THE HAUSDORFF DIMENSION OF CONFORMAL REPELLERS UNDER RANDOM PERTURBATION

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Abstract. We prove that the Hausdorff dimension of a conformal repeller is stable under random perturbations. Our perturbation model uses the notion of a bundle random dynamical system. The main ingredient of our proof is a version of the Bowen-Ruelle formula for expanding almost conformal bundle random dynamical systems.

Introduction

Assume J is a mixing repeller for a $\mathcal{C}^{1+\alpha}$ expanding map F on a Riemannian manifold M. If F is conformal with respect to some Riemannian metric then the Hausdorff dimension $\mathrm{HD}(J)$ of J can be obtained as the zero t_0 of $t \mapsto \pi_F(-t \log \|D_x F\|)$, where π_F denotes the pressure functional. This statement is known as the *Bowen-Ruelle formula* (cf. Bowen [Bow79], Ruelle [Rue82, Rue83]). Moreover, one has that the equilibrium state of $x \mapsto -t_0 \log \|D_x F\|$ is equivalent to the t_0-Hausdorff measure.

In this paper we consider a random perturbation of this situation which is modelled using the notion of a (bundle) random dynamical system (abbreviated RDS henceforth). More precisely, fix an ergodic invertible transformation ϑ of a Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a measurable family $\varphi = \{\varphi(\omega) : M \to M\}$ of $\mathcal{C}^{1+\alpha}$ maps, i. e. $(\omega, x) \mapsto \varphi(\omega)x$ is assumed to be measurable. This determines a differentiable RDS via $\varphi(n, \omega) := \varphi(\vartheta^{n-1}\omega) \circ \ldots \circ \varphi(\omega)$ $(n \in \mathbb{N})$. Further, let $E \subset \Omega \times X$ be a measurable set such that all ω-sections $E_\omega := \{x | (\omega, x) \in E\}$ are compact. If $\mathcal{K}$ denotes the collection of all compact subsets of M endowed with the Hausdorff topology this is equivalent to saying that the $\mathcal{K}$-valued multifunction $\omega \mapsto E_\omega$ is measurable. We refer to E as a *compact bundle* over Ω and to the E_ω's as its fibers. E is said to be invariant with respect to φ if $\varphi(\omega)E_\omega = E_{\vartheta\omega}$ $\mathbb{P}$-a. s. The restriction of φ to an invariant bundle E (or, slightly more general, any measurable family $\{\varphi(\omega) : E_\omega \to E_{\vartheta\omega}\}$ of maps between fibers) is referred to as a bundle RDS. In this situation $\omega \mapsto \mathrm{HD}(E_\omega)$ is a $\mathbb{P}$-a. s. constant $\mathrm{HD}(E)$, say (see e. g. Crauel and Flandoli [CF98, Lemma 4.2]).

The aim of this paper is to make rigorous the statement that if a bundle RDS is close to an expanding map on a repeller then the corresponding Hausdorff dimensions are close.

The paper is organized as follows.

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In section 1 we discuss the pressure functional of a bundle RDS. In particular, we allow random radii in the definition of a Bowen ball and study the effect of this on the definition of the pressure given by Bogenschütz [Bog92].

In section 2 we introduce our model of random perturbation, which has essentially been inspired by a recent result of Liu [Liu98], and formulate a general statement about stochastic stability of the pressure.

Section 3 might be considered as the core of our paper. Here, we introduce the notion of an expanding almost conformal bundle RDS and prove a version of the Bowen-Ruelle formula for such systems. This set-up generalizes the purely topological characterization of conformality which has first been studied by Ochs [Och95]. A first random version of the Bowen-Ruelle formula has been given by Kifer [Kif96] for quite a restricted situation. However, it turned out that the basic techniques of Kifer's proof are suited to be generalized for our purposes.

In section 4, finally, we formulate and prove our main result which, tersely speaking, states that the Hausdorff dimension of a conformal repeller is stochastically stable.

1. The pressure of a bundle RDS

Let ϑ be an ergodic invertible transformation of a Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$. Further, let E be a compact bundle over Ω with fibers in a Polish space M. Let φ be a continuous bundle RDS over ϑ, i.e. $\varphi(\omega) : E_\omega \to E_{\vartheta\omega}$ is continuous $\mathbb{P}$-a.s. Here and in what follows we think of E_ω being equipped with the trace topology, i.e. an open set $A \subset E_\omega$ is of the form $A = B \cap E_\omega$ with some open $B \subset M$.

Let $\mathbb{L}^1_E(\Omega, \mathcal{C}(M))$ denote the collection of all integrable random continuous functions on fibers, i.e. a measurable $f : E \to \mathbb{R}$ is a member of $\mathbb{L}^1_E(\Omega, \mathcal{C}(M))$ if its ω-sections $f(\omega) : E_\omega \to \mathbb{R}$ are continuous and $\int \|f(\omega)\|_\infty d\mathbb{P}(\omega) < \infty$.

1.1. Definition. Consider any measurable $\epsilon : \Omega \to (0,1]$. For any $n \in \mathbb{N}$ define the *random Bowen ball* around $x \in E_\omega$ by

$$B_\omega(n, x, \epsilon) := \left\{ y \in E_\omega \mid d\big(\varphi(k,\omega)x, \varphi(k,\omega)y\big) < \epsilon(\vartheta^k\omega) \text{ for } 0 \le k \le n-1 \right\},$$

where d is the given metric on M.

A subset G of E_ω is said to be (ω, n, ϵ)*-separated* (with respect to φ), if $x, y \in G$, $x \neq y$, implies $y \notin B_\omega(n, x, \epsilon)$.

For $f \in \mathbb{L}^1_E(\Omega, \mathcal{C}(M))$ and $n \ge 1$ we denote $F_n(\omega)x := \sum_{i=0}^{n-1} f \circ \Theta^i(\omega, x) = \sum_{i=0}^{n-1} f(\vartheta^i\omega) \circ \varphi(i,\omega)x$ and put

$$\pi_\varphi(f)(\omega, n, \epsilon) := \sup \left\{ \sum_{x \in G} \exp F_n(\omega)x \mid G \subset E_\omega \text{ is } (\omega, n, \epsilon)\text{-separated} \right\}.$$

One can see as in Bogenschütz [Bog92, Lemma 5.3] that $\omega \mapsto \pi_\varphi(f)(\omega, n, \epsilon)$ is measurable for each $n \in \mathbb{N}$ and any measurable $\epsilon : \Omega \to (0,1]$.

For $f \in \mathbb{L}^1_E(\Omega, \mathcal{C}(M))$ and $\epsilon : \Omega \to (0,1]$ put

$$\pi_\varphi(f)(\epsilon) := \limsup_{n\to\infty} \frac{1}{n} \int \log \pi_\varphi(f)(\omega, n, \epsilon) d\mathbb{P}(\omega).$$

For the sake of distinction, we denote small numbers by δ and use ϵ (possibly with subscripts) exclusively for small random variables in this section. With this, the pressure $\pi_\varphi(f)$ of φ at f may be defined by $\pi_\varphi(f) := \lim_{\delta \downarrow 0} \pi_\varphi(f)(\delta)$.

It seems to be a bit unnatural to first integrate and then go to the limit in the definition of $\pi_\varphi(f)(\epsilon)$. This, however, turned out to be important in proving the variational principle (cf. [Bog92]). A more natural approach is based on the following.

1.2. Lemma. *For $f \in \mathbb{L}^1_E(\Omega, \mathcal{C}(M))$ and $\epsilon : \Omega \to (0,1]$ the limit*

$$\limsup_{n\to\infty} \frac{1}{n} \log \pi_\varphi(f)(\omega, n, \epsilon) =: \pi^*_\varphi(f)(\epsilon)$$

is $\mathbb{P}$-a. s. independent of ω.

Proof. Write $\pi^*_\varphi(f)(\omega,\epsilon) := \limsup_{n\to\infty} \frac{1}{n}\log \pi_\varphi(f)(\omega,n,\epsilon)$. We will show that $\pi^*_\varphi(f)(\omega,\epsilon) \le \pi^*_\varphi(f)(\vartheta\omega,\epsilon)$. From this we conclude that $\pi^*_\varphi(f)(\omega,\epsilon)$ is ϑ-invariant (see Arnold [Arn98, Lemma 3.1.4]) which, by ergodicity, immediately gives the claim.

