in $L^2(\mu)$ has closed range, $R(A_0) = \overline{R(A_0)}$. Then* (1.2) *holds for all g_1 and g_2 in $L^2(\mu)$.*

Moreover, it then holds that $L^2(\mu) = R(A_0) \oplus \mathbb{C}$, and for every $u \in L^2(\mu)$,

$$Pu = d_0^* A_1^{-1} d_0 u \tag{1.23}$$

when $Pu = u - \int u\, d\mu$ denotes the orthogonal projection onto $L^2(\mu) \ominus \mathbb{C}$.

Note also that when g_2 is in $L^2(\mu) \setminus H^1(\mu)$, it is understood in Theorem 1.4 that the right hand side of (1.2) should be read as a duality $\langle \tilde{\mathcal{A}}_1^{-1} dg_1, d\overline{g_2} \rangle_{\tilde{V}\times\tilde{V}'}$ for a certain Hilbert space $\tilde{V}$ with isomorphism $\tilde{\mathcal{A}}_1 \colon \tilde{V} \to \tilde{V}'$ onto its dual. See Section 5.3 for details.

Although the identification of A_1, or rather $A_1|_X$, with a restriction of $\mathcal{A}_1$ is a well-known procedure for operators defined by the Lax–Milgram lemma, it is really the validity of a certain Poincaré inequality for $\tilde{V}$ which makes this possible. Because $\tilde{V}$ is the form domain of $A_1|_X$, one may of course ask directly whether such Poincaré inequalities for the form domain of A_1 or $A_1|_X$ would imply (1.2); moreover, also the exactness of the complex (1.8) may be considered. However, that the Poincaré inequalities are equivalent to e.g. positivity of A_0 and $A_1|_X$, respectively, while exactness is an 'intermediate' condition is proved in Theorem 5.1 below, where the full interrelationship among the various criteria is settled.

By exploiting the possibility in Theorem 1.4 of taking the $g_j \in L^2(\mu)$, one finds as an application that (1.2) implies a generalisation of the inequality in (1.1) from C^1 to $H^1_{\mathrm{loc}}(\mu)$:

Corollary 1.5. *Let $\Phi \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfy $\int e^{-\Phi}\,dx = 1$ and be strictly convex, i.e.*

$$\sum \Phi''_{jk}(x) z_j \overline{z_k} > 0 \quad \textit{for all} \quad x \in \mathbb{R}^n, \quad \textit{all } z \in \mathbb{C}^n \setminus \{0\}. \tag{1.24}$$

Then the inequality (1.1) *holds for all $f \in L^2(\mathbb{R}^n, \mu) \cap H^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

This generalises [Hel98], where Helffer sketched how (1.2) implies (1.1) for *uniformly* strictly convex Φ, when in addition ∇f is bounded. The general case was left open there, although with indications that (1.2), because it is an exact formula, is more powerful. (The reference to (1.2) refines an explanation from 1993, see [Hel95], where A_0 and A_1 were introduced for (1.1) without (1.2).)

For the sake of the proof, it should be recalled from [Hel98] that in the uniformly strictly convex case, say $\Phi''(x) \geq c_0 > 0$ on $\mathbb{R}^n$ for every x, the idea behind (1.2) $\Longrightarrow$ (1.1) is to infer from the formal expression in (1.3) that

$$A_1 \geq \Phi'' \quad \text{and therefore} \quad A_1^{-1} \leq (\Phi'')^{-1} \leq \tfrac{1}{c_0}. \tag{1.25}$$

This actually requires justification, for inclusions between the domains $D(A_1)$ and $D(\Phi'')$ need not exist under the assumption (1.5). Nevertheless one may envisage that $A_1 \geq c_0 > 0$ then holds — so that A_1^{-1} would be well defined — and this is verified in Section 4.5 below. However, the proof of Corollary 1.5 is first completed in Section 5.3, after that of Theorem 1.4.

The identity (1.2) is established independently of compactness of $H^1(\mu) \hookrightarrow L^2(\mu)$, so in particular the essential spectra of A_0 and A_1 need not be empty. In this connection some simple sufficient conditions are given in Section 6, exploiting the Persson–Agmon formula and results due to P. Bolley, M. Dauge and B. Helffer [BDH89].

However, slightly stronger conditions allow A_0 and A_1 to be analysed by means of the Weyl calculus [Hör85, 18.4–6], leading to closed range, essential self-adjointness and positivity. With $v^\alpha := v_1^{\alpha_1} \cdot \dots \cdot v_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$ and $v \in \mathbb{C}^n$, or $v = -i\nabla =: D$, the following results are restatements of Theorems 7.3, 7.4 and 7.5 below:

Theorem 1.6. *Let $\Phi(x) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ satisfy $\int e^{-\Phi(x)}\,dx = 1$; suppose also that $|\Phi'(x)| \to \infty$ for $|x| \to \infty$, that $|D^\beta \Phi(x)| \leq C_\beta (1 + |\Phi'(x)|^2)^{\frac{1}{2}}$ for every multiindex $|\beta| \geq 1$ and that, for some M, any $D^\beta \Phi$ with $|\beta| = M$ is bounded on $\mathbb{R}^n$.*

Then A_0 has closed range in $L^2(\mathbb{R}^n, \mu, \mathbb{C})$, so that the conclusions of Theorem 1.4 are valid, and A_0 is essentially self-adjoint (when considered) on $C_0^\infty(\mathbb{R}^n)$. The operator A_1 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n, \wedge^1 \mathbb{C}^n)$ and has closed range.

Moreover, if there exist $\omega > 0$, $C > 0$ such that $x \cdot \Phi'(x) \geq |x|^{1+\omega}/C$ holds for all $|x| \geq C$, then $A_1 > 0$ on $L^2(\mathbb{R}^n, \mu, \wedge^1 \mathbb{C}^n)$.

The considered class of Φ is larger than those in [Sjö96, Hel98]; e.g. any polynomial Φ of even degree ≥ 2 satisfies the assumptions when the part of highest degree is positive definite; this includes the condition for strict positivity of A_1 (seen as in Example 7.3 below).

Theorem 1.7. *Under the hypothesis for Φ made in the beginning of Theorem 1.6, a function u belongs to $D(A_0)$ precisely when it has the property*

$$(\Phi')^\beta D^\alpha u \in L^2(\mathbb{R}^n, \mu) \quad \textit{for all } \alpha \textit{ and } \beta \textit{ such that} \quad |\alpha + \beta| \leq 2. \tag{1.26}$$

A form $\sum_{l=1}^n v_l\, dx^l$ is in $D(A_1)$ if and only if each v_l satisfies (1.26).

Consequently A_0 and A_1 have compact reolvents, hence their spectra are discrete.

When the condition for $A_1 > 0$ is fulfilled for some $\omega \geq 1$, then $D(A_0)$ contains the set $B_2(\mathbb{R}^n, \mu)$, defined as those u for which $x^\beta D^\alpha u \in L^2(\mu)$ when $|\alpha + \beta| \leq 2$.

Theorem 1.7 may be used for example in the distortion argument for A_1^{-1} in the correlation estimates; see e.g. the proof of [Hel98, Thm. 4.1].

Remark 1.8. For general probability measures μ it is questionable which expressions the associated operators A_0, A_1 can have: already for μ absolutely continuous with respect to Lebesgue measure, Φ equals $+\infty$ in $\mathbb{R}^n \setminus \operatorname{supp} \tilde{\mu}$, and this Borel set may be so irregular that when one attempts a calculation of the d_0^* in (1.7), then $\partial_j e^{-\Phi}$ is unequal to $-\Phi'_j e^{-\Phi}$ (and the latter may even be undefined in $\mathcal{D}'$).

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2. Notation and Preliminairies

For an operator T in a Hilbert space with scalar product $(\cdot \mid \cdot)$ and norm $\|\cdot\|$, the domain, range and kernel is written $D(T)$, $R(T)$ and $Z(T)$, respectively, while $\rho(T)$ and $\sigma(T)$ stand for the resolvent and spectrum of T; the space of bounded operators is denoted by $\mathbb{B}(H)$. The essential spectrum $\sigma_{\text{ess}}(T) = \sigma(T) \setminus \sigma_{\text{disc}}(T)$ consists in the self-adjoint case of those $\lambda \in \mathbb{C}$ for which there is a sequence with $\|x_n\| = 1$ and $(T - \lambda)x_n \to 0$ while $x_n \to 0$ weakly; it is the complement of the λ for which $T - \lambda$ is Fredholm $D(T) \to H$ in the graph topology, i.e. has $Z(T - \lambda)$ of finite dimension and $R(T - \lambda)$ closed (using [Hör85, Prop. 19.3]). Moreover,

$$m(T) = \inf\big\{ \operatorname{Re}(\, Tx \mid x\,) \bigm| \|x\| = 1 \big\} \tag{2.1}$$

is the lower bound of T. Occasionally the norm $\|\cdot\|_X$ in a space X is written $\|\cdot\, |X\|$, to avoid unnecessary subscripts.

Given a triple (H, V, s) consisting of two Hilbert spaces $V \hookrightarrow H$ with bounded, *dense* injection and a bounded sesqui-linear form $s(\cdot, \cdot)$ on V, then *coerciveness* — i.e. existence of $c > 0$ and $k \in \mathbb{R}$ such that

$$\operatorname{Re} s(u, u) \geq c\|u\|_V^2 - k\|u\|_H^2 \quad \text{for all} \quad u \in V, \tag{2.2}$$

gives the following for the operator S defined on $\{\, u \in V \bigm| \exists f \in H \forall v \in V : s(u, v) = (\, f \mid v\,)_H \,\}$ by the formula $Su = f$:

Lemma 2.1. 1° *S is closed in H with $D(S)$ dense in V. The adjoint S^* is the analogous operator defined from $s^*(v, w) := \overline{s(w, v)}$.*

2° *When $s(\cdot, \cdot)$ is positive definite on V (i.e. $s(v, v) > 0$ for $v \neq 0$, e.g. for $k = 0$), then S extends to the isometry $\mathcal{A}$ of $(V, s(\cdot, \cdot))$ onto V^*, its antidual, and $D(S) = \mathcal{A}^{-1}(H)$.*

This result, known as Lax–Milgram's lemma, may be proved straightforwardly in the fashion of J.–L. Lions and E. Magenes [LM68, Sect. 9.1].

$C_0^\infty(\mathbb{R}^n)$ denotes the space of infinitely differentiable functions with compact support, and $\mathcal{D}'(\mathbb{R}^n)$ its dual; $\mathcal{D}'^k$ the subspace of distributions of order k; $\langle\, u, \varphi \,\rangle = u(\varphi)$ for all $u \in \mathcal{D}'$ and $\varphi \in C_0^\infty$. For $L^2(\mathbb{R}^n, \mu)$ the scalar product and norm is written $(\,\cdot \mid \cdot\,)_\mu$ and $\|\cdot\|_\mu$ respectively, although with μ omitted in case of the Lebesgue measure. Similar notation is adopted for the space of k-forms $L^2(\mathbb{R}^n, \mu, \wedge^k\mathbb{C})$; in general, for a space F of functions $\mathbb{R}^n \to \mathbb{C}$, the set $F(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ consists of the differential forms with coefficients therein.

A differential form of degree k with complex C^∞-coefficients has the form

$$f = \textstyle\sum'_{|J|=k} f_J(x)\, dz^J; \tag{2.3}$$

here $\sum'$ indicates summation over increasing k-tuples $J = (j_1, \dots, j_k)$, i.e. strictly increasing maps $\{\, 1, \dots, k \,\} \to \{\, 1, \dots, n \,\}$; and $dz^J := dz^{j_1} \wedge \dots \wedge dz^{j_k}$ stands for the usual k-linear anti-symmetric map $(\mathbb{C}^n)^k \to \mathbb{C}$ derived from the Cartesian coordinates in $\mathbb{C}^n$. By anti-symmetry in the indices, $f_{jL} = f_J \varepsilon_J^{jL}$ where $\varepsilon_J^{jL} = 0$ unless J has the same elements as $jL := (j, l_1, \dots, l_{k-1})$, in which case ε_J^{jL} is the sign of the permutation $\binom{jL}{J}$.

For the distribution space $\mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$, see Appendix A.

3. An Operator Approach

It is shown in this section how Hilbert space methods can provide detailed information about (1.2), using the rewriting given in (1.14). The basic observation is that a similar projection appears in the Closed Range Theorem, at least in the version established below where the self-adjointness of T^*T and TT^* is incorporated for this purpose.

Let in the sequel $T\colon H \to H_1$ be a densely defined, closed operator between Hilbert spaces H and H_1, and let $F \subset H$ and $F_1 \subset H_1$ denote two closed subspaces such that

$$F = \overline{R(T^*)}, \qquad R(T) \subset F_1. \tag{3.1}$$

Here the possibility of taking F_1 different from $\overline{R(T)}$ is adopted from Hörmander's treatment of the $\bar\partial$-complex [Hör66, Ch. 4]; in an analogous way this is useful for the below study of the exterior derivative d in $L^2(\mathbb{R}^n, \mu, \wedge^k\mathbb{C}^n)$ spaces, where $F_1 = Z(d_{k+1})$ is a natural choice when $T = d_k$.

The closedness of T's range is closely connected to the properties of the operators

$$S = T^*T|_F, \qquad S_1 = TT^*|_{F_1} \tag{3.2}$$

and to the orthogonal projection P onto F. In fact one has the next result, which might be folklore, but nevertheless is formulated as a theorem in view of the clarification it gives for (1.2):

Theorem 3.1 (Closed Range Theorem)**.** *When T is an operator as above, and the set-up in* (3.1)–(3.2) *is used, then the following properties are equivalent:*

(i) *$R(T)$ is closed and equal to F_1.*
(ii) *There exists $c > 0$ such that $\| y \,|H_1\| \le c \,\| T^*y \,|H\|$ for $y \in D(T^*) \cap F_1$.*
(iii) *$R(T^*)$ is closed and $F_1^\perp = Z(T^*)$.*
(iv) *S_1 is injective and has closed range.*
(v) *S is injective and has closed range.*
(vi) *S_1 is injective and*

$$Px = T^*S_1^{-1}Tx \quad \textit{for all} \quad x \in D(T). \tag{3.3}$$

In the affirmative case, S and S_1 are unitarily equivalent, that is

$$S = U^*S_1U \tag{3.4}$$

holds for $U = TS^{-\frac{1}{2}}$, which is an isometry $U \in \mathbb{B}(H, H_1)$. Consequently $\sigma(S) = \sigma(S_1)$ and $\sigma_{\rm ess}(S) = \sigma_{\rm ess}(S_1)$.

