Typical configurations for one dimensional random field Kac model *

Marzio Cassandro [1] Enza Orlandi [2] Pierre Picco [3]

Abstract. In this paper we study the typical profiles of a random field Kac model. We give upper and lower bounds of the space scale where the system stays at equilibrium.

1 Introduction

In this paper we consider a one dimensional spin system with a ferromagnetic two body Kac potential and a stochastic external magnetic field. Problems where a stochastic contribution is added to the energy of the system naturally arise in condensed matter physics where the presence of impurities causes the microscopic structure to vary from point to point. A lot of work was dedicated to the subject of spin system with random magnetic field, let us mention [2], [3], [4], [5], [6],[8], [10], [12], [14], [15], [16],[17], [19],[23], [28].

The Kac potentials are functions $J_\gamma(r)$ which depend on the scaling parameter γ as $J_\gamma(r) = \gamma J(\gamma r)$. The equilibrium statistical mechanics of these systems in absence of stochastic external field is well known. In the limit $\gamma \downarrow 0$, it is possible to explicitely derive the thermodynamic potentials, prove the existence of a critical temperature and give a very natural and transparent explanation of the phenomenon of spontaneous magnetization in ferromagnetic systems [18], [20]. It is also possible to analyze the limiting Gibbs states, but since the direct interaction between any two given spins vanishes when $\gamma \downarrow 0$, in order to get non trivial limiting distributions, it is useful to introduce the so called block-spins, that are space average of spins over regions whose size diverges as $\gamma \downarrow 0$ and describe the configurations of the system in term of these magnetization profiles. In the one dimensional case, this analysis [11] for Ising spin and [7] for more general spin, allows to get a satisfactory

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[1] Dipartimento di Fisica, Universitá di Roma "La Sapienza", P.le A. Moro, 00185 Roma, Italy. cassandro@roma1.infn.it

[2] Dipartimento di Matematica, Universitá di Roma Tre, L.go S.Murialdo 1, 00156 Roma, Italy. orlandi@matrm3.mat.uniroma3.it

[3] CPT–CNRS, Luminy, Case 907, F-13288 Marseille Cedex 9, France. picco@cpt.univ-mrs.fr

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description of the typical profiles. The results can be summarized in the following way. The empirical spin average in blocks of size δ/γ, for any positive δ, converges as $\gamma \downarrow 0$, to one of the two thermodynamic magnetizations, uniformly in intervals of size $1/\gamma^p$, for any given $p \geq 1$. Furthermore the intervals where the magnetization is approximately constant have lengths of the order $e^{(\Delta f)/\gamma}$ where Δf is the activation energy of the corresponding Curie-Weiss model.

In this paper we add a stochastic magnetic field and study how the previous picture is modified. This is a particular case of the general problem of stochastic perturbation of random systems. Random walk in random environment is another famous example, [29]. The general theory of such systems is far from being complete, therefore it is important to have examples that can be rigorously treated where the behavior of the perturbed system is radically different from the unperturbed one. The first step in the analysis of such systems is to find the right scale where new phenomena occur. The rigorous analysis is in general delicate even if the heuristic arguments are simple.

In our case, if we consider the system in a volume of order $1/\gamma$ and let $\gamma \downarrow 0$, the model is equivalent to the random field Curie-Weiss model [2], [4], [5], [6], [8], [19],[23], [28]. It is possible to define a critical temperature and if the variance of the magnetic fields is small enough, only two distinct magnetization profiles occur, the relative weight of each one being a random variable. When we take first the infinite volume limit and then the limit $\gamma \downarrow 0$, new phenomena occur that depend on the scale we are considering. If we consider what happens in a large interval, say centered at the origin and of length $\gamma^{-2}[\log 1/\gamma)^p]$ for some $p > 1$, we start seeing new effects of the random magnetic field. The profiles that were approximatively constant on a scale $e^{[(1-\epsilon)\Delta f]/\gamma}$ and made a transition between the two equilibria on a scale $e^{[(1+\epsilon)\Delta f]/\gamma}$ when the random magnetic field was switched off, now make a transition on a scale at most $\gamma^{-2}[(\log 1/\gamma)(\log\log 1/\gamma)^2]$ and are constant on a scale at least $\ell(\gamma) = \gamma^{-2}[\log\log 1/\gamma]^{-1}$. To be a little more precise, for almost all the realizations of the random magnetic fields, for all but a finite number of indices n, if $\gamma = 2^{-n}$, up to a translation of *at most* $\ell(\gamma)$, we meet a constant profile which is constant on an interval which is *at least* $\ell(\gamma)$. Note that for a *given* interval of scale $\ell(\gamma)$, say centered at the origin, the system can be approximatively constant around one of the two equilibria or make just one transition between the two equilibria. That is there is at most one transition in such a fixed interval. Let us note that in a recent paper [9], the Kac-Hopfield model was considered and it was proved that the system made at most one transition in an interval of scale $\gamma^{-2}[\log 1/\gamma]^{-1}$ which is smaller that $\ell(\gamma)$. Here it is possible to get results on a scale $\ell(\gamma)$ mainly because the system we consider is simpler and this allows us to make more accurate estimates. Moreover to get the scale $\gamma^{-2}[(\log 1/\gamma)(\log\log 1/\gamma)^2]$ a very special representation of the system is used. It is possible to get similar results for the Kac-Hopfield model in the regime where the number of patterns is bounded by

$(\log 1/\gamma)/\log 2$. This is just a tedious modification of what is done in this work and no new ideas are needed.

The plan of the paper is the following. In section 2 we introduce notations and state the main results. In section 3 we perform the spin-blocks representation, giving an explicit representation of the random part. A large deviation principle in the strong form , that is with estimates of the subexponential terms, for Hypergeometric random variable is given there. In section 4 we prove the upper bound for the typical length of profiles. A deviation inequality for Lipschitz function of Bernoulli which is more general than the usual ones is given there. In section 5 we prove the lower bound for the typical length of profiles.

2 The model and the main results

Let $(\Omega, \mathcal{F}, I\!\!P)$ be a probability space. Let $h \equiv \{h_i\}_{i\in Z\!\!Z}$ be a family of independent, identically distributed Bernoulli random variables defined on this space, that is $I\!\!P[h_i = +1] = I\!\!P[h_i = -1] = 1/2$. We denote by σ a function $Z\!\!Z \to \{-1,+1\}$ and call $\sigma_i, i \in Z\!\!Z$ the spin at site i. $\mathcal{S}$ is the space of such functions, equipped with the product topology. Given $\Lambda \in Z\!\!Z$, we denote by σ_Λ a function $\Lambda \to \{-1,+1\}$ and the space of such functions is denoted by $\mathcal{S}_\Lambda$. We choose a Kac potential of the form $J_\gamma(i-j) \equiv \gamma J(\gamma|i-j|), \gamma > 0$, where $J(x) = 1\!\!1_{|x|\le 1/2}$. Note that more general ferromagnetic potentials could be used without changing the behavior of the model. The relevant conditions are (1) $J(x) \ge 0$ (i.e. ferromagnetism) (2) $J(x) = J(-x)$ (symmetry) (3) fast decay at infinity, that could be short range or exponential $J(x) = e^{-2|x|}$ as in the original Kac model. We assume that $\int J(x)dx = 1$.
The Hamiltonian in a finite volume $\Lambda \subset Z\!\!Z$ with free boundary conditions is the random variable

$$H_\gamma(\sigma_\Lambda)[\omega] = -\frac{1}{2}\sum_{(i,j)\in\Lambda\times\Lambda} J_\gamma(i-j)\sigma_i\sigma_j - \theta\sum_{i\in\Lambda} h_i[\omega]\sigma_i \tag{2.1}$$

The interaction between the spins in Λ and those outside Λ will be denoted by

$$W_\gamma(\sigma_\Lambda, \sigma_{\Lambda^c}) = -\sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J_\gamma(i-j)\sigma_i\sigma_j \tag{2.2}$$

We will usually drop the ω dependence for all quantities we consider.

The *Gibbs measure* at inverse temperature $\beta > 0$ in the finite region Λ with free boundary conditions is the probability measure valued random variable $\mu_{\beta,\theta,\gamma,\Lambda}$ on $\{-1,+1\}^\Lambda$ defined by

$$\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_{\beta,\theta,\gamma,\Lambda}}\exp\{-\beta H_\gamma(\sigma_\Lambda)\} \tag{2.3}$$

Here $Z_{\beta,\theta,\gamma,\Lambda}$ is the partition function, i.e., the normalization factor to make $\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda)$ into a probability measure on $\mathcal{S}_\Lambda$.

If $\tilde\sigma$ is a spin configuration in $\mathcal{S}$, the Gibbs measure with boundary condition $\tilde\sigma$ is the probability measure valued random variable $\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ on $\{-1,+1\}^\Lambda$ defined by

$$\mu^{\tilde\sigma_{\Lambda^C}}_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}}\exp\left\{-\beta(H_\gamma(\sigma_\Lambda) + W_\gamma(\sigma_\Lambda, \tilde\sigma_{\Lambda^c}))\right\} \tag{2.4}$$

Here $Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$, the partition function in the volume Λ with the boundary condition $\tilde\sigma$, is the normalization factor to make $\mu^{\tilde\sigma_{\Lambda^C}}_{\beta,\theta,\gamma,\Lambda}$ into a probability measure on $\mathcal{S}_\Lambda$.

Given a realization of h, $\forall\gamma > 0$, the infinite volume Gibbs measure $\mu_{\beta,\theta,\gamma}$ is obtained as the unique weak-limit of $\mu_{\beta,\theta,\gamma,\Lambda}$ along a family of volumes $\Lambda_L = [-L,+L], L \in I\!\!N$.

It is also the unique weak-limit of $\mu^{\tilde\sigma_{\Lambda^C}}_{\beta,\theta,\gamma,\Lambda}$ for any $\tilde\sigma$ that could depend on h. Note that different realizations of h give different infinite volume Gibbs measures.

The *free energy* in the volume Λ, with free boundary conditions, is defined by

$$F_\Lambda(\beta,\theta,\gamma) = -\frac{1}{\beta|\Lambda|}\log Z_{\beta,\theta,\gamma,\Lambda} \tag{2.5}$$

The infinite volume limit $F(\beta,\theta,\gamma)$ of the free energy with free boundary conditions, for fixed γ, exists $I\!\!P$-almost surely by standard sub-additive argument. Being measurable with respect to the tail σ-algebra of $\mathcal{F}$, $F(\beta,\theta,\gamma)$ is a non random quantity and it is equal to the limit of the average of $F_\Lambda(\beta,\theta,\gamma)$ with respect to $I\!\!P$.

Given a volume $\Delta\subset Z\!\!\!Z$, we define the sample magnetization in Δ by

$$\tilde m_\Delta(\sigma) = \frac{1}{|\Delta|}\sum_{i\in\Delta}\sigma_i \tag{2.6}$$

A relevant order parameter of this system is the limit when $\Delta\uparrow Z\!\!\!Z$ of the infinite volume Gibbs average of $\tilde m_\Delta$. Note that $\tilde m_\Delta$ can be written as $\tilde m_\Delta(\sigma) = \widehat m_\Delta(+,\sigma)+\widehat m_\Delta(-,\sigma)$ where

$$\widehat m_\Delta(\pm,\sigma) = \frac{1}{|\Delta|}\sum_{i\in\Delta}\sigma_i\left(\frac{1\pm h_i}{2}\right) \tag{2.7}$$

is the local sample magnetization on the random subset of Δ where the magnetic field is positive (resp. negative).

Given $\epsilon>0$ and $(m_1,m_2)\in[-1,+1]^2$, we define the constrained partition function

$$\widehat Z_{\beta,\theta,\gamma,\Lambda}(m_1,m_2,\epsilon) = \frac{1}{2^{|\Lambda|}}\sum_{\sigma_\Lambda\in\mathcal{S}_\Lambda} e^{-\beta H_\gamma(\sigma_\Lambda)} 1\!\!1_{\{|\widehat m_\Lambda(+,\sigma)-m_1|\le\epsilon\}} 1\!\!1_{\{|\widehat m_\Lambda(-,\sigma)-m_2|\le\epsilon\}} \tag{2.8}$$

and the constrained finite volume free energy

$$\widehat F_\Lambda(\beta,\theta,\gamma,m_1,m_2,\epsilon) = -\frac{1}{\beta|\Lambda|}\log \widehat Z_{\beta,\theta,\gamma,\Lambda}(m_1,m_2,\epsilon) \tag{2.9}$$

$I\!\!P$-almost surely, $\lim_{\epsilon\downarrow 0}\lim_{\Lambda\uparrow Z\!\!\!Z}\widehat F_\Lambda(\beta,\theta,\gamma,m_1,m_2,\epsilon) = \widehat F(\beta,\theta,\gamma,m_1,m_2)$ is a non random strictly convex function of (m_1,m_2). Moreover $F(\beta,\theta,\gamma) = \inf_{m_1,m_2}\widehat F(\beta,\theta,\gamma,m_1,m_2)$.

We want to give a precise description of the typical configurations in term of profiles of local magnetizations in a given scale. This leads us naturally to the notion of block spin transformations that will be defined later. This study was done for the one dimensional ferromagnetic Kac model without magnetic field in [11], [7].

For technical reasons that will become clear later, we will not use $\widehat{m}_\Delta(\pm,\sigma)$ to transform our system into a block spins system. We will use an equivalent set of two local averages. The main reason is that the cardinality of the subset of Δ where h is positive is a random number with mean $\Delta/2$. The random fluctuations of this cardinality govern the stochastic fluctuations of the system. We prefer to use another representation of the system in term of a priori less physical quantities. They are the empirical magnetizations over random sets with fixed length equal to $\Delta/2$. In this way the random fluctuations will be easier to control. The effect of the block spin transformation is to transform our microscopic system on $\mathbb{Z}$ into a *macroscopic* system on $\mathbb{R}$. Since the interaction length is γ^{-1}, we consider the system in a macroscopic scale where the interaction length becomes one. The volumes we consider will always be expressed in this macroscopic scale, that is a macroscopic volume $V \subset \mathbb{R}$ corresponds to a microscopic volume $\Lambda = \Lambda(V) = \gamma^{-1}V \cap \mathbb{Z}$. Now, given $0 < \delta^* < 1$, we partition $\mathbb{R}$ into blocks of length δ^*. This will induce a partition of $\mathbb{Z}$ into blocks of length $\delta^*\gamma^{-1}$. We assume for convenience that $\gamma = 2^{-n}$ for some integer n and δ^* is a function of n such that $\delta^*\gamma^{-1}$ is an integer. We denote by $\mathcal{A}(x)$ a block of length δ^* centered at x. This corresponds to a microscopic block of length $\delta^*\gamma^{-1}$, $A(x) \equiv \{i \in \mathbb{Z}, \gamma^{-1}\delta^*(x-1/2) \le i < \gamma^{-1}\delta^*(x+1/2)\}$. We denote by $a^-(x) = \inf\{i : i \in A(x)\}$ and $a^+(x) = \sup\{i : i \in A(x)\}$.

Given a realization of $h : h[\omega] \equiv (h_i[\omega])_{i\in\mathbb{Z}}$, let us call $A^+(x) = \{i \in A(x), h_i[\omega] = +1\}$ and $A^-(x) = \{i \in A(x), h_i[\omega] = -1\}$. We denote by $\lambda(x) \equiv \operatorname{sgn}(|A^+(x)| - (2\gamma)^{-1}\delta^*)$, where sgn is the sign function, with the convention that $\operatorname{sgn}(0) = 0$. Note that if $\delta^*\gamma^{-1}$ is odd, $\lambda(x)$ is a Bernoulli symmetric random variable. However for convenience we assume $\delta^*\gamma^{-1}$ even. In this case the distribution of $\lambda(x)$ have the following mass at zero:

$$\mathbb{P}\left[\lambda(x) = 0\right] = 2^{-\delta^*\gamma^{-1}} \binom{\delta^*\gamma^{-1}}{\delta^*\gamma^{-1}/2} \tag{2.10}$$

We define, if $\lambda(x) = \pm 1$

$$l^\lambda(x) \equiv l^{\lambda(x)}(x) = \inf\{l \ge a^-(x) : \sum_{j=a^-(x)}^{l} \mathbb{1}_{\{A^{\lambda(x)}(x)\}}(j) \ge \delta^*\gamma^{-1}/2\} \tag{2.11}$$

We denote the corresponding subset $B^\lambda(x) = \left\{i \in A^{\lambda(x)}(x), i \le l^\lambda(x)\right\}$ and $B^{-\lambda}(x) = A(x) \setminus B^\lambda(x)$. If $\lambda(x) = 0$ we take $B^+(x) = A^+(x)$ and $B^-(x) = A^-(x)$. Let us call $A^\lambda(x) \setminus B^\lambda(x) \equiv D^\lambda(x)$ Note that with this construction, since we have assumed $\delta^*\gamma^{-1}$ even, we have always $|B^+(x)| = |B^-(x)| = \delta^*\gamma^{-1}/2$.

We define, for $\lambda = \pm 1$

$$m^{\delta^*}(\lambda, x, \sigma) = \frac{2\gamma}{\delta^*} \sum_{i\in B^\lambda(x)} \sigma_i \tag{2.12}$$

Notice that we have still $\frac{\gamma}{\delta^*}\sum_{i\in A(x)}\sigma_i = \frac{1}{2}(m^{\delta^*}(+,x,\sigma)+m^{\delta^*}(-,x,\sigma))$ but now

$$\frac{\gamma}{\delta^*}\sum_{i\in A(x)} h_i\sigma_i = \frac{1}{2}(m^{\delta^*}(+,x,\sigma)-m^{\delta^*}(-,x,\sigma))+\lambda(x)\frac{2\gamma}{\delta^*}\sum_{i\in D^\lambda(x)}\sigma_i \tag{2.13}$$

Given a microscopic volume Λ, we denote by

$$\mathcal{M}_{\delta^*}(\Lambda) \equiv \prod_{x\in\mathcal{C}_{\delta^*}(\Lambda)} \left[-1,-1+\frac{4\gamma}{\delta^*},-1+\frac{8\gamma}{\delta^*},\ldots,1-\frac{4\gamma}{\delta^*},1\right]^2 \tag{2.14}$$

where $\mathcal{C}_{\delta^*}(\Lambda)$ is the set of the centers of the blocks of length $\delta^*\gamma^{-1}$ that we get making a partition of Λ into such blocks. Namely, $\mathcal{M}_{\delta^*}(\Lambda)$ is the set of possible configurations of the pair $(m^{\delta^*}(+,x,\sigma),m^{\delta^*}(-,x,\sigma))$ for $x\in\mathcal{C}_{\delta^*}(\Lambda)$. We call a block spin transformation the random map:

$$\begin{aligned}\{-1,+1\}^\Lambda &\to \mathcal{M}_{\delta^*}(\Lambda)\\ \sigma_\Lambda &\to \Big((m_1^{\delta^*}(+,x),m_2^{\delta^*}(-,x))\Big)_{x\in\mathcal{C}_\Lambda(\delta^*)}\end{aligned} \tag{2.15}$$

By abuse of notations, we denote by $\mu_{\beta,\theta,\gamma,\Lambda}$ the probability measure induced by the Gibbs measure through this map. The infinite volume limit $\lim_{\Lambda\uparrow\mathbb{Z}}\mu_{\beta,\theta,\gamma,\Lambda}$ will be denoted $\mu_{\beta,\theta,\gamma}$.