Let G be an (ω, n, ϵ)-separated set. Then $\hat{G} := \varphi(\omega)G$ is $(\vartheta\omega, n-1, \epsilon)$-separated. Using $F_n(\omega)x = f(\omega)x + F_{n-1}(\vartheta\omega)\varphi(\omega)x$ we obtain

$$\begin{aligned}\sum_{x\in G} \exp F_n(\omega)x &= \sum_{x\in G} e^{f(\omega)x} \exp\big(F_{n-1}(\vartheta\omega)\varphi(\omega)x\big)\\ &\le e^{\|f(\omega)\|} \sum_{x\in\hat{G}} \exp F_{n-1}(\vartheta\omega)x\\ &\le e^{\|f(\omega)\|}\pi_\varphi(f)(\vartheta\omega, n-1, \epsilon).\end{aligned}$$

Since this holds true for all (ω, n, ϵ)-separated sets G we have

$$\pi_\varphi(f)(\omega,n,\epsilon) \le e^{\|f(\omega)\|}\pi_\varphi(f)(\vartheta\omega, n-1,\epsilon)$$

which yields the desired. □

1.3. *Remark.* It can in fact be shown that $\pi_\varphi(f) = \lim_{\delta\downarrow 0} \pi^*_\varphi(f)(\delta)$.

The following result describes a situation where $\pi_\varphi(f)$ can be obtained as the limit of $\pi_\varphi(f)(\epsilon_k)$ for certain sequences (ϵ_k) of random variables with $\epsilon_k \to 0$.

1.4. Proposition. *Assume E is a compact bundle with fibers in a finite dimensional Riemannian manifold M satisfying*

$$\int \log \operatorname{diam} E_\omega d\mathbb{P}(\omega) < \infty, \tag{1}$$

where diam *is measured with respect to the Riemannian distance on M.*

If $(\epsilon_k)_{k\in\mathbb{N}}$ is a sequence of measurable $\epsilon_k : \Omega \to (0,1]$ with

(i) $\log \epsilon_k \in \mathbb{L}^1(\mathbb{P})$ *for all $k \in \mathbb{N}$ and*
(ii) $\lim_{k\to\infty} \epsilon_k = 0$ *uniformly in ω,*

then

$$\lim_{k\to\infty} \pi_\varphi(f)(\epsilon_k) = \pi_\varphi(f).$$

Proof. By Nash's Embedding Theorem M can be isometrically embedded in $\mathbb{R}^N$ for N big enough (cf. Gromov and Rokhlin [GR70]). Therefore it suffices to prove the statement for $M = \mathbb{R}^N$, say.

First note that clearly $\epsilon_1 < \epsilon_2$ implies $\pi_\varphi(f)(\epsilon_1) \ge \pi_\varphi(f)(\epsilon_2)$. With $\delta_k := \operatorname{ess\,sup} \epsilon_k$ we have $\pi_\varphi(f)(\epsilon_k) \ge \pi_\varphi(f)(\delta_k)$. Since $\lim_{k\to\infty}\delta_k = 0$ by (ii), thus,

$$\liminf_{k\to\infty} \pi_\varphi(f)(\epsilon_k) \ge \pi_\varphi(f).$$

We have to show that $\limsup_{k\to\infty} \pi_\varphi(f)(\epsilon_k) \le \pi_\varphi(f)$. For that purpose, we fix ϵ with $\log\epsilon \in \mathbb{L}^1(\mathbb{P})$ and show that there exists a φ-invariant measure μ and a countable partition $\mathcal{Z}$ of E with $H_\mu(\mathcal{Z} \mid \pi^{-1}\mathcal{F}) < \infty$ such that

$$\pi_\varphi(f)(\epsilon) \le h_\mu(\varphi;\mathcal{Z}) + \int f d\mu. \tag{2}$$

Here $h_\mu(\varphi;\mathcal{Z}) = \lim_{n\to\infty} \frac{1}{n}\int H_{\mu_\omega}\big(\bigvee_{i=0}^{n-1}\varphi^{-1}(i,\omega)\mathcal{Z}(\vartheta^i\omega)\big)d\mathbb{P}(\omega)$, where $\mathcal{Z}(\omega)$ denotes the partition of E_ω into the ω-sections of the atoms of $\mathcal{Z}$. The supremum of $h_\mu(\varphi;\mathcal{Z})$ over all countable partitions $\mathcal{Z}$ of E with $H_\mu(\mathcal{Z} \mid \pi^{-1}\mathcal{F}) < \infty$ conincides with the fiber entropy $h_\mu(\varphi)$ which is usually defined using partitions $\mathcal{P}$ of M only (cf. [Bog93]). By the variational principle, thus, (2) yields $\pi_\varphi(f)(\epsilon) \le \pi_\varphi(f)$, hence the claim.

The construction of μ can in fact be carried out as in part (II) of the proof of [Bog92, Theorem 6.1]. Just note that the set of all random probabilities on M with $\mu_\omega(E_\omega) = 1$ is compact in the narrow topology (cf. Crauel [Cra95]). So the Krylov-Bogolioubov procedure needed here still works (cf. [Bog93, Theorem 1.2.8]).

For the choice of $\mathcal{Z}$ we need the Euclidian structure of $M = \mathbb{R}^N$. We proceed as follows. For $k \in \mathbb{N}$ let $\mathcal{W}_k = \{W_i^{(k)}\}_{i\in\mathbb{N}}$ be a covering of $\mathbb{R}^N$ by cubes with diameter $\frac{1}{2k}$ which only intersect at their boundaries. Further, denote by m the marginal of μ on $\mathbb{R}^N$. Then each $W_i^{(k)}$ is contained in a cube $\tilde{W}_i^{(k)}$ with $m(\partial\tilde{W}_i^{(k)}) = 0$ and $\operatorname{diam}\tilde{W}_i^{(k)} < \frac{1}{k}$ [since we cannot have an uncountable collection of disjoint sets of positive measure]. One can check that each compact set $A \subset \mathbb{R}^N$ intersects with at most $[4k \operatorname{diam} A\sqrt{N} + 4]^N$ members of $\tilde{\mathcal{W}}_k = \{\tilde{W}_i^{(k)}\}_{i\in\mathbb{N}}$, where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Write $\Omega_k := \{\frac{1}{k+1} < \epsilon \le \frac{1}{k}\}$. Further, write $P_1^{(k)} := \tilde{W}_1^{(k)}$ and $P_i^{(k)} := \tilde{W}_i^{(k)} \setminus \bigcup_{j=1}^{i-1} P_j^{(k)}$ for $i \ge 2$. The partition

$$\mathcal{Z} = \bigcup_{k\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}\Big\{(\Omega_k \times P_i^{(k+1)}) \cap E\Big\}$$

of E can easily be seen to satisfy

(i) $\operatorname{diam}\mathcal{Z}(\omega) < \epsilon(\omega)$,
(ii) $\mu_\omega(\partial Z_\omega) = 0$ for all $Z \in \mathcal{Z}$ ($\mathbb{P}$-a.s.),
(iii) $\operatorname{card}\mathcal{Z}(\omega) \le [\frac{4\operatorname{diam} E_\omega\sqrt{N}}{\epsilon(\omega)+1} + 4]^N$,
(iv) $\int \log\operatorname{card}\mathcal{Z}(\omega) d\mathbb{P}(\omega) < \infty$,

where (iv) is a consequence of (iii) and (1). Note also that $H_\mu(\mathcal{Z} \mid \pi^{-1}\mathcal{F}) < \infty$ by (iv).

With these properties the calculations in [Bog92, Theorem 6.1] proving (2) can be carried out without substantial changes. □

In the sequel we will be interested in expansiveness.

1.5. Definition. A bundle RDS φ is said to be (positive) *expansive* if there exists a $(0,1)$-valued random variable Δ such that

$$d(\varphi(n,\omega)x, \varphi(n,\omega)y) \le \Delta(\vartheta^n\omega) \qquad \text{for all } n \in \mathbb{N}$$

implies $x = y$.

We call Δ an *expansive constant.*

In studying expansiveness the concept of a generator and the definition of pressure using open coverings is useful.

1.6. Definition. A *generator* of a continuous bundle RDS φ is a family $\mathcal{A} = \{\mathcal{A}(\omega) = (A_i(\omega)) : \mathcal{A}(\omega) \text{ is an open covering of } E_\omega)\}$ with

1. $\mathcal{A}(\omega)$ is finite for all $\omega \in \Omega$,
2. $\omega \mapsto d(x, A_i(\omega))$ is measurable for all $x \in M$ and all $i \in \mathbb{N}$,
3. for each sequence $(A_n)_{n\in\mathbb{N}}$ of sets we have that $A_n \in \mathcal{A}(\vartheta^n\omega)$ for all $n \in \mathbb{N}$ implies that $\bigcap_{n=0}^{\infty} \varphi(n,\omega)^{-1}\bar{A}_n$ contains at most one point.