Proof. Note first that $F_1^\perp \subset Z(T^*)$ because of the assumption on F_1, and that $D(T^*)$ is invariant under projection onto F_1 and $F_1^\perp$; indeed, if $D(T^*) \ni y = f + f^\perp$ where $f \in F_1$ and $f^\perp \in F_1^\perp$, then $f^\perp \in Z(T^*)$ and $f \in D(T^*)$. Moreover, S_1 is densely defined in F_1: if $D(TT^*) \ni y_k \to y \in F_1$, one may split $y_k = f_k + f_k^\perp$, where $f_k \in F_1 \cap D(TT^*) = D(S_1)$ while $\|y - f_k\| \le \|y - y_k\| \searrow 0$; the self-adjointness of TT^* then carries over to S_1. Because the roles of T and T^* may be interchanged, also S is self-adjoint.

(i) $\iff$ (ii) is proved in [Hör66, Lem. 4.1.1]. When (ii) holds, then $Z(T^*) \ni z = f + f^\perp$ (with $f^{(\perp)}$ as above) implies $T^*f = 0$, hence $f = 0$, so $Z(T^*) = F_1^\perp$; hence $R(T^*)$ equals $R(T^*|_{F_1\cap D(T^*)})$, and the latter is closed by (ii) so (iii) is obtained. To deduce (ii) from (iii), it suffices to consider $T^*\colon F_1 \cap D(T^*) \to H$ as a bounded operator in the graph norm and apply the Open Mapping Theorem. Because (i) and (iii) are equivalent, so are (v) and (iv) provided (i) $\iff$ (iv) is proved.

From (iii) injectivity follows since $TT^*z = 0$ yields $z \in Z(T^*) = F_1^\perp$; clearly $R(S_1) \subset R(T)$, but any $y \in R(T)$ equals Tx for some $x \in D(T) \ominus Z(T) \subset Z(T)^\perp = R(T^*)$; thus $R(S_1)$ equals $R(T)$, hence is closed. This shows (iv).

When (iv) holds, S_1^{-1} is defined on F_1 (because $R(S_1) = Z(S_1) = \{0\}$), and $S_1^{-1} \in \mathbb{B}(F_1)$ since S_1 is closed. S_1^{-1} maps into $D(TT^*)$, so for every $x \in D(T)$ it is obvious that $x \in D(T^*S_1^{-1}T)$ and that

$$x - T^*S_1^{-1}Tx \in Z(T); \tag{3.5}$$

since $1 - P$ is the projection onto $Z(T)$, this entails (3.3), for

$$0 = P(x - T^*S_1^{-1}Tx) = Px - T^*S_1^{-1}Tx. \tag{3.6}$$

Finally, (vi) $\implies$ (i), for injectivity of S_1 yields $F_1 = \overline{R(T)}$, while (3.3) shows that

$$R(T) \subset D(S_1^{-1}) = R(S_1) \subset R(TT^*) \subset R(T); \tag{3.7}$$

hence $R(T) = R(TT^*)$, and since this implies that $R(T)$ is closed, (i) is obtained.

However, for completeness' sake an elementary proof of the just mentioned implication shall be given. When $R(T) = R(TT^*)$, then one can pass to a domain consideration for the operator $\bar{T} = (T|_F)^{-1}$ and use that

$$\bar{T} = \bar{T}(I + \bar{T}^*\bar{T})^{-1} + \bar{T}\bar{T}^*\bar{T}(I + \bar{T}^*\bar{T})^{-1}. \tag{3.8}$$

Since $\bar{T}^*\bar{T} = (TT^*|_F)^{-1}$, it is self-adjoint ≥ 0 in F_1 with a square root $Q = Q^* = (\bar{T}^*\bar{T})^{1/2} \geq 0$ fulfilling

$$D(Q) = D(Q^2) = D(\bar{T}). \tag{3.9}$$

Indeed, $D(Q^2) = D((TT^*)^{-1}) = R(T) = D(\bar{T})$ so it remains to be shown that $D(Q) = D(\bar{T})$. But if (x_k) is a sequence in $D(Q^2) = D(\bar{T})$,

$$\|Q(x_k - x_m)\|^2 = (\, Q^2(x_k - x_m) \mid x_k - x_m \,) = \|\bar{T}(x_k - x_m)\|^2, \tag{3.10}$$

whence the closures of this set in the graph topologies on $D(Q)$ and $D(\bar{T})$ coincide.

Combining the boundedness of the resolvent with (3.9) it follows that

$$(I + \bar{T}^*\bar{T})^{-1}\colon F_1 \to D(\bar{T}^*\bar{T}) = D(Q^2) \tag{3.11}$$

$$Q(I + \bar{T}^*\bar{T})^{-1}\colon F_1 \to D(Q) = D(Q^2) \tag{3.12}$$

$$Q^2(I + \bar{T}^*\bar{T})^{-1}\colon F_1 \to D(Q) = D(\bar{T}), \tag{3.13}$$

and it is seen from the first of these lines and (3.9) that $\bar{T}(I+\bar{T}^*\bar{T})^{-1}$ belongs to $\mathbb{B}(F_1, F)$; then the third line gives that $\bar{T}\bar{T}^*\bar{T}(I+\bar{T}^*\bar{T})^{-1} \in \mathbb{B}(F_1, F)$, and (3.8) implies that $R(T) = D(\bar{T}) = F_1$, which is closed.

Given that (i)–(vi) hold, then (v) gives both that $S^{\frac{1}{2}}$ is injective and that $S^{-\frac{1}{2}} \in \mathbb{B}(F)$, because it is closed and everywhere defined, and similarly $U := TS^{-\frac{1}{2}} \in \mathbb{B}(F, F_1)$.

For any $x \in D(S)$ it holds that $S^{-\frac{1}{2}}x \in D(S) \subset D(T^*T)$, so

$$\|Ux\|^2 = (\, S^{-\frac{1}{2}}T^*TS^{-\frac{1}{2}}x \mid x\,) = \|x\|^2, \tag{3.14}$$

This extends to all $x \in F$, and $TS^{-\frac{1}{2}}$ maps onto F_1 by (i), for the fact that $D(S^{\frac{1}{2}}) = D(T|_F)$ (shown analogously to (3.9)) shows that $S^{-\frac{1}{2}}$ maps onto $D(T) \ominus Z(T)$. Hence U is unitary.

The identity (3.4) holds on vectors $x \in D(S^2)$, for $S^2x = (T^*T)^2x = T^*S_1Tx$, so that

$$(S^{-\frac{1}{2}}T^*)S_1(TS^{-\frac{1}{2}})x = S^{-\frac{1}{2}}S^2S^{-\frac{1}{2}}x = I_{D(S^{1/2})}Sx = Sx; \tag{3.15}$$

note that $S^{-\frac{1}{2}}T^* \subset U^*$. Now any $x \in D(S)$ may be approximated in the graph norm by $x_k \in D(S^2)$: then $Sx_k \to Sx$ while

$$S_1Ux_k = USx_k \to USx \quad \text{and} \quad Ux_k \to Ux; \tag{3.16}$$

therefore $Ux \in D(S_1)$ with $S_1Ux = USx$, from where (3.4) follows for x by application of U^*. Conversely, note that U^*S_1U acts as $U^*TS^{\frac{1}{2}}$, hence has its domain contained in $D(TS^{\frac{1}{2}}) = D(S)$ (since $U^* \in \mathbb{B}(F_1, F)$); therefore U^*S_1Ux can only be defined when Sx is so, and then we have already seen that $U^*S_1Ux = Sx$.

Finally, for $\lambda \in \sigma(S)$ there is $x_k \in D(S)$ with $\|x_k\| = 1$ and $(S-\lambda)x_k \to 0$, and $y_k = Ux_k$ is also normalised while

$$(S_1 - \lambda)y_k = U(U^*S_1U - \lambda)x_k \to 0 \quad \text{for} \quad k \to \infty; \tag{3.17}$$

moreover, $y_k \to 0$ weakly if the x_k do so, hence also $\sigma_{\text{ess}}(S) \subset \sigma_{\text{ess}}(S_1)$, and the opposite inclusions are equally easy. □

The requirement in (iv) is equivalent to 0 belonging to the resolvent set of S_1, and by the minimax principle this may, of course, be replaced by strict positivity of S_1. Applied to the complex (1.8) this yields, because of (1.12):

Corollary 3.2. *The conclusions of Theorem 1.3 are valid.*

For $k = 0$ this almost gives the main part of Theorem 1.4, for clearly $A_0 = d_0^*d_0$ has kernel $\mathbb{C}$ so that P_0 must equal $u - \int u\,d\mu$; whence (1.23).

The goal is not yet attained, however. First of all we shall in Section 4 below give a definition of A_0 and A_1 using sesqui-linear forms, and then verify in (4.13) and (4.21) below that this coincides with the A_k in (1.7) above and gives a meaning to (1.3)

Secondly the formula for P_k in Theorem 1.3 is obtained for $H^1(\mu, \wedge^k\mathbb{C}^n)$ only, whereas for Theorem 1.4 it is necessary to make sense of the right hand side of (1.23) when the u there is arbitrary in $L^2(\mu)$. This is based on the Lax–Milgram definition in Section 4, and is carried out in Section 5.

Remark 3.3. Corollary 3.2 and the remark following it entails

$$\sigma(A_0) = \{0\} \cup \sigma(A_1|_{X_1}). \tag{3.18}$$

Earlier it was obtained by Sjöstrand [Sjö96] that the gap between the first two eigenvalues of A_0 is larger than A_1's first eigenvalue. This also follows immediately from the above formula when the spectra are known to be discrete as (it follows from the stronger assumptions on Φ posed) in [Sjö96].

4. The d-complex in Weighted L^2 Spaces

4.1. **The Operators.** To avoid cumbersome justification of integration by parts, it is worthwhile to define $A_0, A_1, \dots$ variationally, i.e. by Lax–Milgram's lemma (in contrast with [Hel95, Sjö96, Hel98] were the Friedrichs extension was used). This is based on the weighted space $L^2(\mu) := L^2(\mathbb{R}^n, \mu, \mathbb{C})$ with measure $\mu := e^{-\Phi(x)}dx$ and scalar product

$$(\, u_1 \,|\, u_2 \,)_\mu := \int_{\mathbb{R}^n} u_1(x)\overline{u_2(x)}\, d\mu(x) \tag{4.1}$$

and its analogue for k-forms $L^2(\mathbb{R}^n, \mu, \wedge^k\mathbb{C}^n)$, where $(\, v \,|\, w \,)_\mu$ is defined instead by integration of $\sum' v_J(x)\overline{w_J(x)}$, with prime denoting summation over increasing k-tuples J, see Section 2.

For each k the operator A_k is defined in $H_k := L^2(\mathbb{R}^n, \mu, \wedge^k\mathbb{C}^n)$ by means of the triple (H_k, V_k, a_k), where

$$a_k(v, w) = (\, d_k v \,|\, d_k w \,)_\mu + (\, d^*_{k-1} v \,|\, d^*_{k-1} w \,)_\mu \qquad (d^*_{-1} \equiv 0) \tag{4.2}$$

$$V_k = \{\, v \in L^2(\mu, \wedge^k\mathbb{C}^n) \;\big|\; d_k v \in H_{k+1}, \quad d^*_{k-1} v \in H_{k-1} \,\}, \tag{4.3}$$

with $a_k(\cdot,\cdot)$ defined on V_k and d_k equal to the exterior differential from $D(d_k) \subset H_k$ to H_{k+1} (with derivatives calculated in $\mathcal{D}'(\mathbb{R}^n)$), while d^*_k denotes the Hilbert space adjoint with respect to $(\cdot\,|\,\cdot)_\mu$, see (4.11) below for the expression. Recall that in this way A_k is defined as follows:

$$D(A_k) = \{\, u \in V_k \;\big|\; \exists f \in H_k : a_k(u, w) = (\, f \,|\, w \,)_\mu \ \forall w \in V_k \,\} \tag{4.4}$$

$$A_k u = f \quad \text{for } u \in D(A_k). \tag{4.5}$$

To substantiate this, note that V_k in (4.3), in view of d^*_k's closedness and differentiation's continuity in $\mathcal{D}'(\mathbb{R}^n)$, is a Hilbert space with

$$\begin{aligned}(\, v \,|\, w \,)_{V_k} &:= (\, v \,|\, w \,)_{H_k} + (\, d_k v \,|\, d_k w \,)_{H_{k+1}} + (\, d^*_k v \,|\, d^*_k w \,)_{H_{k-1}} \\ &= (\, v \,|\, w \,)_\mu + a_k(v, w);\end{aligned} \tag{4.6}$$

the sesqui-linear form is clearly bounded

$$|a_k(v,w)| \le \| v\,|V_k\| \, \| w\,|V_k\| \quad \forall v,w\in V_k. \tag{4.7}$$

Since $a_k(\cdot,\cdot)$ is symmetric and (4.6) yields

$$\operatorname{Re} a_k(v,v) \ge \| v\,|V_k\|^2 - \| v\,|H_k\|^2 \ge 0, \tag{4.8}$$

A_k is well defined, self-adjoint with spectrum in $[0,\infty[$ and with domain $D(A_k)$ dense in V_k. When obtaining this from Lemma 2.1, it is important to have density of $V_k \subset H_k$, but more than that holds in the present set-up:

Lemma 4.1. *The set $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ is dense in each of the domains $D(d_k)$, $D(d_{k-1}^*)$ and $D(d_k)\cap D(d_{k-1}^*) = V_k$ with respect to their graph topologies (for V_k this is given by* (4.6)*).*

This may be proved by a cut-off and convolution procedure, as in [Hör66, Lem. 4.1.3] mutatis mutandem. (Specifically one should let $\Omega = \mathbb{R}^n$, $S = d_k$ and $T^* = d_{k-1}^*$ there, while $\varphi_1, \varphi_2, \varphi_3$ should equal our Φ; the required inequality (4.1.6) there is redundant for we may take $\eta_\nu(x) = \eta(\nu^{-1}x)$ on $\mathbb{R}^n$.)