If $\lim_{\gamma\downarrow 0}\delta^*(\gamma)=0$, the induced Gibbs measure $\mu_{\beta,\theta,\gamma}$ will have a support in the subset $\mathcal{T}$ of $L^\infty(\mathbb{R},dx)\times L^\infty(\mathbb{R},dx)$ of all measurable functions $(m_1(x),m_2(x)), x\in\mathbb{R}$ such that $\max(|m_1(x)|,|m_2(x)|)\le 1$. $\mathcal{T}$ is a compact convex set with respect to the weak L^2-loc topology.

We want to study the block spin profiles which are typical with respect to the Gibbs measure $\mu_{\beta,\theta,\gamma}$ when $\gamma\downarrow 0$. However since the Gibbs measure is a random variable defined on Ω, we have also to specify in what $\mathbb{P}$-probabilistic sense this is true. In this paper we consider results that are true $\mathbb{P}$-almost surely.

These typical configurations will have a spatial structure that will critically depend on the values of the parameters β,θ and on the length scale we are considering. As in all Kac models, the local behavior is related to the one of the corresponding Curie-Weiss model. In our case it is the Random Field Curie-Weiss model (RFCW). This model is well studied [2], [5], [19], [23], [28] for various distributions of the random field $h[\omega]$. The Bernoulli and the Gaussian distributions are the most commonly used. Note that even if parameters similar to the $\widehat{m}(\pm,x)$ were already introduced in [28], in all the previous mentioned references, the results were given for the measure induced by the Gibbs measure through the magnetization. Since our approach is slightly different, let us state some results for the RFCW model in term of the parameters $m(\pm,x)$.

The Random Field Curie-Weiss Model.

This is the case where we assume $\Lambda = \frac{1}{\gamma} \equiv N$; so that the thermodynamic limit and the limit $\gamma \downarrow 0$ are not independent. The Hamiltonian of the Random Field Curie-Weiss model is given by

$$H(\sigma_N)[\omega'] = -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - \theta \sum_{i=1}^{N} h_i[\omega'] \sigma_i \tag{2.16}$$

The partition function is $Z_N(\beta,\theta) = \sum_{\sigma \in \mathcal{S}_N} e^{-\beta H(\sigma_N)}$ and the finite volume free energy is $f_N(\beta,\theta) = -\frac{1}{\beta N} \log Z_N(\beta,\theta)$. Using the same construction that leads to (2.12), with $\delta^* = 1$, we denote by $Z_N(\beta,\theta,m_1,m_2,\epsilon)$ the constrained partition function and by $f_N(\beta,\theta,m_1,m_2,\epsilon) = -\frac{1}{\beta N} \log Z_N(\beta,\theta,m_1,m_2,\epsilon)$ the associated free energy.

It is easy to check that $I\!\!P$-almost surely, uniformly with respect to $(m_1,m_2) \in [-1,+1]^2$, we have

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \lim_{N \uparrow \infty} f_N(\beta,\theta,m_1,m_2,\epsilon) &= \frac{-(m_1+m_2)^2}{8} - \theta(m_1 - m_2) - \frac{1}{2\beta}\left(I(m_1) + I(m_2)\right) \\ &\equiv f_{\beta,\theta}(m_1,m_2) \end{aligned} \tag{2.17}$$

here $I(m) = -\frac{(1+m)}{2}\log(\frac{1+m}{2}) - \frac{(1-m)}{2}\log(\frac{1-m}{2})$. The function $f_{\beta,\theta}(m_1,m_2)$ is called the canonical free energy. Moreover, it can be checked that, $I\!\!P$ almost surely,

$$\lim_{N \uparrow \infty} f_N(\beta,\theta) = f(\beta,\theta) = \inf_{(m_1,m_2) \in [-1,+1]^2} f_{\beta,\theta}(m_1,m_2) \tag{2.18}$$

Our first result relates the free energy of the random field Kac model to the one of the random field Curie-Weiss model.

Theorem 2.1 *For all positive β, for all positive θ, $I\!\!P$-almost surely we have*

$$\lim_{\gamma \downarrow 0} \lim_{\Lambda \uparrow \mathbb{Z}} F_\Lambda(\beta,\theta,\gamma) = f(\beta,\theta) \tag{2.19}$$

The proof of this result being straightforward however lengthy, will not be given here.

To state our next results, we need some results on the RFCW model. The critical points of $f_{\beta,\theta}(m_1,m_2)$ are the 2-dimensional vectors $m = (m_1,m_2)$ solutions of the system of equations

$$\begin{aligned} m_1 &= \tanh\left(\beta\frac{(m_1+m_2)}{2} + \beta\theta\right) \\ m_2 &= \tanh\left(\beta\frac{(m_1+m_2)}{2} - \beta\theta\right) \end{aligned} \tag{2.20}$$

We assume throughout this paper that $\beta > 1$ and $\beta\theta$ satisfies

$$\tanh \beta\theta \leq \min(1/\sqrt{3}, (1-\beta^{-1})^{1/2}) \tag{2.21}$$

This implies that the system (2.20) has only three solutions, two of them being absolute minima and one the local maximum of $f_{\beta,\theta}(m_1, m_2)$. This can be proved easily by considering the equation

$$m = \frac{1}{2} \tanh \beta(m+\theta) + \frac{1}{2} \tanh \beta(m-\theta) \tag{2.22}$$

The previous condition implies that the derivate at the origin of the function on the right hand side of (2.22) is bigger than one, and the function is concave on the positive real, convex on the negative real number. Moreover if $\tilde{m}_\beta$ is the largest positive solution of (2.22), then the two absolute minima of $f_{\beta,\theta}(m_1, m_2)$ are of the form $m_\beta = (m_{\beta,1}, m_{\beta,2})$ and $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$ where $m_{\beta,1} = \tanh \beta(\tilde{m}_\beta + \theta)$ and $m_{\beta,2} = \tanh \beta(\tilde{m}_\beta - \theta)$

It is easy to see that the function $f_{\beta,\theta}(m_1, m_2)$ is quadratic around its minima. Moreover, there exists a constant $c(\beta, \theta)$ such that for all $m = (m_1, m_2)$

$$f_{\beta,\theta}(m) - f_{\beta,\theta}(m_\beta) \geq c(\beta,\theta) \min(\|m - m_\beta\|_2^2, \|m - Tm_\beta\|_2^2) \tag{2.23}$$

here $\|.\|_2$ is the Euclidean norm in $I\!\!R^2$.

Our second result is the analog of the Lebowitz- Penrose theorem [20], [24]. It relates the canonical free energy of the Random Field Kac model to the convex envelope of the canonical free energy of the Random Field Curie-Weiss model. Recall that the convex envelope of a function f is the largest convex function that is smaller than f and it will be denoted by $Conv(f)$.

Theorem 2.2 *For all positive β, for all positive θ, $I\!\!P$-almost surely, uniformly with respect to $(m_1, m_2) \in [-1, +1]$ we have*

$$\lim_{\epsilon \downarrow 0} \lim_{\gamma 0} \lim_{\Lambda \uparrow \mathbb{Z}} \widehat{F}_\Lambda(\beta, \theta, \gamma, m_1, m_2, \epsilon) = Conv(f_{\beta,\theta})(m_1, m_2) \tag{2.24}$$

The proof of this theorem will not be given here. It is a consequence of the estimates that will be done later and the standard proofs of the Lebowitz and Penrose, see [20].

To describe the asymptotic properties of the support of the measure $\mu_{\beta,\theta,\gamma}$, we need to introduce another scale. To avoid possible confusion, we emphasize that we do not make a block spin transformation on this scale. Given $\delta > \delta^*$ and assuming that $\delta = k\delta^*$ for some positive integer $k \geq 2$, for $l \in \mathbb{Z}$, we denote by $C_\delta(l)$ the set of centers of those blocks of length δ that are in the macroscopic interval $[l - \frac{1}{2}, l + \frac{1}{2}[$ and given $r \in C_\delta(l)$ we denote by $C_{\delta^*/\delta}(r)$ the set of centers of those blocks of length δ^* that are in the interval of length

δ indexed by r. To define the notion of being near an equilibrium with tolerance ζ, where $0<\zeta\le m_2^+(\beta)$, we define for $l\in\mathbb{Z}$, the random variable

$$\eta^{\delta,\zeta}(l)=\begin{cases}1 & \text{if } \forall_{u\in C_\delta(l)}\ \frac{\delta^*}{\delta}\sum_{x\in\mathcal{C}_{\delta^*/\delta}(u)}\|m^{\delta^*}(x)-m_\beta\|_1\le\zeta;\\ -1 & \text{if } \forall_{u\in C_\delta(l)}\ \frac{\delta^*}{\delta}\sum_{x\in\mathcal{C}_{\delta^*/\delta}(u)}\|m^{\delta^*}(x)-Tm_\beta\|_1\le\zeta;\\ 0 & \text{otherwise.}\end{cases}\tag{2.25}$$

where $\|\cdot\|_1$ is the L^1 norm. In analogy with [11], we expect that when $\gamma\downarrow 0$, the typical profiles will be described by runs of $\eta^{\delta,\zeta}=1$ followed by runs of $\eta^{\delta,\zeta}=-1$. It was proved in [11] that, for the ferromagnetic Kac model, the profiles make runs of $\eta^{\delta,\zeta}=1$ on a scale which is of order $\exp(\Delta f/\gamma)$ where Δf is the activation energy of the Curie-Weiss model that is the difference between the value of the canonical free energy at its saddle point and at its minima. Roughly speaking this means that on a scale $e^{((1+\epsilon)\Delta f/\gamma)}$ the profiles are non constant if $\epsilon>0$ and are constant if $\epsilon<0$.

As we will see, the presence of the random magnetic field makes the profiles non constant on a much smaller scale. To be more precise, given $\tau\in\{-1,+1\}$, $\ell_1\in\mathbb{Z}$, $\ell_2\in\mathbb{Z}$ with $\ell_1<\ell_2$ we define

$$\mathcal{R}^{\delta,\zeta}(\ell_1,\ell_2,\tau)=\left\{m^{\delta^*}:\eta^{\delta,\zeta}(\ell)=\tau,\forall\ell;\quad \ell_1\le\ell\le\ell_2\right\}\tag{2.26}$$

and

$$\mathcal{R}^{\delta,\zeta}(\ell_1,\ell_2)=\mathcal{R}^{\delta,\zeta}(\ell_1,\ell_2,+)\cup\mathcal{R}^{\delta,\zeta}(\ell_1,\ell_2,-)\tag{2.27}$$

that is the set of profiles that between ℓ_1 and ℓ_2, are near the equilibrium m_β, Tm_β, respectively for $\tau=\pm1$, with tolerance ζ.

Given positive constants $\hat c$, $\tilde c$, $p>1$, L_1, we denote by $N_\gamma=[\frac{\tilde c}{\hat c}(\log\frac{1}{\gamma})^p(\log\log\frac{1}{\gamma})]$, where $[x]$ is the integer part of x, by $\ell_{\hat c}(\gamma)=\frac{\hat c}{\gamma\log\log\frac{1}{\gamma}}$ and by

$$\mathcal{R}^{\delta,\zeta}(L_1,\hat c,\frac{\tilde c(\log\frac{1}{\gamma})^p}{\gamma})\equiv\bigcup_{k=-N_\gamma}^{N_\gamma}\mathcal{R}^{\delta,\zeta}(k\ell_{\hat c}(\gamma),L_1+k\ell_{\hat c}(\gamma))\tag{2.28}$$

That is the set of profiles that in an interval of length $2\frac{\tilde c(\log\frac{1}{\gamma})^p}{\gamma}$, centered at the origin, have at least one interval of length L_1 that it is rigid. We have the following result.

Theorem 2.3 *Given $\tilde c>0$, $\beta>1$, $p>1$, $\rho>0$, $c_0>0$, $\beta\theta$ small enough, for all $x>0$, $\delta>\delta^*=c_0\log\log\frac{1}{\gamma}$ there exist an absolute constant $c>0$ and a positive constant $\hat c=\hat c(\beta,\theta,x)$ such that if $\gamma=2^{-n}$, $\mathbb{P}$-almost surely, for all but a finite number of indices n, if*

$$L_1\ge\frac{(\log\frac{1}{\gamma})(\log\log\frac{1}{\gamma})^{2+\rho}}{\gamma}\left[\frac{c(x,\rho,\gamma)}{(\beta\theta)^2(m_{\beta,1}+m_{\beta,2})^2}\right]\tag{2.29}$$

where $c(x,\rho,\gamma) = \frac{2(4+x)^2}{1+(2+\frac{3\rho}{4})\frac{\log\log\log\frac{1}{\gamma}}{\log\log\frac{1}{\gamma}}}$ *then*

$$\mu_{\beta,\theta,\gamma}\left[\mathcal{R}^{\delta,\zeta}(L_1,\hat{c},\frac{\tilde{c}(\log\frac{1}{\gamma})^p}{\gamma})\right] \le e^{-\beta x\gamma^{-1}} \tag{2.30}$$

provided that for some function $g_2(1/\zeta)$, *with* $\lim_{\zeta\downarrow 0} g_2(1/\zeta) = \infty$, *slowly varying at infinity,* $\lim_{\zeta\downarrow 0}\sqrt{\zeta}g_2(1/\zeta) = 0$ *and* $\zeta g_2(1/\zeta) < \beta\theta(m_{\beta,1}+m_{\beta,2})^2\frac{c}{p+2}$.

To make the previous theorem meaningful, we need a result in the opposite direction. That is to prove that the system is rigid with the *same* tolerance ζ on a scale smaller that L_1. As we will see later, this will give a constraint from below on ζ. For reasons that will become clear later we need to introduce two different tolerance parameters that we call ζ_4 and ζ_1 and the corresponding δ_4 and δ_1. ζ_4 will play the role of ζ in the previous theorem.

Given $\ell_1 \in \mathbb{Z}$, $\ell_2 \in \mathbb{Z}$ with $\ell_1 < \ell_2$, $\delta_4 > 0$, $\zeta_4 > 0$, $\delta_1 > 0$, $\zeta_1 > 0$, $R_1 \in \mathbb{R}$, $x \in [\ell_1 + R_1 + 1, \ell_2 - 3R_1 - 1]$ and $\tau \in \{-1,+1\}$, we define a front starting at the equilibrium τ at the point x by

$$\begin{aligned}\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2,\tau,x) = \big\{m^{\delta^*} : \forall\ell\in[\ell_1+R_1,x],\ \eta^{\delta_4,\zeta_4}(\ell) = \tau = \eta^{\delta_1,\zeta_1}(x),\\ \forall\ell\in[x+R_1+1,\ell_2-R_1[,\ \eta^{\delta_4,\zeta_4}(\ell) = -\tau = \eta^{\delta_1,\zeta_1}(x+R_1)\big\}\end{aligned} \tag{2.31}$$

and $\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2,\tau) = \bigcup_{\ell_1+R_1+1\le x\le\ell_2-3R_1-1}\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2,\tau,x)$. We denote the set of fronts that occur within $[\ell_1,\ell_2]$ by

$$\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2) = \cup_{\tau\in\{+1,-1\}}\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2,\tau) \tag{2.32}$$

Moreover, to short notation we set, see (2.27),

$$\mathcal{R}^{\delta_4,\zeta_4}(\ell_1,\ell_2,R_1) \equiv \mathcal{R}^{\delta_4,\zeta_4}(\ell_1+2R_1,\ell_2-2R_1) \tag{2.33}$$

We have the following result.

Theorem 2.4 *Given* $\beta > 1$, $\rho > 0$ *and* $\widehat{c}$ *there exists an* ϵ_0 *such that if* $\beta\theta \le \epsilon_0$, *we can find* $\gamma_0 > 0$, $c_0 > 0$ *and constants* $c_i = c_i(\beta,\theta)$ *for* $i = 1,2,3$, *such that for all* $\gamma \le \gamma_0$, *for all* $\zeta_4 > \zeta_1 > 0$, $\delta_4 > \delta_1 > \delta^* = c_0\gamma\log\log\frac{1}{\gamma}$ *that satisfy*

$$\delta_4\zeta_4^3 \ge c_1\left(\sqrt{\frac{1}{\log\log\frac{1}{\gamma}}} \vee \zeta_1\right) \tag{2.34}$$

for $R_1 = c_2(\delta_1\zeta_1^3)^{-1}$, *for any interval* $I = [\ell_1, \ell_2]$ *such that* $4R_1 \le |\ell_1 - \ell_2| \le \frac{\widehat{c}}{\gamma \log\log \frac{1}{\gamma}}$, *there exists* $\Omega_1 = \Omega_1(\beta, \theta, \ell_1, \ell_2, \gamma)$ *such that* $I\!P[\Omega_1] \ge 1 - e^{-(\log\log\frac{1}{\gamma})(1+2\rho)}$ *and on* Ω_1

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{R}^{\delta_4,\zeta_4}(\ell_1, \ell_2, R_1) \bigcup \mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1, \ell_2)\right) \ge 1 - e^{-\frac{c_3\delta_4\zeta_4^3}{\gamma}} \tag{2.35}$$

in particular, if $\gamma = 2^{-n}$, $I\!P$*-almost surely, (2.35) occurs for all but a finite number of indices* n.

Roughly speaking, inside an interval of length $\hat{c}(\gamma \log\log \frac{1}{\gamma})^{-1}$ centered, for example, at the origin, the typical profiles are rigid with a tolerance ζ_4 around one of the two equilibria or make only one transition between the two equilibria. Note also that we have allowed a fuzzy region of length $2R_1$ around the extremes of the interval we have considered and also a region R_1 around the front. As it will be proved in the chapter 5, this R_1 corresponds to the longest runs of $\eta^{\delta_1,\zeta_1} = 0$ which is typical with respect to the Gibbs measure. Note that it is possible to take ζ_4 in the Theorem 2.4 and ζ in the Theorem 2.3 equal.