Going along the arguments of Walters [Wal82, § 5.6] for each ω separately one can check that a bundle RDS is expansive if and only if it possesses a generator. Moreover, each generator satisfies $\lim_{k\to\infty} \operatorname{diam} \bigvee_{i=0}^{k-1} \varphi(i,\omega)^{-1}\mathcal{A}(\vartheta^i\omega) = 0$ for $\mathbb{P}$-a. a. ω (cf. Walters [Wal82, Theorem 5.21]).

One may give a definition of the pressure of a bundle RDS using open coverings. However, to make the arguments of Walters [Wal82, § 9.1] work here one needs the following notion.

1.7. Definition. A generator $\mathcal{A}$ of φ is called a *strong generator* if

$$\lim_{k\to\infty} \operatorname{diam} \bigvee_{i=0}^{k-1} \varphi(i,\omega)^{-1}\mathcal{A}(\vartheta^i\omega) = 0 \qquad \text{uniformly in } \omega.$$

An expansive RDS is said to be *strongly expansive* if it possesses a strong generator.

1.8. Example. In a recent paper of Gundlach and Kifer [GK98] the following uniformity assumption on an expansive RDS appears:

(3) To each $p \in \mathbb{N}$ there is $N_p \in \mathbb{N}$ independent of ω such that

$$d(\varphi(n,\omega)x, \varphi(n,\omega)y) \le \Delta(\vartheta^n\omega) \qquad \text{for } n = 0, \dots, N_p - 1$$

implies $d(x,y) < \frac{\Delta(\omega)}{p}$.

This implies strong expansiveness. In fact, let (x_i) be a countable dense subset of M and put $\mathcal{A}(\omega) := \{E_\omega \cap B(x_i, \frac{\Delta(\omega)}{2}) : i = 1, \dots, l(\omega)\}$ with $l(\omega) := \inf\{k : \bigcup_{i=1}^{k} B(x_i, \frac{\Delta(\omega)}{2}) \supset E_\omega\}$. One easily checks that $\mathcal{A}$ defined this way is a generator and (3) yields $\operatorname{diam} \bigvee_{i=0}^{N_p-1} \varphi(i,\omega)^{-1}\mathcal{A}(\vartheta^i\omega) < \frac{\Delta(\omega)}{p} < \frac{1}{p}$ for all $\omega \in \Omega$.

1.9. Theorem. *Let φ be a strongly expansive bundle RDS with expansive constant Δ. Further, let $f \in \mathbb{L}^1_E(\Omega, \mathcal{C}(M))$ and let ϵ be a $(0,1)$-valued random variable with $\epsilon < \frac{\Delta}{4}$. If $\lim_{k\to\infty} \pi_\varphi(f)(\frac{\epsilon}{k}) = \pi_\varphi(f)$ then*

$$\pi_\varphi(f) = \lim_{n\to\infty} \frac{1}{n} \log \pi_\varphi(f)(\omega, n, \epsilon) \qquad \mathbb{P}\text{-a. s.}$$

In the situation of proposition 1.4, thus, the above convergence holds if $\log\epsilon \in \mathbb{L}^1(\mathbb{P})$.

Proof. The proof can be obtained by adapting the argument of Walters [Wal82, Theorem 9.6] and will not be carried out here. We just remark that by reasoning for $\mathbb{P}$-a. e. ω separately one ends up with

$$\pi_\varphi(f) \le \liminf_{n\to\infty} \frac{1}{n} \log \pi_\varphi(f)(\omega, n, \epsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log \pi_\varphi(f)(\omega, n, \epsilon) = \pi^*_\varphi(f)(\epsilon).$$

The assumption $\lim_{k\to\infty} \pi_\varphi(f)(\frac{\epsilon}{k}) = \pi_\varphi(f)$ is needed to ensure $\lim_{k\to\infty} \pi^*_\varphi(f)(\frac{\epsilon}{k}) = \pi_\varphi(f)$ (which again follows using a randomized result of Walters [Wal82, Theorem 9.2]); hence by monotonicity $\pi^*_\varphi(f)(\epsilon) \le \pi_\varphi(f)$. □

2. Structural stability implies stochastic stability

Let $U \subset M$ be an open subset of the Polish space M with $\bar{U} \in \mathcal{K}$ and let $\mathcal{C}(U,M)$ denote the space of all continuous mappings from U to M endowed with the compact open topology.

2.1. Definition. Assume $F \in \mathcal{C}(U,M)$ and $\Xi \in \mathcal{K}$ with $F\Xi = \Xi$. A family $\{\varphi_\epsilon\}_{\epsilon>0}$ of $\mathcal{C}(U,M)$-valued random variables is called a *random perturbation of F on Ξ* if

(i) $\lim_{\epsilon\to 0} \varphi_\epsilon = F$ in probability,
(ii) there exists a family $\{\Xi_\epsilon\}_{\epsilon>0}$ of $\mathcal{K}$-valued random variables such that
- for each $\epsilon > 0$ we have that $\mathbb{P}$-a. s. $\varphi_\epsilon(\omega)\Xi_\epsilon(\omega) = \Xi_\epsilon(\vartheta\omega)$,
- $\lim_{\epsilon\to 0} \Xi_\epsilon = \Xi$ in probability.

$\{\varphi_\epsilon\}_{\epsilon>0}$ is said to be *structurally stable* if there exists a family $\{h_\epsilon\}_{\epsilon>0}$ of $\mathcal{C}(\Xi,M)$-valued random variables such that

(iii) for each $\epsilon > 0$ we have that $\mathbb{P}$-a. s. $h_\epsilon(\omega)$ maps Ξ homeomorphically onto $\Xi_\epsilon(\omega)$ and
$$\varphi_\epsilon(\omega) \circ h_\epsilon(\omega) = h_\epsilon(\vartheta\omega) \circ F,$$
(iv) $\lim_{\epsilon\to 0} h_\epsilon = \mathrm{id}$ in probability.

2.2. Example. Let M be a finite dimensional Riemannian manifold and let $\mathcal{C}^k(U,M)$ denote the space of all $\mathcal{C}^k$ maps from U to M $(k > 1)$. Further, let $\mathrm{Emb}^k(U,M)$ denote the set of those members of $\mathcal{C}^k(U,M)$ which are diffeomorphisms onto their images. A recent result of Liu [Liu98] yields that if $F \in \mathrm{Emb}^k(U,M)$ and Ξ is an Axiom A basic set, then there exists a neighbourhood $\mathcal{U}(F)$ of F in $\mathrm{Emb}^k(U,M)$ such that each family $\{\varphi_\epsilon\}_{\epsilon>0}$ of $\mathcal{U}(F)$-valued random variables with $\lim_{\epsilon\to 0} \varphi_\epsilon = F$ in probability is a structurally stable random perturbation of F on Ξ.

If $F \in \mathcal{C}^k(U,M)$ is expanding on Ξ this statement can be seen to hold true with $\mathcal{U}(F)$ replaced by a neighbourhood of F in $\mathcal{C}^k(U,M)$. Note that here each φ_ϵ determines an expanding bundle RDS; if F is conformal, however, the same will not be true for φ_ϵ in general, i. e. we will not be in the situation of Kifer [Kif96, Theorem 4.8]. We will discuss this particular situation in more detail in section 4.

Let us finally remark that if we assume M to be compact and put $\Xi = U = M$, and if we assume $\operatorname{supp} \varphi_\epsilon\mathbb{P} \subset B_\epsilon(F)$, we are in a situation which is quite well understood (cf. Baladi [Bal97], Baladi, Kondah, and Schmitt [BKS96], Bogenschütz [Bog96]).

Put $X = \bar{U}$ and let $\mathcal{C}^\alpha(X,\mathbb{R})$ denote the space of all Hölder continuous functions on X with Hölder exponent α. We endow $\mathcal{C}^\alpha(X,\mathbb{R})$ with the usual norm $\|\cdot\|_\alpha := \|\cdot\|_\infty + |\cdot|_\alpha$, where $\|\cdot\|_\infty$ is the sup-norm and $|\cdot|_\alpha$ is the least Hölder constant, i. e. $|f|_\alpha := \sup_{x,y\in X, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^\alpha}$.