As a second application of Lemma 4.1 we have a characterisation of A_k:

Lemma 4.2. *A_k equals the Friedrichs extension from $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$.*

Proof. Let S denote A_k's restriction to C_0^∞ and let T be the Friedrichs extension (using that $A_k \ge 0$). The completion of C_0^∞ with respect to

$$\| u\,|V_S\|^2 := (\, Su + (1-m(S))u \,|\, u\,)_\mu \tag{4.9}$$

is a Hilbert space $V_S \subset L^2(\mu, \wedge^k\mathbb{C}^n)$. If $\varphi\in C_0^\infty$ and $c = 1 + m(S)_-$,

$$\| \varphi\,|V_S\|^2 = a_k(\varphi,\varphi) + (1-m(S))\|\varphi\|_\mu^2 \le c\,\| \varphi\,|V_k\|^2 \tag{4.10}$$

so by Lemma 4.1 it is found that $D(A_k)\subset V_k \subset V_S$. Because T is the only self-adjoint extension of S with domain contained in V_S, this yields $A_k = T$. □

As a part of the above-suggested proof of Lemma 4.1 it is found that d_{k-1}^* has the following action on forms $f = \sum'_{|J|=k} f_J\, dz^J$, cf. Section 2, when the derivatives are calculated in the distribution sense:

$$d_{k-1}^* f = \sum_L{}' \Big(\sum_{j=1}^n (\Phi'_j - \partial_j) f_{jL}\Big)\, dz^L. \tag{4.11}$$

For later reference the argument is recalled: if $f \in H_k$ and $w \in C_0^1(\mathbb{R}^n, \wedge^{k-1}\mathbb{C}^n)$,

$$
\begin{aligned}
(\, f \mid dw\,)_\mu &= \int \sum_J{}' f_J \sum_{j=1}^n \sum_L{}' \partial_j \overline{w_L} \cdot \varepsilon_J^{jL} e^{-\Phi}\, dx \\
&= \sum_L{}' \langle \sum_{j,J} -\partial_j (e^{-\Phi} f_J) \varepsilon_J^{jL},\, \overline{w_L}\,\rangle \\
&= \sum_L{}' \langle \sum_j (\Phi'_j - \partial_j) f_{jL},\, e^{-\Phi}\overline{w_L}\,\rangle .
\end{aligned}
\tag{4.12}
$$

This should be justified since w is not C^∞, but by reading $\langle \cdot, \cdot \rangle$ as the duality of $\mathcal{D}'^1$ and C_0^1, the f_J may be approximated from $C_0^\infty(\mathbb{R}^n)$ so that Leibniz' rule may be applied together with the continuity of $\partial_j \colon \mathcal{D}'^0 \to \mathcal{D}'^1$ in the last line.

Taking $f \in D(d_{k-1}^*)$ and w such that $w_L = \delta_{LK}\varphi$ for arbitrary K and $\varphi \in C_0^\infty(\mathbb{R}^n)$, the left hand side equals $\langle\, (d_{k-1}^* f)_K,\, e^{-\Phi}\overline{\varphi}\,\rangle$ so that (4.11) results.

Using Lemma 4.1 once more, we now observe (as a first step in the verification of (1.7)) two different identifications for the domain of d_{k-1}^*:

Lemma 4.3. *For the exterior differential d going from H_{k-1} to H_k, whereby H_j stands for the space $L^2(\mathbb{R}^n, \mu, \wedge^j\mathbb{C}^n)$, and its formal adjoint d^* given by* (4.11)*, the minimal and maximal realisation coincide, i.e. $d_{\min} = d_{\max}$ and $(d^*)_{\min} = (d^*)_{\max}$.*

Indeed, for d itself this is just a reformulation of Lemma 4.1, so by duality $(d^*)_{\max} = (d^*)_{\min} = d_{k-1}^*$. For this reason the true adjoint d_k^* may be abbreviated as d^*.

When it is understood that d and d^* act in the distribution sense (as opposed to their maximal realisations in H_k, viz. d_k and d_{k-1}^*, which have subscripts), it is now easy to infer that A_k acts according to the formula

$$A_k = d^*d + dd^*. \tag{4.13}$$

For this it is advantageous to test against $w = e^\Phi \varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$:

$$a_k(v, w) = (\, A_k v \mid w\,)_\mu = (\, A_k v \mid \varphi\,) \tag{4.14}$$

if $v \in D(A_k)$, while taking $f = d_k v \in H_{k+1}$ in (4.12) yields

$$(\, d_k v \mid d_k w\,)_\mu = \langle\, d^* dv,\, \overline{\varphi}\,\rangle. \tag{4.15}$$

Strictly speaking the right hand side should be read as a sum (over $|J| = k$) of distributions acting on φ_J, cf. (4.12), for the dual of $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ is not considered here. Using the compact support of w it follows from a calculation analogous to (4.12) that, since d^*v is in $\mathcal{D}'^0$ (or rather has coefficients there),

$$\langle\, dd^*v,\ \overline{\varphi}\,\rangle = \sum_J{}'\langle \sum_{j,L} \partial_j (d^*v)_L \varepsilon_J^{jL},\ e^{-\Phi}\overline{w}_J\,\rangle = (\, d^*_{k-1}v \mid d^*_{k-1}w\,)_\mu. \tag{4.16}$$

From the definition of a_k this shows (4.13).

Combining (4.13) with (4.11) a calculation now yields an explicit formula for A_k's action. The details of this will be given partly to verify the expressions for A_0 and A_1 in the introduction, and partly because such formulae may be of interest in their own right.

Since $dv = \sum_K' \sum_{m,M} \partial_m v_M \varepsilon_K^{mM}\, dz^K$, where $|K| = |M| + 1$,

$$\begin{aligned} d^*dv &= \sum_J{}'(\sum_j (\Phi'_j - \partial_j) \sum_K{}' \varepsilon_K^{jJ} (dv)_K)\, dz^J \\ &= \sum_J{}' \sum_{j,m,M} (\Phi'_j - \partial_j)\partial_m v_M \varepsilon_{jJ}^{mM}\, dz^J, \end{aligned} \tag{4.17}$$

while the other contribution becomes, with $|L| = |J| - 1$,

$$dd^*v = \sum_J{}'(\sum_{j,l,L} \partial_j(\Phi'_l - \partial_l) v_{lL}\varepsilon_J^{jL})\, dz^J. \tag{4.18}$$

Taking $j = m$ in (4.17) for fixed J, only $j \notin J = M$ gives a non-trivial term, viz. $(\Phi'_j - \partial_j)\partial_j v_J$; and for $j \neq m$ there are contributions when $j \in M$ and $m \in J$, in which case deletion of j from M and of m from J gives the same tuple, say L, so that

$$\varepsilon_{jJ}^{mM} = \varepsilon_{jmL}^{mM} \cdot \varepsilon_J^{mL} = -\varepsilon_J^{mL} \cdot \varepsilon_{jL}^{M} \tag{4.19}$$

and hence $\partial_m v_M \varepsilon_{jJ}^{mM} = -\partial_m v_{jL}\varepsilon_J^{mL}$. Therefore

$$d^*dv = \sum_J{}'(\sum_{j\notin J}(\Phi'_j - \partial_j)\partial_j v_J + \sum_{j\neq m}\sum_L{}' \varepsilon_J^{mL}(\Phi'_j - \partial_j)\partial_m v_{jL})\, dz^J. \tag{4.20}$$

When $j = l$ in (4.18), then $\varepsilon_J^{jL} = 0$ unless j *belongs* to J, so $\sum_L'$ has only one term $\neq 0$; hence $\sum_{J\ni j}' \sum (\Phi'_j - \partial_j)\partial_j v_J\, dz^J$ is the contribution. For $j \neq l$ there appears a term present in (4.20) with opposite sign, plus one involving Φ''.

Therefore, when $v = \sum_J' v_J\, dz^J$ the action of $d^*d + dd^*$, and hence of A_k, is altogether given by the relatively simple formula

$$(d^*d + dd^*)v = \sum_J{}'((-\Delta + \Phi'\cdot\nabla)v_J + \sum_{j\in J}\sum_{l=1}^{n} \Phi''_{jl} v_{J_{j\to l}})\, dz^J, \tag{4.21}$$

where $J_{j\to l}$ means J with j replaced by l. Note that the degree k of v really only enters in the determination of how Φ'' acts on v. (This formula was also given by Sjöstrand [Sjö96] for the Witten Laplacians ensuing after the transformation in Section 4.2 below.)

In particular, if 1-forms are identified with vector functions,

$$A_1 v = (-\Delta + \Phi'\cdot\nabla) \otimes Iv + \Phi'' \cdot v \tag{4.22}$$

as claimed in the introduction. Note that for $k = 0$ or n the action of A_k is given by (4.17) or (4.18), respectively, that is

$$A_0 u = (-\Delta + \Phi' \cdot \nabla) u \quad \text{for} \quad u \in D(A_0) \tag{4.23}$$

$$A_n f = (-\Delta + \Phi' \cdot \nabla) f + (\Delta\, \Phi) f, \tag{4.24}$$

when f in $L^2(\mathbb{R}^n, \mu, \wedge^n \mathbb{C})$ is (considered as) a function in $D(A_n)$.

4.2. **Unitary Transformation.** The operators A_k are easily transformed into the Witten Laplacians denoted by $\Delta_\Phi^{(k)}$ in [Sjö96]. E.g. multiplication by $e^{-\Phi/2}$ defines a unitary $U\colon L^2(\mu) \to L^2(\mathbb{R}^n)$ transforming A_0 into (a realisation of) $B_0 = -\Delta + \frac14 |\Phi'|^2 - \frac12 \Delta\, \Phi$.

For later reference, this is done consisely here. Writing $d = \sum \partial_j \, dz_j \wedge$ for the differential on k-forms $v = \sum_J' v_J \, dz^J$, where v_J is in $\mathcal{D}'(\mathbb{R}^n)$ in general, the formal adjoint with respect to $(\cdot \,|\, \cdot)_\mu$ on $L^2(\mathbb{R}^n, \mu, \wedge^k \mathbb{C}^n)$ is

$$d^* = \sum_{j=1}^n (\Phi'_j - \partial_j) \, dz_j \rfloor \tag{4.25}$$

whereby $dz_j \rfloor$ either removes dz_j when present (and anti-commuted to the left) or gives zero. When denoting (with subscript k if necessary)

$$U v = \sum_J{}' e^{-\Phi/2} v_J \, dz^J \tag{4.26}$$

$$d_\Phi = \sum_{j=1}^n (\partial_j + \tfrac12 \Phi'_j) \, dz_j \wedge, \qquad d_\Phi^* = \sum_{j=1}^n (-\partial_j + \tfrac12 \Phi'_j) \, dz_j \rfloor, \tag{4.27}$$

then d_Φ^* is the formal adjoint of d_Φ on $L^2(\mathbb{R}^n, \wedge^k \mathbb{C}^n)$ and (on $\mathcal{D}'^1$ forms)

$$U_{k+1} d_k = d_{k,\Phi} U_k, \qquad U_k d_k^* = d_{k,\Phi}^* U_{k+1}. \tag{4.28}$$

Using this, $v \in D(A_k)$ with $v_1 = A_k v$ if and only if for all $w \in V_k$,

$$(\, U v_1 \,|\, U w \,) = (\, d_\Phi U v \,|\, d_\Phi U w \,) + (\, d_\Phi^* U v \,|\, d_\Phi^* U w \,). \tag{4.29}$$

Hence $U A_k U^*$ equals the operator B_k for the tripel $(L^2(\mathbb{R}^n, \wedge^k \mathbb{C}^n), V_{k,\Phi}, b_k)$ where $b_k(\cdot, \cdot) = (\, d_\Phi \cdot \,|\, d_\Phi \cdot \,) + (\, d_\Phi^* \cdot \,|\, d_\Phi^* \cdot \,)$ and $V_{k,\Phi} := U V_k$ equals $D(d_{k,\Phi}) \cap D(d_{k-1,\Phi}^*)$ as subspaces of $L^2(\mathbb{R}^n, \wedge^k \mathbb{C}^n)$; cf. Lemma 2.1.

Using (4.28) it follows that the B_k acts as $\Delta_\Phi^{(k)}$.

4.3. **Identification with the Hodge Laplacian.** Denoting by X_{k+1} the closure of d_k's range in $L^2(\mathbb{R}^n, \mu, \wedge^{k+1} \mathbb{C}^n)$, that is $X_{k+1} = \overline{R(d_k)}$ and $X_0 = \mathbb{C}$ as in (1.9),

$$L^2(\mathbb{R}^n, \mu, \wedge^{k+1} \mathbb{C}^n) = H_{k+1} = X_{k+1} \oplus Z(d_k^*). \tag{4.30}$$

It is now elementary to see that A_k commutes with the orthogonal projections onto the summands, and exploiting Lemma 4.1 once more it also follows that (4.13) may be read with d and d^* as the respective unbounded operators associated with the complex (1.8):

Lemma 4.4. *If P_k is the orthogonal projection onto X_k,*

$$P_k V_k \subset V_k, \qquad P_k A_k \subset A_k P_k. \tag{4.31}$$

Furthermore, the restriction $A_k|_{X_k}$ is injective, and $A_k = d_k^* d_k + d_{k-1} d_{k-1}^*$ holds as a formula for unbounded operators, i.e. with $D(A_k) = D(d_k^* d_k) \cap D(d_{k-1} d_{k-1}^*)$.

