The Griffiths Singularity Random Field

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Abstract

We consider a spin system on sites of a d-dimensional cubic lattice ($d \geq 2$), with the values $0, 1$ or -1. It is built over the Bernoulli site percolation model, with spins taking the value 0 on empty sites, and taking values ± 1 on occupied sites according to the ferromagnetic Ising model

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distribution on the occupied clusters. The Hamiltonian corresponds to the nearest neighbor interaction under external field h, at inverse temperature β, and the boundary conditions for clusters are free. When the probability p for a site to be occupied is small enough, so that a.s. all the clusters of non-0 spins are finite, this description gives rise to a unique random field. We show that it is non-Markovian, and when p is small, β is large and $h = 0$, it is even non-Gibbsian, but only almost Gibbsian. This provides another example of a non-Gibbsian, but almost Gibbsian, random field which emerges naturally in a Gibbsian context. Our random field is directly related to, and motivated by, the model studied by Griffiths in connection to what became known as the phenomenon of Griffiths' singularities.

1. Introduction

In this note we show a new connection between the phenomenon of Griffiths singularities and the general theory of Gibbs random fields (Gibbs measures). We construct a Griffiths Singularity Random Field – or *GriSing random field* – with a three-valued single-site space (describing plus, minus or empty sites), based on the famous Griffiths example of a random Ising model with a quenched dilution disorder. We investigate the possibility of describing this field by means of a model with annealed disorder. More precisely, we investigate if it is possible to represent this random field as a Gibbs random field. We find that the answer is negative: our random field is *not* a Gibbs field. Thus it can not be described by a "well-behaved" interaction, although it is an "almost Gibbs" field.

We remark that ideas and techniques developed for the study of disordered statistical mechanics models in the Griffiths singularity regime [8] or the random field Ising model [3], have already been applied to the study both of non-Gibbsian fields [2, 9, 10, 14], and also of Gibbs fields with significant many-body interactions [1].

2. Definition of the GriSing field

Let $0 < p < 1$ be fixed. Let $\eta_t, t \in \mathbb{Z}^d$ be the Bernoulli random field of independent random variables, taking values 1 with probability p and 0 with probability $1-p$. We denote by $\mathbb{P}(\eta)$ the corresponding probability distribution.

Consider now for each configuration η the Ising model on $\mathbb{Z}^d$, defined by the formal Hamiltonian

$$H_\eta(\sigma) = -\sum_{|s-t|=1} (\eta_s \eta_t)\, \sigma_s \sigma_t - h \sum_s \eta_s \sigma_s. \tag{2.1}$$

Let us denote by $\mathbb{P}_{\beta,h,\eta}(\sigma)$ the Gibbs state of this model defined by the Hamiltonian (2.1) at inverse temperature β with empty (free) boundary conditions. By this we mean the following. For every site $s \in \mathbb{Z}^d$ we define the cluster $C_\eta(s) \subset \mathbb{Z}^d$ as the set of all sites $t \in \mathbb{Z}^d$, for which there exists a chain $s_1, ...s_k$ of sites such that $s_1 = s$, $s_n = t$, and $\prod_{i=1}^n \eta_i = 1$. By definition, $s \in C_\eta(s)$. Let $\mathcal{C}_\eta$ be the partition of the lattice $\mathbb{Z}^d$ into these clusters of various sites. Then by definition

$$\mathbb{P}_{\beta,h,\eta}(\sigma) = \prod_{C \in \mathcal{C}_\eta} \mathbb{P}_{\beta,h,C}(\sigma|_C),$$

where by $\mathbb{P}_{\beta,h,V}(\sigma)$, $\sigma \in \Omega_V$ we denote the Gibbs state in the box $V \subset \mathbb{Z}^d$ defined by the standard Hamiltonian

$$H_V(\sigma) = -\sum_{\substack{s,t \in V, \\ |s-t|=1}} \sigma_s \sigma_t - h \sum_{s \in V} \sigma_s \tag{2.2}$$

and empty boundary conditions. No further comments are needed if the box V is finite. In the case of an infinite box V we define

$$\mathbb{P}_{\beta,h,V} = \lim_{n \to \infty} \mathbb{P}_{\beta,h,V_n}. \tag{2.3}$$

Here V_n is the intersection of the box V with the cube B_n of side $2n$, centered at the origin. The existence of the last limit follows from the second Griffiths inequality.