2.3. Proposition. *Let $\{\varphi_\epsilon\}_{\epsilon>0}$ be a structurally stable random perturbation of F on Ξ. Further, let $\{f_\epsilon\}_{\epsilon>0}$ be a family of $\mathcal{C}^\alpha(X,\mathbb{R})$-valued random variables satisfying*
$$\lim_{\epsilon\to 0} \|f_\epsilon - f\|_\alpha = 0 \qquad \text{in } \mathbb{L}^1(\mathbb{P})$$
for some $f \in \mathcal{C}^\alpha(X,\mathbb{R})$. Then
$$\lim_{\epsilon\to 0} \pi_{\varphi_\epsilon}(f_\epsilon) = \pi_F(f).$$

Proof. First note that since φ_ϵ and F are topologically conjugate (definition 2.1(iii)) for each $\epsilon > 0$ we have $\pi_{\varphi_\epsilon}(f_\epsilon) = \pi_F(f_\epsilon \circ h_\epsilon)$ (cf. Bogenschütz [Bog93, Proposition 3.1.3]). Further, we have

$$\|f_\epsilon \circ h_\epsilon - f\|_\infty \le \|f_\epsilon \circ h_\epsilon - f_\epsilon\|_\infty + \|f_\epsilon - f\|_\infty.$$

The second summand of the right hand side tends to zero as $\epsilon \to 0$ by assumption. For the first summand we have

$$\begin{aligned}\|f_\epsilon \circ h_\epsilon - f_\epsilon\|_\infty &= \sup_{x\in X}\{|f_\epsilon(\omega, h_\epsilon(\omega)x) - f_\epsilon(\omega,x)|\}\\ &\le |f_\epsilon(\omega)|_\alpha \cdot d(h_\epsilon(\omega),\mathrm{id})^\alpha.\end{aligned}$$

Note that $\omega \mapsto d(h_\epsilon(\omega),\mathrm{id})$ is bounded uniformly in ϵ by $\operatorname{diam} X$, hence $\lim_{\epsilon\to 0} d(h_\epsilon(\cdot),\mathrm{id})^\alpha = 0$ in $\mathbb{L}^1(\mathbb{P})$ by definition 2.1(iv). With this, and since $\lim_{\epsilon\to 0} |f_\epsilon(\cdot)|_\alpha = |f|_\alpha$ in $\mathbb{L}^1(\mathbb{P})$ by assumption, we conclude

$$\lim_{\epsilon\to 0} \|f_\epsilon \circ h_\epsilon - f\|_\infty = 0 \qquad \text{in } \mathbb{L}^1(\mathbb{P}).$$

From this the claim follows by the continuity of the pressure. □

In view of Bogenschütz [Bog96, Proposition 2.2] this has the following consequence.

2.4. Corollary. *In the situation of proposition 2.3, assume that for each $\epsilon > 0$ the entropy map $\mu \mapsto h_\mu(\varphi_\epsilon)$ is upper semi-continuous. Then the following holds true:*

Assume $\{\mu_\epsilon\}_{\epsilon>0}$ is a family of random probabilities on M such that each μ_ϵ is an equilibrium state for f_ϵ with respect to φ_ϵ, i. e. $\pi_{\varphi_\epsilon}(f_\epsilon) = h_{\mu_\epsilon}(\varphi_\epsilon) + \int f_\epsilon d\mu_\epsilon$. Then the marginal on M of each accumulation point of $\{\mu_\epsilon\}$ is an equilibrium state for f with respect to F. In particular, thus, if f possesses a unique equilibrium state μ_f then $\lim_{\epsilon\to 0}\mu_\epsilon = \mathbb{P}\otimes\mu_f$.

2.5. *Remark.* Consider the situation described in example 2.2. Using the methods of Liu [Liu98] one can derive that, by making $\mathcal{U}(F)$ smaller if necessary, φ_ϵ has the expansiveness property and hence $\mu \mapsto h_\mu(\varphi_\epsilon)$ is upper semi-continuous. Applying the above results to $f_\epsilon \equiv 0$, thus, we obtain stochastic stability of the topological entropy and the measure of maximal entropy.

One may also derive (again using [Liu98]) stochastic stability of the generalized SRB measure μ_{SRB} of F. This is defined as the equilibrium state for $f^{(u)}(x) := -\log|\det D_xF|_{E^u(x)}|$, where $E^u(x) \subset T_xM$ denotes the unstable subspace of F at x. Stochastic stability here means $\lim_{\epsilon\to 0}\mu_\epsilon = \mathbb{P}\otimes\mu_{SRB}$ where μ_ϵ is an equilibrium state for $f_\epsilon^{(u)}(\omega,x) := -\log|\det D_x\varphi_\epsilon(\omega)|_{E^u_\epsilon(\omega,x)}|$.

3. A random version of the Bowen-Ruelle formula

The classical Bowen–Ruelle formula connects the Hausdorff dimension of a conformal repeller of a deterministic $\mathcal{C}^{1+\alpha}$ map with topological pressure. In the sequel we will generalize this result to a class of random invariant compact sets of a random dynamical system which we will call "almost conformal repellers". A map is called conformal, if the expansion of its linearization at each point is the same in every direction. We will weaken this assumption by requiring certain lower and upper bounds of expansion in different directions. Repeller in our sense means that the distance of nearby points "increases in average". Formally this looks as follows.

3.1. Definition. Let φ be a continuous bundle RDS on a compact bundle E with fibers in some complete separable metric space (M,d).

(i) φ is said to be (λ, η) *almost conformal* if there exist measurable functions $\lambda, \eta : E \to \mathbb{R}_+$, and $K : \Omega \to \mathbb{R}_+$, with $\log^+ K \in \mathbb{L}^1(\mathbb{P})$, $\log \lambda, \log \eta \in \mathbb{L}^1_E(\Omega, \mathcal{C}(M))$, and $\lambda(\omega, \cdot), \eta(\omega, \cdot)$ Hölder continuous in x with uniform exponent α, $0 < \alpha \le 1$ such that
$$\lambda(\omega, x) - K(\omega) d(x,y)^\alpha \le \frac{d(\varphi(\omega)x, \varphi(\omega)y)}{d(x,y)} \le \eta(\omega, x) + K(\omega) d(x,y)^\alpha$$
for every $\omega \in \Omega$ and $x \neq y \in E_\omega$.

(ii) φ is said to be *expanding* on E (or: E is called an *almost conformal repeller* for φ) if, in addition, for almost every $\omega \in \Omega$ there exists $n = n(\omega) \ge 1$ with
$$\Lambda_n(\omega) := \min_{x \in E_\omega} \lambda_n(\omega, x) > 1,$$
where $\lambda_n(\omega, x) := \prod_{k=0}^{n-1} \lambda(\vartheta^k \omega, \varphi(k, \omega)x)$.

3.2. *Remark.* λ and η can be interpreted as bounds for the "local expansion rate" of φ. In particular, in the case $\lambda = \eta$ part (i) of the definition gives a characterization of conformality without using a differential structure (cf. Ochs [Och95] for the deterministic case).

3.3. Lemma. *Let φ be an expanding (λ, η) almost conformal bundle RDS on E. There exist positive random variables ϵ, R, and $\tilde{R}$ such that the following holds:*

(i) *ϵ is an expansive constant for φ with $\log \epsilon \in \mathbb{L}^1(\mathbb{P})$.*

(ii) *For every $n \ge 0$ we have that $\mathbb{P}$-a. s.*
$$B_\omega(n+1, x, \epsilon) \subset B\left(x, \frac{e^{R(\vartheta^n \omega)}}{\lambda_n(\omega, x)}\right) \qquad \textit{for every } x \in E_\omega.$$

(iii) *For every $n \in \mathbb{N}$ and $0 < \delta \le 1$ we have that $\mathbb{P}$-a. s.*
$$B\left(x, \frac{\delta \tilde{R}(\vartheta^n \omega)}{\eta_n(\omega, x)}\right) \subset B_\omega(n, x, \delta) \qquad \textit{for every } x \in E_\omega.$$
Here $\eta_n(\omega, x) := \prod_{k=0}^{n-1} \eta(\vartheta^k \omega, \varphi(k, \omega)x)$.

Proof. Take α, λ_n, Λ_n, and K from definition 3.1. Since $-\log \Lambda_n$ defines a subadditive sequence of random variables, there exists $\beta > 0$ with
$$\lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(\omega) = \sup_{n \ge 0} \frac{1}{n} \log \Lambda_n(\omega) = \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(\vartheta^{-n}\omega) = \beta \quad \mathbb{P}\text{–a.s.} \tag{4}$$
With this, we define
$$\epsilon(\omega) := \min\left\{1, \left(\frac{(1 - e^{-\frac{\beta}{2}})\Lambda_1(\omega)}{K(\omega)}\right)^{\frac{1}{\alpha}}\right\}.$$
Clearly, $\log \epsilon \in L^1(\mathbb{P})$.