Proof. Omitting some k's for simplicity, it follows from (4.30) that $PV \subset V$, since $V = D(d^*) \cap D(d)$. Using this one finds: if $v \in D(A_k)$ and $w \in V$, then $d^*(1-P) \equiv 0$ and $dP \equiv 0$ so that

$$a_1(Pv, w) = (\, d^* P v \mid d^* w\,)_\mu = (\, d^* v \mid d^* P w\,)_\mu = (\, A_k v \mid P w\,)_\mu = (\, P A_k v \mid w\,)_\mu;$$

consequently $PA_k \subset A_k P$. If $v \in X \cap Z(A_k)$, then $0 = (\, A_k v \mid v\,)_\mu = \|d^* v\|_\mu^2$, so v is also in $Z(d_{k-1}^*) = X^\perp$, whence $v = 0$.

If $v \in D(A_k)$ and $A_k v = v_1$ then $(\, P v_1 \mid w\,)_\mu = (\, d^* P v \mid d^* w\,)_\mu$ and $(\,(1-P)v_1 \mid w\,) = (\, d(1-P)v \mid dw\,)_\mu$ for all w in V. Because C_0^∞ is dense with respect to the graph norms in $D(d)$ and $D(d^*)$, this gives by closure that $d^* P v \in D(d)$ with $dd^* P v = P v_1$ and that $d(1-P)v \in D(d^*)$ with $d^* d(1-P)v = (1-P)v_1$; hence that $v \in D(d_1^* d_1) \cap D(d_0 d_0^*)$ with $dd^* v = P v_1$ and $d^* dv = (1-P)v_1$. Conversely $d_k^* d_k + d_{k-1} d_{k-1}^* \subset A_k$ follows easily from (4.2)–(4.3). □

Note that by (4.31) the subspaces X_k and $Z(d_{k-1}^*)$ are invariant under A_k, and that the terms $d_k^* d_k$ and $d_{k-1} d_{k-1}^*$ vanish there, respectively.

Also both $d_k A_k$ and $A_{k+1} d_k$ are defined on $D(d_{k+1} d_k^* d_k)$, so in this way we have the intertwining properties

$$A_{k+1} d_k = d_k A_k, \qquad A_{k-1} d_{k-1}^* = d_{k-1}^* A_k \tag{4.32}$$

on $D(d_k d_k^* d_k)$ and $D(d_{k-1}^* d_{k-1} d_{k-1}^*)$, respectively, for the unbounded operators, as well as in general in the distribution sense.

Since Lemma 4.4 shows that the A_k of this section coincide with (1.7) above, it is clear that Theorem 1.3 holds for the operators given in (4.2)–(4.5) and (4.21).

4.4. **A direct H^1-proof.** The injectiveness of $A_1|_X$ shown in Lemma 4.4 may be used for a short proof of Theorem 1.4's essential parts. This is done in the spirit of [Hel98, Sjö96], but now for our general Φ and with much sharper assumptions:

Proposition 4.5. *Suppose* (1.5) *holds and that A_0 defined above satisfies:*

$$R(A_0) = \overline{R(A_0)} \quad in \quad H_0. \tag{4.33}$$

Then it holds true for all $g_1,\ g_2 \in H^1(\mu)$ that

$$(g_1 - \langle g_1\rangle \mid g_2 - \langle g_2\rangle)_\mu = (A_1^{-1} dg_1 \mid dg_2)_\mu. \tag{4.34}$$

Proof. Since $A_0 = A_0^*$, with closed range by (4.33) and $Z(A_0) = \mathbb{C}$ by (4.4)–(4.5), there is a decomposition $L^2(\mu) = R(A_0) \oplus \mathbb{C}$; hence $g_1 - \langle g_1\rangle = A_0 f$ for some $f \in D(A_0)$, so according to (4.4),

$$(g_1 - \langle g_1\rangle \mid g_2 - \langle g_2\rangle)_\mu = a_0(f, g_2 - \langle g_2\rangle) = (df \mid dg_2)_\mu. \tag{4.35}$$

In the distribution sense $d^*df = A_0 f$, since f is picked in $D(A_0)$; therefore we moreover have for $w \in C_0^\infty(\mathbb{R}^n, \wedge^1\mathbb{C}^n)$,

$$a_1(df, w) = 0 + (A_0 f \mid -\operatorname{div}(e^{-\Phi} w)) = \lim_{k\to\infty} \langle dA_0 f, \overline{\varphi_k} \rangle = (dg_1 \mid w)_\mu \tag{4.36}$$

when $\varphi_k \in C_0^\infty$ tends to $e^{-\Phi} w$ in the V_1-topology.

By completion (4.36) also holds for every $w \in V_1$, cf. the density in Lemma 4.1 and (4.7), so it follows that $df \in D(A_1)$ with $A_1^{-1} dg_1 = df$ (using the injectivity of Lemma 4.4). This and (4.35) yields the proof. □

In addition to the above, we may observe that using (4.34), partial integration gives for each test function φ

$$(Pu \mid \varphi)_\mu = (Pu \mid P\varphi)_\mu = (d^* A_1^{-1} du \mid \varphi)_\mu, \tag{4.37}$$

which shows (1.23) when $u \in H^1(\mu)$. Since P is bounded in $L^2(\mu)$, we can extend $d^* A_1^{-1} d$ by continuity such that (1.23) holds by definition on $L^2(\mu)$.

A closer analysis given in Section 5.3 below will show that each of the individual factors in $d^* A_1^{-1} du$ are well defined too, when $u \in L^2(\mu)$.

4.5. **Brascamp–Lieb's inequality.** When Φ is strictly convex, Corollary 1.5 may now be proved for $f \in H^1(\mu)$, i.e.

$$\| f - \langle f\rangle \,|L^2(\mu)\|^2 \le ((\Phi'')^{-1} df \mid df)_\mu. \tag{4.38}$$

To begin with it is first assumed that $\Phi''(x) \ge c_0 > 0$ in $\mathbb{C}^n$ for each $x \in \mathbb{R}^n$. Partial integration shows that on C_0^∞ we have

$$(A_1 v \mid v)_\mu = a_1(v, v) = \sum_{j,k=1}^n \|\partial_j v_k\|_\mu^2 + (\Phi'' v \mid v)_\mu \ge c_0 \|v\|_\mu^2 \tag{4.39}$$

so $m(A_1) \ge c_0 > 0$ follows by Lemma 4.2. Then Theorem 1.3 yields that A_0 restricted to $\mathbb{C}^\perp$ has 0 in the resolvent, hence that (4.33) and (4.34) hold.

Because A_1^{-1} is symmetric ≥ 0, Cauchy–Schwarz' inequality (for such operators) applied to $(A_1^{-1}v \mid A_1 w)_\mu = (v \mid w)_\mu$ for $v \in L^2(\mu, \wedge^1\mathbb{C})$ and $w \in D(A_1)$ yields

$$|(v \mid w)_\mu|^2 \leq (A_1^{-1}v \mid v)_\mu (w \mid A_1 w)_\mu \tag{4.40}$$

with equality if $v = A_1 w$. Hence

$$(A_1^{-1}v \mid v)_\mu = \sup\Big\{ \frac{|(v \mid w)_\mu|^2}{(A_1 w \mid w)_\mu} \;\Big|\; w \in D(A_1) \setminus \{0\} \Big\}, \tag{4.41}$$

and analogously for Φ'', so (4.39) and the density of $D(A_1)$ and C_0^∞ in V give

$$(A_1^{-1}v \mid v)_\mu = \sup\Big\{ \frac{|(v \mid w)_\mu|^2}{a_1(w, w)} \;\Big|\; w \in C_0^\infty(\mathbb{R}^n, \wedge^1\mathbb{C}^n) \setminus \{0\} \Big\} \leq ((\Phi'')^{-1}v \mid v)_\mu \tag{4.42}$$

[regardless of whether C_0^∞ is dense in $D(\Phi'')$], which proves (4.38) in this case.

In general this applies for $0 < \varepsilon < 1$ to

$$\Phi_\varepsilon(x) = \Phi(x) + \varepsilon|x|^2 + \log C_\varepsilon \quad \text{with} \quad C_\varepsilon = \int e^{-\Phi(x)-\varepsilon|x|^2}\,dx, \tag{4.43}$$

which is uniformly strictly convex with $\int d\mu_\varepsilon = 1$; note that $C_\varepsilon \nearrow 1$ for $\varepsilon \searrow 0$. Clearly (4.38) holds with μ_ε instead of μ; because $e^{-\varepsilon|x|^2-\log C_\varepsilon} \leq C_1^{-1}$,

$$\int |f - \textstyle\int f e^{-\Phi_\varepsilon}\,dx|^2 e^{-\Phi_\varepsilon}\,dx \longrightarrow \|f - \langle f\rangle\|_\mu^2 \tag{4.44}$$

by majorised convergence for $\varepsilon \searrow 0$. Indeed, in this way $\int f e^{-\Phi_\varepsilon}\,dx$ tends to $\langle f\rangle$ and the whole left hand side is controlled by $(|f(x)| + \| f \,|L^1(\mu)\|)^2 e^{-\Phi(x)}$.

Being positive, $I_\varepsilon := \nabla f^T (\Phi_\varepsilon'')^{-1}\overline{\nabla f}$ always has an integral; if this is finite for $\varepsilon = 0$ then (4.38) must be proved. But then I_0 itself may serve as a majorant, and because $\Phi_\varepsilon''(x) \geq \Phi''(x)$ in $\mathbb{B}(\ell^2(\{1, \dots, n\}))$ for all x,

$$I_\varepsilon(x)e^{-\varepsilon|x|^2-\log C_\varepsilon} \leq I_0(x)/C_1. \tag{4.45}$$

Pointwise convergence is clear from the norm continuity of inversion. This completes the proof for $f \in H^1(\mu)$.

5. Extensions to Integrable Functions

5.1. **Sufficient Conditions Revisited.** Instead of merely establishing estimates sufficient for the full proof of Theorem 1.4, their relation to the other conditions treated is given below, for they all fit so well together that a discussion should be of interest in its own right; the following ten conditions are elucidated (in and) after the proof. Note that subscripts are suppressed on d and d^* when the context makes it clear what the domain is.

Theorem 5.1. *For $A_k = d_k^* d_k + d_{k-1} d_{k-1}^*$ with $k > 0$ it holds true that*

$$\begin{gathered}(1),\ (2) \text{ and } (3) \text{ are equivalent}\\ (3) \implies (4) \implies (5)\\ (5),\ (6),\ (7),\ (8),\ (9) \text{ and } (10) \text{ are equivalent},\end{gathered} \tag{5.1}$$

when the properties (1)–(10) *are as follows:*

(1) $0 < m(A_k) := \inf\{\, (A_k v \mid v)_\mu \mid v \in D(A_k),\ \|v\|_\mu = 1, \}$;

(2) *the norms* $a_k(\cdot,\cdot)^{1/2}$ *and* $\|\cdot \mid V_k\|$ *are equivalent on* V_k;

(3) $\|v\|_\mu^2 \le c^2(\|d^* v\|_\mu^2 + \|dv\|_\mu^2)$ *for all k-forms* $v \in D(d^*) \cap D(d)$;

(4) $\|v\|_\mu^2 \le c^2(\|d^* v\|_\mu^2 + \|dv\|_\mu^2)$ *for all k-forms* $v \in D(d^*) \cap Z(d)$;

(5) $\|v\|_\mu^2 \le c^2(\|d^* v\|_\mu^2 + \|dv\|_\mu^2)$ *for all k-forms* $v \in D(d^*) \cap X_k$;

(6) $\|v\|_\mu \le c\|d^* v\|_\mu$ *for all k-forms* $v \in D(d^*) \cap X_k$;

(7) *the range of* $d\colon D(d_{k-1}) \to L^2(\mu, \wedge^k \mathbb{C})$ *is closed, i.e.* $R(d_{k-1}) = X_k$;

(8) *the range* $R(A_{k-1}|_{Z(d)^\perp})$ *is closed in* $L^2(\mu, \wedge^{k-1})$;

(9) *the norms* $\|d^* \cdot \|_\mu$ *and* $\|\cdot \mid V_k\|$ *are equivalent on* $V_k \cap X_k$;

(10) $0 < m(A_k|_{X_k}) := \inf\{\, (A_k v \mid v)_\mu \mid v \in D(A_k) \cap X_k,\ \|v\|_\mu = 1, \}$.

In the affirmative case $m(A_k) = m(A_k|_{X_k}) \ge c^{-2}$, *where c is any of the constants in* (3)–(6).

Moreover, the closed forms in $L^2(\mu, \wedge^k \mathbb{C}^n)$ *belong to* $R(d_{k-1})$, *i.e.* $Z(d_k) = R(d_{k-1})$, *when any of* (1)–(4) *holds.*

Proof. (1) $\iff$ (2) because they are both equivalent to (3) in view of $D(A_k)$'s density in V_k and (4.2)–(4.6). In addition $m(A_1) \ge c^{-2}$ is found.

Now (3) $\implies$ (4) $\implies$ (5) $\iff$ (6) by shrinking of the set F_1 of v's from $D(d)$ to X_k (and since $d \equiv 0$ on X). That (6) $\iff$ (7) $\iff$ (8) follows from Theorem 3.1 with $F_1 = X_k$; this moreover gives that $R(d_{k-1}) = Z(d_k)$ when (4) holds.