We now define the joint distribution $\mathbb{P}_{\beta,h}(\sigma, \eta)$ by the relation

$$\mathbb{P}_{\beta,h}(d\sigma, d\eta) = \mathbb{P}_{\beta,h,\eta}(d\sigma)\, \mathbb{P}(d\eta), \tag{2.4}$$

provided we know the measurability of the measure $\mathbb{P}_{\beta,h,\eta}(d\sigma)$ as a function of η. This measurability is a consequence of the continuity properties proven below.

The *GriSing random field* ξ is now defined by

$$\xi_t = \sigma_t \eta_t.$$

In words, we first specify which sites are empty and which are occupied by the Ising spins. The empty/occupied events happen independently at each site with probabilities $1-p$ and p. The empty sites get values $\xi = 0$. We then specify the values of the Ising spins σ_t at the occupied sites t according to the Gibbs distributions on the occupied clusters, corresponding to the Hamiltonian (2.2) with empty boundary conditions. We then put $\xi_t = \sigma_t$.

It is very tempting to believe that the random field ξ is actually a Markov field with nearest-neighbor dependence. But this appearance is deceptive.

Before explaining this, let us formulate another notable property of our field, which follows from the result of Griffiths. As an example, consider the function

$$m\left(h\right) = \int \xi_0 \mathbb{P}_{\beta,h}\left(d\xi\right) \equiv \int_{\eta_0=1} \left\langle \sigma_0 \right\rangle_{\beta,h,C_0(\eta)} \mathbb{P}(d\eta), \tag{2.5}$$

with $\langle\cdot\rangle_{\beta,h,C_0(\eta)}$ the expectation in the Ising model on the cluster $C_0(\eta)$ of the origin. This last expression coincides of course with the usual double average in a quenched disordered system of a function of the spin degrees of freedom (here the spin at the origin) with a random interaction (disorder described by η). At low temperatures, as Griffiths showed, [12], thermodynamic quantities like free energy density and magnetization (2.5) are non-analytic functions of the magnetic field h, at $h=0$, no matter how small the probability p is. This result already implies that the random field under consideration cannot be a Markov field (Gibbs field with nearest neighbour interaction). To see this, recall that from the Möbius inversion formula (see for instance Chapter 12 of [13]) we would then obtain expressions for the nearest neighbour potential which would be analytic functions of the parameters of our model (p, β and h). Moreover Dobrushin's uniqueness condition would be satisfied, when p is small enough, and hence ([7]) we would obtain analyticity of the magnetization as a function of h (with p and β fixed). Moreover, our random field cannot be a Gibbs field for any interaction whose many-body terms are small enough, so that at low densities (complete) analyticity properties hold ([7, 5, 6]).

Going back to the Markov property in some more detail, let us compute the conditional probability of the event $\xi_0 = +1$ under condition that the configuration $\bar{\xi}_t, t \neq 0$ is fixed away from the origin, in such a way that the cluster $C_{\left(\bar{\xi}\cup\xi_0\right)}(0) = \Lambda$ is finite. Then

$$\mathbb{P}_{\beta,h} \quad \left(\xi_0 = +1 | \bar{\xi}_t\right) = \frac{p\mathbb{P}_{\beta,h,\Lambda}\left((\xi_0 = +1) \cup \bar{\xi}_t\right)}{p\mathbb{P}_{\beta,h,\Lambda}\left(\bar{\xi}_t\right) + (1-p)\,\mathbb{P}_{\beta,h,\Lambda\setminus 0}\left(\bar{\xi}_t\right)} \tag{2.6}$$

$$= \frac{p\exp\left\{\beta\left(\sum_{|t|=1}\bar{\xi}_t+h\right)\right\}}{p\left(\exp\left\{\beta\left(\sum_{|t|=1}\bar{\xi}_t+h\right)\right\}+\exp\left\{-\beta\left(\sum_{|t|=1}\bar{\xi}_t+h\right)\right\}\right)+(1-p)\,\frac{Z_{\beta,h,\Lambda}}{Z_{\beta,h,\Lambda\setminus 0}}},$$

where $Z_{\beta,h,\Lambda}$ is the partition function in Λ with empty boundary conditions. Since the ratio $\frac{Z_{\beta,h,\Lambda}}{Z_{\beta,h,\Lambda\setminus 0}}$ does depend on Λ (as we shall see below), the Markov property is missing.