Assume $d(x,y) \le \epsilon(\omega)$. Then, by almost conformality, we have
$$\begin{aligned} d(\varphi(\omega)x, \varphi(\omega)y) &\ge (\lambda(\omega, x) - K(\omega)\epsilon(\omega)^\alpha) d(x,y) \\ &\ge (\lambda(\omega,x) - (1 - e^{-\frac{\beta}{2}})\lambda(\omega, x)) d(x,y) = e^{-\frac{\beta}{2}} \lambda(\omega, x) d(x,y). \end{aligned}$$
With $\gamma(\omega, x) := e^{-\frac{\beta}{2}} \lambda(\omega, x)$, thus, we have
$$d(x,y) \le \epsilon(\omega) \qquad \Rightarrow \qquad d(\varphi(\omega)x, \varphi(\omega)y \ge \gamma(\omega, x) d(x,y). \tag{5}$$
But $\limsup_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} \gamma(\vartheta^k \omega, \varphi(k, \omega)x) \ge \frac{\beta}{2} > 0$, which proves statement (i).

To prove (ii), take $x \in E_\omega$ and $y \in B_\omega(n+1, x, \epsilon)$, i.e.

$$d(\varphi(k,\omega)x, \varphi(k,\omega)y) \leq \epsilon(\vartheta^k\omega) \text{ for } k = 0, ..., n.$$

By (5), then, we have

$$\begin{aligned} 1 &\geq d(\varphi(n,\omega)x, \varphi(n,\omega)y) \\ &\geq \left(\prod_{i=k}^{n-1} \gamma(\vartheta^i\omega, \varphi(i,\omega)x)\right) d(\varphi(k,\omega)x, \varphi(k,\omega)y) \qquad \text{for } 0 \leq k \leq n. \end{aligned}$$

With $\Gamma_n(\omega) := e^{-\frac{n\beta}{2}}\Lambda_n(\omega) = \min_{x\in E_\omega} \prod_{k=0}^{n-1} \gamma(\vartheta^k\omega, \varphi(k,\omega)x)$ this rewrites as

$$d(\varphi(k,\omega)x, \varphi(k,\omega)y) \leq \frac{1}{\Gamma_{n-k}(\vartheta^k\omega)} \qquad \text{for } 0 \leq k \leq n. \tag{6}$$

Using the fact that $\log a - \log b \leq \frac{a-b}{\min\{a,b\}}$ if $a, b > 0$, definition 3.1(i) thus yields

$$\begin{aligned} \log \lambda(\vartheta^k\omega, \varphi(k,\omega)x) &- \log \frac{d(\varphi(k+1,\omega)x, \varphi(k+1,\omega)y)}{d(\varphi(k,\omega)x, \varphi(k,\omega)y)} \\ &\leq \frac{K(\vartheta^k\omega)}{\Gamma_1(\vartheta^k\omega)} \cdot \left(\Gamma_{n-k}(\vartheta^k\omega)\right)^{-\alpha} \qquad \text{for } k = 0, ...n-1. \end{aligned}$$

Summing this over $k = 0, ...n-1$ yields

$$\log \lambda_n(\omega, x) - \log \frac{d(\varphi(n,\omega)x, \varphi(n,\omega)y)}{d(x,y)} \leq \sum_{k=1}^{n} \frac{K(\vartheta^{n-k}\omega)}{\Gamma_1(\vartheta^{n-k}\omega)} \cdot \left(\Gamma_k(\vartheta^{n-k}\omega)\right)^{-\alpha}. \tag{7}$$

Set $R_k(\omega) := \frac{K(\vartheta^{-k}\omega)}{\Gamma_1(\vartheta^{-k}\omega)} \cdot \left(\Gamma_k(\vartheta^{-k}\omega)\right)^{-\alpha}$ so that the right hand side of (7) reads $\sum_{k=1}^n R_k(\vartheta^n\omega)$.

Since $\log K$ and $\log \Gamma_1$ are integrable we have

$$\lim_{k\to\infty} \frac{1}{k} \log R_k(\omega) = -\alpha \lim_{k\to\infty} \frac{1}{k} \log \Gamma_k(\vartheta^{-k}\omega) = -\frac{\alpha\beta}{2} \quad \mathbb{P}\text{–a.s.}$$

Therefore $R_k(\omega) \leq \left(\exp\left(-\frac{\alpha\beta}{4}\right)\right)^k$ for all exept finitely many k (depending on ω) so

$$R(\omega) := \sum_{k=1}^{\infty} R_k(\omega) < \infty \quad \mathbb{P}\text{–a.s.}$$

By (7), thus, we have

$$\log \lambda_n(\omega, x) - \log \frac{d(\varphi(n,\omega)x, \varphi(n,\omega)y)}{d(x,y)} \leq R(\vartheta^n\omega)$$

for any $y \in B_\omega(n+1, x, \epsilon)$, which immediately proves (ii).

In order to prove (iii) we set $\tilde{K}(\omega) := \frac{K(\omega)}{\Lambda_1(\omega)^{1+\alpha}}$ so that

$$d(\varphi(\omega)x, \varphi(\omega)y) \leq \eta(\omega, x) d(x,y) \left(1 + \tilde{K}(\omega) d(\varphi(\omega)x, \varphi(\omega)y)^\alpha\right)$$

for every $x, y \in E_\omega$. [This is obviously true if $d(\varphi(\omega)x, \varphi(\omega)y) \leq \eta(\omega, x)d(x,y)$ and follows otherwise from definition 3.1(i) if we replace $d(x,y)^\alpha$ by $\left(\frac{d(\varphi(\omega)x, \varphi(\omega)y)}{\eta(\omega,x)}\right)^\alpha$ using $\eta(\omega, x) \geq \Lambda_1(\omega)$.]

Hence for any $\rho > 0$ we have

$$d(\varphi(\omega)x, \varphi(\omega)y) \geq \rho \qquad \Rightarrow \qquad d(x,y) \geq \frac{\rho}{\eta(\omega,x)(1+\tilde{K}(\omega)\rho^{\alpha})}. \tag{8}$$

[Recall that $\alpha \leq 1$.]

Now set

$$\zeta_n(\omega) := \prod_{k=1}^{n} \left[1 + \tilde{K}(\vartheta^{-k}\omega)\left(\Lambda_{k-1}(\vartheta^{-k+1}\omega)\right)^{-\alpha}\right]^{-1}$$

and $\zeta(\omega) := \inf_{n \geq 0} \zeta_n(\omega) = \lim_{n\to\infty} \zeta_n(\omega)$. We have $\zeta(\omega) > 0$.

Indeed, because $\lim_{k\to\infty} \frac{1}{k} \log \left[\tilde{K}(\vartheta^{-k}\omega)\left(\Lambda_{k-1}(\vartheta^{-k+1}\omega)\right)^{-\alpha}\right] = -\alpha\beta < 0$ we have $\sum_{k=1}^{\infty} \tilde{K}(\vartheta^{-k}\omega)\left(\Lambda_{k-1}(\vartheta^{-k+1}\omega)\right)^{-\alpha} < \infty$ which immediately implies $\zeta(\omega) > 0$.

Next, let $k \in \{0, ..., n-1\}$ and $0 < \delta \leq 1$. We can apply (8) with $\rho = \delta \frac{\zeta_k(\vartheta^n\omega)}{\eta_k(\vartheta^{n-k}\omega, \varphi(n-k,\omega)x)} \leq \frac{1}{\Lambda_k(\vartheta^{n-k}\omega)}$. Thus

$$d(\varphi(n-k,\omega)x, \varphi(n-k,\omega)y) \geq \delta \frac{\zeta_k(\vartheta^n\omega)}{\eta_k(\vartheta^{n-k}\omega, \varphi(n-k,\omega)x)}$$

implies

$$\begin{aligned} &d(\varphi(n-k-1,\omega)x, \varphi(n-k-1,\omega)y) \\ &\quad\geq \delta \frac{\zeta_k(\vartheta^n\omega)}{\eta_{k+1}(\vartheta^{n-k-1}\omega, \varphi(n-k-1,\omega)x)\left(1+\tilde{K}(\vartheta^{n-k-1}\omega)(\Lambda_k(\vartheta^{n-k}\omega))^{-\alpha}\right)} \\ &\quad= \delta \frac{\zeta_{k+1}(\vartheta^n\omega)}{\eta_{k+1}(\vartheta^{n-k-1}\omega, \varphi(n-k-1,\omega)x)}. \end{aligned}$$

We may conclude that for $j = 1, ..., n$ we have

$$d(x,y) < \delta \frac{\zeta(\vartheta^n\omega)}{\eta_n(\omega,x)} \qquad \Rightarrow \qquad d(\varphi(j,\omega)x, \varphi(j,\omega)y) < \frac{\delta}{\eta_{n-j}(\vartheta^j\omega, \varphi(j,\omega)x)}.$$

With $\underline{\Lambda}(\omega) := \inf_{n\geq 0} \Lambda_n(\vartheta^{-n}\omega) > 0$ and $\tilde{R}(\omega) := \zeta(\omega)\underline{\Lambda}(\omega)$, and using $\underline{\Lambda}(\vartheta^n\omega) \leq \eta_{n-j}(\vartheta^j\omega, \varphi(j,\omega)x)$ this immediately gives (iii). □

Now we are prepared to prove the following version of the Bowen-Ruelle formula.