The Theorem 2.4 suggests that the good notion of rigidity is not to fix the whole intervals where the profiles are at equilibrium with a given tolerance but to allow those intervals of rigidity to have a fuzzy zone of length $2R_1$ at the extremes. The system can make a front in this fuzzy zone, if it more likely to do it. To describe the typical profiles, we want to put together the two previous Theorems. We can expect to give an upper and lower bound on the distance between two fronts for the typical profiles in an interval of length, say $\gamma^{-1}(\log 1/\gamma)^p$ for some $p > 1$ since it is the scale where we know by the Theorem 2.3 that such fronts exists. This correspond also to give an upper and lower bound on the number of transitions from one equilibrium to the other one in such an interval. To be more precise, we need some more definitions. Given an interval $\mathcal{J} = [-j_1, j_1]$, centered at the origin, and positives integers k and L, we define, for $\ell_1, \ell_2 \in Z\!\!Z$, $\tau \in \{-1, +1\}$, $\mathcal{R}^{\delta_4,\zeta_4}(\ell_1, \ell_2, R_1, \tau) \equiv \mathcal{R}^{\delta_4,\zeta_4}(\ell_1 + 2R_1, \ell_2 - 2R_1, \tau)$, $\mathcal{T}^{\delta_4,\zeta_4}(L, 1, \tau, \mathcal{J}) \equiv \mathcal{R}^{\delta_4,\zeta_4}(-j_1, j_1, R_1, \tau)$ and

$$\mathcal{T}^{\delta_4,\zeta_4}(L, k, \tau, \mathcal{J}) = \bigcup_{\ell_1=-j_1}^{j_1} \bigcup_{\substack{\ell_2>\ell_1 \\ \ell_2-\ell_1>L}}^{j_1} \dots \bigcup_{\substack{\ell_k>\ell_{k-1} \\ \ell_k-\ell_{k-1}>L}}^{j_1} \bigcap_{k_1=1}^{k} \mathcal{R}^{\delta_4,\zeta_4}\left(\ell_{k_1}, \ell_{k_1+1}, R_1, (-1)^{k_1+1}\tau\right) \tag{2.36}$$

that is the profiles in $\mathcal{T}^{\delta_4,\zeta_4}(L, k, \tau, \mathcal{J})$ change exactly $k-1$ times equilibrium, starting from τ somewhere within $[-j_1, -j_1 + 2R_1]$ and remaining in a given equilibrium for a length at least L. We define also

$$\mathcal{T}^{\delta_4,\zeta_4}(L, \leq k, \tau, \mathcal{J}) = \bigcup_{k_2=1}^{k} \mathcal{T}^{\delta_4,\zeta_4}(L, k_2, \tau, \mathcal{J}) \tag{2.37}$$

and $\mathcal{T}^{\delta_4,\zeta_4}(L, \leq k, \mathcal{J}) = \mathcal{T}^{\delta_4,\zeta_4}(L, \leq k, +, \mathcal{J}) \cup \mathcal{T}^{\delta_4,\zeta_4}(L, \leq k, -, \mathcal{J})$. The profiles in $\mathcal{T}^{\delta_4,\zeta_4}(L, \leq k, \mathcal{J})$ change equilibrium at most $k-1$ times, starting from one equilibrium somewhere within $[-j_1, -j_1 + 2R_1]$ and remaining in a given equilibrium for a length at least L.

Theorem 2.5 *Given $\beta > 1$ and $\rho > 0$ there exists $\epsilon_0 > 0$ such that for all $\beta\theta \leq \epsilon_0$, we can find $p = p(\beta\theta) > 1$, $\bar{\zeta}_4(\beta\theta) > 0$, $\gamma_0 > 0$, $c_0 > 0$, $\hat{c} > 0$ and constants $c_i = c_i(\beta,\theta)$ for $i = 1,2,3$, such that for all $\gamma \leq \gamma_0$, for all $\bar{\zeta}_4(\beta\theta) \geq \zeta_4 > \zeta_1 > 0$, $\delta_4 > \delta_1 > \delta^* = c_0\gamma \log\log 1/\gamma$ that satisfy (2.34), L_1 that satisfies (2.29) and for $R_1 = c_2(\delta_1\zeta_1^3)^{-1}$, for all given interval $\mathcal{J}$ of length $\tilde{c}(\log\frac{1}{\gamma})^p\gamma^{-1}$, for some positive constant $\tilde{c}$, if $\gamma = 2^{-n}$, $I\!P$-almost surely, for all but a finite number of indices n,*

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{T}^{\delta_4,\zeta_4}(\ell_{\hat{c}}(\gamma), \leq \frac{\tilde{c}}{\hat{c}}(\log\frac{1}{\gamma})^p \log\log\frac{1}{\gamma}, \mathcal{J}) \setminus \mathcal{T}^{\delta_4,\zeta_4}(L_1, \leq \frac{\tilde{c}(\log\frac{1}{\gamma})^{p-1}}{\hat{c}(\log\log\frac{1}{\gamma})^{2+\rho}}, \mathcal{J})\right)$$
$$\geq 1 - e^{-\frac{c_3\delta_4\zeta_4^3}{\gamma}} \tag{2.38}$$

Our estimates give the scaling relation $p(\beta\theta) = \epsilon_0^2/(\beta\theta)^2$. Following a typical profile starting from the left end of the interval $\mathcal{J}$, we meet at least one transition, within a scale $L_1 \approx \frac{1}{\gamma}[(\log 1/\gamma)(\log\log 1/\gamma)^{2+\rho}]$, then after this transition, we are near an equilibrium on a scale which is at least $\frac{1}{\gamma}[\log\log 1/\gamma]^{-1}$ and at most L_1, then we meet another transition within a scale L_1 and so on. This implies that the number of oscillations between the equilibria in the interval $\mathcal{J}$ is bounded from above by $(\log 1/\gamma)^p \log\log 1/\gamma$ and from below by $(\log 1/\gamma)^{p-1}(\log\log 1/\gamma)^{-(2+\rho)}$.

3 The analysis of the block spin representation

In this section we perform the block spin transformation on the scale δ^* mentioned in the previous chapter and we make a rather precise analysis of the stochastic contribution in order to prove our theorems.

Given a macroscopic interval $I \equiv [i^-, i^+[\subset I\!R$ with $i^\pm \in Z\!\!Z$, let us denote by $\Sigma_I^{\delta^*}$ the σ-algebra of $\mathcal{S}$ generated by the variables $\left(m^{\delta^*}(x,\sigma)\right)_{x\in\mathcal{C}_{\delta^*}(I)}$ where $m^{\delta^*}(x,\sigma) = (m^{\delta^*}(+,x,\sigma), m^{\delta^*}(-,x,\sigma))$. For such an interval I we denote by $\partial^+ I \equiv \{x \in I\!R, i^+ \leq x < i^+ + 1\}$ and $\partial^- I \equiv \{x \in I\!R, i^- - 1 \leq x < i^-\}$ the two macroscopic intervals of length 1, that are on the right and on the left of I. We call $\partial I = \partial^+ I \cup \partial^- I$.

If F^{δ^*} is a $\Sigma_I^{\delta^*}$-measurable bounded function, we define the conditional expectation of F^{δ^*}, given the σ-algebra $\Sigma_{\partial I}^{\delta^*}$, as the real $\Sigma_{\partial I}^{\delta^*}$-measurable function that associates to $m^{\delta^*}(\partial I) \equiv \left\{m^{\delta^*}(x), x \in \mathcal{C}_{\delta^*}(\partial I)\right\}$ the value

$$\mu_{\beta,\theta,\gamma}\left(F^{\delta^*} \mid \Sigma_{\partial I}^{\delta^*}\right)(m^{\delta^*}(\partial I)) = \frac{1}{Z_{\beta,\gamma,\theta,I}(m^{\delta^*}(\partial I))} \sum_{\sigma_{\gamma^{-1}I} \in \mathcal{S}_{\gamma^{-1}I}} F^{\delta^*}(\sigma_{\gamma^{-1}I}) e^{-\beta\left[H(\sigma_I) + W(\sigma_I | m^{\delta^*}(\partial I))\right]} \tag{3.1}$$

where

$$W(\sigma_I | m^{\delta^*}(\partial I)) \equiv \frac{\delta^*}{\gamma} \sum_{i\in\gamma^{-1}I} \sum_{x\in\mathcal{C}_{\delta^*}(\partial I)} J_\gamma(i - \delta^*\gamma^{-1}x)\sigma_i \widetilde{m}^{\delta^*}(x) \tag{3.2}$$

with $\widetilde{m}^{\delta^*}(x) = (m_1^{\delta^*}(x) + m_2^{\delta^*}(x))/2$ and $Z_{\beta,\gamma,\theta,I}(m^{\delta^*}(\partial I))$ is the normalization factor that gives $\mu_{\beta,\theta,\gamma}\left(1 | \Sigma_{\partial I}^{\delta^*}\right) = 1$.

Given a configuration $(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*})$, let us denote by

$$E(m_I^{\delta^*}) \equiv -\frac{\delta^*}{2} \sum_{(x,y)\in\mathcal{C}^2_{\delta^*}(I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y) \tag{3.3}$$

and

$$E(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*}) \equiv -\delta^* \sum_{x\in\mathcal{C}_{\delta^*}(I)} \sum_{y\in\mathcal{C}_{\delta^*}(\partial^\pm I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y) \tag{3.4}$$

On the set $M^{\delta^*}(m^{\delta^*}(I)) \equiv \{\sigma : m^{\delta^*}(x,\sigma) = m^{\delta^*}(x)\, \forall_{x\in\mathcal{C}_{\delta^*}(I)}\}$, we have

$$\sup_{\sigma_I \in M^{\delta^*}} \left| H(\sigma_I) + \theta \sum_{i\in\gamma^{-1}I} h_i\sigma_i - \frac{1}{\gamma}E(m_I^{\delta^*})\right| \leq \delta^*\gamma^{-1}|I| \tag{3.5}$$

here $|I|$ is the length of the macroscopic interval I. Moreover we have also

$$\sup_{\sigma_I\in M^{\delta^*}}\left|W(\sigma_I|m^{\delta^*}(\partial^\pm I))-\frac{1}{\gamma}E(m_I^{\delta^*},m_{\partial^\pm I}^{\delta^*})\right|\le\delta^*\gamma^{-1} \tag{3.6}$$

Therefore, using (2.13), we can write

$$\begin{aligned}\mu_{\beta,\theta,\gamma}\left(F^{\delta^*}\mid\Sigma_{\partial I}\right)(m^{\delta^*}(\partial I))=&\frac{e^{(\pm\delta^*\gamma^{-1}|I|)}}{Z_{\beta,\theta,\gamma,I}(m^{\delta^*}(\partial I))}\\ \sum_{m^{\delta^*}(I)\in\mathcal{M}_{\delta^*}(I)}&F^{\delta^*}(m^{\delta^*})e^{-\frac{\beta}{\gamma}\left(E(m_I^{\delta^*})+E(m_I^{\delta^*},m_{\partial I}^{\delta^*})+\frac{\theta\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}(I)}(m_1^{\delta^*}(x)-m_2^{\delta^*}(x))\right)}\\ &\sum_{\sigma_{\gamma^{-1}I}}1\!\!\mathrm{I}_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x),\forall x\in\mathcal{C}_{\delta^*}(I)\}}\prod_{x\in\mathcal{C}_{\delta^*}(I)}e^{2\beta\theta\lambda(x)\sum_{i\in D^{\lambda}(x)}\sigma_i}\end{aligned} \tag{3.7}$$

where, this equality has to be interpreted as an upper bound for $\pm=1$ and a lower bound for $\pm=-1$ and the first sum is over $m^{\delta^*}(x)_{x\in\mathcal{C}_{\delta^*}(I)}\in\mathcal{M}_{\delta^*}(I)$.

Note that the random terms appear only in the last product $\prod_{x\in\mathcal{C}_{\delta^*}(I)}$ and that the last sum in (3.7) factors into a product over the intervals of length $\delta^*\gamma^{-1}$ indexed by $\mathcal{C}_{\delta^*}(I)$. For all $x\in\mathcal{C}_{\delta^*}(I)$, we introduce on $\{-1,+1\}^{\delta^*\gamma^{-1}}=\mathcal{S}_{\delta^*\gamma^{-1}}$ the measure denoted the *canonical* measure in physics literature

$$I\!\!E^{\delta^*}_{x,m^{\delta^*}(x)}(\varphi)=\frac{\sum_{\sigma\in\mathcal{S}_{\delta^*\gamma^{-1}}}\varphi(\sigma)1\!\!\mathrm{I}_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}}}{\sum_{\sigma\in\mathcal{S}_{\delta^*\gamma^{-1}}}1\!\!\mathrm{I}_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}}} \tag{3.8}$$

The denominator in (3.8) is

$$\binom{|B^+|}{\frac{1+m_1^{\delta^*}(x)}{2}|B^+|}\binom{|B^-|}{\frac{1+m_2^{\delta^*}(x)}{2}|B^-|} \tag{3.9}$$

where $|B^\pm|=|B|=\delta^*(2\gamma)^{-1}$. We set

$$\begin{aligned}\widehat{\mathcal{F}}(m_I^{\delta^*},m_{\partial I}^{\delta^*})=&E(m_I^{\delta^*})+E(m_I^{\delta^*},m_{\partial I}^{\delta^*})\\ &+\frac{\theta\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}(I)}(m_1^{\delta^*}(x)-m_2^{\delta^*}(x))\\ &+\delta^*\sum_{x\in\mathcal{C}_{\delta^*}(I)}\frac{\gamma}{\beta\delta^*}\log\binom{|B^+|}{\frac{1+m_1^{\delta^*}(x)}{2}|B^+|}\binom{|B^-|}{\frac{1+m_2^{\delta^*}(x)}{2}|B^-|}\end{aligned} \tag{3.10}$$

We introduce the moment generating function

$$L^{\delta^*}_{x,m^{\delta^*}(x)}(\lambda(x)\beta\theta, D^\lambda(x)) \equiv I\!\!E^{\delta^*}_{x,m^{\delta^*}(x)}(e^{2\beta\theta\lambda(x)\sum_{i\in D^{\lambda}(x)}\sigma_i}) \tag{3.11}$$

and the cumulant generating function

$$\mathcal{G}_{x,m^{\delta^*}(x)}\,(\lambda(x)) \equiv \log L^{\delta^*}_{x,m^{\delta^*}(x)}(\lambda(x)\beta\theta, D^\lambda(x)) \tag{3.12}$$

then (3.7) becomes

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\left(F^{\delta^*}\;\middle|\;\Sigma_{\partial I}\right)(m^{\delta^*}(\partial I)) \\ &= \frac{e^{(\pm\delta^*\gamma^{-1}|I|)}}{Z_{\beta,\theta,\gamma,I}(m^{\delta^*}(\partial I))} \sum_{m^{\delta^*}(I)\in\mathcal{M}_{\delta^*}(I)} F^{\delta^*}(m^{\delta^*}) e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m_I^{\delta^*}, m_{\partial I}^{\delta^*})+\gamma\mathcal{G}(m_I^{\delta^*})\right\}} \end{aligned} \tag{3.13}$$

where

$$\mathcal{G}(m_I^{\delta^*}) \equiv \sum_{x\in\mathcal{C}_{\delta^*}(I)} \mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) \tag{3.14}$$

$i.e$ up to the error terms $e^{(\pm c\delta^*\gamma^{-1}|I|)}$, we have been able to describe our system in term of the block spin variables giving a rather explicit form to the deterministic and the stochastic part.

Note that the stochastic dependence is given only by the fluctuations of the magnetic fields on each block $\lambda(x) = \mathrm{sgn}\,(\sum_{i\in A(x)} h_i)$ and by $|D^\lambda(x)| = \frac{\lambda(x)}{2}\sum_{i\in A(x)} h_i$.

As we will see, we need precise estimates, when γ small, of $\mathcal{G}_{x,m^{\delta^*}(x)}\,(\lambda(x))$, see (3.12); i.e. of the cumulant generating function of an Hypergeometric. To get them we need large deviation estimates in the strong form with good control of the polynomial prefactors. We have to consider all the possible behaviors of the fluctuations of an Hypergeometric. It is well known that there are three possible regimes, namely a gaussian one, a binomial and a poissonian one. Classical results are usually given in terms of convergence in distribution. Since we are interested in the cumulant generating function and in controlling the error terms, we need some extra work. Because it can be of independent interest, we give a short proof of the estimates we need. The statements of them are given in Proposition 3.4, for the gaussian regime, and in Proposition 3.5 for the binomial and poissonian regimes.

If $\lambda(x) = +1$, then $D^{\lambda(x)}$ is a subset of A^- (the subset of sites where the magnetic field is negative) and therefore the sum over the sites in A^+ factors out and it is cancelled by the first combinatorial factor in (3.9), (if $\lambda(x) = -1$ it is the second term in (3.9)). In particular this means that if $\lambda(x) = +1$, we have

$$L^{\delta^*}_{x,m^{\delta^*}(x)}(\beta\theta, D^1) = I\!\!E^{\delta^*}_{x,m_2^{\delta^*}(x)}(e^{2\beta\theta\sum_{i\in D^1}\sigma_i}) \tag{3.15}$$

which depends only on the second coordinate of $m^{\delta^*}(x)$; while if $\lambda(x) = -1$,

$$L^{\delta^*}_{x,m^{\delta^*}(x)}(\beta\theta, D^{-1}) = I\!\!E^{\delta^*}_{x,m_1^{\delta^*}(x)}(e^{-2\beta\theta \sum_{i\in D^{-1}} \sigma_i}) \tag{3.16}$$

which depends only on the first coordinate of $m^{\delta^*}(x)$.