Finally (6) trivially gives (9), and (9) $\implies$ (10) is clear. When (10) holds, the inequality in (6) is valid in the subspace $D(A_k) \cap X_k$. Therefore, if $v_0 \in V_k \cap X_k$ then $v_m \underset{m}{\to} v_0$ in V_k for some $v_m \in D(A_k)$ where $P_k v_m \to v_0$ in $L^2(\mu, \wedge^k \mathbb{C}^n)$ with $P_k v_m \in D(A_k) \cap X_k$ by Lemma 4.4. Using (10) and $d^*(1 - P) \equiv 0$,

$$\|v_0\|_\mu \le c \lim_m \|d^* P v_m\|_\mu = c \lim_m \|d^* v_m\|_\mu = c\|d^* v_0\|_\mu. \tag{5.2}$$

Consequently (6) holds. □

While (8) is the central point because of its implications for Theorem 1.4, (1) and (10) are probably the most convenient to verify for a given $\Phi(x)$; but if this can be done, then A_k^{-1} and $A_k|_{X_k}^{-1}$ extend automatically, by the Poincaré inequalities (2), (9) and 2° of Lemma 2.1, to bounded operators from certain spaces of distributions of order 1 (or of forms with such coefficients).

Since (3) $\implies$ (4) $\implies$ (5), it is clear that (4) is an intermediate property in comparison with the two situations described by (1)–(3) on the one hand and (5)–(10) on the other.

Actually condition (4) is equivalent to exactness of the d-complex at $L^2(\mu, \wedge^k\mathbb{C}^n)$: $Z(d_k)$ is closed in $L^2(\mu, \wedge^k\mathbb{C}^n)$ because d has the maximal domain there and is continuous in $\mathcal{D}'$, so Theorem 3.1 with $F_1 = Z(d_k)$ shows that (4) holds if and only if $R(d_{k-1}) = F_1$.

Injectiveness of A_k is furthermore a consequence of (4). For by the Lax–Milgram definition of A_k, cf. (4.2),

$$Z(A_k) = Z(d^*_{k-1}) \cap Z(d_k), \tag{5.3}$$

so (4) implies $Z(A_k) = \{0\}$. In addition, if (5) holds but (4) doesn't, then there is some $v \in Z(d_k) \setminus R(d_{k-1})$; writing this as $v = x + x^\perp$ with $x \in X_k$ and $x^\perp \in X_k^\perp$, then $x \in R(d_{k-1})$ since (5) implies (7), whence $0 \neq x^\perp \in Z(d_k) \cap Z(d^*_{k-1})$. So when A_k is injective, then either (4), i.e. exactness, holds or $R(A_{k-1}|_{Z(d)^\perp})$ is unclosed.

5.2. **Proof preparations.** As mentioned, (2) and (9) imply the extendability of A_1^{-1} and $A_1|_X^{-1}$ to larger spaces than just the L^2-forms, which is crucial for Theorem 1.4:

Corollary 5.2. 1° *Let* (2) *in Theorem 5.1 hold. One has then* $A_k \subset \mathcal{A}_k$*, when* $\mathcal{A}_k \colon V_k \xrightarrow{\sim} V_k'$ *is the linear isometry from* $(V_k, a_k(\cdot,\cdot))$ *onto its dual* V' *given by*

$$\langle \mathcal{A}_k v, \cdot \rangle_{V_k'\times V_k} = a_k(\cdot, \overline{v}) \quad \textit{for} \quad v \in V_k; \tag{5.4}$$

and $D(A_k) = \mathcal{A}_k^{-1}(H_k)$ *holds.*

2° *When* (9) *holds, then* $A_k|_{X_k} \subset \tilde{\mathcal{A}}_k$*, when* $\tilde{\mathcal{A}}_k$ *is the isomorphism* $\tilde{V}_k \xrightarrow{\sim} (\tilde{V}_k)'$*, with* $\tilde{V}_k$ *denoting* $V_k \cap X_k$ *normed by* $(\, d^*v \mid d^*v\,)_\mu^{1/2}$.

Observe that when also $H_k \simeq H_k'$ in *this* manner (i.e. $g \mapsto (\,\cdot \mid \overline{g}\,)_\mu$ instead of $(\,\cdot \mid g\,)$), then there are inclusions $V_k \overset{\iota}{\hookrightarrow} H_k \overset{\iota'}{\hookrightarrow} V_k'$ with dense ranges and ι' equal to the transpose of ι:

$$\langle \iota' f,\, v \rangle_{V_k'\times V_k} = (\, \iota v \mid \overline{f}\,)_\mu = (\, f \mid \overline{\iota v}\,)_\mu \quad \text{for all } f \in H_k, v \in V_k. \tag{5.5}$$

Similarly $\tilde{V}_k \overset{\iota}{\hookrightarrow} X_k \overset{\iota'}{\hookrightarrow} (\tilde{V}_k)'$. Thus it is meaningful to state the last part of 1° or the corresponding fact that $D(A_k|_{X_k}) = \tilde{\mathcal{A}}_k^{-1}(X_k)$.

Proof. $a_k(u, \overline{w}) = (\, A_k u \mid \overline{w}\,)_\mu$ $\forall w \in V_k$, so (5.5) gives 1°. For 2°, identify $A_k|_{X_k}$ with the operator $\tilde{A}_k$ defined from $(X_k, \tilde{V}_k, (\, d^* \cdot \mid d^* \cdot)_\mu)$; completeness of $\tilde{V}_k$ and denseness carry over from (H_k, V_k, a_k) by means of P_k in Lemma 4.4. While $A_k|_{X_k} \subset \tilde{A}_k$ is clear, any $v \in D(\tilde{A}_k)$ gives $(\, \tilde{A}_k v \mid w\,)_\mu = (\, d^*v \mid d^*w\,)_\mu$ for all w in V_k since $\tilde{A}_k v \perp (1 - P_k)w$ and $d^*(1 - P_k)w = 0$. Since $v \in X_k$ we first get $(\, \tilde{A}_k v \mid w\,)_\mu = a_k(v, w)$, thereafter $\tilde{A}_k \subset A_k$. $\square$

When applying this we shall need that V_k' or $\tilde{V}_k'$ can receive the image $d(H_{k-1})$. To establish this it is necessary to make a precise identification of H_k, V_k' and $\tilde{V}_k'$ with subspaces of $\mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$, which roughly speaking consists of forms $\sum' u_J\,dz^J$ with $u_J \in \mathcal{D}'(\mathbb{R}^n)$.

In Appendix A below this is introduced concisely by means of a direct approach based on the finite dimension of $\wedge^k\mathbb{C}^n$ and the simplicity of the manifold $\mathbb{R}^n$. This should provide the reader with an alternative to the general and vast expositions of G. de Rham [dR55] and L. Schwartz [Sch59] and [Sch66, Ch. 9].

By the continuity of $\mathcal{J}$ in (A.1), the linear form $\Lambda_f(\varphi) := \sum' \int f_J\varphi_J\,dx$, defined on $\varphi \in C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$, gives injections $L^2(\mu, \wedge^k\mathbb{C}^n) \subset L^2_{\mathrm{loc}}(\wedge^k\mathbb{C}^n) \overset{\Lambda}{\hookrightarrow} \mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ for which

$$\langle\, \Lambda_f,\, \varphi\,\rangle_{\mathcal{D}'\times C_0^\infty} = \int_{\mathbb{R}^n} f\varphi\,dx = (\, f \mid \overline{e^\Phi\varphi}\,)_\mu = \langle\, \iota' f,\, e^\Phi\varphi\,\rangle_{V_k'\times V_k}, \tag{5.6}$$

so Λ is the transpose of $M_{e^{-\Phi}}\colon C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n) \to H_k$, multiplication by $e^{-\Phi}$.

From the last identity above $\Lambda\colon H_k \to \mathcal{D}'$ is seen to be continuous in the topology induced by V_k', hence it extends to V_k' by the density of $H_k \subset V_k'$:

Proposition 5.3. *The operator* Λ *introduced above* (5.6) *extends by continuity to an embedding* $\Lambda\colon V_k' \hookrightarrow \mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ *and for this it holds that*

$$\langle\, v',\, e^\Phi\varphi\,\rangle_{V_k'\times V_k} = \langle\, v',\, \varphi\,\rangle_{\mathcal{D}'\times C_0^\infty}. \tag{5.7}$$

for every $v' \in V_k'$ *and* $\varphi \in C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ *(when* Λ *is suppressed).*

Proof. By taking closures, (5.7) clearly follows from the left- and rightmost parts of (5.6). Because M_{e^Φ} has dense range in V_k', the extended Λ is an injection. □

The point of this proposition and (5.7) is of course to note the factor e^Φ.

From the boundedness of $d^*\colon V_k \to H_{k-1}$ follows the existence of a bounded transpose $d^{*\prime}\colon H_{k-1} \to V_k'$, and by means of (5.7) and (4.11) this is seen to be a realisation of the distributional differential d: indeed for $f \in H_{k-1}$ and elements of the dense subset $C_0^2 \subset V_k$ of the form $e^\Phi\varphi$ with φ in C_0^∞,

$$\begin{aligned}\langle\, d^{*\prime} f,\, e^\Phi\varphi\,\rangle_{V_k'\times V_k} &= \int \sum_J{}' f_J \sum_{j=1}^n (\Phi_j' - \partial_j)(e^\Phi\varphi_{jJ})e^{-\Phi}\,dx \\ &= \sum_{j,J} \langle\, \partial_j f_J,\, \varphi_{jJ}\,\rangle = \langle\, d_{k-1} f,\, \varphi\,\rangle,\end{aligned} \tag{5.8}$$

where the last identity uses (A.5), (A.7); by (5.7) this means that $d^{*\prime} f = df$.

The space $\tilde{V}_k'$ is normed by $\|\cdot\, |V_k\|$ and moreover a closed subspace of V_k'; indeed $\tilde{V}_k' = P_k' V_k'$ because $P_k \in \mathbb{B}(V_k)$ (the last fact follows from $d^*(1 - P_k) \equiv 0$).

For one thing this gives an embedding $\tilde{V}_k' \hookrightarrow \mathcal{D}'$ by the above construction for V_k', and for another that $d(H_{k-1}) \subset \tilde{V}_k'$. Indeed, d_{k-1}^* restricts to a continuous map $\tilde{V}_k \to H_{k-1}$, so d_{k-1}

extends to a bounded operator $H_{k-1} \to \tilde{V}_k'$; however, this is not a proper extension since $\tilde{V}_k' \subset V_k'$. Altogether we have:

Proposition 5.4. *The distributional differential is bounded $d\colon H_{k-1} \to V_k'$; it is the transpose of $d^*\colon V_k \to H_{k-1}$, so for $f \in H_{k-1}$ and $\varphi \in C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$*

$$\langle df, \varphi \rangle_{\mathcal{D}'\times C_0^\infty} = \langle df, e^{\Phi}\varphi \rangle_{V_k'\times V_k} = (\, f \mid d^*(e^{\Phi}\varphi)\,)_\mu. \tag{5.9}$$

Moreover, the range $d(H_{k-1})$ is contained in the subspace $(V_k \cap X_k)' = P_k' V_k'$.

Remark 5.5. Considering $\mathbb{R}^n$ as a manifold, it would be possible to use the C^2-density furnished by the measure $\mu = e^{-\Phi}\,dx$ (see e.g. [Hör85, Ch. 6] for the notions), but it is preferable to use the Lebesgue integral, for this gives an extension of the usual embeddings, such as $L^2(\mu) \subset L^2_{\mathrm{loc}}(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n)$.

5.3. **Continuation of Proofs.** When $R(A_0)$ is closed the implication (8) $\Longrightarrow$ (9) of Theorem 5.1 allows a renorming of $\tilde{V} := V \cap X$ such that $A_1|_X \subset \tilde{\mathcal{A}}_1$; cf. 2° of Corollary 5.2. From Proposition 5.4 it follows that $d^*\tilde{\mathcal{A}}_1^{\,-1} d$ is bounded

$$L^2(\mu) \longrightarrow (\tilde{V})' \longrightarrow \tilde{V} \longrightarrow L^2(\mu), \tag{5.10}$$

and it coincides with $d^*A_1^{-1}d$ in $H^1(\mu)$ by the remark after (5.5) and therefore with $u \mapsto u - \int u\,d\mu$ by (4.37); extension by continuity gives (1.23) for all $u \in L^2(\mu)$. Closure of (4.34) similarly yields (1.2): indeed, for $g_j \in H^1(\mu)$ one can take $v = A_1|_{X_1}^{-1} dg_1$ and $f = d\overline{g_2}$ in formula (5.5) so (4.34) (and the obvious interpretation of gradients as differentials) gives

$$\begin{aligned}(\, A_1^{-1}\nabla g_1 \mid \nabla g_2\,)_\mu &= (\, A_1|_{X_1}^{-1} dg_1 \mid dg_2\,)_\mu = \langle d\overline{g_2}, \tilde{\mathcal{A}}_1^{-1} dg_1 \rangle_{\tilde{V}'\times\tilde{V}} \\ &= (\, Pg_1 \mid Pg_2\,)_\mu.\end{aligned} \tag{5.11}$$

The last equality extends to all g_j in $L^2(\mu)$ in view of the density of $H^1(\mu)$ and the continuity of $d\colon L^2(\mu) \to \tilde{V}'$ and $\tilde{\mathcal{A}}_1^{-1} d\colon L^2(\mu) \to \tilde{V}$; cf. Proposition 5.4.

For the Brascamp–Lieb inequality (4.38) with $f \in L^2(\mu) \cap H^1_{\mathrm{loc}}$, (4.39) still shows that $A_1 > 0$ in the uniformly strictly convex case; from (1) $\Longrightarrow$ (8) of Theorem 5.1 it follows that (1.2) is available for $g_j = f$. From (1) $\Longrightarrow$ (2) we see that 1° of Corollary 5.2 applies. Since $\mathcal{A}_1$ is an isometry, $a_1(\mathcal{A}_1^{-1}\cdot, \mathcal{A}_1^{-1}\cdot)$ is an inner product on V' inducing the norm, so for $v \in V'$,

$$\langle v, \overline{\mathcal{A}_1^{-1} v} \rangle_{V'\times V} = \| v \,|V'\|^2 = \sup\Big\{ \frac{|\langle v, \overline{w} \rangle|^2}{a_1(w, w)} \;\Big|\; w \in C_0^\infty(\mathbb{R}^n, \wedge^1\mathbb{C}^n) \setminus \{0\} \Big\}. \tag{5.12}$$

Because of the H^1_{loc}-condition, $(\nabla f)^T(\Phi'')^{-1}\overline{\nabla f}$ is a well defined function belonging to L^1_{loc}; if it has finite integral, then $v = df$ is in $D(\Phi'') \subset L^2(\mu, \wedge^1\mathbb{C}^n)$ so that (4.39) may be invoked as in the argument for (4.42), which hence also holds in this case.