3.4. Theorem. *Let φ be an expanding (λ, η) almost conformal bundle RDS on E.*

(i) *There exists a unique real number $t_1 \geq 0$ with $\pi_\varphi(-t_1 \log \eta) = 0$.*
For $\mathbb{P}$ almost every ω we have

$$t_1 \leq \mathrm{HD}(E_\omega).$$

(ii) *There exists a unique real number $t_2 \geq 0$ with $\pi^*_\varphi(-t_2 \log \lambda)(\epsilon/2) = 0$, where ϵ is the expansive constant from lemma 3.3.*
For $\mathbb{P}$ almost every ω we have

$$\mathrm{HD}(E_\omega) \leq t_2.$$

Proof. (i) First observe that $t \mapsto \pi_\varphi(-t\log\eta)$ is strictly decreasing. Indeed, for every φ–invariant measure μ we have

$$\int -t\log\eta d\mu \leq \int -t\log\Lambda_1(\omega)d\mu(\omega,x) = -t\int \Lambda_1 d\mathbb{P} \leq -t\beta,$$

where β is from (4). By the variational principle, thus,

$$\pi_\varphi(-(t+s)\log\eta) = \sup_\mu \left(h_\mu(\varphi) - \int (t+s)\log\eta d\mu\right) \leq \pi_\varphi(-t\log\eta) - s\beta.$$

Let m be an equilibrium state with respect to $-t_1\log\eta$ with disintegration $dm(\omega,x) = dm_\omega(x)d\mathbb{P}(\omega)$ so that $h_m(\varphi) - \int t_1 \log\eta d\mu = 0$. The Brin–Katok formula (cf. Kifer [Kif96, Formula (4.42)]) yields for m–a.e. $(\omega,x)\in E$

$$\text{(9)} \qquad -\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\log m_\omega(B_\omega(n,x,\delta)) = h_m(\varphi)$$
$$= t_1 \int \log\eta dm = t_1 \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\eta(\vartheta^k\omega,\varphi(k,\omega)x).$$

With this we will show

$$\text{(10)} \qquad \liminf_{r\to 0}\frac{\log m_\omega(B(x,r))}{\log r} \geq t_1 \qquad \text{for } m\text{–a.e. } (\omega,x)\in E,$$

which implies $HD(E_\omega) \geq HD(m_\omega) \geq t_1$ (cf. Young [You82, Proposition 2.1]). The reasoning will be for each $(\omega,x)\in E$ separately, the dependence on (ω,x) of some constants below will not be made explicit.

So fix $(\omega,x)\in E$ satisfying (9). Because $m_\omega(B(x,r))$ decreases monotonically in r it suffices to consider a sequence (r_j) with $\lim_{j\to\infty} r_j = 0$ and $\lim_{j\to\infty}\frac{\log r_{j+1}}{\log r_j} = 1$.

In order to define r_j choose $\tilde{R} > 0$ with $\mathbb{P}\{\tilde{R}(\omega) \geq \tilde{R}\} > 0$, where is $\tilde{R}(\omega)$ from lemma 3.3(iii). For a.e. ω there exists a sequence $(n(j))_{j\in\mathbb{N}}$ of integers with $\lim_{j\to\infty} n(j) = \infty$, $\lim_{j\to\infty}\frac{n(j+1)}{n(j)} = 1$, and $\tilde{R}(\vartheta^{n(j)}\omega) \geq \tilde{R}$ for all $j\in\mathbb{N}$.

Fix $t < t_1$. By (9) we can find $\delta > 0$ such that

$$\lim_{n\to\infty}\frac{1}{n}\log m_\omega(B_\omega(n,x,\delta)) \leq -\frac{t}{t_1}h_m(\varphi).$$

Set $r_j := \frac{\delta\tilde{R}}{\eta_{n(j)}(\omega,x)}$ and $\Sigma_j := \sum_{k=0}^{n(j)-1}\log\eta(\vartheta^k\omega,\varphi(k,\omega)x)$. By (9) we have

$$\lim_{j\to\infty}\frac{1}{n(j)}\Sigma_j = \frac{h_m(\varphi)}{t_1}.$$

This gives $\lim_{j\to\infty}\frac{\log r_{j+1}}{\log r_j} = 1$. [Just write $\frac{\log r_{j+1}}{\log r_j} = \frac{-\Sigma_{j+1}}{-\Sigma_j} = \frac{n(j+1)}{n(j)}\cdot\frac{\frac{1}{n(j+1)}\Sigma_{j+1}}{\frac{1}{n(j)}\Sigma_j}$.]

Furthermore, by lemma 3.3(iii), we have

$$\log m_\omega(B(x,r_j)) \geq \log m_\omega\left(B\left(x,\frac{\delta\tilde{R}(\vartheta^{n(j)}\omega)}{\eta_{n(j)}(\omega,x)}\right)\right) \geq \log m_\omega\left(B_\omega\left(n(j),x,\delta\right)\right).$$

Now we can calculate

$$\liminf_{r\to 0}\frac{\log m_\omega(B(x,r))}{\log r} = \liminf_{j\to\infty}\frac{\frac{1}{n(j)}\log m_\omega(B(x,r_j))}{\frac{1}{n(j)}\log r_j}$$
$$\geq \frac{\lim_{j\to\infty}\frac{1}{n(j)}\log m_\omega(B_\omega(n(j),x,\delta))}{\lim_{j\to\infty}\frac{1}{n(j)}(-\Sigma_j + \log\delta\tilde{R})} = \frac{-\frac{t}{t_1}h_m(\varphi)}{-h_m(\varphi)/t_1} = t.$$

Because $t < t_1$ was arbitrary, this completes the proof of (10), hence (i).

(ii) By the definition of π^* and $\lambda_n(\omega,x) \geq \Lambda_n(\omega)$ we have $\mathbb{P}$ a.s. for $s,t\geq 0$

$$\begin{aligned}\pi^*_\varphi(-(t+s)\log\lambda)(\epsilon/2) &\leq \pi^*_\varphi(-t\log\lambda)(\epsilon/2) + \limsup_{n\to\infty}\frac{1}{n}\log\Lambda_n(\omega)^{-s}\\ &= \pi^*_\varphi(-t\log\lambda)(\epsilon/2) - s\beta.\end{aligned}$$

Here λ_n, Λ_n are from definition 3.1 and β from (4). This shows [since $\pi^*_\varphi(0)(\epsilon/2)\geq 0$] that there exists a unique $t_2\geq 0$ with $\pi^*_\varphi(-t_2\log\lambda)(\epsilon/2)=0$.

Take $R(\omega)$ from lemma 3.3(ii) and fix $R<\infty$ with $\mathbb{P}\{R(\omega)\leq R\}>0$. For a.e. ω there exists a sequence $(n(j))_{j\in\mathbb{N}}$ of integers with $\lim_{j\to\infty} n(j)=\infty$ and $R(\vartheta^{n(j)}\omega)\leq R$ for all $j\in\mathbb{N}$. For every j choose a maximal $(\omega, n(j)+1,\epsilon/2)$–separated set $G_j\subset E_\omega$ so that

$$\bigcup_{x\in G_j} B_\omega(n(j)+1,x,\epsilon) = E_\omega.$$

Furthermore, $\operatorname{diam} B_\omega(n(j)+1,x,\epsilon)\leq \frac{2e^R}{\lambda_{n(j)}(\omega,x)}\leq\frac{2e^R}{\Lambda_{n(j)}(\omega)}$ by lemma 3.3(ii). Thus for $t>t_2$ we have

$$\begin{aligned}\sum_{x\in G_j}\left(\operatorname{diam} B_\omega(n(j)+1,x,\epsilon)\right)^t &\leq 2^t e^{Rt}\sum_{x\in G_j}\exp\left(-t\sum_{k=0}^{n(j)-1}\log\lambda(\vartheta^k\omega,\varphi(k,\omega)x)\right)\\ &\leq 2^t e^{Rt}\pi_\varphi(-t\log\lambda)(\omega,n(j)+1,\epsilon/2)\end{aligned}$$

for sufficiently large j, because

$$\lim_{j\to\infty}\frac{1}{n(j)+1}\log\pi_\varphi(-t\log\lambda)(\omega,n(j)+1,\epsilon/2) = \pi^*_\varphi(-t\log\lambda)(\epsilon/2)<0.$$

Since $\sup_{x\in G_j}\operatorname{diam} B_\omega(n(j)+1,x,\epsilon)$ tends to 0 as $j\to\infty$ the t–dimensional Hausdorff measure of E_ω is finite for every $t>t_2$. Hence we have $HD(E_\omega)\leq t_2$ for $\mathbb{P}$–a.e. ω. □

We can improve statement (ii) as soon as we are in the situation of theorem 1.9, in that then we may look at the zero of $t\mapsto\pi_\varphi(-t\log\eta)$. The following, thus, is just a reformulation of theorem 3.4 for the special case which we will consider later on.