To simplify notations, we drop out the x and δ^* dependence of the quantities we consider and we denote

$$L_m(\lambda\beta\theta, D) \equiv I\!\!E_m(e^{2\lambda\beta\theta \sum_{i\in D} \sigma_i}) \tag{3.17}$$

Let us first consider the case where $|m|$ is bounded away from 1. Let us denote by $I\!\!E_{\sigma_B}$ the normalized symmetric Bernoulli measure on $\{-1,+1\}^{|B|}$ and for $D \subset B$ we have, see (3.8),

$$L_m(\lambda\beta\theta, D) = \frac{I\!\!E_{\sigma_B}\left[e^{2\lambda\beta\theta \sum_{i\in D} \sigma_i} 1\!\!1_{\{m_B(\sigma)=m\}}\right]}{I\!\!E_{\sigma_B}\left[1\!\!1_{\{m_B(\sigma)=m\}}\right]} \tag{3.18}$$

Let $I\!\!E_\nu$ be the grand canonical measure with chemical potential ν, defined on $\{-1,+1\}^{|B|}$

$$I\!\!E_\nu(\varphi) = \frac{I\!\!E_{\sigma_B}\left[\varphi(\sigma) e^{\nu \sum_{i\in B} \sigma_i}\right]}{I\!\!E_{\sigma_B}\left[e^{\nu \sum_{i\in B} \sigma_i}\right]} \tag{3.19}$$

Following Yau [1][31], we introduce two different chemical potentials and we write the following identity, for all $\nu_1, \nu_2 \in I\!\!R$,

$$\begin{aligned} L_m(\lambda\beta\theta, D) =& \frac{I\!\!E_{\nu_2}\left[e^{2b\theta\lambda \sum_{i\in D} \sigma_i} 1\!\!1_{\{m_B(\sigma)=m\}}\right]}{I\!\!E_{\nu_2}\left[e^{2b\theta\lambda \sum_{i\in D} \sigma_i}\right]} \frac{1}{I\!\!E_{\nu_1}\left[1\!\!1_{\{m_B(\sigma)=m\}}\right]} \\ & e^{\{m(\nu_1-\nu_2)|B|\}} \frac{(\cosh(\nu_2))^{|B\setminus D|} (\cosh(\nu_2 + 2\lambda\beta\theta))^{|D|}}{(\cosh(\nu_1))^{|B|}} \end{aligned} \tag{3.20}$$

We choose $\nu_1 \equiv \nu_1(m)$ such that $m = \tanh \nu_1$ and $\nu_2 \equiv \nu_2(m, \alpha)$ such that

$$m = \alpha \tanh(\nu_2 + 2\lambda\beta\theta) + (1-\alpha)\tanh \nu_2 \tag{3.21}$$

where $\alpha = \frac{|D|}{|B|}$, $0 \le \alpha \le 1$. Then since $\{m_B(\sigma) = m\} = \{(|B|)^{-1/2} \sum_{i\in B}(\sigma_i - m) = 0\}$, we can estimate the two first ratios in (3.20) as done in Yau [31], using a *Local* Central Limit Theorem (LCLT). Therefore denoting

$$\Psi_{\lambda\beta\theta,\alpha,m} \equiv \frac{I\!\!E_{\nu_2}\left[e^{2\beta\theta\lambda \sum_{i\in D} \sigma_i} 1\!\!1_{\{(|B|)^{-1/2} \sum_{i\in B}(\sigma_i - m)=0\}}\right]}{I\!\!E_{\nu_2}\left[e^{2b\theta\lambda \sum_{i\in D} \sigma_i}\right]} \tag{3.22}$$

[1] We thank Enzo Olivieri for bringing the results of [31] to our attention.

and

$$e^{\phi(m,\lambda\beta\theta,\alpha)} \equiv e^{\{m(\nu_1-\nu_2)|B|\}} \frac{(\cosh(\nu_2))^{|B\setminus D|}(\cosh(\nu_2+2\lambda\beta\theta))^{|D|}}{(\cosh(\nu_1))^{|B|}} \tag{3.23}$$

we have

$$L_m(\lambda\beta\theta, D) = \frac{\Psi_{\lambda\beta\theta,\alpha,m}}{\Psi_{0,0,m}} e^{\phi(m,\lambda\beta\theta,\alpha)} \tag{3.24}$$

The next step is to estimate each term separately. We start with $\Psi_{\theta,\alpha,m}$. We give first a lower bound for the variance of $m_B(\sigma)$ under $I\!E_{\nu_2}$.

Lemma 3.1 *Let ν_2 be a solution of (3.21), and $\sigma_{\lambda\beta\theta}$ given by*

$$\sigma^2_{\lambda\beta\theta} = \alpha\frac{1}{\cosh^2(\nu_2+2\lambda\beta\theta)} + (1-\alpha)\frac{1}{\cosh^2(\nu_2)} \tag{3.25}$$

then for all m such that $|m|<1$, for all $\beta>1$, for all θ such that $\beta\theta<\epsilon$, for some $\epsilon>0$ small enough, for all $\alpha\in[0,1]$

$$\sigma^2_{\lambda\beta\theta} > (1-m^2)(1-c(\beta\theta)^2) \tag{3.26}$$

for some positive constant c.

Proof:

We have $\sigma^2_{\lambda\beta\theta} = 1-m^2-\alpha(1-\alpha)(\tanh(\nu_2+2\lambda\beta\theta)-\tanh(\nu_2))^2$. Now calling $\nu_2-\nu_1\equiv\Delta$, using $m=\tanh\nu_1$, it is easy to see that

$$\tanh(\nu_2+2\lambda\beta\theta)-\tanh(\nu_2) = \frac{(1-m^2)(\tanh(2\lambda\beta\theta+\Delta)-\tanh(\Delta))}{(1+m\tanh(2\lambda\beta\theta+\Delta))(1+m\tanh(\Delta))} \tag{3.27}$$

On the other hand, since $\nu_2=\nu_2(\theta)$ and $\nu_2(0)=\nu_1$, and see (3.21),

$$\frac{d\nu_2}{d\theta} = \frac{-2\lambda\beta\theta\alpha}{\sigma^2_{\lambda\beta\theta}\cosh^2(\nu_2+2\lambda\beta\theta)} \tag{3.28}$$

after an easy computation, we get

$$\nu_2-\nu_1 = \int_0^\theta \frac{d\nu_2}{d\theta}\,d\theta' = -2\lambda\beta\int_0^\theta \frac{\alpha\cosh^2(\nu_2(\theta'))}{\alpha\cosh^2(\nu_2(\theta'))+(1-\alpha)\cosh^2(2\lambda\beta\theta+\nu_2(\theta'))}\,d\theta' \tag{3.29}$$

From which it is easy to get

$$|\nu_2-\nu_1| \le 2\beta\theta \tag{3.30}$$

Therefore, we have $(\tanh(2\lambda\beta\theta+\Delta)-\tanh(\Delta)) \le [1-\tanh^2(2\beta\theta)]^{-1}\tanh(2\beta\theta)$ and $(1+m\tanh(2\lambda\beta\theta+\Delta)) \ge 1-\tanh(4\beta\theta)$. Collecting we get the lemma. ■

Proposition 3.2 (LCLT) *There exists an $\epsilon > 0$ such that if $\beta\theta \le \epsilon$, for all given m such that $|m| < 1$, $\alpha = |D|/|B|$ and $\sigma_{\lambda\beta\theta}$ given by (3.25)*

$$\Psi_{\lambda\beta\theta,\alpha,m} = \frac{1}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\left(1 \pm \frac{3}{|B|\sigma^2_{\lambda\beta\theta}}\right) \tag{3.31}$$

provided $|B|$ is large enough. Moreover, for all m, such that $|m| \le 1 - \frac{g(|B|)}{|B|}$ for some g that satisfies $\lim_{x\uparrow\infty} g(x) = \infty$ but $\lim_{x\uparrow\infty} \frac{g(x)}{x} = 0$, we have

$$\Psi_{\lambda\beta\theta,\alpha,m} = \frac{1}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\left(1 \pm \frac{c}{g(|B|)}\right) \tag{3.32}$$

for some positive constant c.

Remark:

The only reason to prove this proposition is to get in the error term the explicit dependence on α through $\sigma_{\lambda\beta\theta}$ and the $g(|B|)$ dependence in (3.32). The proof is rather standard and follows the usual strategy to get asymptotic expansions in the LCLT. We have been influenced by Yau [31], see also the Renyi's book [25], pg 460-466.

Proof: We start with the following simple

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\,dk = \frac{\sin \pi x}{\pi x} = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \ne 0, x \in \mathbb{Z}. \end{cases} \tag{3.33}$$

which implies after some algebra,

$$\Psi_{\lambda\beta\theta,\alpha,m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikm|B|}\Phi(\lambda\beta\theta,\alpha,k)\,dk \tag{3.34}$$

where

$$\Phi(\lambda\beta\theta,\alpha,k) \equiv \left[\frac{\cosh(2\lambda\beta\theta+\nu_2+ik)}{\cosh(2\lambda\beta\theta+\nu_2)}\right]^{|D|}\left[\frac{\cosh(\nu_2+ik)}{\cosh(\nu_2)}\right]^{|B\backslash D|} \tag{3.35}$$

Introducing the variable $e^x(2\cosh x)^{-1}$, using $1 - y \le e^{-y}$, $\forall y \in \mathbb{R}$ and $1 - \cos k \ge k^2/2$, $\forall k \in \mathbb{R}$, it is easy to check that for all $(x,k) \in \mathbb{R}^2$,

$$\left|\frac{\cosh(x+ik)}{\cosh(x)}\right| \le \exp\{-\frac{k^2}{2\cosh^2(x)}\} \tag{3.36}$$

Then, we easily get

$$|\Phi(\lambda\beta\theta,\alpha,k)| \le e^{-|B|\frac{k^2}{2}\sigma^2_{\lambda\beta\theta}} \tag{3.37}$$

If we denote

$$\mathcal{E}_\rho(m) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\!\!\mathrm{I}_{\{\rho<|k|\leq\pi\}}\Phi(\lambda\beta\theta,\alpha,k)e^{ikm|B|}\,dk \tag{3.38}$$

then, after some standard tail gaussian estimates, we get, for all $\rho > 0$

$$|\mathcal{E}_\rho(m)| \leq \frac{1}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\left(\frac{8}{3\sqrt{2\pi}(1+\rho\sigma_{\lambda\beta\theta}\sqrt{|B|})}e^{-\frac{\rho^2}{2}\sigma^2_{\lambda\beta\theta}|B|}\right) \tag{3.39}$$

The equation (3.39) suggests to take $\rho = (\sigma_{\lambda\beta\theta}\sqrt{|B|})^{-1}f(|B|)$ for some $f(|B|)$ that diverges with $|B|$ but it is such that

$$\lim_{|B|\uparrow\infty}\frac{f(|B|)}{\sqrt{g(|B|)}} = 0 \tag{3.40}$$

and we get

$$|\mathcal{E}_\rho| \leq \frac{1}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\left(\frac{8}{3\sqrt{2\pi}(1+f(|B|))}e^{-\frac{f^2(|B|)}{2}}\right) \tag{3.41}$$

It remains to estimate

$$\Psi_{\lambda\beta\theta,\alpha,m}(\rho) \equiv \frac{1}{2\pi}\int_{-\rho}^{\rho} e^{ikm|B|}\Phi(\lambda\beta\theta,\alpha,k)\,dk \tag{3.42}$$

Since we restricted the domain of $|k| \leq \rho$ and ρ goes to zero when $|B| \uparrow \infty$, using the Taylor formula with an integral rest for the term in k^4, cancelling the linear term in k, we get

$$\Psi_{\lambda\beta\theta,\alpha,m}(\rho) = \frac{1}{2\pi}\int_{-\rho}^{\rho} e^{|B|\{-\frac{k^2\sigma^2_{\lambda\beta\theta}}{2}-\frac{ik^3}{3}\mathcal{R}_\alpha(3)-\frac{k^4}{3}\mathcal{R}_\alpha(4,k)\}}\,dk \tag{3.43}$$

with $|\mathcal{R}_\alpha(4,k)| \leq (1+16\rho e^\rho)\sigma^2_{\lambda\beta\theta}$ and $|\mathcal{R}_\alpha(3)| \leq \sigma^2_{\lambda\beta\theta}$

Therefore if $\lim_{|B|\uparrow\infty}\frac{f^3(|B|)}{\sqrt{g(|B|)}} = 0$, f satisfies (3.40) as well and the terms of order k^3 and k^4 in the exponent in (3.43) go to zero.

Therefore, using $|e^{ix}-1-ix| \leq x^2/2$ for all $x \in I\!\!R$ for the term of order three in k, and $|e^x - 1| \leq |x|e^{|x|}$ for the term of order 4 in (3.43), we get after gaussian estimates

$$\begin{aligned}&\left|\Psi_{\lambda\beta\theta,\alpha,m}(\rho) - \frac{1}{2\pi}\int_{-\rho}^{\rho} e^{\{-\frac{k^2\sigma^2_{\lambda\beta\theta}|B|}{2}\}}\left[1-\frac{ik^3|B|}{3}\mathcal{R}_\alpha(3)\right]\,dk\right| \\ &\leq \frac{1}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\left(\frac{(1+32\rho e^\rho)}{|B|\sigma^2_{\lambda\beta\theta}}\right)\end{aligned} \tag{3.44}$$

The point is that the term in k^3 in the left hand side of (3.44) cancel by symmetry. It is now easy to get (3.31) by taking for example $f(|B|)=|B|^s$ with s small as we want.

To get (3.32), we just take $f(|B|)=\sqrt{2\log g(|B|)}$ and we have

$$|\mathcal{E}_\rho| \le \frac{1}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}} \frac{8}{3\sqrt{2\pi}g(|B|)(1+\sqrt{2\log g(|B|)})} \tag{3.45}$$

and (3.32) is immediate. ■

We come back to (3.24) and we estimate the second factor. We have from (3.23)

$$\phi(m,\lambda\beta\theta,\alpha)=|B|\left((\nu_1-\nu_2)m+\alpha\log\frac{\cosh(\nu_2+2\lambda\beta\theta)}{\cosh(\nu_1)}+(1-\alpha)\log\frac{\cosh(\nu_2)}{\cosh(\nu_1)}\right) \tag{3.46}$$

From (3.18) it is evident that $|\phi(m,\alpha)|/|D|$ is bounded from above by $2\beta\theta$. Therefore there are some important cancellations that occur in (3.46) in order to make it proportional to $|D|$ instead of $|B|$ as it looks at first sight. To achieve this we first prove

Lemma 3.3 *Let ν_2 be a solution of (3.21), ν_1 a solution of $m=\tanh\nu_1$, and $\sigma_{\lambda\beta\theta}$ given by (3.25), then there exists a constant c such that for all m such that $|m|\le 1$, for all $\beta>1$, for all θ such that $\beta\theta<\epsilon$ for some $\epsilon>0$ small enough*

$$|\nu_2-\nu_1| \ \le 2(\beta\theta)^2\alpha(1+c\beta\theta) \tag{3.47}$$

for some positive constant c

Proof: The proof is easy starting from (3.28), using the estimate (3.30) we get

$$\tanh(\nu_2+2\lambda\beta\theta)=m+(1-m^2)p(m,\Delta) \tag{3.48}$$

with $|p(m,\Delta)|\le 4\beta\theta(1+c\beta\theta)$ for some positive constant c. Therefore, using (3.28) and (3.26) we have

$$\left|\frac{d\nu_2}{d\theta}\right| \le 2\beta\theta\alpha(1+c\beta\theta) \tag{3.49}$$

from which we get (3.47). ■

With this result, using Taylor formula with an integral rest, we expand around ν_1 the last two terms in (3.46). Using $m=\tanh(\nu_1)$, we get,

$$\begin{aligned}\frac{\phi(m,\alpha)}{|B|} &= 2\lambda\beta\theta\alpha m+\alpha(\nu_2-\nu_1+2\lambda\beta\theta)^2\int_0^1\frac{(1-\xi)}{\cosh^2(\nu_1+\xi(\nu_2-\nu_1+2\lambda\beta\theta))}d\xi\\ &+(1-\alpha)(\nu_2-\nu_1)^2\int_0^1\frac{(1-\xi)}{\cosh^2(\nu_1+\xi(\nu_2-\nu_1))}d\xi\end{aligned} \tag{3.50}$$

The only term which is not evidently proportional to α is the last one, but using (3.47) and defining $b(\lambda\beta\theta,\alpha) \equiv \frac{\nu_2-\nu_1}{2\lambda\beta\theta\alpha}$ we have $|b(\lambda\beta\theta,\alpha)| \le \theta(1+c\beta\theta))$.

We denote by

$$\begin{aligned}\widehat{\varphi}(m,\lambda\beta\theta,\alpha) \equiv 2\lambda\beta\theta(1+\alpha b(\lambda\beta\theta,\alpha))^2 \int_0^1 \frac{(1-\xi)}{\cosh^2(\nu_1+\xi(\nu_2-\nu_1+2\lambda\beta\theta))}\, d\xi \\ + 2\lambda\beta\theta\alpha(1-\alpha)b^2(\lambda\beta\theta,\alpha)\int_0^1 \frac{(1-\xi)}{\cosh^2(\nu_1+\xi(\nu_2-\nu_1))}\, d\xi\end{aligned} \tag{3.51}$$

and we have $|\widehat{\varphi}(m,\lambda\beta\theta,\alpha)| \le \alpha\beta\theta(1+c\beta\theta)$, for some positive constant c.

At last $\phi(m,\lambda\beta\theta,\alpha) = 2\lambda\beta\theta|D|\,[m+\widehat{\varphi}(m,\lambda\beta\theta,\alpha)]$.

Therefore, the formula (3.17) takes the form

$$L_m(\lambda\beta\theta,D) = \frac{\Psi_{\lambda\beta\theta,\alpha,m}}{\Psi_{0,0,m}} e^{2\lambda\beta\theta|D|\left[m+\widehat{\varphi}(m,\lambda\beta\theta,\alpha)\right]} \tag{3.52}$$

Collecting what we have done, recalling (3.12) and (3.24), we have proved the following.

Proposition 3.4 *There exist an $\epsilon>0$ and positive constants c_1,c_2 such that if $\beta\theta<\epsilon$, for all $m^{\delta^*}(x)$ such that $|m^{\delta^*}(x)|<1$, if $i(x)=(3+\lambda(x))/2$ then*

$$\begin{aligned}\mathcal{G}_{x,m^{\delta^*}}(\lambda(x)) = 2\lambda(x)\beta\theta|D(x)|\left[m^{\delta^*}_{i(x)}(x)+\widehat{\varphi}(m^{\delta^*}_{i(x)}(x),\lambda(x)\beta\theta,\alpha)\right] \\ + \log\frac{\Psi_{\lambda(x)\beta\theta,\alpha,m^{\delta^*}_{i(x)}(x)}}{\Psi_{0,0,m^{\delta^*}_{i(x)}(x)}}\end{aligned} \tag{3.53}$$

with $\sup_{m^{\delta^*}(x):|m^{\delta^*}(x)|<1}|\widehat{\varphi}(m^{\delta^*}_{i(x)}(x),\lambda(x)\beta\theta)| \le \beta\theta(1+c_1\beta\theta)$*. Moreover, for all $g(n)$ such that* $\lim_{n\uparrow\infty} g(n)=\infty$ *but* $\lim_{n\uparrow\infty}\frac{g(n)}{n}=0$*, for all $m^{\delta^*}(x)$ such that* $|m^{\delta^*}_{i(x)}(x)| \le 1-\frac{g(|B|)}{|B|}$,

$$\left|\frac{\Psi_{\lambda(x)\beta\theta,\alpha,m^{\delta^*}_{i(x)}(x)}}{\Psi_{0,0,m^{\delta^*}_{i(x)}(x)}}-1\right| \le c_2(\beta\theta)^2+\frac{25}{g(|B|)} \tag{3.54}$$

Therefore it remains to control the cases where m^{δ^*} is such that $1\ge|m^{\delta^*}|\ge 1-\frac{g(|B|)}{|B|}$.