When Φ is merely strictly convex, the reduction to the uniform case carries over verbatim.

6. Criteria for Closed Range

Because $Z(A_0)$ has finite dimension, the closed range requirement in (4.33) is satisfied when $0 \notin \sigma_{\mathrm{ess}}(A_0)$, which holds when $\Phi(x)$ is well behaved near ∞:

Proposition 6.1. *If Φ in addition to* (1.5) *satisfies*

(I) $\frac{1}{2}|\Phi'|^2 - \Delta\,\Phi \ge c > 0$ *in a neighbourhood of* ∞,

then $0 \notin \sigma_{\mathrm{ess}}(A_0)$, so (4.33) *and the projection identity* (4.34) *hold.*

Proof. A_0 is equivalent to $B_0 = -\Delta + \frac{1}{4}|\Phi'|^2 - \frac{1}{2}\Delta\,\Phi$, cf. Section 4.2. The latter is essentially self-adjoint by Kato's result [Kat73], so the Persson–Agmon formula [Per60], [Agm78, Thm. 3.2] may without ambiguity be used for the estimate:

$$
\begin{aligned}
\inf \sigma_{\mathrm{ess}}(A_0) &= \sup_{K \subset\subset \mathbb{R}^n} \inf\big\{ \, (B_0\varphi \mid \varphi) \bigm| \varphi \in C_0^\infty(\mathbb{R}^n \setminus K), \quad \|\varphi\| = 1 \,\big\} \\
&\ge \inf\big\{ \, ((\tfrac{1}{4}|\Phi'|^2 - \tfrac{1}{2}\Delta\,\Phi)\varphi \mid \varphi) \bigm| \varphi \in C_0^\infty(\mathbb{R}^n \setminus K_0), \ \|\varphi\| = 1 \,\big\} \\
&\ge c/2 > 0
\end{aligned}
\tag{6.1}
$$

when (I) holds in $\mathbb{R}^n \setminus K_0$. This yields (4.33). □

In addition, (I) combined with a growth condition implies the stronger fact that $\sigma_{\mathrm{ess}}(A_0) = \emptyset$, as shown below. Note that whenever $0 < \eta < 1$ and $\theta \in \,]\eta, 1[$,

$$
\theta|\Phi'|^2 - \Delta\,\Phi = (\eta|\Phi'|^2 - \Delta\,\Phi) + (\theta - \eta)|\Phi'|^2, \tag{6.2}
$$

so if $|\Phi'(x)| \to \infty$ for $|x| \to \infty$ and the first term on the right hand side is known to have finite infimum, consequently the left hand side tends to ∞ for $x \to \infty$. Taking $\eta = 1/2$, this shows that (I) together with $\lim_{|x|\to\infty} |\Phi'(x)| = \infty$ implies condition (II) below. Similarly one finds that (II) is more general than the results one would obtain from the techniques of J.-M. Kneib and F. Mignot in [KM94, Lem. 5] (where the proof contains a minor flaw).

Proposition 6.2. *If Φ satisfies the following condition*

(II) $\exists \theta \in \,]0,1[\colon\ \lim_{|x|\to\infty} \theta|\Phi'(x)|^2 - \Delta\,\Phi(x) = \infty$,

then $H^1(\mu) \hookrightarrow L^2(\mu)$ is compact, and consequently $\sigma_{\mathrm{ess}}(A_0) = \emptyset$.

That (II) is sufficient may be proved along the lines of P. Bolley, Dauge and Helffer [BDH89] (even directly, that is without the unitary transformation in Section 4.2); because of this reference's inaccessibility we shall supply the details.

Proof. Introducing the vector fields $X_j = \partial_j$ and their formal adjoints $X_j^* = -\partial_j + \Phi'_j$, one has when $u \in C_0^\infty(\mathbb{R}^n)$ for their sum and commutator

$$
(X_j + X_j^*)u = \Phi'_j u, \qquad [X_j, X_j^*]u = \Phi''_{jj} u. \tag{6.3}
$$

Now it is straightforward to see that

$$([X_j, X_j^*]u \mid u)_\mu = \|X_j^* u\|_\mu^2 - \|X_j u\|_\mu^2 \tag{6.4}$$

$$\|(X_j + X_j^*)u\|_\mu^2 \le (1 + \tfrac{1}{\varepsilon})\|X_j u\|_\mu^2 + (1+\varepsilon)\|X_j^* u\|_\mu^2 \quad \forall \varepsilon > 0, \tag{6.5}$$

so that a linear combination of these formulae gives for any $\varepsilon > 0$

$$((|\Phi'|^2 - (1+\varepsilon)\,\Delta\,\Phi)u \mid u)_\mu \le (2 + \varepsilon + \varepsilon^{-1})(\|X_1 u\|_\mu^2 + \cdots + \|X_n u\|_\mu^2). \tag{6.6}$$

Because $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mu)$, this inequality is valid for all $u \in H^1(\mu)$. Indeed, letting $\mu' = (|\Phi'|^2 - (1+\varepsilon)\,\Delta\,\Phi)\mu$, we infer from (6.6) that a fundamental sequence in $H^1(\mu)$ also converges in $L^2(\mathbb{R}^n, \mu')$, and necessarily to the same limit since both spaces are embedded into $\mathcal{D}'(\mathbb{R}^n)$.

If $u_k \to u$ weakly in $H^1(\mu)$, assumption (II) implies that $\Psi := |\Phi'|^2 - (1+\varepsilon)\,\Delta\,\Phi$ is positive in a neighbourhood of ∞ when $\theta = (1+\varepsilon)^{-1}$, so by (6.6),

$$\begin{aligned}\int |u_k|^2 e^{-\Phi}\,dx &\le \int_{|x|<R} |u_k|^2 e^{-\Phi}\,dx + \int_{|x|\ge R} \frac{\Psi |u_k|^2}{\inf_{\mathbb{R}^n \setminus B(0,R)} \Psi} e^{-\Phi}\,dx \\ &\le C_\Phi \,\|\, u_k \,|L^2(B(0,R))\|^2 + \frac{C_\varepsilon \,\|\, u_k \,|H^1(\mu)\|^2}{\inf\{\, \Psi(y) \mid |y| \ge R \,\}}.\end{aligned} \tag{6.7}$$

Hence (II) and the compactness of $H^1(B(0,R)) \hookrightarrow L^2(B(0,R))$ show that a subsequence of u_k tends to 0 in $L^2(\mu)$.

If $\lambda \in \sigma_{\mathrm{ess}}$ there is $u_k \in D(A_0)$ such that $\|u_k\|_\mu = 1$ while $u_k \to 0$ weakly and $\|(A_0 - \lambda)u_k\|_\mu \to 0$. Since

$$\begin{aligned}\|\, u_k \,|H^1(\mu)\| &= \|\, u_k \,|L^2(\mu)\| + a_0(u_k, u_k) \\ &\le 1 + \|(A_0 - \lambda)u_k\|_\mu \|u_k\|_\mu + |\lambda| \|u_k\|_\mu^2 \le 1 + |\lambda| + \mathcal{O}(1),\end{aligned} \tag{6.8}$$

the sequence (u_k) is bounded in $H^1(\mu)$, but without convergent subsequences in $L^2(\mu)$, so the embedding is non-compact. Thus $\sigma_{\mathrm{ess}}(A_0) = \emptyset$ is shown. □

7. A Pseudo-differential View Point

As shown in the following, a few extra assumptions on $\Phi(x)$ lead to domain characterisations, essential self-adjointness of the A_j and positivity of A_1 (in addition to closed ranges).

Actually C^∞-smoothness with a little control of the higher order derivatives is enough to invoke the calculus in [Hör85, 18.4–6], and in this framework A_0 and A_1 are easily seen to be Fredholm operators if $|\Phi'|$ tends to ∞ at infinity. Therefore it is assumed in this section that

(III) $\Phi \in C^\infty(\mathbb{R}^n, \mathbb{R})$,

(IV) $|\Phi'(x)| \to \infty$ for $|x| \to \infty$,

(V) for $|\alpha| \geq 1$ there are constants C_α such that

$$|D^\alpha \Phi(x)| \leq C_\alpha (1 + |\Phi'(x)|^2)^{1/2}, \tag{7.1}$$

(VI) $D^\beta \Phi$ is bounded on $\mathbb{R}^n$ when β has a fixed length, say $M \in \mathbb{N}$.

This implies that $\Phi(x)$ is slowly increasing, $\Phi \in \mathcal{O}_M(\mathbb{R}^n)$, so Φ of, say exponential growth is ruled out; thus the stronger conclusions of this section have their price.

7.1. **An auxiliary Schrödinger operator.** To exploit (III)–(VI) above, we shall henceforth work in the unweighted space $L^2(\mathbb{R}^n)$ and with the Witten-Laplacians ensuing after the unitary transformation in Section 4.2; that is

$$\Delta^{(0)}_\Phi = -\Delta + \tfrac{1}{4}|\Phi'|^2 - \tfrac{1}{2}\,\Delta\,\Phi \tag{7.2}$$

$$\Delta^{(1)}_\Phi = (-\Delta + \tfrac{1}{4}|\Phi'|^2 - \tfrac{1}{2}\,\Delta\,\Phi) \otimes I + \Phi''. \tag{7.3}$$

These operators are more precisely given their maximal domains in $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n, \wedge^1\mathbb{C}^n)$, respectively, in the distribution sense.

For convenience one can here study the auxiliary operator

$$P = -\Delta + V_0, \quad \text{where} \quad V_0(x) = \tfrac{1}{4}|\Phi'|^2, \tag{7.4}$$

with the domain

$$D(P) = \left\{\, u \in H^2(\mathbb{R}^n) \;\middle|\; V_0 \cdot u \in L^2(\mathbb{R}^n) \,\right\}. \tag{7.5}$$

To analyse this, let the pseudo-differential operators $p(x, D)$ and $q(x, D)$ have symbols

$$p(x,\xi) = |\xi|^2 + V_0(x), \qquad q(x,\xi) = ((1-\chi(x,\xi))p(x,\xi) + \chi(x,\xi))^{-1} \tag{7.6}$$

where $\chi \in C_0^\infty(\mathbb{R}^{2n})$ is positive and $\equiv 1$ on a compact set K such that

$$(x,\xi) \notin K \implies p(x,\xi) \geq 1. \tag{7.7}$$

This makes $q(x,\xi)$ well defined in $C^\infty(\mathbb{R}^{2n})$.

The calculus in [Hör85, Ch. 18.4–6] applies to this case with

$$p(x,\xi) \in S(m^2, g), \qquad q(x,\xi) \in S(m^{-2}, g) \tag{7.8}$$

when the weight and metric equal, respectively,

$$m(x,\xi) = (1 + |\xi|^2 + |\Phi'(x)|^2)^{1/2}, \qquad g = |dx|^2 + \frac{|d\xi|^2}{m(x,\xi)^2}; \tag{7.9}$$

Hörmander's notation and terminology is used here and below. When applying this theory, condition (VI) is posed in order to show that g is σ-temperate.

From the calculus we next infer that $q(x, D)$ acts as a parametrix of $p(x, D)$, i.e.

$$p(x,D)q(x,D) = 1 - K_1(x,D), \qquad q(x,D)p(x,D) = 1 - K_2(x,D) \tag{7.10}$$

where $K_j \in \mathrm{OPS}(m^{-1}, g)$. By [Hör85, Thm. 18.6.6] the K_j are compact in $L^2(\mathbb{R}^n)$ because m^{-1} according to (IV) tends to 0 at infinity, for with the choice of g made above we have $g \le g^\sigma$.

To see the latter fact, note that by definition

$$g^\sigma_{x,\xi}(y,\eta) = \sup\Big\{ \frac{|\langle \eta, \hat{y}\rangle - \langle y, \hat{\eta}\rangle|^2}{g_{x,\xi}(\hat{y},\hat{\eta})} \;\Big|\; (\hat{y},\hat{\eta}) \in \mathbb{R}^{2n} \setminus \{(0,0)\} \Big\} \tag{7.11}$$

so the isometry of the Hilbert space $(\mathbb{R}^{2n}, g_{x,\xi}(\cdot,\cdot))$ onto its dual gives

$$\begin{aligned} g^\sigma_{x,\xi}(y,\eta) &= \sup_{(\hat{y},\hat{\eta})\neq 0} \left| \frac{g_{x,\xi}((\eta, -m(x,\xi)^2 y), (\hat{y},\hat{\eta}))}{|(\hat{y},\hat{\eta})|_{g_{x,\xi}}} \right|^2 \\ &= (|(\eta, -m(x,\xi)^2 y)|_{g_{x,\xi}})^2 \\ &= |\eta|^2 + m(x,\xi)^2 |y|^2 = m(x,\xi)^2 g_{x,\xi}(y,\eta). \end{aligned} \tag{7.12}$$

This shows for one thing the claim that $g \le g^\sigma$, because $m \ge 1$, and for another that

$$h(x,\xi) := \sup (g_{x,\xi}/g^\sigma_{x,\xi})^{1/2} = m(x,\xi)^{-1}. \tag{7.13}$$

Using [Hör85, 18.5.10] we can pass to the Weyl calculus and conclude that

$$\begin{aligned} p(x,D) &= a^W(x,D), \qquad a(x,\xi) = e^{-i\langle D_x, D_\xi\rangle/2} p(x,\xi) \\ q(x,D) &= b^W(x,D), \qquad b(x,\xi) = e^{-i\langle D_x, D_\xi\rangle/2} q(x,\xi) \end{aligned} \tag{7.14}$$

with the remainder information that, since $h \cdot m^2 = m$,

$$a(x,\xi) = p(x,\xi) + R_1(p), \qquad a \in S(m^2, g),\ R_1(p) \in S(m,g) \tag{7.15}$$

$$b(x,\xi) = q(x,\xi) + R_1(q), \qquad b \in S(m^{-2}, g),\ R_1(q) \in S(m^{-3}, g) \quad . \tag{7.16}$$

Moreover, by [Hör85, 18.4.3] and [Hör85, 18.5.4],

$$R_1(p)q + pR_1(q) \in S(m^{-1}, g) \tag{7.17}$$

$$R_1(a,b) \in S(hm^{-2}m^2, g) = S(m^{-1}, g). \tag{7.18}$$

This gives finally, by the compact support of χ,

$$pq = 1 + \chi(p-1)((1-\chi)p + \chi)^{-1} \in S(m^{-1}, g), \tag{7.19}$$

and hence the relations

$$\begin{aligned} p(x,D)q(x,D) &= (a\#b)^W(x,D) \\ &= (ab + R_1(a,b))^W(x,D) \\ &= (pq + R_1(p)q + pR_1(q) + R_1(a,b))^W(x,\xi) \\ &= 1 - K_1(x,\xi) \end{aligned} \tag{7.20}$$

with $K_1(x,\xi)$ in $S(m^{-1}, g)$ as claimed. Similarly, because $R_1(b,a) \in S(m^{-1}, g)$, one finds that $q(x,\xi)p(x,\xi) - 1 \in \mathrm{OPS}(m^{-1}, g)$.