3.5. Corollary. *Let φ be an expanding (λ,η) almost conformal bundle RDS on a compact bundle E with fibers in a finite dimensional Riemannian manifold M. Assume*

(i) $\int\log\operatorname{diam} E_\omega d\mathbb{P}(\omega)<\infty$,
(ii) *φ is strongly expansive.*

Then there exist unique real numbers $t_2\geq t_1\geq 0$ with $\pi_\varphi(-t_1\log\eta)=0=\pi_\varphi(-t_2\log\lambda)$. We have

$$t_1\leq \mathrm{HD}(E_\omega)\leq t_2 \qquad \mathbb{P}\text{-}a.s.$$

Note that this gives an exact formula for the Hausdorff dimension in the conformal case, i. e. when $\lambda=\eta$.

3.6. *Remark.* A first random version of the Bowen–Ruelle formula has been given by Kifer [Kif96, Theorem 4.8]. Our proof of theorem 3.4 is based on the ideas of Kifer's proof: Using coverings by random Bowen balls to prove the upper bound and using the Brin-Katok formula to prove the lower bound.

The main differences between theorem 3.4 and Kifer [Kif96, Theorem 4.8] are:

- We do not assume a mixing condition on φ.
- Our expansive constant ϵ is allowed to be a random variable, which makes it possible to work with systems with weaker uniformity properties. [Consider the following example: Let $\varphi(\omega) : S^1 \to S^1, z \mapsto z^{n(\omega)}$ for some unbounded random variable $n \in L^1(\Omega, \mathbb{N})$, $n(\omega) \geq 2$. Then the RDS generated by φ is not expansive for any deterministic constant $\delta > 0$.]
- φ is not assumed to be conformal. This makes it possible to apply theorem 3.4 to general $C^{1+\alpha}$ perturbations of a conformal repeller instead of perturbations by conformal maps.
- Our setup allows bundles of compact metric spaces and does not assume that the system lives on a smooth manifold.

4. Stability of the Hausdorff dimension under random perturbations

In this section we apply theorem 3.4 to random perturbations of a conformal repeller.

4.1. Definition. A compact subset J of a smooth Riemannian manifold M is said to be a *conformal repeller* for a $\mathcal{C}^{1+\alpha}$ map $F : U \to M$, where $U \subset M$ is an open neighborhood of J, if the following holds:

(i) $FJ = J$,
(ii) $J = \bigcap_{n \geq 0} F^{-n}(U)$,
(iii) $DF(x)$ is conformal for every $x \in J$, i. e. $DF(x) = \lambda(x)O(x)$, where $\lambda(x) \in \mathbb{R}$ and $O(x) \in \mathcal{O}(d, \mathbb{R})$ is orthogonal,
(iv) There exists $n \geq 1$ such that $|\lambda(F^{n-1}(x))...\lambda(x)| > 1$ for every $x \in J$.

Conformal repellers persist under random perturbations. Recall that $\mathcal{K}$ denotes the collection of all compact subsets of M endowed with the Hausdorff topology.

4.2. Proposition. *Assume J is a conformal repeller for a $\mathcal{C}^{1+\alpha}$ map $F : U \to M$. There exists a $\mathcal{C}^1$ neighborhood $\mathcal{U}(F) \subset \mathcal{C}^1(U, M)$ of F such that the following holds:*

(i) *For every random variable $\varphi : \Omega \to \mathcal{U}(F)$ there exists a $\mathcal{K}$-valued random variable J_φ satisfying $J_\varphi(\omega) \subset U$ and $\varphi(\omega)J_\varphi(\omega) = J_\varphi(\vartheta\omega)$, and a $\mathcal{C}^0(U, M)$-valued random variable h such that each $h(\omega)$ is a homeomorphism between J and $J_\varphi(\omega)$ and $\varphi(\omega) \circ h(\omega) = h(\vartheta\omega) \circ F$ on J.*
(ii) *If $(\varphi_\epsilon : \Omega \to \mathcal{U}(F))_{\epsilon>0}$ is a family of random variables with $\lim_{\epsilon\to 0} \varphi_\epsilon = F$ in probability (with respect to the $\mathcal{C}^1$ distance), then $\lim_{\epsilon\to 0} h_\epsilon = \mathrm{id}$ in probability and thus $\lim_{\epsilon\to 0} J_\epsilon = J$ in probability. Here h_ϵ and J_ϵ are the corresponding objects associated to φ_ϵ by (i).*

In other words, each family $\{\varphi_\epsilon\}_{\epsilon>0}$ of $\mathcal{U}(F)$-valued random variables with $\lim_{\epsilon\to 0} \varphi_\epsilon = F$ in probability is a structurally stable random perturbation of F on J.

Proof. (i) can be shown analogously to the corresponding assertions in Theorem 1.1 of [Liu98].

To prove (ii) one has to observe that $h(\omega) \in \mathcal{C}(J, M)$ depends continuously on $(\varphi(\vartheta^k\omega))_{k\in\mathbb{Z}} \in \mathcal{U}(F)^{\mathbb{Z}}$ (equipped with product topology). This follows in the same way as the corresponding statement of [Liu98, Theorem 1.1]. ∎

Here is our final result.

4.3. Theorem. *Assume J is a conformal repeller for a $\mathcal{C}^{1+\alpha}$ map $F : U \to M$. There exists a $\mathcal{C}^1$ neighborhood $\mathcal{U}(F) \subset \mathcal{C}^1(U, M)$ of F such that the assertions of proposition 4.2 hold with the following additional property.*

If $(\varphi_\epsilon : \Omega \to \mathcal{U}(F) \cap \mathcal{C}^{1+\alpha}(U, M))_{\epsilon>0}$ is a random perturbation of F with

$$\lim_{\epsilon\to 0} \varphi_\epsilon = F \qquad \text{in } \mathbb{L}^1(\Omega, \mathcal{C}^{1+\alpha}(U, M)),$$

then

$$\lim_{\epsilon\to 0} \mathrm{HD}(J_\epsilon(\omega)) = \mathrm{HD}(J) \qquad \mathbb{P}\text{-a. s.}$$

Moreover, if $L \subset J$ is compact with $FL = L$, then $\lim_{\epsilon\to 0} \mathrm{HD}(h_\epsilon(\omega)L) = \mathrm{HD}(L)$ a. s.

Proof. Choose an open neighborhood V of J such that $\bar{V}$ is a compact subset of U. Then $\mathcal{U}(F)$ can be chosen in such a way that $J_\epsilon(\omega) \subset \bar{V}$ for every $\epsilon > 0$, $\omega \in \Omega$.

Fix $\epsilon > 0$. For $(\omega, x) \in \Omega \times \bar{V}$ set

$$\eta_\epsilon(\omega, x) := \|D\varphi_\epsilon(\omega)(x)\| \text{ and } \lambda_\epsilon(\omega, x) := \|(D\varphi_\epsilon(\omega)(x))^{-1}\|^{-1}.$$

(1) We claim that φ_ϵ is $(\lambda_\epsilon, \eta_\epsilon)$ almost conformal.

Choose $r_0 > 0$ such that $A := \{x : \mathrm{dist}(x, \bar{V}) \le r_0\} \subset U$. Define

$$K_0(\omega) := |D\varphi_\epsilon(\omega)|_{\alpha,A} = \sup\left\{ \frac{\|D\varphi_\epsilon(\omega)(x) - D\varphi_\epsilon(\omega)(y)\|}{d(x,y)^\alpha} : x, y \in A, x \ne y \right\}.$$

We restrict ourselves to the case when M is an open subset of $\mathbb{R}^d$. For the general case, similar calculations can be done via local coordinates. We let $|\cdot|$ denote the Euclidian norm on $\mathbb{R}^d$ and write φ instead of $\varphi_\epsilon(\omega)$ for convenience.