Proposition 3.5 *There exist an $\epsilon>0$ and a positive constant c_1 such that if $0<\beta\theta<\epsilon$, for all $g(n)$ such that* $\lim_{n\uparrow\infty} g(n)=\infty$ *but* $\lim_{n\uparrow\infty}\frac{g^2(n)}{n}=0$*, for all $m^{\delta^*}(x)$ such that* $|m^{\delta^*}_{i(x)}(x)|\ge 1-\frac{g(|B|)}{|B|}$*, if $\iota(x)=(3+\lambda(x))/2$ we have* [1]

$$\mathcal{G}_{x,m^{\delta^*}}(\lambda(x)) = 2\lambda(x)\beta\theta|D(x)|\left[m^{\delta^*}_{\iota(x)}(x)+\widehat{\varphi}(m^{\delta^*}_{\iota(x)}(x),\lambda(x)\beta\theta,\alpha)\right] \tag{3.55}$$

[1] The $\widehat{\varphi}$ here is different from that one in (3.53)

with

$$\sup_{m^{\delta^*}(x):|m^{\delta^*}(x)|\geq 1-\frac{g(|B|)}{|B|}} |\widehat{\varphi}(m^{\delta^*}_{\iota(x)}(x), \lambda(x)\beta\theta, \alpha)| \leq c_1 \left(\frac{g(|B|)}{|B|}\beta\theta + \frac{g^2(|B|)}{\beta\theta|B|} \right) \tag{3.56}$$

Proof: It is simpler to start directly from the explicit expression of $L_m(\lambda\beta\theta, |D|)$ given by (3.18). By symmetry, it is enough to consider the case where $m^{\delta^*} \geq 1 - \frac{g(|B|)}{|B|}$. As before we drop out the δ^* dependence. Moreover to simplify the formulae, it is better to set $m = 1 - \frac{2k}{|B|}$ and use the variable k instead of m. We set $L_{(1-\frac{2k}{|B|})}(\lambda\beta\theta, |D|) \equiv L_k(\lambda\beta\theta, |D|)$. We assume that $1 \leq k \leq g(|B|)$ and we denote $|B| \equiv B$ and $|D| \equiv D$. It is easy to check that

$$L_k(\lambda\beta\theta, |D|) = e^{2\lambda\beta\theta|D|} \binom{B}{k}^{-1} \sum_{\ell=0}^{k\wedge|D|} e^{-4\lambda\beta\theta\ell} \binom{B-D}{k-\ell}\binom{D}{\ell} \tag{3.57}$$

The first case to consider is when $k \leq D$. We are in the binomial regime. We use the following standard estimates

$$\frac{(B-D-k)^{k-\ell}}{(k-\ell)!} \leq \binom{B-D}{k-\ell} \leq \frac{(B-D)^{k-\ell}}{(k-\ell)!} \tag{3.58}$$

On the one hand, using the right part of (3.58) and some easy algebra, we get

$$L_k(\lambda\beta\theta, |D|) \leq \frac{B^k(B-k)!}{B!} e^{2\lambda\beta\theta|D|} \left((1-\alpha) + \alpha e^{-4\lambda\beta\theta}\right)^k \tag{3.59}$$

where as before $\alpha = D/B$. Using $(1-x)^{-1} \leq e^{x(1+x)}$ if $0 \leq x \leq 1/2$, we get

$$\frac{B^k(B-k)!}{B!} \leq \left(1 - \frac{(k-1)}{B}\right)^{-k+1} \leq \exp\left(\frac{k^2}{B}(1+\frac{k}{B})\right) \tag{3.60}$$

and this entails

$$L_k(\lambda\beta\theta, |D|) \leq e^{2\lambda\beta\theta|D|} \left((1-\alpha) + \alpha e^{-4\lambda\beta\theta}\right)^k e^{\frac{g^2(B)}{B}(1+\frac{g(B)}{B})} \tag{3.61}$$

On the other hand, using the left part of (3.58) and calling $\rho_k = k/B$, we get

$$L_k(\lambda\beta\theta, |D|) \geq \frac{B^k(B-k)!}{B!} e^{2\lambda\beta\theta|D|} \left((1-\alpha) + \alpha e^{-4\lambda\beta\theta}\right)^k \left(1 - \rho_k \frac{1+e^{-4\lambda\beta\theta}}{1-\alpha+\alpha e^{-4\lambda\beta\theta}}\right)^k \tag{3.62}$$

Using $1-x \geq e^{-x(1+x)}$ if $0 \leq |x| \leq 1/2$, the left part of (3.60) and some easy estimates, we get

$$L_k(\lambda\beta\theta, D) \geq e^{2\lambda\beta\theta D}\left((1-\alpha)+\alpha e^{-4\lambda\beta\theta}\right)^k e^{-\frac{g^2(B)(1+e^{4\beta\theta})}{B}} \tag{3.63}$$

After some computations, we get

$$e^{2\lambda\beta\theta D}\left((1-\alpha)+\alpha e^{-4\lambda\beta\theta}\right)^k = e^{2\lambda\beta\theta D\left[m+(1-m)\tilde{f}(\lambda\beta\theta,\alpha)\right]} \tag{3.64}$$

with $\left|(1-m)\tilde{f}(\lambda\beta\theta,\alpha)\right| \leq \frac{g(B)}{B} 2\beta\theta e^{\beta\theta}(1+c\beta\theta)$.

Collecting (3.61), (3.63) and (3.64), we get

$$L_m(\lambda\beta\theta, D) = e^{2\lambda\beta\theta D\left[m+\widehat{\varphi}_b(m,\lambda\beta\theta,\alpha)\right]} \tag{3.65}$$

with

$$\sup_{m:|m|\geq 1-\frac{g(B)}{B}} |\widehat{\varphi}_b(m,\lambda\beta\theta,\alpha)| \leq c_1\left(\frac{g(B)}{B}\beta\theta + \frac{g^2(B)}{\beta\theta B}\right) \tag{3.66}$$

It remains to consider the case where $D \leq k \leq g(B)$. This is the poissonian regime. It can be checked that

$$\begin{aligned} L_k(\lambda\beta\theta, D) &\leq e^{2\lambda\beta\theta D}\left(\frac{1-\alpha}{1-\rho_k}\right)^k \sum_{\ell=0}^{D} \frac{1}{\ell!} e^{-4\lambda\beta\theta\ell}\left(\frac{Dk}{B-D}\right)^\ell \\ &\leq e^{2\lambda\beta\theta D} e^{\frac{\alpha k}{1-\alpha}e^{-4\lambda\beta\theta}}\left(\frac{1-\alpha}{1-\rho_k}\right)^k \end{aligned} \tag{3.67}$$

The last factor in (3.67) is here to make a nice cancellation that will give the correct behavior when $\beta\theta \downarrow 0$. We have

$$\left(\frac{1-\alpha}{1-\rho_k}\right)^k \leq e^{-\alpha k} e^{\frac{g^2(B)}{B}} \tag{3.68}$$

Therefore after some computations, we get

$$L_m(\lambda\beta\theta, D) \leq \exp\left[2\lambda\beta\theta D\left(m+\hat{\varphi}_p(\lambda\beta, m, \alpha)\right)\right] \tag{3.69}$$

with $|\hat{\varphi}_p(\lambda\beta, m, \alpha)| \leq \frac{2g^2(B)}{B} + 2\beta\theta\frac{g(B)}{B}$.

For the lower bound, we have

$$L_k(\lambda\beta\theta, D) \geq e^{2\lambda\beta\theta D}\left(\frac{B-D-k}{B}\right)^k \sum_{\ell=0}^{D} \frac{1}{\ell!} e^{-4\lambda\beta\theta\ell} \frac{(D-\ell)^\ell (k-\ell)^\ell}{(B-D-k+\ell)^\ell} \tag{3.70}$$

Keeping the first two terms in the previous sum gives

$$L_k(\lambda\beta\theta, D) \geq e^{2\lambda\beta\theta D} \left(\frac{B-D-k}{B}\right)^k \left(1 + e^{-4\lambda\beta\theta} \frac{(D-1)(k-1)}{(B-D-k)}\right) \tag{3.71}$$

After some computations we get

$$L_k(\lambda\beta\theta, D) \geq e^{2\lambda\beta\theta D\left[m+\widehat{\varphi}_p(\lambda\beta\theta,m,\alpha)\right]} \tag{3.72}$$

with $|\widehat{\varphi}_p(\lambda\beta\theta, m, \alpha)| \leq c\frac{g^2(B)}{B}e^{8\beta\theta}$. ∎

4 Proof of the Theorem 2.3 and some probability estimates

In this section we prove the Theorem 2.3. To study the properties of the system, uniformly on an interval V of length $\tilde{c}\frac{(\log 1/\gamma)^p}{\gamma}$, $p > 1$, we start considering a region $V_1 \subset V$ of scale $L_1 \approx \frac{1}{\gamma}(\log 1/\gamma)(\log\log 1/\gamma)^{2+\rho}$, with $\rho > 0$, and divide it in smaller intervals of scale $\ell(\gamma) = \frac{1}{\gamma \log 1/\gamma}$. We reduce the proof to the estimate of the upper bound of the ratio of two constrained partition functions over one of these intervals. We then write this ratio as the product of two stochastic contributions and with $I\!\!P$-probability one, we prove the following

i) there is at *least* one interval of scale $\ell(\gamma)$ such that the first factor of the stochastic part is smaller than $e^{-\frac{c}{\gamma}}$, $c > 0$.

ii) *for all* above mentioned intervals the contribution of the second factor is negligible

iii) this can be done uniformly with respect to the choices of V_1 in V.

The choice of the relative sizes of the intervals involved is suggested by two conflicting conditions: the existence of a large enough fluctuation of the magnetic field, in at least one small interval, for the first factor and the uniform control of the second factor over *all* intervals contained in V.

In step (ii) we need a deviation inequality for a Lipschitz function of symmetric Bernoulli random variables, but our construction of the stochastic part, in section 3, does not allow to check the convexity hypothesis assumed in [21] or in [30]. Therefore we give a simple proof of such deviation inequality without any convexity hypothesis.

We start the proof of the Theorem 2.3. Given $\tilde{c} > 0$, $p > 1$, it is enough to prove that

$$\mu_{\beta,\theta,\gamma}\left[\mathcal{R}^{\delta,\zeta}([k\ell_{\hat{c}}(\gamma), L_1 + k\ell_{\hat{c}}(\gamma)], \tau)\right] \le e^{-\beta x \gamma^{-1}} \tag{4.1}$$

simultaneously for $\tau = 1$ *and* $\tau = -1$, and for any k such that $|k| \le \frac{\tilde{c}}{\hat{c}}(\log 1/\gamma)^p \log\log 1/\gamma$, where $\hat{c}$ is a constant to be determined later. We take $I_{12} \equiv [\ell_1, \ell_2] \subset [k\ell_{\hat{c}}(\gamma), L_1 + k\ell_{\hat{c}}(\gamma)]$ and we start estimating $\mu_{\beta,\theta,\gamma}\left[\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, \tau)\right]$, with ℓ_1 and ℓ_2 such that $|\ell_1 - \ell_2| = \frac{\hat{c}}{\gamma \log\log 1/\gamma} = \ell_{\hat{c}}(\gamma)$.

The first remark is that if Λ_1 and Λ_2 are two blocks of macroscopic length 1, then $\sup_{\sigma_{\Lambda_1 \cup \Lambda_2}} |W_\gamma(\sigma_{\Lambda_1}, \sigma_{\Lambda_2})| \le \gamma^{-1}$, this follows from $\int J(x)\, dx = 1$. Therefore, cutting all the interactions between $[\ell_1, \ell_2]$ and its complementary, we have the estimate

$$\mu_{\beta,\theta,\gamma}\left[\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, \tau)\right] \le e^{4\beta\gamma^{-1}} \mu_{\beta,\theta,\gamma}\left(\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, \tau) \,\big|\, \Sigma_{\partial I_{12}}\right)(0) \tag{4.2}$$

We bound from below the partition function $Z_{\beta,\gamma,\theta,[\ell_1,\ell_2]}(0)$, see (3.1), by restricting the sum over all the spin configurations in $\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, -\tau)$. Taking into account that the two normalization factors cancel, we have

$$\mu_{\beta,\theta,\gamma}\left[\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, \tau)\right] \le e^{4\beta\gamma^{-1}} \frac{\mu_{\beta,\theta,\gamma}\left(\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, \tau) \,\big|\, \Sigma_{\partial I_{12}}\right)(0)}{\mu_{\beta,\theta,\gamma}\left(\mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, -\tau) \,\big|\, \Sigma_{\partial I_{12}}\right)(0)} \tag{4.3}$$

For simplicity, let us denote $\mathcal{R}(\tau) \equiv \mathcal{R}^{\delta,\zeta}(\ell_1, \ell_2, \tau)$. Performing a block spins transformation on the scale $\delta^* \gamma^{-1}$ and using (3.13) we get

$$\mu_{\beta,\theta,\gamma}(\mathcal{R}(\tau)) \le e^{\beta\gamma^{-1}[\delta^*|\ell_2-\ell_1|+4]} \frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} \tag{4.4}$$

where

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} \equiv \frac{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} \mathbb{1}_{\{\mathcal{R}(\tau)\}} e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\mathcal{G}(m^{\delta^*}_{I_{12}})\right\}}}{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} \mathbb{1}_{\{\mathcal{R}(-\tau)\}} e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\mathcal{G}(m^{\delta^*}_{I_{12}})\right\}}} \tag{4.5}$$

We denote by T, the linear bijection on $\mathcal{M}_{\delta^*}(I_{12})$ defined by

$$T(m_1(x), m_2(x)) = (-m_2(x), -m_1(x)) \quad \forall x \in \mathcal{C}_{\delta^*}(I_{12}) \tag{4.6}$$

then $T\mathcal{R}(\tau) = \mathcal{R}(-\tau)$. Moreover from (3.10), it is immediate to check that $\widehat{\mathcal{F}}(Tm^{\delta^*}_{I_{12}}, 0) = \widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}}, 0)$ by using the symmetry properties of the combinatorial factors. Therefore, performing the change of variables induced by T in the denominator in (4.5), we get

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} \equiv \frac{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} \mathbb{1}_{\{\mathcal{R}(\tau)\}} e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\mathcal{G}(m^{\delta^*}_{I_{12}})\right\}}}{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} \mathbb{1}_{\{\mathcal{R}(\tau\}} e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\mathcal{G}(Tm^{\delta^*}_{I_{12}})\right\}}} \tag{4.7}$$

By construction we note that changing h_i into $-h_i$ makes the following changes: $\lambda(x) \to -\lambda(x)$, $B^+ \to B^-$ while $|D(x)|$ is left invariant. Therefore we get the following

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})}(-h) = \frac{Z_{-\tau}(I_{12})}{Z_\tau(I_{12})}(h) \tag{4.8}$$

which implies the non trivial fact that $\log \frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})}(h)$ is a symmetric random variable and therefore has mean zero. The next step is to extract what we expect to be the leading term of the stochastic part coming in (4.7). Recalling (3.14), we introduce

$$\Delta\mathcal{G}(m^{\delta^*}_{\beta,I_{12}}, \tau) \equiv \tau\left[\mathcal{G}(m^{\delta^*}_{\beta,I_{12}}) - \mathcal{G}(Tm^{\delta^*}_{\beta,I_{12}})\right] \tag{4.9}$$

where $m^{\delta^*}_{\beta,I_{12}}$ is the configuration of $m^{\delta^*}(x) = m^{\delta^*}_\beta \, \forall x \in I_{12}$ and $m^{\delta^*}_\beta$ is any point in $[-1, -1+4\gamma(\delta^*)^{-1}, \dots, 1]^2$ which is among the nearest to m_β defined before (2.23). We write:

$$\frac{Z_\tau(I_{12})}{Z_{-\tau}(I_{12})} = e^{\Delta\mathcal{G}(m^{\delta^*}_{\beta,I_{12}},\tau)} \frac{Z_{\tau,0}(I_{12})}{Z_{-\tau,0}(I_{12})} \tag{4.10}$$

where

$$\frac{Z_{\tau,0}(I_{12})}{Z_{-\tau,0}(I_{12})} \equiv \frac{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} 1\!\!I_{\{\mathcal{R}(\tau)\}} e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\Delta^\tau_0\mathcal{G}(m^{\delta^*}_{I_{12}})\right\}}}{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} 1\!\!I_{\{\mathcal{R}(\tau)\}} e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\Delta^\tau_0\mathcal{G}(Tm^{\delta^*}_{I_{12}})\right\}}} \tag{4.11}$$

and

$$\Delta^\tau_0\mathcal{G}(m^{\delta^*}_{I_{12}}) \equiv \mathcal{G}(T^{\frac{1-\tau}{2}} m^{\delta^*}_{I_{12}}) - \mathcal{G}(T^{\frac{1-\tau}{2}} m^{\delta^*}_{\beta,I_{12}}) \tag{4.12}$$

with $T^0 = 1\!\!I$, the identity.

Recalling that $m_{\beta,1}$ and $m_{\beta,2}$ which are defined before (2.23) are bounded away from 1. For $\beta > 1$ and $\beta\theta$ small enough, we can use the Proposition 3.4 to control $\Delta\mathcal{G}(m^{\delta^*}_{\beta,I_{12}})$. Recall that this term has mean zero. Using (3.53) and the definition of Tm_β given before (2.23), we can write $\Delta\mathcal{G}(m^{\delta^*}_{\beta,I_{12}},\tau) = -\tau\sum_{x\in\mathcal{C}_{\delta^*}(I_{12})} X(x)$ with

$$X(x) \equiv -2\beta\theta\lambda(x)|D(x)|\left[m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + \Xi(x,\beta\theta,\alpha)\right] - \lambda(x)\log\frac{\Psi_{\beta\theta,\alpha(x),m^{\delta^*}_{\beta,2}}\Psi_{0,0,m^{\delta^*}_{\beta,1}}}{\Psi_{\beta\theta,\alpha(x),m^{\delta^*}_{\beta,1}}\Psi_{0,0,m^{\delta^*}_{\beta,2}}} \tag{4.13}$$

and

$$\Xi(x,\beta\theta,\alpha) \equiv \left[\widehat{\varphi}(m^{\delta^*}_{\beta,1},\lambda(x)\beta\theta,\alpha) - \widehat{\varphi}(m^{\delta^*}_{\beta,2},\lambda(x)\beta\theta,\alpha)\right] \tag{4.14}$$

The next step is to get a lower bound for the probability of $\{\tau\gamma\sum_{x\in\mathcal{C}_{\delta^*}(I_{12})} X(x) > u\}$. We follow De Acosta [1] and write this sum as a sum over $|\mathcal{C}_{\delta^*}(I_{12})|/N$ blocks, each block having N elements, $1 \le N \le \frac{|I_{12}|}{\delta^*}$.