Note that in a similar fashion one has:

Proposition 7.1. *Let g be a σ-temperate metric fulfilling $g \le g^\sigma$ and $g_{x,\xi}(t,\tau) \equiv g_{x,\xi}(t,-\tau)$, and let m_1 and m_2 be σ, g-temperate weights. Then*

$$\mathrm{OPS}(m_1, g) \cdot \mathrm{OPS}(m_2, g) \subset \mathrm{OPS}(m_1 m_2, g). \tag{7.21}$$

Proof. One may use [Hör85, Th. 18.5.10] and the remark thereafter to express $p(x,D)q(x,D)$ as $(a\#b)^W(x,D)$ and then apply 18.5.4 and 18.5.10. □

Obviously the maximal domain of $p(x,D)$ as an operator in $L^2(\mathbb{R}^n)$ is the set $D(p)$ consisting of those $u \in L^2$ for which also $p(x,D)u \in L^2$. However this equals $D(P)$ in (7.5): for if $u \in D(p)$ then (7.10) yields, with $f = p(x,D)u$ in L^2,

$$u - K_2(x,D)u = q(x,D)f \tag{7.22}$$

and by application of $1 + K_2(x,D)$

$$u = (1 + K_2(x,D))q(x,D)f + K_2(x,D)^2 u; \tag{7.23}$$

by the proposition both $K_2(x,D)^2$ and $(1 + K_2(x,D))q(x,D)$ are in $\mathrm{OPS}(m^{-2}, g)$, so that $D^\alpha u$ and $V_0 u$ are in L^2 when $|\alpha| \le 2$ by combined application of Proposition 7.1 and [Hör85, 18.6.3]. Since also $(\Phi'_{j_1}(x))^{k_1}(\Phi'_{j_2}(x))^{k_2}\xi^\alpha$ is in $S(m^2, g)$ when $k_1+k_2+|\alpha| \le 2$, it is seen that $(\Phi')^\beta D^\alpha u \in L^2$, so that we for later use have the more precise result:

Lemma 7.2. *Both $D(P)$ and $D(p(x,D)_{\max})$ coincide with the space of u for which*

$$(\Phi')^\beta D^\alpha u \in L^2(\mathbb{R}^n) \tag{7.24}$$

for all α and $\beta \in \mathbb{N}_0^n$ for which $|\alpha| + |\beta| \le 2$; and $(\sum_{|\alpha+\beta|\le 2} |(\Phi')^\beta D^\alpha u|^2)^{1/2}$ is equivalent to the graph norm of P.

Thereby $P = p(x,D)_{\max}$, so by duality and symmetry of P,

$$p(x,D)_{\min} \supset P^* \supset P \supset p(x,D)_{\max} \tag{7.25}$$

so that $D(P)$ is both the minimal and maximal domain of $p(x,D)$. Consequently P is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

Furthermore $D(P)$ is a Hilbert space in P's graph norm with P and $q(x,D)$ belonging to $\mathbb{B}(D(P), L^2)$ and $\mathbb{B}(L^2, D(P))$, respectively. For this reason $\dim\operatorname{coker} P \le \dim\operatorname{coker}(1 - K_1) < \infty$, where the last inequality is obtained from the compactness of K_1, and hence $R(P)$ is necessarily closed; cf. [Hör85, 19.1.1].

Returning to $\Delta_\Phi^{(0)}$, the perturbation $-\Delta\,\Phi(x)$ of P is easily handled; first we show that the domain of $\Delta_\Phi^{(0)}$ equals $D(P)$. Clearly it contains $D(P)$ in view of (V) (when this is used to get the elementary estimate $|\Delta\Phi| \le c(1+|\Phi'|^2)^{1/2} \le c(1+|\Phi'|^2)$, which entails $|u\,\Delta\Phi| \le c(1+|\Phi'|^2)|u| \in L^2$), and if $f := \Delta_\Phi^{(0)}\,u$ is in L^2 for some $u \in L^2$, one has in $\mathcal{S}'(\mathbb{R}^n)$

$$q(x,D)f = (1 - K_2(x,D) - \tfrac{1}{2}q(x,D)\,\Delta\Phi(x))u \tag{7.26}$$

where

$$K_2'(x,D) = K_2(x,D) + \tfrac{1}{2}q(x,D)\,\Delta\Phi(x) \in \mathrm{OPS}(m^{-1},g). \tag{7.27}$$

By application of $1 + K_2'(x,D)$

$$u = (1 + K_2'(x,\xi))q(x,D)f + K_2'(x,D)^2u, \tag{7.28}$$

so, like for P above, we find that u is in H^2 with $|\Phi'|^2u$ in L^2.

For the range we get that

$$\Delta_\Phi^{(0)}\,q(x,D) = p(x,D)q(x,D) - \tfrac{1}{2}\,\Delta\Phi(x)q(x,D) = 1 - K_1'(x,D) \tag{7.29}$$

where $K_1'(x,D) \in \mathrm{OPS}(m^{-1},g)$ since both $K_1(x,D)$ and $(\Delta\Phi)q(x,D)$ are so; the latter fact is by Proposition 7.1 because $\Delta\Phi$ is in $S(m,g)$ in view of condition (V). Using (IV) as above $K_1'(x,D)$ is compact in L^2, and since $q(x,D)$ maps all of $L^2(\mathbb{R}^n)$ into $D(\Delta_\Phi^{(0)})$ by the just shown characterisation of this set, $\Delta_\Phi^{(0)}$ has closed range.

Using that $D(\Delta_\Phi^{(0)}) = D(P)$, it is straightforward to see that $\Delta_\Phi^{(0)}$ is self-adjoint: for if $u \in D(\Delta_\Phi^{(0)})$, then $d_\Phi u$ is in L^2 by (V) and the just shown characterisation of $D(-\Delta_\Phi^{(0)})$, and in addition we may by Lemma 4.1 approximate any v in the domain of d_Φ by functions $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ and get

$$(\,d_\Phi u \mid d_\Phi v\,) = \lim_{k\to\infty}\langle\, -\Delta\,u + \tfrac{|\Phi'|^2}{4}u - \tfrac{\Delta\,\Phi}{2}u,\ \overline{\varphi_k}\,\rangle = (\,-\Delta_\Phi^{(0)}\,u \mid v\,), \tag{7.30}$$

so that $\Delta_\Phi^{(0)}$ coincides with the self-adjoint Lax–Milgram operator B_0 defined in Section 4.2. Because of this it is moreover essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

Altogether we have the following:

Theorem 7.3. *Let $\Phi(x)$ have the properties in* (1.5) *and* (III)–(VI) *above. Then A_0 has closed range in $L^2(\mathbb{R}^n,\mu,\mathbb{C})$, so that the conclusions of Theorem 1.4 are valid, and A_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.*

Moreover, $D(A_0)$ consists of the functions u for which $(\Phi')^\beta D^\alpha u$ belongs to $L^2(\mathbb{R}^n,\mu)$ for all α and $\beta \in \mathbb{N}_0^n$ such that $|\alpha+\beta| \le 2$.

Observe that the closed range, and even $\sigma_{\mathrm{ess}}(A_0) = \emptyset$, is an immediate consequence of Proposition 6.2 since (IV)–(V) imply condition (II) there.

However, the density of $C_0^\infty(\mathbb{R}^n)$ in the graph norm has only been obtained because the pseudo-differential techniques made an analysis of the maximal domains possible.

7.2. **Applications to A_1.** Using the same line of thought as for A_0 one finds:

Theorem 7.4. *Let $\Phi(x)$ satisfy* (1.5) *together with* (III)–(VI) *above. Then the domain of A_1 is given by*

$$D(A_1) = \Big\{ \sum_{j=1}^{n} v_j dz^j \;\Big|\; \forall j\colon |\alpha+\beta| \le 2 \implies (\Phi')^{\beta} D^{\alpha} v_j \in L^2(\mu, \mathbb{R}^n) \Big\}, \tag{7.31}$$

and A_1 is essentially self-adjoint from $C_0^\infty(\mathbb{R}^n, \wedge^1\mathbb{C}^n)$ and has closed range.

Indeed, that $D(\Delta_\Phi^{(1)})$ contains $D(\Delta_\Phi^{(0)}) \times \cdots \times D(\Delta_\Phi^{(0)})$ (cf. the set in (7.31)) is clear by (7.3) and (V); if conversely $w := \Delta_\Phi^{(1)} v$ is in $L^2(\mathbb{R}^n, \wedge^1\mathbb{C}^n)$ for some v there, the procedure in (7.26)–(7.28) gives, when $q(x, D)$ is tensored with I,

$$v = ((1 + \tilde{K}_2(x, D))q(x, D)) \otimes I w + \tilde{K}_2(x, D)^2 v, \tag{7.32}$$

where $\tilde{K}_2(x, D)$ equals $K_2'(x, D) \otimes I - q(x, D) \otimes I \cdot \Phi''$ and has all of its entries in $\mathrm{OPS}(m^{-1}, g)$. Therefore the inclusion from the left to the right in (7.31) follows.

When applying $q(x, \xi) \otimes I$ as a right parametrix we find

$$\Delta_\Phi^{(1)}(q(x, D) \otimes I) = (1 - K_1'(x, D)) \otimes I + \Phi''(q(x, D) \otimes I) =: I - \tilde{K}_1(x, D) \tag{7.33}$$

where each entry of $\tilde{K}_1(x, D)$ is in $\mathrm{OPS}(m^{-1}, g)$, and hence compact in $L^2(\mathbb{R}^n)$. Now $q \otimes I$ sends $L^2(\mathbb{R}^n, \wedge^1\mathbb{C}^n)$ into $D(\Delta_\Phi^{(1)})$, so this shows that $\Delta_\Phi^{(1)}$ has closed range; cf. the argument for P above.

To show the self-adjointness one can identify $\Delta_\Phi^{(1)}$ with $U A_1 U^*$, where $U = U \otimes I$; cf. Section 4.2. Now d_Φ and d_Φ^* map any $v \in D(\Delta_\Phi^{(1)})$ into L^2 by the characterisation of $D(\Delta_\Phi^{(1)})$, so as in (7.30) one finds that $\Delta_\Phi^{(1)}$ is contained in the self-adjoint Lax–Milgram operator defined from $(L^2(\mathbb{R}^n, \wedge^1\mathbb{C}^n), V_\Phi, b_1)$. Hence $\Delta_\Phi^{(1)}$ equals this as well as the minimal realisation of the expression in (7.3). Consequently $\Delta_\Phi^{(1)}$ and A_1 are essentially self-adjoint.

Note that $D(A_1) \hookrightarrow H^1(\mu)^n$ by Lemma 7.2, so that $\sigma_{\mathrm{ess}}(A_1)$ is empty.

Injectivity of A_1 may be obtained in the set-up above as soon as (IV) is strengthened to a specific growth rate at infinity; that is when (IV) is replaced by:

(IV_ω) There exist $\omega > 0$ and $C > 0$ such that

$$x \cdot \Phi'(x) \ge \tfrac{1}{C}|x|^{1+\omega} \quad \text{for} \quad |x| \ge C. \tag{7.34}$$

Since $C|\Phi'| \ge |x|^\omega$ holds a fortiori, A_1 is then moreover strictly positive because of the closed range obtained in Theorem 7.4:

Theorem 7.5. *Let Φ satisfy* (1.5), (III)–(VI) *and* (IV_ω). *Then $A_1 > 0$ on $L^2(\mathbb{R}^n, \mu, \wedge^1\mathbb{C}^n)$.*

Proof. As remarked it suffices to show injectivity of $\Delta_\Phi^{(1)}$, and for this it is enough that

$$Z(\Delta_\Phi^{(1)}) \subset \Big\{ \sum v_j \, dz_j \;\Big|\; v_j \in \mathcal{S}(\mathbb{R}^n) \,\forall j \Big\} =: \mathcal{S}(\mathbb{R}^n, \mathbb{C}^n). \tag{7.35}$$

Indeed, given $v \in Z(\Delta_\Phi^{(1)}) \cap \mathcal{S}(\mathbb{R}^n, \mathbb{C}^n)$, condition ($\mathrm{IV}_\omega$) will imply that

$$f(x) = \int_0^1 e^{-(\Phi(x)-\Phi(tx))/2} x \cdot v(tx)\, dt \tag{7.36}$$

defines an element $f(x)$ of $L^2(\mathbb{R}^n)$ for which $d_\Phi f = v$; since $v \in Z(\Delta_\Phi^{(1)}) \subset Z(d_\Phi^*) = R(d_\Phi)^\perp$ this will give $v = 0$ as desired.