In fact, this random field is even "worse". Our main result is the following

Theorem 2.1. *Let p be small and β be large enough. Then for $h = 0$ the GriSing random field is* **not a Gibbs random field**.

On the positive side of things, the GriSing random field is an almost Gibbs random field for all p below $p_c(d)$ (the percolation threshold for Bernoulli site percolation on $\mathbb{Z}^d$) and all h and β.

We remind the reader that by an *almost Gibbs random field* we mean a random field which has conditional distributions which are continuous functions a.s.. The difference between these random fields and the usual Gibbs fields is explained in [14]. In particular, every almost Gibbs random field can be described by a Gibbs potential, so that the DLR equation (3.5) below is satisfied, with however the setback of the convergence property (3.1) to hold only almost everywhere.

The proof of almost Gibbsianness is straightforward. Indeed, let Ω^{cont} be the set of all configurations ξ for which all the clusters $C_\xi(\cdot) \equiv C_{\eta(\xi)}(\cdot)$ are finite. Then for p small enough Ω^{cont} has measure one, while the conditional probabilities are clearly continuous on Ω^{cont}.

The statement about non-Gibbsianness requires more work. The idea of the proof is to show that the conditional distributions of the GriSing random field have *essential discontinuities* ('bad' configurations of [14]), presented below, while conditional distributions of Gibbs fields do not have such singularities.

3. Gibbs fields and their continuity

3.1. Definitions

In this subsection we repeat some well known definitions, see for example [4] or [11].

Let S be a finite set. For $V \subseteq \mathbb{Z}^d$ we denote by Ω_V the set of configurations $\zeta_V \in S^V$. A *Gibbsian potential* $\mathcal{U} = (U_A(\zeta_A), A \subset \mathbb{Z}^d, 0 < |A| < \infty)$ is a system of real-valued functions $U_A(\zeta_A)$ of $\zeta_A \in S^A$, labelled by the system of all finite nonempty subsets $A \subset \mathbb{Z}^d$. We assume that for any $t \in \mathbb{Z}^d$ the following series absolutely converges:

$$\sum_{A \subset \mathbb{Z}^d : t \in A, |A| < \infty} \max_{\zeta_A \in S^A} |U_A(\zeta_A)| < \infty. \tag{3.1}$$

For any finite $V \subset \mathbb{Z}^d$, any configuration $\bar{\zeta}_{\mathbb{Z}^d \setminus V} \in S^{\mathbb{Z}^d \setminus V}$, called a *boundary condition*, and any $\zeta_V \in S^V$ consider the *relative energy*

$$E_V^{\mathcal{U}}(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V}) = \sum_{A \subseteq V, A \neq \emptyset} U_A(\zeta_A) + \sum_{A \subset \mathbb{Z}^d : A \cap V \neq \emptyset, A \cap (\mathbb{Z}^d \setminus V) \neq \emptyset, |A| < \infty} U_A(\zeta_{A \cap V} \cup \bar{\zeta}_{A \cap (\mathbb{Z}^d \setminus V)}). \tag{3.2}$$

The condition (3.1) guarantees the convergence of the series in (3.2) for all boundary conditions $\bar{\zeta}_{\mathbb{Z}^d \setminus V}$ and configurations ζ_V. Let

$$p_V^{\mathcal{U}}(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V}) = \frac{\exp\{-E_V^{\mathcal{U}}(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V})\}}{Z_V^{\mathcal{U}}(\bar{\zeta}_{\mathbb{Z}^d \setminus V})}, \tag{3.3}$$

where the *partition function*

$$Z_V^{\mathcal{U}}(\bar{\zeta}_{\mathbb{Z}^d \setminus V}) = \sum_{\zeta_V \in S^V} \exp\{-E_V^{\mathcal{U}}(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V})\}. \tag{3.4}$$

The *transition function* $p_V(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V})$ is defined for all ζ_V and all boundary conditions $\bar{\zeta}_{\mathbb{Z}^d \setminus V}$. The system $p^{\mathcal{U}} = \{p_V^{\mathcal{U}}(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V}), \zeta_V \in S^V, \bar{\zeta}_{\mathbb{Z}^d \setminus V} \in S^{\mathbb{Z}^d \setminus V}, |V| < \infty\}$ of transition functions is called the *Gibbs specification with the potential* $\mathcal{U}$.