Calling $V^2(N) = V^2(N(w)) \equiv \sum_{x\in N(w)} I\!\!E[X^2(x)]$ for $1 \le w \le |I_{12}|/(\delta^* N)$, we require that N satisfies also

$$\gamma\sum_{x\in\mathcal{C}_{\delta^*}(I_{12})} X(x) = \frac{N}{|\mathcal{C}_{\delta^*}(I_{12})|}\sum_{w=1}^{\frac{|\mathcal{C}_{\delta^*}(I_{12})|}{N}}\frac{1}{V(N)}\sum_{x\in N(w)} X(x) \tag{4.15}$$

Assuming that such N can be found, then we have

$$\left\{\tau\gamma\sum_{x\in\mathcal{C}_{\delta^*}(I_{12})} X(x) > u\right\} \supset \bigcap_{w=1}^{\frac{|\mathcal{C}_{\delta^*}(I_{12})|}{N}}\left\{\frac{\tau}{V(N)}\sum_{x\in N(w)} X(x) > u\right\} \tag{4.16}$$

Using the fact that the events in the right hand side are independent we apply the Central Limit Theorem to estimate their individual probabilities. To check that we can find an $1 \le N \le \frac{|I_{12}|}{\delta^*}$ such that (4.15) is true, we note that on the one hand

$$I\!\!E[X^2(x)] \ge \left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})\right)^2 I\!\!E[(\sum_{i=1}^{\delta^*\gamma^{-1}} h_i)^2] = \left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})\right)^2\frac{\delta^*}{\gamma} \tag{4.17}$$

and on the other hand, using the Proposition 3.4, we get if $g(|B|)$ is large enough

$$\mathbb{E}[X^2(x)] \leq \left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + c\beta\theta)\right)^2 \frac{\delta^*}{\gamma}\left(1 + 2c\frac{\gamma}{\delta^*}(\beta\theta)^2\right) \tag{4.18}$$

for some positive constant c. It is easy to check that N must satisfy

$$\left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})\right)^2 \frac{\gamma|I_{12}|^2}{\delta^*} \leq N \leq \left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})\right)^2 \frac{\gamma|I_{12}|^2}{\delta^*}\left(1 + 2c\frac{\gamma}{\delta^*}(\beta\theta)^2\right) \tag{4.19}$$

Therefore, $N \leq |I_{12}|/\delta^*$ provided $|I_{12}| \leq \frac{1}{\gamma}\left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})\right)^{-2}\left(1 + 2c\frac{\gamma}{\delta^*}(\beta\theta)^2\right)^{-1}$. Obviously $N \geq 1$ provided

$$|I_{12}| \geq \left(\frac{\delta^*}{\gamma}\right)^{1/2} \frac{1}{\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})} \tag{4.20}$$

To continue, using (4.17) and (4.19) we have

$$V^2(N) \geq \left(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2})\right)^4 |I_{12}|^2 \tag{4.21}$$

Since $\lim_{\gamma\downarrow 0}|I_{12}| = \infty$, it is clear that we are in the domain of application of the Central Limit Theorem. Therefore, we have, for all $\tilde{\epsilon} > 0$ and $u > 0$

$$\mathbb{P}\left[\frac{\tau}{V(N)}\sum_{x\in N(w)} X(x) > u\right] \geq \mathbb{P}\left[u(1+\tilde{\epsilon}) \geq \frac{\tau}{V(N)}\sum_{x\in N(w)} X(x) > u\right] \geq e^{-\frac{u^2(1+\tilde{\epsilon})}{2}} \tag{4.22}$$

and, using the lower bound for N, see (4.19), and for $V(N)$, see (4.21), together with (4.16), we get

$$\mathbb{P}\left[\tau\gamma\sum_{x\in\mathcal{C}_{\delta^*}(I_{12})} X(x) \geq u\right] \geq \exp\left(-\frac{u^2(1+\tilde{\epsilon})}{2(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2}))^2\gamma|I_{12}|}\right) \tag{4.23}$$

Now to end the proof of the Theorem 2.3, first we use (4.23), for M consecutive blocks of length $\ell_{\hat{c}}(\gamma)$, that we denote by $L(1), \ldots L(M)$. Using independence over disjoint blocks and $1 - x \leq e^{-x}$, considering the two cases $\tau = 1$ and $\tau = -1$ separately, we get

$$\mathbb{P}\left[\inf_{\tau\in\{-1,+1\}}\sup_{1\leq\ell\leq M}\tau\gamma\sum_{x\in\mathcal{C}_{\delta^*}(L(\ell))} X(x) \geq u\right] \geq 1 - 2\exp\left[-Me^{-\frac{u^2(1+\tilde{\epsilon})}{2(\beta\theta(m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2}))^2\gamma\ell_{\hat{c}}(\gamma)}}\right] \tag{4.24}$$

Moreover it follows from the next proposition, see (4.27), that for all $\epsilon > 0$, provided $g_2(1/\zeta)$ is a diverging, slowly varying function at infinity, $\lim_{\zeta\downarrow 0}\sqrt{\zeta}g_2(1/\zeta) = 0$, then

$$I\!P\left[\sup_{1\le\ell\le M}\left|\log\frac{Z_{+,0}(L(\ell))}{Z_{-,0}(L(\ell))}\right| \le \frac{\epsilon}{\gamma}\right] \ge 1 - 2M\exp\left(-\frac{\epsilon^2}{212\gamma\ell_{\hat c}(\gamma)\beta\theta\zeta g_2(1/\zeta)}\right) \tag{4.25}$$

Given $\rho > 0$ and $x > 0$ we make the following choice of parameters.

$$\begin{aligned} c(x,\rho,\gamma) &= \frac{2(4+x)^2\beta^2}{1+(2+\frac{3\rho}{4})\frac{\log\log\log\frac{1}{\gamma}}{\log\log\frac{1}{\gamma}}} \\ \hat c &= \left[\frac{(1+\tilde\epsilon)}{(\beta\theta)^2(m_{\beta,1}+m_{\beta,2})^2}c(x,\rho,\gamma)\right] \\ u &= 2\beta(4+x+c_0\hat c) \\ M &= (\log\frac{1}{\gamma})(\log\log\frac{1}{\gamma})^{3+\rho} \\ \epsilon &= \frac{(4+x)\beta}{2} \\ \zeta g_2(1/\zeta) &\le \frac{1}{8\times 212(p+2+\tilde\epsilon)}\beta\theta(m_{\beta,1}+m_{\beta,2})^2 \end{aligned} \tag{4.26}$$

an easy computation show that the right hand side of (4.24) is bounded below by $1 - e^{-(\log\log\frac{1}{\gamma})^{1+\frac{\rho}{4}}}$, and the one of (4.25) by $1-\frac{1}{(\log\frac{1}{\gamma})^{p+2+\tilde\epsilon}}$. By (4.3), (4.4) and (4.10) we obtain the (4.1). Moreover, it is immediate to see that we have also the uniformity with respect to the $2\frac{\tilde c}{\hat c}(\log 1/\gamma)^p\log\log 1/\gamma$ possible choices of k in (2.28). Using the first Borel-Cantelli lemma and the fact that $\gamma = 2^{-n}$, we get the Theorem 2.3. ■

Proposition 4.1 *Given $\beta > 1$, $\beta\theta < \epsilon_0$ for some ϵ_0, let ζ small enough and $g_2(1/\zeta)$ be a real function such that $\lim_{\zeta\downarrow 0} g_2(1/\zeta) = \infty$, slowly varying at infinity that satisfies $\lim_{\zeta\downarrow 0}\sqrt{\zeta}g_2(1/\zeta) = 0$, then for all $\epsilon > 0$, for all integers ℓ_1, ℓ_2, if γ is small enough*

$$I\!P\left[\left|\log\frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})}\right| \ge \frac{\epsilon}{\gamma}\right] \le \exp\left(-\frac{\epsilon^2}{212\gamma|\ell_1-\ell_2|\beta\theta\zeta g_2(1/\zeta)}\right) \tag{4.27}$$

The proof of this proposition is rather long and technical. We first remark that using the explicit expression (3.11), (3.12) and the fact that $T(m_{\beta,1}, m_{\beta,2}) = (-m_{\beta,2}, -m_{\beta,1})$, we get $I\!E\left[\mathcal{G}(m^{\delta^*}_{\beta,I_{12}}) - \mathcal{G}(Tm^{\delta^*}_{\beta,I_{12}})\right] = 0$, using (4.8), we have also $I\!E\left[\log\frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})}\right] = 0$.
Let us prove the above mentioned deviation inequality.

Lemma 4.2 *Let N be a positive integer and F be a real function on $\Omega = \{-1,+1\}^N$ and for all $i \in \{1,\ldots,N\}$ let*

$$\|\partial_i F\|_\infty = \sup_{(h,\tilde{h}):h_j=\tilde{h}_j,\forall j\neq i} \frac{\left|F(h) - F(\tilde{h})\right|}{|h_i - \tilde{h}_i|} \tag{4.28}$$

If $\mathbb{P}$ is the symmetric Bernoulli measure and $\|\partial(F)\|_\infty^2 = \sum_{i=1}^N \|\partial_i(F)\|_\infty^2$ then, for all $t > 0$

$$\mathbb{P}\left[F - \mathbb{E}(F) \geq t\right] \leq e^{-\frac{t^2}{4\|\partial(F)\|_\infty^2}} \tag{4.29}$$

and also

$$\mathbb{P}\left[F - \mathbb{E}(F) \leq -t\right] \leq e^{-\frac{t^2}{4\|\partial(F)\|_\infty^2}} \tag{4.30}$$

Proof: We prove (4.29), the proof of (4.30) is exactly the same. As usually in this kind of estimates, we start with the exponential Markov inequality. For all $\lambda > 0$, we have

$$\mathbb{P}\left[F - \mathbb{E}(F) \geq t\right] \leq e^{-\lambda t}\mathbb{E}\left[e^{\lambda(F-\mathbb{E}(F))}\right] \tag{4.31}$$

To estimate the last term, we introduce the family of increasing σ-algebra:

$$(\emptyset,\Omega) = \Sigma_0 \subset \Sigma_1 = \sigma(h_1) \subset \Sigma_2 = \sigma(h_1,h_2) \subset \ldots \subset \Sigma_N = \sigma(h_1,h_2,\ldots,h_N) \tag{4.32}$$

and the martingale difference sequences, $\forall\ 1 \leq k \leq N$, $\Delta_k(F) = \mathbb{E}\left[F|\Sigma_k\right] - \mathbb{E}\left[F|\Sigma_{k-1}\right]$. If we prove that

$$\mathbb{E}\left[e^{\lambda\sum_{k=1}^N \Delta_k(F)}\right] \leq e^{\lambda^2\|\partial(F)\|_\infty^2} \tag{4.33}$$

then (4.29) follows from (4.31) by taking $\lambda = t(2\|\partial(F)\|_\infty^2)^{-1}$. To prove (4.33), we perform the integrations in the right hand side of (4.33) starting from h_N. The only term that depends on h_N is $\Delta_N(F) = F(h_{<N},h_N) - \int F(h_{<N},\tilde{h}_N)\,\mathbb{P}(d\tilde{h}_N)$ where $h_{<N} \equiv (h_1,h_2,\ldots,h_{N-1})$. Therefore, using the Jensen inequality, we get

$$\int e^{\lambda\Delta_N(F)}\,\mathbb{P}(dh_N) \leq \int e^{\lambda\left[F(h_{<N},h_N)-F(h_{<N},\tilde{h}_N)\right]}\mathbb{P}(dh_N)\mathbb{P}(d\tilde{h}_N) \tag{4.34}$$

For all fixed $h_{<N}$ the term into the exponential is the symmetrized of F with respect to the last variable. Then if we expand the exponential and integrate with respect to the

product measure $I\!P(dh_N)I\!P(d\tilde{h}_N)$, all the odd terms vanish and we get $\forall h_{<N}$

$$\begin{aligned}\int e^{\lambda\Delta_N(F)}\, I\!P(dh_N) &\le \sum_{n=0}^{\infty} \frac{(\lambda)^{2n}}{(2n)!}\int \left[F(h_{<N},h_N)-F(h_{<N},\tilde{h}_N)\right]^{2n} I\!P(dh_N)I\!P(d\tilde{h}_N)\\ &\le \sum_{n=0}^{\infty} \frac{(\lambda\|\partial_N F\|_\infty)^{2n}}{(2n)!}\int |h_N-\tilde{h}_N|^{2n} I\!P(dh_N)I\!P(d\tilde{h}_N)\\ &= \sum_{n=0}^{\infty} \frac{(\lambda\|\partial_N F\|_\infty)^{2n}}{(2n)!} 2^{(2n-1)^+} \le e^{\lambda^2\|\partial_N(F)\|_\infty^2}\end{aligned} \tag{4.35}$$

where $(x)^+=\max(x,0)$ and we have used $\frac{2^{(2n-1)^+}}{(2n)!}\le\frac{1}{n!}$.

There is a little difference for the successive integrations which is just the way to use the Jensen inequality. We perform the next h_{N-1} integration, since the term $\Delta_{N-1}(F)$ is the only one that comes into play, we use Jensen inequality as follows

$$\begin{aligned}&\int e^{\lambda\Delta_{N-1}(F)} I\!P(dh_{N-1})\\ &\le \int e^{\lambda\int\left[F(h_{<N-1},h_{N-1},\hat{h}_N)-F(h_{<N-1},\tilde{h}_{N-1},\hat{h}_N)\right]I\!P(d\hat{h}_N)} I\!P(dh_{N-1})I\!P(d\tilde{h}_{N-1})\end{aligned} \tag{4.36}$$

Now we can make exactly the same computations since for fixed $h_{<N-1}$,

$$\int \left[F(h_{<N-1},h_{N-1},\hat{h}_N)-F(h_{<N-1},\tilde{h}_{N-1},\hat{h}_N)\right] I\!P(d\hat{h}_N) \tag{4.37}$$

is a symmetric random variable under $I\!P(dh_{N-1})I\!P(d\tilde{h}_{N-1})$ and we can use (4.28) to get

$$\int e^{\lambda\Delta_{N-1}(F)} I\!P(dh_{N-1}) \le e^{\lambda^2\partial_{N-1}(F)} \tag{4.38}$$

Iterating we get (4.33). ■

It is clear that we have to estimate the corresponding Lipschitzian factors, see (4.28),

$$\left\|\partial_i \log\frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})}\right\|_\infty \tag{4.39}$$

for all $i\in\gamma^{-1}(I_{12})$. Here, there is a difficulty that comes from the fact that the definition (2.25) of $\eta^{\delta,\zeta}$ is given in term of a Cesaro average of blocks of length δ^* that are contained in a block δ of ℓ_1 norm. So we cannot assume that *all* the blocks of length δ^* are near an equilibrium, some but certainly not all blocks of length δ^* can have $m_i^{\delta^*}(x)$ very near one. On the other hand the correction to the leading behavior of $\Delta_0 \mathcal{G}_{x,m^{\delta^*}}$ is dependent on the values of m^{δ^*} and here we have to estimate a Lipschitz norm which certainly becomes more and more singular as $m_i^{\delta^*}(x)$ approaches 1. To solve this problem we localize the blocks which are near equilibrium (the good ones) and their complementary (the bad ones). We show that the fraction of the bad blocks can be neglected provided we increase the "tolerance" ζ.

We need to introduce some definitions. Given $i \in \gamma^{-1} I_{12}$, let $x(i)$ be the index of the block of length δ^* that contains the microscopic site i. Let $u(i)$ be the index of the block of length δ that contains $x(i)$, let $\mathcal{C}_{\delta/\delta^*}(u(i)) \equiv \mathcal{C}_{\delta/\delta^*}(i)$ be the set of the centers of blocks of length δ^* that are in the blocks of length δ indexed by $u(i)$. We have to estimate

$$\log \frac{Z_{+,0}(I_{12})(h)}{Z_{+,0}(I_{12})(\tilde{h}_i)} - \log \frac{Z_{-,0}(I_{12})(h)}{Z_{-,0}(I_{12})(\tilde{h}_i)} \tag{4.40}$$

Where the only discrepancy between h and $\tilde{h}_i$ is at the site i. To continue we need a simple lemma, its proof is similar to the Markov inequality

Lemma 4.3 *If*

$$\sum_{x \in \mathcal{C}_{\delta/\delta^*}(i)} \|m^{\delta^*} - m_\beta\|_1 \leq \frac{\delta}{\delta^*} \zeta \tag{4.41}$$

then given $g_1(\zeta)$ *such that* $\lim_{\zeta \downarrow 0} g_1(\zeta) = 0$ *but* $\frac{\zeta}{g_1(\zeta)} < 1$ *if* $\zeta \leq 1$, *we have*

$$\sum_{x \in \mathcal{C}_{\delta/\delta^*}(i)} 1\!\!\mathrm{I}_{\{\|m^{\delta^*}(x) - m_\beta\|_1 \leq g_1(\zeta)\}} \geq \frac{\delta}{\delta^*}\left(1 - \frac{\zeta}{g_1(\zeta)}\right) \tag{4.42}$$

This suggest to make a partition of $\mathcal{C}_{\delta/\delta^*}(i)$ into two sets,

$$\mathcal{A}(m^{\delta^*}) \equiv \{x \in \mathcal{C}_{\delta/\delta^*}(i) : \|m^{\delta^*}(x) - m_\beta\|_1 \leq g_1(\zeta), \sup(|m_1^{\delta^*}(x)|, |m_2^{\delta^*}(x)|) \leq 1 - \frac{g(|B|)}{|B|}\} \tag{4.43}$$

and $\mathcal{B}(m^{\delta^*}) = \mathcal{C}_{\delta/\delta^*}(i) \backslash \mathcal{A}(m^{\delta^*})$. Let us call $\Delta(m_\beta) = 1 - m_{\beta,1}$, recalling that $m_{\beta,2} \leq m_{\beta,1}$. We assume that the parameters ζ, δ, δ^* and the functions $g_1(\zeta)$ and $g(|B|)$ are all chosen in such a way that for the given pair (β, θ) we have

$$g_1(\zeta) + \frac{g(|B|)}{|B|} \leq \Delta(m_\beta) \tag{4.44}$$

This will imply that $\sup(|m_1^{\delta^*}(x)|, |m_2^{\delta^*}(x)|) \leq g_1(\zeta) + 1 - \Delta(m_b) \leq 1 - \frac{g(|B|)}{|B|}$ and therefore the second condition in the definition of $\mathcal{A}$ is automatically satisfied. Let us note that since the two terms in the left hand side of (4.44) go to zero, we can assume that (4.44) is satisfied by taking ζ and γ small enough.

let $\ell(i)$ be the index of the block of length 1 containing the microscopic site i. For all $m^{\delta^*} \equiv m^{\delta^*}_{\ell(i)}$ we write

$$\mathbb{1}_{\{\eta^{\delta,\zeta}(\ell(i))=1\}}(m^{\delta^*}) = \sum_{X\subset\mathcal{C}_{\delta/\delta^*}(i)} \mathbb{1}_{\{\mathcal{A}=X\}}(m^{\delta^*})\mathbb{1}_{\{\mathcal{B}=X^c\}}(m^{\delta^*})\mathbb{1}_{\{\eta^{\delta,\zeta}(\ell(i))=1\}}(m^{\delta^*}) \tag{4.45}$$

where the sum is over all the subsets of $\mathcal{C}_{\delta/\delta^*}(i)$ and $X^c \equiv \mathcal{C}_{\delta/\delta^*}(i)\setminus X$. Note that it follows from the previous lemma, that $\eta_{\delta,\zeta}(\ell(i)) = 1$ and $|X| \leq \frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})$ are incompatible, therefore we can impose that $|X| \geq \frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})$ in (4.45).