It is straightforward to see that $f \in C^0(\mathbb{R}^n)$ for such v, and (7.36) gives, cf. (4.26),

$$U^* f(x) = \int_0^1 x \cdot U^* v(tx)\, dt \tag{7.37}$$

hence that $dU^* f = U^* v$; therefore $f = e^{-\Phi/2} U^* f$ is in $C^\infty(\mathbb{R}^n)$ by (7.35) and (III). This also yields $v = U d U^* f = d_\Phi U U^* f = d_\Phi f$ as claimed.

That f is in $L^2(\mathbb{R}^n)$ follows if $|x|^k f(x)$ is bounded for large k. Using (IV_ω) we get, for $|x| > C$ and $t|x| > C$,

$$\Phi(x) - \Phi(tx) = \int_t^1 x \cdot \Phi'(sx)\, ds \geq C^{-1} |x|^{1+\omega} \int_t^1 s^\omega \, ds = \tfrac{|x|^{1+\omega}}{C(1+\omega)} (1 - t^{1+\omega}), \tag{7.38}$$

so the fact that $(1-t)^{1+\omega} \leq 1 - t^{1+\omega}$ on $[0, 1]$ (since these functions are convex and concave, respectively, on this interval) leads to the conclusion that

$$\Phi(x) - \Phi(tx) \geq \frac{|x - tx|^{1+\omega}}{C(1+\omega)} \quad \text{for} \quad |x|,\ |tx| \geq C. \tag{7.39}$$

Next one may for each $k \in \mathbb{N}$ deduce the existence of a constant C_1 such that when $|x| \geq C$, then (with $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ in the rest of this proof)

$$\langle x - y \rangle^k e^{-(\Phi(x)-\Phi(y))/2} \leq C_1 \quad \text{for all} \quad y \in \operatorname{ch}\{0, x\}. \tag{7.40}$$

when $\operatorname{ch} A$ denotes the convex hull of A. Indeed, if $|y| \geq C$ the inequality in (7.39) yields that the left hand side, when $r := \langle x - y \rangle \in \mathbb{R}_+$, is estimated by

$$(r^\beta e^{-r})^\gamma \quad \text{for some} \quad \beta,\ \gamma > 0; \tag{7.41}$$

when $|y| < C$ one can let $z = \frac{C}{|x|} x$ and reduce to the case $|y| \geq C$, using the inequalities

$$\langle x - y \rangle^k \leq \langle x \rangle^k \leq \langle z \rangle^k \langle x - z \rangle^k \leq (1 + C^2)^k \langle x - z \rangle^k \tag{7.42}$$

$$\exp(\Phi(y) - \Phi(z)) \leq \exp(\sup_{|y| \leq C} \Phi - \inf_{|z| \leq C} \Phi) < \infty. \tag{7.43}$$

For $f(x)$ this now gives, since $|x| < C$ is easy,

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} |x|^k |f(x)| \leq \sup_{x \in \mathbb{R}^n} \int_0^1 & \langle x - tx \rangle^{k+1} e^{-(\Phi(x)-\Phi(tx))/2}\, dt \\ & \times \sup\Big\{ \langle y \rangle^{k+1} |v(y)| \;\Big|\; y \in \mathbb{R}^n \Big\} < \infty. \end{aligned} \tag{7.44}$$

It remains, therefore, to show (7.35). If $v \in Z(\Delta_\Phi^{(1)})$, then $v = \tilde{K}_2(x,D)v$, hence

$$v = \tilde{K}_2(x,D)^N v \quad \text{for every} \quad N \in \mathbb{N}; \tag{7.45}$$

cf. (7.32) ff. Since $\tilde{K}_2(x,D)^N$ by Proposition 7.1 has entries in $\operatorname{OPS}(m^{-N}, g)$, it follows that $|\Phi'(x)|^N D^\beta v_j(x)$ is in $L^2(\mathbb{R}^n)$ for all multiindices β and all j and N. Moreover,

$$\langle x\rangle^N \le C^{-1}|\Phi'(x)|^{N/\omega} \le C^{-1}\langle\Phi'(x)\rangle^{N'} \le C'(1+|\Phi'(x)|^{N'}) \tag{7.46}$$

when $N' \ge N/\omega$; cf. (IV_ω). It is thus shown that v has coefficients in $\mathcal{S}(\mathbb{R}^n)$, and altogether this shows the theorem. □

7.3. **Example.** Consider the potential, as done by Helffer [Hel98, Hel97a] and many others,

$$\Phi(x) = \tfrac{1}{h}\sum_{j=1}^{n}(\tfrac{\lambda}{12}x_j^4 + \tfrac{\nu}{2}x_j^2) + \tfrac{1}{h}\tfrac{\mathcal{I}}{2}\sum_{j=1}^{n}|x_j - x_{j+1}|^2 \tag{7.47}$$

whereby $x_{n+1} = x_1$ as a convention. Here $h > 0$ and $\mathcal{I} > 0$ while

$$\lambda > 0 > \nu. \tag{7.48}$$

Therefore $\Phi(x)$ is not convex, so the Brascamp–Lieb inequality does not apply to this case. The condition $\int e^{-\Phi}\,dx = 1$ may be fulfilled by adding an h-dependent constant. Moreover, (III) and (V) clearly hold. Concerning (IV_ω), Euler's formula gives

$$x\cdot\Phi'(x) \ge \tfrac{1}{h}\sum(\tfrac{\lambda}{3}x_j^4 + \nu x_j^2) = \tfrac{|x|^4}{h}(\tfrac{\lambda}{3}(\tfrac{1}{n})^2 - |\nu||x|^{-2}) \ge C^{-1}|x|^4 \tag{7.49}$$

when $|x| \ge C$ for some sufficiently large $C = C(h,n,\lambda,\nu)$; here it is used that $|x_j|^2 \ge |x|^2/n$ for some $j \in \{1,\dots,n\}$. Consequently (IV_ω) holds for all $\omega \in \,]0,3]$ for the above $\Phi(x)$. (By comparison, the assumptions in [Sjö96] are unfulfilled since the Φ''_{jk} are unbounded on $\mathbb{R}^n$.)

Because of this, the corresponding operators A_0 and A_1 have the properties given in Theorems 1.4, 7.3, 7.4 and 7.5, in particular (1.2) holds because $A_1 > 0$.

The lower bound $m(A_1)$ can moreover, for certain h and $\mathcal{I}$, be estimated in various ways, see for example [Hel98, Hel97a].

8. Final Remark

The essential self-adjointness of A_0 and A_1 (or A_k) holds in a greater generality than that established in Section 7. For scalar Schrödinger operators this is well known from works of T. Kato [Kat73] and S. Agmon [Agm78], but especially C. Simader's note [Sim78] appears useful for an extension to 'systems' like $\Delta_\Phi^{(k)}$. In fact, Simader's argument for $-\Delta + V$ specialised to the case $V \in C^0(\mathbb{R}^n, \mathbb{R})$ appears in a recent lecture note [Hel99, Thm. 9.4.1], and in this form it is straightforward to carry over to $-\Delta\otimes I + V$ with $V \in C^0(\mathbb{R}^n, \mathbb{R}^{n^2})$, when this operator is positive on $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, hence to $\Delta_\Phi^{(1)}$ and A_1.

However, the domain characterisations and the corollary on the compact resolvent (in particular of A_1) should in any case motivate the given applications of the Weyl calculus.

Appendix A. Forms with distributions as coefficients

The general framework for distribution-valued differential forms, so-called currents, is given by G. de Rham [dR55] and L. Schwartz [Sch59]. However, the definition of E-valued distributions as continuous linear maps $\mathcal{D}(\Omega) \to E$ given in [Sch59, Ch. 1 §2] leads to severe difficulties (cf. the introduction of [Sch59]) in the proof that $\mathcal{D}'(\Omega, E)$ is the dual of $C_0^\infty(\Omega, E')$; for the finite-dimensional example $E = \wedge^k\mathbb{C}^n$, a much more direct approach is given in Schwartz' book [Sch66, Ch. 9] where differential forms on manifolds are treated.

In the present article where $\Omega = \mathbb{R}^n$ is a flat, oriented manifold, further simplifications are given below for the reader's sake. The definition of $\mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ as the dual of $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ is a little unconventional (testfunctions valued in $\wedge^{n-k}\mathbb{C}^n$ is common), but this choice is consistent with the made identification of $L^2(\mathbb{R}^n, \mu, \wedge^k\mathbb{C}^n)$ and its dual.

For precision, $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ denotes the compactly supported, infinitely differentiable maps $\mathbb{R}^n \to \wedge^k\mathbb{C}^n$ (i.e. into the space of anti-symmetric k-linear forms on $\mathbb{C}$). The canonical coordinates $z_1, \dots, z_n$ in $\mathbb{C}^n$ lead to a basis for $\wedge^k\mathbb{C}^n$ consisting of $dz^J := dz_{j_1} \wedge \dots \wedge dz_{j_k}$, where $J = (j_1, \dots, j_k)$ is an increasing k-tuple. Therefore any $\varphi \in C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ equals $\sum' \varphi_J \, dz^J$ with unique $\varphi_J \in C_0^\infty(\mathbb{R}^n)$. Thus there is a bijection

$$\mathcal{J}\colon C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n) \to \prod_{1 \le j \le \binom{n}{k}} C_0^\infty(\mathbb{R}^n); \tag{A.1}$$

and there is a unique topology on the domain which makes $\mathcal{J}$ a homeomorphism, when the codomain has the product topology. (For brevity, indexation on $\prod$ is suppressed below.)

The dual of $\mathcal{J}$'s codomain is isomorphic to $\prod \mathcal{D}'(\mathbb{R}^n)$, for any continuous linear functional F acts on $\varphi = (\varphi_j)$ as $F(\varphi) = \sum F \circ I_j(\varphi_j)$ where I_j sends φ_j into $(0, \dots, \varphi_j, \dots, 0)$; since $F \circ I_j$ is continuous it is in $\mathcal{D}'(\mathbb{R}^n)$.

Now $\mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ may be defined as the dual of $C_0^\infty(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$; equipping dual spaces with their w*-topologies, there is by transposition a linear homeomorphism

$$\mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n) \xleftarrow{\mathcal{J}'} \prod \mathcal{D}'(\mathbb{R}^n), \tag{A.2}$$

so for (u_J) in $\prod \mathcal{D}'(\mathbb{R}^n)$ and $\varphi = \sum' \varphi_J \, dz^J$ in $C_0^\infty(\mathbb{R}^n \wedge^k \mathbb{C}^n)$,

$$\langle \mathcal{J}'(u_J), \varphi \rangle = \langle (u_J), \mathcal{J}\varphi \rangle = \sum{}' \langle u_J, \varphi_J \rangle. \tag{A.3}$$

Indeed, $\mathcal{J}'$ is surjective because any $u \in \mathcal{D}'(\mathbb{R}^n, \wedge^k\mathbb{C}^n)$ gives rise to the continuous linear functional $u \circ \mathcal{J}^{-1}$, which is in $\prod \mathcal{D}'(\mathbb{R}^n)$, in view of (A.1) ff, so that for some (u_J) it holds for all

(φ_J) in $\prod C_0^\infty(\mathbb{R}^n)$ that

$$\langle\, u,\ \mathcal{J}^{-1}(\varphi_J)\,\rangle = u\circ\mathcal{J}^{-1}(\varphi_J) = \langle\,(u_J),\ (\varphi_J)\,\rangle = \langle\,\mathcal{J}'(u_J),\ \mathcal{J}^{-1}(\varphi_J)\,\rangle. \tag{A.4}$$

Therefore $u=\mathcal{J}'(u_J)$; the rest of (A.2) is straightforward.

It is natural to write $u=\sum{}' u_J\, dz^J$ instead of $u=\mathcal{J}'(u_J)$, and thereby (A.3) attains the following more intuitive form,

$$\langle\, u,\ \varphi\,\rangle = \langle\,\sum{}' u_J\, dz^J,\ \sum{}'\varphi_J\, dz^J\,\rangle = \sum{}'\langle\, u_J,\ \varphi_J\,\rangle. \tag{A.5}$$

The usual denseness of C_0^∞ in $\mathcal{D}'$ carries over to the k-form-valued spaces by (A.1) and (A.2), and therefore multiplication, M_ψ, by $\psi\in C^\infty(\mathbb{R}^n)$ and the operators ∂_j extend in a unique way to $\mathcal{D}'(\mathbb{R}^n,\wedge^k\mathbb{C}^n)$ as usual. More precisely, M_ψ and ∂_j are both continuous on $C_0^\infty(\mathbb{R}^n,\wedge^k\mathbb{C}^n)$ because their definitions show that they act on each φ_J (i.e. they commute with $\mathcal{J}$), so the transposed operators $(M_\psi)'$ and ∂_j' act as M_ψ and $-\partial_j$ on each u_J in (A.5), respectively, and they are therefore denoted by the latter symbols hereafter.

In this way one finds that $\mathcal{D}'(\mathbb{R}^n,\wedge^k\mathbb{C}^n)$ is a $C^\infty(\mathbb{R}^n)$-module and that any differential operator $P(\partial)$ with coefficients in $C^\infty(\mathbb{R}^n)$ is well defined by its action on each u_J (and independent of the canonical choice of dz^J); transposition moreover follows the usual rule. In particular this hold for the exterior derivative d_k, and for this the identies

$$d^2 := d_{k+1}\circ d_k \equiv 0 \quad\text{on}\quad \mathcal{D}'(\mathbb{R}^n,\wedge^k\mathbb{C}^n), \tag{A.6}$$

$$d(\sum{}' f_J\, dz^J) = \sum_J{}'\sum_{j=1}^n \partial_j f_J\, dz^j\wedge dz^J \tag{A.7}$$

are obtained by transposition and by closure from $C_0^\infty(\mathbb{R}^n,\wedge^k\mathbb{C}^n)$, respectively.

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