For any subset $W \subseteq \mathbb{Z}^d$ we consider the smallest σ-algebra of subsets of the space $S^{\mathbb{Z}^d}$ with respect to which all the coordinate functions $\zeta_t, t \in W$, are measurable. This σ-algebra is denoted by $\mathfrak{B}_W$. A probability measure $\mathbb{P}$ on the measurable space $(S^{\mathbb{Z}^d}, \mathfrak{B}_{\mathbb{Z}^d})$ is called *consistent with the Gibbs specification* $p^{\mathcal{U}}$, if for any finite $V \subset \mathbb{Z}^d$, any function $\phi(\zeta_V)$ of $\zeta_V \in S^V$ and any subset $B \in \mathfrak{B}_{\mathbb{Z}^d \setminus V}$

$$\int_B \phi(\zeta_V) \mathbb{P}(d\zeta) = \int_B \left(\sum_{\zeta_V \in S^V} \phi(\zeta_V) p_V^{\mathcal{U}}(\zeta_V / \bar{\zeta}_{\mathbb{Z}^d \setminus V}) \right) \mathbb{P}_{\mathbb{Z}^d \setminus V}(d\bar{\zeta}_{\mathbb{Z}^d \setminus V}), \tag{3.5}$$

where $\mathbb{P}_{\mathbb{Z}^d \backslash V}$ is the restriction of the measure $\mathbb{P}$ to the σ-algebra $\mathfrak{B}_{\mathbb{Z}^d \backslash V}$. If this is the case then $\mathbb{P}$ is called also *Gibbs random field with potential* $\mathcal{U}$. The probability measure $\mathbb{P}$ is called "*non-Gibbsian*", if there exists no potential $\mathcal{U}$ satisfying the condition (3.1), such that the measure $\mathbb{P}$ is consistent with the Gibbs specification $p^{\mathcal{U}}$.

3.2. Continuity property

The following statement is straightforward; a similar statement appeared already in [4].

Lemma 3.1. *Consider the conditional probabilities of the random field* $\mathbb{P}$:

$$q_{V,W}\left(\zeta_V|\zeta_W\right) = \frac{\mathbb{P}\left(\zeta_V \cup \zeta_W\right)}{\mathbb{P}\left(\zeta_W\right)},$$

where V, W are nonintersecting finite domains. Let $\mathbb{P}$ be a Gibbs random field (with some potential $\mathcal{U}$). Take

$$W = W_\rho\left(V\right) = \left\{t \in \mathbb{Z}^d \backslash V : \operatorname{dist}\left(t, V\right) \le \rho\right\}, \ \rho > 0.$$

Then the family of functions $q_{V,W_\rho(V)}\left(\zeta_V|\cdot\right)$ has the following uniform continuity property in $\zeta_{W_\rho(V)}$: there exist functions $C_V\left(r\right) \to 0$ as $r \to \infty$, such that for every ρ' and ρ'' and any two configurations $\zeta'_{W_{\rho'}(V)}, \zeta''_{W_{\rho''}(V)}$, coinciding on the set $W_\rho\left(V\right)$ with $\rho \le \min\left(\rho', \rho''\right)$, we have

$$\left|q_{V,W_{\rho'}(V)}\left(\zeta_V|\zeta'\right) - q_{V,W_{\rho''}(V)}\left(\zeta_V|\zeta''\right)\right| \le C_V\left(\rho\right). \tag{3.6}$$

Proof. Let $W = W_\rho\left(V\right), W' = W_{\rho'}\left(V\right), W'' = W_{\rho''}\left(V\right)$. It follows from the definitions (3.3), (3.5), that

$$\left|\frac{q_{V,W'}\left(\zeta_V|\zeta'\right)}{q_{V,W''}\left(\zeta_V|\zeta''\right)}\right| \le$$