Let us call

$$\mathcal{N}(\zeta) = \sum_{X\subset\mathcal{C}_{\delta/\delta^*}(i)} \mathbb{1}_{\{|X|\geq\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})\}} = \sum_{k=\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})}^{\frac{\delta}{\delta^*}} \binom{\frac{\delta}{\delta^*}}{k} \tag{4.46}$$

Then (4.40) is also equivalent to

$$\log\frac{Z_{+,0}(I_{12})(h)}{\mathcal{N}(\zeta)Z_{+,0}(I_{12})(\tilde{h}_i)} - \log\frac{Z_{-,0}(I_{12})(h)}{\mathcal{N}(\zeta)Z_{-,0}(I_{12})(\tilde{h}_i)} \tag{4.47}$$

The two terms are estimated in the same way. We consider the first one. It is easy to see that, with self explained notation

$$\frac{Z_{+,0}(I_{12})(h)}{\mathcal{N}(\zeta)Z_{+,0}(I_{12})(\tilde{h}_i)} = \frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\left[e^{\Delta_0\mathcal{G}^h_{x(i)}-\Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}}\right] \tag{4.48}$$

where $\mathcal{Q}$ is the probability measure

$$\mathcal{Q}(\Psi) = \frac{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})}\Psi(m^{\delta^*})\mathbb{1}_{\{\mathcal{R}(+)\}}e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\Delta_0\mathcal{G}^{\tilde{h}_i}(m^{\delta^*}_{I_{12}})\right\}}}{\sum_{m^{\delta^*}(I_{12})\in\mathcal{M}_{\delta^*}(I_{12})}\mathbb{1}_{\{\mathcal{R}(+)\}}e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{I_{12}},0)+\gamma\Delta_0\mathcal{G}^{\tilde{h}_i}(m^{\delta^*}_{I_{12}})\right\}}} \tag{4.49}$$

Inserting (4.45) in (4.48), we get

$$\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\left[e^{\Delta_0\mathcal{G}^h_{x(i)}-\Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}}\right] = \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\geq\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})}}\mathcal{Q}\left[e^{\Delta_0\mathcal{G}^h_{x(i)}-\Delta_0\mathcal{G}^{\tilde{h}_i}_{x(i)}}\mathbb{1}_{\{\mathcal{A}=X\}}\mathbb{1}_{\{\mathcal{B}=X^c\}}\right] \tag{4.50}$$

and note that *if* we have an estimate of the form

$$|\Delta_0\mathcal{G}^h_{x(i)} - \Delta_0\mathcal{G}^{\tilde h_i}_{x(i)}| \le f_1(\zeta)\mathbb{1}_{\{i\in\mathcal{A}\}} + f_2(\zeta)\mathbb{1}_{\{i\in\mathcal{B}\}} \tag{4.51}$$

then on the one hand, we get

$$\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\left[e^{\Delta_0\mathcal{G}^h_{x(i)}-\Delta_0\mathcal{G}^{\tilde h_i}_{x(i)}}\right] \le \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\ge\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})}}\left[e^{f_1(\zeta)}\mathbb{1}_{\{i\in X\}} + e^{f_2(\zeta)}\mathbb{1}_{\{i\in X^c\}}\right] \tag{4.52}$$

and on the other hand

$$\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}\left[e^{\Delta_0\mathcal{G}^h_{x(i)}-\Delta_0\mathcal{G}^{\tilde h_i}_{x(i)}}\right] \ge \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\ge\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})}} e^{-f_1(\zeta)}\mathbb{1}_{\{i\in X\}} \tag{4.53}$$

It is simple to check that

$$1-\frac{\zeta}{g_1(\zeta)} \le \frac{1}{\mathcal{N}(\zeta)}\sum_{\substack{X\subset\mathcal{C}_{\delta/\delta^*}(i)\\ |X|\ge\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})}} \mathbb{1}_{\{i\in X\}} \le 1 \tag{4.54}$$

Therefore, coming back to (4.48) and using (4.52) and (4.53), we get

$$\left|\log\frac{Z_{+,0}(I_{12})(h)}{\mathcal{N}(\zeta)Z_{+,0}(I_{12})(\tilde h_i)}\right| \le f_1(\zeta) + \frac{\zeta}{g_1(\zeta)}e^{|f_2(\zeta)-f_1(\zeta)|} \tag{4.55}$$

Therefore, recalling (4.51), even if we have a very poor bound $f_2(\zeta)$ on the set $\mathcal{B}$, (4.55) implies that by choosing $g_1(\zeta)$ in such a way that $\frac{\zeta}{g_1(\zeta)}\downarrow 0$, we recover something small coming from the prefactor in second term in (4.55).

Let us prove something similar to (4.51). There are two cases to consider, the first one is when $\lambda^h = -\lambda^{\tilde h_i}$ and the second one is when $\lambda^h = \lambda^{\tilde h_i}$. In the first case, it is easy to check that we have $|D^h| = |D^{\tilde h_i}| = 1$. In this case, it is simpler to use directly (3.57), and after an easy computation we get, if $|D(x)| = 1$

$$\mathcal{G}_{x,m^{\delta^*}}(\lambda(x)) = \log\cosh(2\beta\theta) + \log\Big(1+\lambda(x)m^{\delta^*}(x)\tanh(2\beta\theta)\Big) \tag{4.56}$$

from which it is immediate that, if $|D(x)| = 1$ and $\beta\theta$ is small enough

$$\begin{aligned}\left|\Delta_0\mathcal{G}(m^{\delta^*}(x)) - \Delta_0\mathcal{G}^{\tilde h_i}(m^{\delta^*}(x))\right| &\le \frac{4\tanh(2\beta\theta)\|m^{\delta^*}(x)-m_\beta\|_1}{1-m_{\beta,1}\tanh(2\beta\theta)}\\ &\le c(\beta,\theta)\|m^{\delta^*}(x)-m_\beta\|_1\end{aligned} \tag{4.57}$$

and this estimate is valid for all values of m^{δ^*}.

In the second case, it is a rather long task to make all the estimates. We have **Proposition 4.4** *There exists an $\epsilon > 0$ and an absolute constant c such that if $\beta\theta \le \epsilon$, for all $g(n)$ such that $\lim_{n\uparrow\infty} g(n) = \infty$ but $\lim_{n\uparrow\infty} g(n)/n = 0$,*

$$
\begin{aligned}
&|\Delta_0 \mathcal{G}^h[m^{\delta^*}(x(i))] - \Delta_0 \mathcal{G}^{\tilde{h}_i}[m^{\delta^*}(x(i))]| \\
&\le 2\beta\theta \left(1 + 16\beta\theta + \frac{|B|}{g^2(|B|)}\right) \|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1 + \frac{c}{g(|B|)\sqrt{\log g(|B|)}}
\end{aligned}
\tag{4.58}
$$

on the set $\left\{|m^{\delta^*}(x(i))| \le 1 - \frac{g(|B|)}{|B|}\right\}$.

While,

$$
|\Delta_0 \mathcal{G}^h[m^{\delta^*}(x(i))] - \Delta_0 \mathcal{G}^{\tilde{h}_i}[m^{\delta^*}(x(i))]| \le 2\beta\theta \|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1 + c\left(\frac{g^2(|B|)}{|B|}\right) \tag{4.59}
$$

on the set $\left\{|m^{\delta^*}(x(i))| \ge 1 - \frac{g(|B|)}{|B|}\right\}$.

Proof: The formula (4.59) is immediate from the Proposition 3.5. To prove (4.58), remembering (3.53), we have to study three terms. The first one is the simplest:

$$
\Delta_0^1 \mathcal{G}[m^{\delta^*}(x(i))] \equiv 2\beta\theta \left(\lambda^h |D^h| - \lambda^{\tilde{h}_i} |D^{\tilde{h}_i}|\right) \left[m^{\delta^*}_{\iota(x(i))}(x(i)) - m^{\delta^*}_{\beta,\iota(x(i))}(x(i))\right] \tag{4.60}
$$

and using

$$
\left|\lambda^h |D^h| - \lambda^{\tilde{h}_i} |D^{\tilde{h}_i}|\right| = 1 \tag{4.61}
$$

we get

$$
|\Delta_0^1 \mathcal{G}[m^{\delta^*}(x(i))]| \le 2\beta\theta \|m^{\delta^*}(x(i)) - m_\beta^{\delta^*}\|_1 \tag{4.62}
$$

The next one corresponds to $\widehat{\varphi}$ and we start from (3.46) and cancel from it the previously estimated term. That is we consider

$$
\begin{aligned}
\Delta_0^2 \mathcal{G}[m^{\delta^*}(x(i))] \equiv\ & \phi(m^{\delta^*}_{\iota(x(i))}(x(i)), \lambda^h(x)\beta\theta, \alpha^h) - \phi(m^{\delta^*}_{\iota(x(i))}(x(i)), \lambda^{\tilde{h}_i}(x)\beta\theta, \alpha^{\tilde{h}_i}) \\
& - \left(\phi(m^{\delta^*}_{\beta,\iota(x(i))}(x(i)), \lambda^h(x)\beta\theta, \alpha^h) - \phi(m^{\delta^*}_{\beta,\iota(x(i))}(x(i)), \lambda^{\tilde{h}_i}(x)\beta\theta, \alpha^{\tilde{h}_i})\right) \\
& - \Delta_0^1 \mathcal{G}[m^{\delta^*}(x(i))]
\end{aligned}
\tag{4.63}
$$

A simple way to estimate this term is to compute the double integral of its second derivative with respect to α and m. After easy estimates we get

$$
\left|\Delta_0^2 \mathcal{G}[m^{\delta^*}(x(i))]\right| \le (32\beta^2\theta^2) \|m^{\delta^*} - m_\beta\|_1 \tag{4.64}
$$

It remains to consider the last term in (3.53). We use that $\Psi_{0,0,m}$ does not depend on α and define

$$\begin{aligned}\Delta_0^3\mathcal{G}(m^{\delta^*}(x(i))) \equiv \quad & \log\Psi_{\lambda^h(x(i))\beta\theta,\alpha^h,m^{\delta^*}_{\iota(x(i))}}(x(i)) - \log\Psi_{\lambda^{\tilde h_i}(x(i))\beta\theta,\alpha^{\tilde h_i},m^{\delta^*}_{\iota(x(i))}}(x(i)) \\ & - \left(\log\Psi_{\lambda^h(x(i))\beta\theta,\alpha^h,m^{\delta^*}_{\beta,\iota(x(i))}}(x(i)) - \log\Psi_{\lambda^{\tilde h_i}(x(i))\beta\theta,\alpha^{\tilde h_i},m^{\delta^*}_{\beta,\iota(x(i))}}(x(i))\right)\end{aligned} \tag{4.65}$$

The estimates are done in two different ways depending on the fact that the blocks we consider belongs to $\mathcal{B}$ or to $\mathcal{A}$. In the first case, recalling (4.55), we do not need a sharp estimate. We use simply (3.54), bounding the difference in (4.65) by a sum of four terms, we get immediately

$$\left|\Delta_0^3\mathcal{G}(m^{\delta^*}(x(i)))\right| \le c(\beta\theta)^2 + \frac{200}{g(|B|)} \tag{4.66}$$

for some positive constant c.

In the second case, as it becomes clear in a moment, we need to use the fact that $\|m^{\delta^*} - m^{\delta^*}_\beta\|_1 \le g_1(\zeta)$ and this makes the computations more involved.

Lemma 4.5 *There exists an $\epsilon > 0$ and an absolute constant c such that if $\beta\theta \le \epsilon$, for all $g(n)$ such that $\lim_{n\uparrow\infty} g(n) = \infty$ but $\lim_{n\uparrow\infty} \frac{g(n)}{n} = 0$ for all m such that $|m| \le 1 - \frac{g(|B|)}{|B|}$,*

$$\begin{aligned}\left|\Delta_0^3\mathcal{G}(m^{\delta^*}(x(i)))\right| \le & \|m^{\delta^*}_{\iota(x(i))} - m^{\delta^*}_{\beta,\iota(x(i))}\|_1 \frac{c\beta\theta|B|}{g^2(|B|)}\left(1 + \frac{c}{\sqrt{g(|B|)}}\right) \\ & + \frac{c}{g(|B|)\sqrt{\log g(|B|)}}\end{aligned} \tag{4.67}$$

Proof:

We use first (3.38) and (3.42) to write

$$\log\Psi_{\lambda\beta\theta,\alpha,m} = \log\Psi_{\lambda\beta\theta,\alpha,m}(\rho) + \log\left(1 + \frac{\mathcal{E}_{\lambda\beta\theta,\alpha,m}(\rho)}{\Psi_{\lambda\beta\theta,\alpha,m}(\rho)}\right) \tag{4.68}$$

with $\rho = (\sigma_{\lambda\beta\theta}\sqrt{|B|})^{-1}\sqrt{2\log g(|B|)}$ and we use (3.41), setting $f(|B|) = \sqrt{2\log g(|B|)}$ together with (3.31) to control the last term. This leads to

$$\left|\frac{\mathcal{E}_{\lambda\beta\theta,\alpha,m}(\rho)}{\Psi_{\lambda\beta\theta,\alpha,m}(\rho)}\right| \le \frac{5}{3\pi g(|B|)(1 + \sqrt{2\log g(|B|)})} \tag{4.69}$$

Therefore the four terms of this type in (4.65) will give a contribution which corresponds to the last term in (4.67). For the remaining terms, we proceed as before, starting with

$$\begin{aligned}\Delta_0^3\mathcal{G}(m^{\delta^*}(x(i)),\rho) = &\int_{\alpha^{\tilde{h}_i}}^{\alpha^h}\int_{m^{\delta^*}_{\beta,\iota(x(i))}}^{m^{\delta^*}_{\iota(x(i))}(x(i))} \frac{\partial^2\Psi_{\lambda\beta\theta,\alpha,m}(\rho)}{\partial\alpha\partial m}\frac{1}{\Psi_{\lambda\beta\theta,\alpha,m}(\rho)}\, d\alpha dm \\ &-\int_{\alpha^{\tilde{h}_i}}^{\alpha^h}\int_{m^{\delta^*}_{\beta,\iota(x(i))}}^{m^{\delta^*}_{\iota(x(i))}(x(i))} \frac{\partial\Psi_{\lambda\beta\theta,\alpha,m}(\rho)}{\partial\alpha}\frac{\partial\Psi_{\lambda\beta\theta,\alpha,m}(\rho)}{\partial m}\frac{1}{\Psi^2_{\lambda\beta\theta,\alpha,m}(\rho)}\, d\alpha dm\end{aligned} \tag{4.70}$$

We estimate separately the last two lines of (4.70). We start from (3.42). We derivate the integral with respect to m. This gives a term proportional to $|B|$ which is bad. Using Taylor formula with an integral rest, we expand in k up to the order 1 the term in the integrand that comes from derivating $\Phi(\lambda\beta\theta,\alpha,k)$. Then making computations similar to the ones that we did in (3.44), being aware of the cancellation of the previous linear term in k, we get the leading term of order $|B|k^2$. Performing the gaussian integral, we get

$$|\partial_m\Psi_{\lambda\beta\theta,\alpha,m}(\rho)| \le \frac{(1+c\rho e^{\rho})e^{\rho_4}}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\frac{2}{\sigma^2_{\lambda\beta\theta}}\left(1+\frac{4}{\sqrt{g(|B|)}}\right) \tag{4.71}$$

Let us note that in the denominator the term $\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}$ will be cancelled out by the corresponding term in $\Psi_{\lambda\beta\theta,\alpha,m}(\rho)$, see (3.32) when estimating the ratios in (4.70).

For the derivative with respect to α, we proceed in a similar way. It can be checked that the linear term in k is not present here and the result is:

$$|\partial_\alpha\Psi_{\lambda\beta\theta,\alpha,m}(\rho)| \le \frac{(1+c\epsilon e^{\rho})8\beta\theta e^{\rho_4}}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\frac{c}{\sigma^2_{\lambda\beta\theta}}\left(1+\frac{1}{\sqrt{g(|B|)}}\right) \tag{4.72}$$

For the second order derivative, we get a term proportional to $|B|$ and another to $|B|^2$. This last one being really dangerous. The one proportional to $|B|$ is treated as previously. For the one proportional to $|B|^2$, we expand up to the fourth order in k all the integrand but the exponential terms. By making explicit computations, similar to the one we did in (3.44), all the terms of order strictly less than 4 in k gives a zero contribution. The result is

$$|\partial_m\partial_\alpha\Psi_{\lambda\beta\theta,\alpha,m}(\rho)| \le \frac{\beta\theta}{\sqrt{2\pi|B|}\sigma_{\lambda\beta\theta}}\frac{c}{\sigma^4_{\lambda\beta\theta}}\left(1+\frac{c}{\sqrt{g(|B|)}}\right) \tag{4.73}$$

for some positive constant c. Recalling (4.70), and using (4.71), (4.72), (4.73) together with (3.32) we get, for some positive constant c

$$\Delta_0^3\mathcal{G}(m^{\delta^*}(x(i)),\rho) \le \left|\alpha^h-\alpha^{\tilde{h}_i}\right| \|m^{\delta^*}_{\iota(x(i))}-m^{\delta^*}_{\beta,\iota(x(i))}\|_1\frac{c\beta\theta}{\sigma^4_{\lambda\beta\theta}}\left(1+\frac{c}{\sqrt{g(|B|)}}\right) \tag{4.74}$$

Using now the fact that $|\alpha^h - \alpha^{\tilde{h}_i}| \le |B|^{-1}$, and that $\sigma^2_{\lambda\beta\theta} \ge cg(|B|)|B|^{-1}$, we have

$$\frac{1}{|B|\sigma^4_{\lambda\beta\theta}} \le \frac{|B|}{g^2(|B|)} \tag{4.75}$$

therefore

$$\Delta_0^3 \mathcal{G}(m^{\delta^*}(x(i)), \rho) \le \|m^{\delta^*}_{\iota(x(i))} - m^{\delta^*}_{\beta,\iota(x(i))}\|_1 \frac{c\beta\theta|B|}{g^2(|B|)}\left(1 + \frac{c}{\sqrt{g(|B|)}}\right) \tag{4.76}$$

and this ends the proof of the Lemma (4.5). ■

With the Proposition 4.4, we get easily an estimate like (4.52) with

$$f_1(\zeta) \le \|h - \tilde{h}_i\| \left[2\beta\theta g_1(\zeta)\left(1 + 16\beta\theta + \frac{|B|}{g^2(|B|)}\right) + \frac{c}{g(|B|)\sqrt{\log g(|B|)}}\right] \tag{4.77}$$

and recalling (4.66)

$$f_2(\zeta) \le \|h - \tilde{h}_i\| \left[8\beta\theta\left(1 + 17\beta\theta + \frac{200}{g(|B|)} + c\frac{g^2(|B|)}{|B|}\right) + \frac{c}{g(|B|)\sqrt{\log g(|B|)}}\right] \tag{4.78}$$

for some positive constant c.