For $x, y \in J_\epsilon(\omega)$ with $0 < |x - y| < r_0$ we put $e := \frac{y-x}{|y-x|}$ and obtain

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &= \left| \int_0^{|y-x|} D\varphi(x + te) e dt \right| \le \int_0^{|y-x|} \|D\varphi(x + te)\| dt \\
&\le |y - x| \cdot \sup\{\|D\varphi(x + z)\| : |z| \le |y - x|\} \\
&\le |y - x| \left(\|D\varphi(x)\| + K_0(\omega)|y - x|^\alpha \right).
\end{aligned}$$

Thus, $\frac{|\varphi(y)-\varphi(x)|}{|y-x|} \le \eta_\epsilon(\omega, x) + K_0(\omega)|y - x|^\alpha$.

On the other hand,

$$\begin{aligned}
&\frac{1}{|y - x|} \left| \int_0^{|y-x|} D\varphi(x + te) e dt \right| \\
&\quad \ge \inf\{|Ae| : A \in \text{“convex hull of } D\varphi(x + te), 0 \le t \le |y - x|\text{”}\} \\
&\quad \ge |D\varphi(x)e| \\
&\qquad - \sup\{|Ae| : A \in \text{“convex hull of } (D\varphi(x + te) - D\varphi(x)), 0 \le t \le |y - x|\text{”}\} \\
&\quad \ge \lambda_\epsilon(\omega, x) - K_0(\omega)|y - x|^\alpha.
\end{aligned}$$

Here we used $|D\varphi(x)e| \ge \lambda_\epsilon(\omega, x)$ [which is immediate from the definition of λ_ϵ].

Thus φ_ϵ is $(\lambda_\epsilon, \eta_\epsilon)$ almost conformal with

$$K(\omega) = \max\left\{ K_0(\omega), \frac{\mathrm{diam}\, \bar{V}}{r_0}, \frac{\max\{\|D\varphi_\epsilon(\omega)(x)\| : x \in \bar{V}\}}{r_0^\alpha} \right\}.$$

(2) We claim that φ_ϵ is expanding in the sense of definition 3.1(ii). To prove this we have to show that there exists $n = n(\omega) \geq 1$ with $\Lambda_n^\epsilon(\omega) := \min_{x\in J_\epsilon(\omega)} \prod_{k=0}^{n-1} \lambda_\epsilon(\vartheta^k\omega, \varphi_\epsilon(k,\omega)x) > 1$.

Let λ and n be from definition 4.1(iii) and (iv). Observe that $\lim_{\epsilon\to 0} \log \Lambda_n^\epsilon(\cdot) = \min_{x\in J} \log \lambda(F^{n-1}x) + ... + \log \lambda(x) > 0$ in probability.

By making $\mathcal{U}(F)$ smaller if necessary we have that $|\log(|D\varphi(x)e|)|$ is uniformly bounded for all $\varphi \in \mathcal{U}(F)$, $x \in \bar{V}$, and $e \in T_xM$ with $|e| = 1$. Thus $\lim_{\epsilon\to 0} \log \Lambda_n^\epsilon(\cdot) = \min_{x\in J} \log \lambda(F^{n-1}x) + ... + \log \lambda(x) > 0$ also in $\mathbb{L}^1$.

Hence for sufficiently small ϵ the subadditive ergodic theorem yields

$$0 < \sup_{n\geq 1} \frac{1}{n} \int \log \Lambda_n^\epsilon d\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \int \log \Lambda_n^\epsilon d\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n^\epsilon(\omega) \quad \mathbb{P}\text{–a.s.},$$

i.e. for a.e. ω there exists $n(\omega)$ with $\log \Lambda_{n(\omega)}^\epsilon(\omega) > 0$.

(3) We claim that φ_ϵ is strongly expansive. Since $F|_J$ is expanding there exist a neighborhood V of J, a constant $c > 0$, and an integer $n \geq 1$ such that $|DF^n(x)| \geq 1+c$ for every $x \in V$. We can choose $\mathcal{U}(F)$ in a way that $|D(\varphi_n \circ ... \circ \varphi_1)(x)| \geq 1 + \frac{c}{2}$ whenever $\varphi_1, ..., \varphi_n \in \mathcal{U}(F)$ and $x \in V$, and that $J_\epsilon(\omega) \subset V$ for every $\epsilon > 0$ and $\omega \in \Omega$. Then φ_ϵ is uniformly expanding and thus strongly expansive on the bundle $(J_\epsilon(\omega))_{\omega\in\Omega}$.

(4) Now let $L \subseteq J$ be compact with $FL = L$. We apply corollary 3.5 to the bundle RDS φ_ϵ on $E_\epsilon := \{(\omega, x) : x \in h_\epsilon(\omega)L\}$. Denoting the pressure functional of φ_ϵ restricted to E_ϵ by π_ϵ, thus, we have that there exist $t_2^\epsilon \geq t_1^\epsilon \geq 0$ with

$$\pi_\epsilon(-t_1^\epsilon \log \eta_\epsilon) = 0 = \pi_\epsilon(-t_2^\epsilon \log \lambda_\epsilon)$$

and

$$t_1^\epsilon \leq \mathrm{HD}(h_\epsilon(\omega)L) \leq t_2^\epsilon \qquad \mathbb{P}\text{-a.s.}$$

Furthermore, by the classical Bowen-Ruelle formula the Hausdorff dimension of L is the unique $t_0 \geq 0$ with $\pi_{F|_L}(-t_0 \log \lambda) = 0$.

(5) Next, we apply proposition 2.3. Just note that

$$\lim_{\epsilon\to 0} \|\log \lambda_\epsilon - \log \lambda\|_\alpha = 0 = \lim_{\epsilon\to 0} \|\log \eta_\epsilon - \log \lambda\|_\alpha \qquad \text{in } \mathbb{L}^1(\mathbb{P})$$

because $|\log \|(D\varphi_\epsilon(\omega)(x))^{\pm 1}\||$ is uniformly bounded. Hence

$$\lim_{\epsilon\to 0} \pi_\epsilon(-t \log \lambda_\epsilon) = \lim_{\epsilon\to 0} \pi_\epsilon(-t \log \eta_\epsilon) = \pi_{F|_L}(-t \log \lambda) \qquad \text{for every } t \geq 0.$$

(6) To complete the proof, let $\delta > 0$ be given. Because $t \mapsto \pi_{F|_L}(-t \log \lambda)$ is strictly decreasing there exists $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$

$$\begin{aligned} &\pi_\epsilon(-(t_0+\delta)\log \lambda_\epsilon) < 0 < \pi_\epsilon(-(t_0-\delta)\log \lambda_\epsilon) \quad \text{and} \\ &\pi_\epsilon(-(t_0+\delta)\log \eta_\epsilon) < 0 < \pi_\epsilon(-(t_0-\delta)\log \eta_\epsilon). \end{aligned}$$

Thus $t_0 - \delta < t_1^\epsilon \leq \mathrm{HD}(h_\epsilon(\omega)L) \leq t_2^\epsilon < t_0 + \delta$, finishing the proof. □

4.4. *Remark.* Ruelle [Rue83] showed that if J is a mixing conformal repeller for F and $t = \mathrm{HD}(J)$ then the equilibrium state m of $-t \log |DF|$ is the unique invariant probability measure which is equivalent to the t–dimensional Hausdorff measure.

Corollary 2.4 implies that m is stochastically stable in the following sense. Let $(\varphi_\epsilon)_{\epsilon>0}$ satisfy the assumptions of theorem 3.4 and let μ_ϵ be an equilibrium state for $-t_1^\epsilon \log \eta_\epsilon$ (or $-t_2^\epsilon \log \lambda_\epsilon$) with respect to φ_ϵ. Then $\lim_{\epsilon\to 0} \mu_\epsilon = \mathbb{P} \otimes m$.

If in addition the $\varphi_\epsilon(\omega)$ are conformal then $\eta_\epsilon = \lambda_\epsilon$. "Randomizing" Ruelle's proof one should be able to show that μ_ϵ is the unique φ_ϵ invariant measure whose disintegrations on the fibers $J_\epsilon(\omega)$ are almost surely equivalent to the $t_1^\epsilon = t_2^\epsilon = \mathrm{HD}(J_\epsilon(\omega))$ dimensional Hausdorff measure.

4.5. *Remark.* Combining techniques of McCluskey and Manning [MM83] with that of this paper it should be possible to derive stochastic stability of the Hausdorff dimension of "horseshoes", where a horseshoe is meant to be a hyperbolic invariant set of a surface diffeomorphism with one–dimensional stable and unstable manifold, respectively.

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