$$\sup_{\widetilde{\zeta}_V, \bar{\zeta}^1_{\mathbb{Z}^d\backslash(V\cup W')}, \bar{\zeta}^2_{\mathbb{Z}^d\backslash(V\cup W'')}} \frac{\exp\{-E^{\mathcal{U}}_V(\zeta_V \cup \zeta'_{W'}/\bar{\zeta}^1_{\mathbb{Z}^d\backslash(V\cup W')})\}\exp\{-E^{\mathcal{U}}_V(\widetilde{\zeta}_V \cup \zeta''_{W''}/\bar{\zeta}^2_{\mathbb{Z}^d\backslash(V\cup W'')})\}}{\exp\{-E^{\mathcal{U}}_V(\widetilde{\zeta}_V \cup \zeta'_{W'}/\bar{\zeta}^1_{\mathbb{Z}^d\backslash(V\cup W')})\}\exp\{-E^{\mathcal{U}}_V(\zeta_V \cup \zeta''_{W''}/\bar{\zeta}^2_{\mathbb{Z}^d\backslash(V\cup W'')})\}}$$

$$\le \ \exp\left\{4 \sum_{\substack{A: A\cap V \ne \emptyset,\\ \operatorname{diam}(A\backslash V)\ge\rho}} \max_{\zeta_A \in S^A} \left|U_A(\zeta_A)\right|\right\}.$$

Now the application of condition (3.1) completes the proof. ■

4. Non-Gibbsian behavior of the GriSing model

In this section we show that property (3.6) is violated for the GriSing random field. In other words, we will present some points where the uniform continuity of the family of the function $q_{V,W_\rho(V)}(\zeta_V|\cdot)$ for our field fails. In what follows, we consider the case $h = 0$.

We start with a definition. We will call an infinite subset $R \subset \mathbb{Z}^d$ *wide enough,* if the following property holds: for any $s, t \in R$ we have $\langle \sigma_s \sigma_t \rangle_{\beta,R} > c_R > 0$, uniformly in s, t. Here $\langle \cdot \rangle_{\beta,R}$ is the Gibbs state of the Ising model at inverse temperature β in the box R with empty boundary conditions, and we suppose that β is large enough. As an easy Peierls argument shows, such boxes R do exist: for example, the whole lattice $\mathbb{Z}^d$ (in case $d \geq 2$), or its halves: $\mathbb{Z}^d_\pm = \left\{ s \in \mathbb{Z}^d : s_1 \lesseqgtr 0 \right\}$, or the set $\left\{ s \in \mathbb{Z}^d : s^{(1)} > \left(s^{(2)} \right)^2 \right\}$ are wide enough. However, the 1D slabs: $\left\{ s \in \mathbb{Z}^d : \left| s^{(i)} \right| < l_i, i = 1, 2, ..., d-1 \right\}$ are not.

Suppose now that we have two subsets $R^1, R^2 \subset \mathbb{Z}^d$, which are wide enough, and which have the following extra set $\mathfrak{C}$ of four properties (satisfied, for example, for the pair $\mathbb{Z}^d_+, \mathbb{Z}^d_-$):
i) $\operatorname{dist}(R^1, R^2) > 1$,
ii) $0 \notin R^1 \cup R^2$,
iii) each of the sets R^1, R^2 contains exactly one nearest neighbor of the origin, say, x_0 and y_0,
iv) there exist a sequence of points $z_i \in \mathbb{Z}^d \setminus (R^1 \cup R^2)$, $z_i \to \infty$ as $i \to \infty$, such that each of the sets R^1, R^2 contains exactly one nearest neighbor of every site z_i, say x_i and y_i.

Let now R^1, R^2 be a $\mathfrak{C}$-pair of subsets. Denote by $\tilde{R}^1_n, \tilde{R}^2_n$ the connected components of the sets $R^i_n = R^i \cap B_n$, containing the sites x_0, y_0, and let $\Omega^{disc}_n(R^1, R^2) \subset \Omega_{B_n}$ be the set of configurations ξ with the property that the two sets $\tilde{R}^1_n, \tilde{R}^2_n$ are among their clusters.

Lemma 4.1. *Let $\xi_n \in \Omega^{disc}_n(R^1, R^2)$, and the configuration $\xi^{z_i}_n$ is obtained from ξ by changing it from the value 0 to the value +1 at z_i. Then for β large enough*

$$|q(\xi_0|\xi_n) - q(\xi_0|\xi^{z_i}_n)| > c_{R_1,R_2} > 0 \tag{4.1}$$

for every i, provided $n > n(i)$ is large enough.

Proof. We begin by passing to finite volumes. So we denote by η_n the restriction of the configuration η to the cubic box B_n, and we introduce the measure $\mathbb{P}_\beta^{(n)}$ by

$$\mathbb{P}_\beta^{(n)}(d\sigma, d\eta) = \mathbb{P}_{\beta,\eta_n}(d\sigma)\,\mathbb{P}(d\eta).$$

By (2.3), (2.4) it follows immediately that

$$\lim_{n\to\infty} \mathbb{P}_\beta^{(n)} = \mathbb{P}_\beta.$$

To check (4.1) it is enough to prove its analogue for the conditional distributions of the random fields $\mathbb{P}_\beta^{(n)}$, uniformly in n. Until the end of this proof the value of the parameter n will be fixed, and in order to save on notation we will use $q(\cdot|\cdot)$ to denote the conditional distribution of the field $\mathbb{P}_\beta^{(n)}$.

Let us denote by Λ^1, Λ^2 the sets $\tilde{R}_n^1, \tilde{R}_n^2$, and let $\Lambda = \Lambda^1 \cup \Lambda^2$. Inspecting the relation (2.6), we see that it is enough to show the following estimate:

$$\frac{Z_{\beta,\Lambda\cup 0\cup z_i}\, Z_{\beta,\Lambda}}{Z_{\beta,\Lambda\cup z_i}\, Z_{\beta,\Lambda\cup 0}} - 1 \geq c'_{R_1,R_2} > 0. \tag{4.2}$$

Note however, that a straightforward calculation shows that

$$\frac{Z_{\beta,\Lambda\cup 0}}{Z_{\beta,\Lambda}} = \left\langle 2\cosh\beta\left(\sigma_{x_0} + \sigma_{y_0}\right)\right\rangle_{\beta,\Lambda}.$$

If we put $a = \cosh 2\beta - 1, b = \cosh 2\beta + 1$, then we have the identity $2\cosh\beta(\sigma_1 + \sigma_2) = a\sigma_1\sigma_2 + b$, so we proceed by:

$$\frac{Z_{\beta,\Lambda\cup 0}}{Z_{\beta,\Lambda}} = \left\langle a\sigma_{x_0}\sigma_{y_0} + b\right\rangle_{\beta,\Lambda} = a\left\langle \sigma_{x_0}\right\rangle_{\beta,\Lambda^1}\left\langle \sigma_{y_0}\right\rangle_{\beta,\Lambda^2} + b = b, \tag{4.3}$$

since the expectations $\langle \sigma.\rangle_{\beta,.}$ with free boundary conditions vanish. In the same way we have

$$\begin{aligned} \frac{Z_{\beta,\Lambda\cup 0\cup z_i}}{Z_{\beta,\Lambda}} &= \left\langle \left(a\sigma_{x_0}\sigma_{y_0} + b\right)\left(a\sigma_{x_i}\sigma_{y_i} + b\right)\right\rangle_{\beta,\Lambda} \\ &= a^2\left\langle \sigma_{x_0}\sigma_{x_i}\right\rangle_{\beta,\Lambda^1}\left\langle \sigma_{y_0}\sigma_{y_i}\right\rangle_{\beta,\Lambda^2} + b^2, \end{aligned} \tag{4.4}$$

due to the independence of what is happening in the boxes Λ^1, Λ^2. Rewriting the ratio in the lhs of (4.2) as

$$\frac{\frac{Z_{\beta,\Lambda\cup 0\cup z_i}}{Z_{\beta,\Lambda}}}{\frac{Z_{\beta,\Lambda\cup z_i}}{Z_{\beta,\Lambda}}\frac{Z_{\beta,\Lambda\cup 0}}{Z_{\beta,\Lambda}}},$$

we obtain (4.2) from (4.3), (4.4), with $c'_{R_1,R_2} = \frac{a^2}{2b^2} c_{R_1} c_{R_2}$. $\blacksquare$

Note that intuitively the underlying mechanism of the proof of violation of continuity (quasilocality) is the presence of a phase transition with 4 extremal states when the two regions are disjoint; this 4-fold degeneracy is partially lifted by connecting the two regions. When the two regions are already connected far away, adding an extra connection does not have the same effect, as this degeneracy to some extent was already lifted.

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