The presence of both terms $|B|/g^2(|B|)$ and $g^2(|B|)/|B|$ suggests to take $g(|B|) = \sqrt{\frac{|B|}{g_2(1/\zeta)}}$ for some function $g_2(x)$ that diverges with x but is slowly varying at infinity. Assuming that ζ is such that $1/\sqrt{|B|} \le g_1(\zeta)\sqrt{g_2(1/\zeta)}$ and choosing $g_1(\zeta) = \sqrt{\frac{\zeta}{2\beta\theta g_2(1/\zeta)}}$, recalling (4.55), we get, if g_2 satisfies also $\lim_{\zeta\downarrow 0}\sqrt{\zeta}g_2(1/\zeta) = 0$.

$$\left\|\partial_i \log \frac{Z_{+,0}(I_{12})}{Z_{-,0}(I_{12})}\right\|_\infty \le 8\sqrt{2\beta\theta\zeta g_2(1/\zeta)} \tag{4.79}$$

Then we apply the Lemma 4.2 and we end the proof of the Proposition 4.1. ■

5 Some deviations estimates and proof of Theorem 2.4 and 2.5

In the previous chapter, we have used the fact that the difference between the stochastic contributions computed on the profiles constantly equal to one minimum and the one computed on the other minimum, has mean zero. In this chapter, we consider profiles that are non constant and make arbitrary oscillations so that in general we loose the mean zero property. Roughly speaking, there are basically three kinds of possible oscillations that we expect to be unlikely. The first one is when the system stays out of the equilibria for a too long interval. The second one is when the system jumps from one equilibrium to the other one, stays there for a too short interval and comes back to the first equilibrium. The third one is when the system makes too many oscillations around one equilibrium without reaching the other one. We have to be careful since without "too long", "too short" and "too many" the previous oscillations could be typical for the Gibbs measure.

To prove the Theorem 2.4, we first consider the case where such oscillations occur on macroscopic intervals Δ that are not bigger than $\sqrt{\log\log 1/\gamma}$. In this case, our estimates will be true on a subset, say $\widehat{\Omega} \subset \Omega$ of $I\!\!P$-probability one, uniformly with respect to all the possible positions of such intervals Δ inside a bigger interval $\mathcal{J}$ centered at the origin, of macroscopic length γ^{-k}, for any given k. A priori we have to consider only the case $|\mathcal{J}| \approx \gamma^{-1}(\log 1/\gamma)^p$, $p > 1$, however, when it is possible, we consider $|\mathcal{J}| = \gamma^{-2}$, that is in microscopic unit γ^{-3}. But while for the first and third type of oscillations it will be enough to estimate them in an interval not bigger than $\sqrt{\log\log 1/\gamma}$, since being outside of equilibria or fluctuating around one for "too long" is very unlikely and it can be detected already in the scale $\sqrt{\log\log 1/\gamma}$, for the second type we must be more careful. Namely we have to distinguish when being close to one equilibrium is typical and when it is not. This requires to analyze the system over longer intervals and to control the contribution of the magnetic field and the entropy terms over intervals where the estimates used in the scale $\sqrt{\log\log 1/\gamma}$ will give a too large contribution.

Let Δ_R be a macroscopic interval of length $R \in I\!\!N$ and δ_1, ζ_1 be two positive real numbers. Let $\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R) \equiv \left\{\eta^{\delta_1,\zeta_1}(\ell) = 0, \quad \forall \ell \in \Delta_R \cap Z\!\!\!Z\right\}$, then our first result is

Proposition 5.1 *There exists an absolute positive constant c such that given $\beta > 1$ and $\beta\theta$ that satisfies (2.21), there exists a positive constant $c(\beta,\theta)$, such that for all $\delta_1 > \delta^* > 0$, $\zeta_1 > 0$ and $z_1 > 0$, we can find $\Omega_1 = \Omega_1(\gamma, \delta^*, \delta_1, \zeta_1, z_1, \Delta_R) \subset \Omega$ such that on Ω_1*

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R)\right) \le e^{-\frac{\beta}{\gamma}\left[c(\beta,\theta)\zeta_1^3\delta_1 R - 4 - 2cR(\delta^* + \frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}) - 2R\theta\sqrt{\frac{\gamma}{\delta^*}} - \sqrt{R\gamma}4\theta z_1\right]} \tag{5.1}$$

and $I\!\!P[\Omega_1] \ge 1 - e^{-\frac{z_1^2}{64}}$.

Proof: By the very same argument that leads to (4.2) we have

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R)\right) \le e^{4\frac{\beta}{\gamma}}\mu_{\beta,\theta,\gamma}\left(\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R) \mid \Sigma_{\partial\Delta_R}\right)(0) \tag{5.2}$$

Performing a block spin transformation on the scale δ^*, recalling (3.13), we have

$$\begin{aligned}&\mu_{\beta,\theta,\gamma}\left(\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R)\mid\Sigma_{\partial\Delta_R}\right)(0)\\&=\frac{e^{(\pm\beta\delta^*\gamma^{-1}R)}}{Z_{\beta,\theta,\gamma,\Delta_R}(0)}\sum_{m^{\delta^*}(\Delta_R)\in\mathcal{M}_{\delta^*}(\Delta_R)}\mathbb{1}_{\{\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R)\}}e^{-\frac{1}{\gamma}\left\{\beta\widehat{\mathcal{F}}(m^{\delta^*}_{\Delta_R},0)+\gamma\mathcal{G}(m^{\delta^*}_{\Delta_R})\right\}}\end{aligned}\tag{5.3}$$

To estimate the stochastic part, we make a rough upper bound, see (3.15) and (3.16) $\left|\mathcal{G}_{x,m^{\delta^*}}(x)(\lambda(x))\right|\le 2\beta\theta|D(x)|$ which corresponds to the situation where all the spins in $D^\lambda(x)$ are equal to $-\lambda(x)$. This gives us a factor

$$\Xi(2\beta\theta,\Delta_R)\equiv\exp\{\sum_{x\in\mathcal{C}_{\delta^*}(\Delta_R)}2\beta\theta|D(x)|\}\tag{5.4}$$

that we extract from the numerator in the left hand side of (5.3). To estimate the combinatorial factor that appears in $\widehat{\mathcal{F}}$, see (3.10), we use the Stirling formula in the form given by Robbins [26] which is, $\forall N\ge 1$, $N!=\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}e^{\epsilon_N}$ with $1/12N\le\epsilon_N\le 1/(12N+1)$. Let us denote

$$\begin{aligned}\widetilde{\mathcal{F}}(m^{\delta^*}_{\Delta_R})=&\frac{\delta^*}{2}\sum_{(x,y)\in\mathcal{C}^2_{1/\delta^*}(\Delta_R)}J_{\delta^*}(x-y)\left[\tilde m^{\delta^*}(x)-\tilde m^{\delta^*}(y)\right]^2\\&+\delta^*\sum_{x\in\mathcal{C}_{\delta^*}(\Delta_R)}f_{\beta,\theta}(m^{\delta^*}(x))\end{aligned}\tag{5.5}$$

where $f_{\beta,\theta}$ is the canonical free energy of the RFCW model, see (2.17). It is easy to see that restricting the configurations to those that are constantly equal to $m^{\delta^*}_\beta$, where $m^{\delta^*}_\beta$ is the nearest point to m_β belonging to the set $\left[-1,-1+\frac{4\gamma}{\delta^*},-1+\frac{8\gamma}{\delta^*},\dots,1-\frac{4\gamma}{\delta^*},1\right]^2$, we get a lower bound for the normalization factor $Z_{\beta,\theta,\gamma,\Delta_R}(0)$. On the other hand using the fact that

$$\sum_{(m^{\delta^*}(\pm1,x))_{x\in\mathcal{C}_{\delta^*}(\Delta_R)}}1\le\left(\frac{\delta^*}{2\gamma}\right)^{\frac{2R}{\delta^*}}=e^{\frac{2R}{\delta^*}\log\frac{\delta^*}{2\gamma}}\tag{5.6}$$

to control the number of terms that occurs in the sum in (5.3), after the cancellation of some constants we get

$$\begin{aligned}\mu_{\beta,\theta,\gamma}\left(\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R)\mid\Sigma_{\partial\Delta_R}\right)(0)\le&e^{\frac{\beta}{\gamma}(R\delta^*+4+2R\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma})}\Xi(4\beta\theta,\Delta_R)\\&e^{-\frac{\beta}{\gamma}\inf_{m^{\delta^*}_{\Delta_R}\in\mathcal{O}_0^{\delta_1,\zeta_1}}\left\{\mathcal{F}(m^{\delta^*}_{\Delta_R})\right\}}\end{aligned}\tag{5.7}$$

where $\mathcal{F}(m^{\delta^*}_{\Delta_R}) \equiv \widetilde{\mathcal{F}}(m^{\delta^*}_{\Delta_R}) - \widetilde{\mathcal{F}}(m^{\delta^*}_{\beta,\Delta_R})$.

To give a lower bound on the previous infimum, we use the fact that if x_i are positive numbers, bounded from above by a constant c then if the arithmetic mean of N terms x_i is bounded from below by some $\zeta_1 \le c$ then there are at least $\frac{N\zeta_1}{2c-\zeta_1}$ terms x_i among the N, such that $x_i > \frac{\zeta_1}{2}$. Using (2.25) we get after some easy computations,

$$\inf_{m^{\delta^*}_{\Delta_R} \in \mathcal{O}_0^{\delta_1,\zeta_1}} \left\{ \mathcal{F}(m^{\delta^*}_{\Delta_R}) \right\} \ge Rc(\beta,\theta)\zeta_1^3 \delta \frac{1}{4(4-\zeta_1)^2} \ge Rc_1(\beta,\theta)\zeta_1^3\delta \tag{5.8}$$

It remains to estimate $\Xi(4\beta\theta, \Delta_R)$. Let us denote $X(\Delta_R) \equiv 4\gamma \sum_{x\in\mathcal{C}_{\delta^*}(\Delta_R)} |D(x)|$. It is easy to see that $I\!\!E(X(\Delta_R) \le cR\sqrt{\gamma/\delta^*}$. Using the Lemma 4.2, setting $t = 2\sqrt{R\gamma z_1}$, where z_1 is a positive real number, and regrouping, we get (5.1). ■

With the Proposition 5.1 we can control the Gibbs-Probability to have a run of $\eta^{\delta_1,\zeta_1} = 0$ anywhere on intervals that are rather long. However their lengths depend on the parameters $\delta_1, \zeta_1, \delta^*$.

Corollary 5.2 *Given $\beta > 1$ and $\beta\theta$ that satisfies (2.21), then there exists a constant $\tilde{c} = \tilde{c}(\beta,\theta)$ such that, if $\delta^* \log\frac{1}{\gamma} \downarrow 0$ when $\gamma \downarrow 0$, for all $\delta_1 > \delta^* > 0$, $\zeta_1 > 0$, that satisfy*

$$\delta_1\zeta_1^3 \ge \tilde{c}(\beta,\theta) \left(\sqrt{\frac{\gamma}{\delta^*}} \vee \delta^* \right) \tag{5.9}$$

for all $x > 0$, for all intervals Δ_R of macroscopic length R that are included in a macroscopic interval I containing the origin, with $|I| \le \gamma^{-2}$ and satisfy

$$R \ge R_1 \equiv \frac{4\beta(1+x)}{c(\beta,\theta)\delta_1\zeta_1^3} \tag{5.10}$$

if $\gamma = 2^{-n}$, with $I\!\!P$-probability one, for all but a finite number of indices n,

$$\mu_{\beta,\theta,\gamma}\left(\exists_{R:R_1\le|R|\le|I|}\ \exists_{\Delta_R\subset I}\ : \mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R) \right) \le e^{-\frac{4\beta x}{\gamma}} \tag{5.11}$$

Proof:

Let us first remark that for a given R, the number of intervals Δ_R that are included in I, is bounded from above by $|I|^2$, therefore if we take $z_1 = \sqrt{64(5+\epsilon)\log\frac{1}{\gamma}}$ for some positive ϵ, we get easily

$$I\!\!P\left[\sup_{R:R_1\le|R|\le|I|} \sup_{\Delta_R\subset I} \frac{1}{\sqrt{R}} \left(X(\Delta_R) - I\!\!E\left[X(\Delta_R)\right]\right) \ge \sqrt{64(5+\epsilon)\gamma\log(\frac{1}{\gamma})} \right] \le \gamma^{1+\epsilon} \tag{5.12}$$

and the $I\!P$-probabilistic statement follows from the first Borel-Cantelli Lemma. Let us consider the term into the bracket in the exponent in the right hand side of (5.1). Notice first that, since $\frac{\delta^*}{\gamma}\uparrow\infty$ when $\gamma\downarrow 0$, $\sqrt{\frac{\gamma}{\delta^*}}\geq\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}$, if γ small enough we can ignore the corresponding term in (5.1) and keep just the square root. To get a negative term in this exponent, we impose, since $\beta\theta$ is small,

$$c(\beta,\theta)\zeta_1^3\delta_1-4(\delta^*\vee\sqrt{\frac{\gamma}{\delta^*}})-256(5+\epsilon)\beta\theta\sqrt{\gamma\log\frac{1}{\gamma}}\geq 0 \tag{5.13}$$

Using $\delta^*\log\frac{1}{\gamma}\downarrow 0$ when $\gamma\downarrow 0$, this becomes $c(\beta,\theta)\zeta_1^3\delta_1-4c(\delta^*\vee\sqrt{\frac{\gamma}{\delta^*}})\geq 0$ by enlarging the constant c if necessary. To cancel the constant term 4β, in (5.1) and get the factor x in (5.11) we just impose (5.10). ■

The second family of events we consider are roughly speaking those one having two blocks, far apart but not too much, at the same equilibrium and somewhere between them there is a block of macroscopic length at least 1, close to the other equilibrium.

Let $\Delta_L=[\ell_1,\ell_2]$ with $\ell_i\in Z\!\!Z$ for $i=1,2$ be a macroscopic interval of length L, and $\delta_2>0$, $\zeta_2>0$ be two real positive numbers, let us define for $\eta=+1$ or $\eta=-1$

$$\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,\eta)\equiv\left\{\eta^{\delta_2,\zeta_2}(\ell_1)=\eta^{\delta_2,\zeta_2}(\ell_2)=\eta,\ \exists\tilde\ell,\ \ell_1<\tilde\ell<\ell_2\quad\eta^{\delta_2,\zeta_2}(\tilde\ell)=-\eta\right\} \tag{5.14}$$

and $\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L)\equiv\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,+)\cup\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,-)$. Our second result is

Proposition 5.3 *Given $\beta>1$ and $\beta\theta$ that satisfies (2.21), $\delta_2>\delta^*>0$, $\zeta_2>0$ and $z_2>0$, then there exists $\Omega_2=\Omega_2(\gamma,\delta^*,\delta_2,\zeta_2,z_2,\Delta_L)\subset\Omega$ such that on Ω_2*

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L)\right)\leq e^{-\gamma^{-1}\left[\Delta\mathcal{F}-4\beta\zeta_2-2L(\delta^*+\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma})-2L\beta\theta\sqrt{\frac{\gamma}{\delta^*}}-\sqrt{L\gamma}4\beta\theta z_2\right]} \tag{5.15}$$

for a strictly positive constant $\Delta\mathcal{F}=\Delta\mathcal{F}(\beta,\theta)$ and $I\!P[\Omega_1]\geq 1-e^{-\frac{z_2^2}{64}}$.

Proof: The proof is similar to the proof of the Proposition (5.1), we point out only the main differences. Let us call $\Delta_L^-=[\ell_1+1,\ell_2-1]$, and for $\eta=\pm1$

$$m_{\beta,\eta,\partial\Delta_l}=\left\{m^{\delta^*}(x);\ \forall x\in\mathcal{C}_{\delta^*}(\partial\Delta_L)\quad m^{\delta^*}(x)=T^{\frac{1-\eta}{2}}m_\beta^{\delta^*}\right\} \tag{5.16}$$

where if $m=(m_1,m_2)$, $T^0m=m$ and $T^1m=Tm=(-m_2,-m_1)$. An easy computation, using the fact that $\eta^{\delta_2,\zeta_2}=\eta$, leads to

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,\eta)\right)\leq e^{4\beta c\zeta_2}\mu_{\beta,\theta,\gamma}\left(\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,\eta)\;\middle|\;\Sigma_{\partial\Delta_L}\right)(m_{\beta,\eta,\partial\Delta_L}) \tag{5.17}$$

Then making a block spin transformation on the scale δ^* inside the volume Δ_L^-, denoting

$$\mathcal{F}(m^{\delta^*}_{\Delta_L^-}, m^{\delta^*}_{\partial\Delta_L}) = \mathcal{F}(m^{\delta^*}_{\Delta_L^-}) + \frac{\delta^*}{2} \sum_{\substack{x \in \mathcal{C}_{\delta^*}(\Delta_L^-) \\ y \in \mathcal{C}_{\delta^*}(\partial\Delta_L)}} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y) \tag{5.18}$$

and using the same arguments that leads to (5.7) give

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,\eta) \;\middle|\; \Sigma_{\partial\Delta_L}\right)(m_{\beta,\eta,\partial\Delta_L}) \le e^{\frac{\beta}{\gamma}(L\delta^*+4\zeta_2+2L\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma})}\Xi(4\beta\theta,\Delta_L)$$
$$\cdot\, e^{-\frac{\beta}{\gamma}\inf_{m^{\delta^*}_{\Delta_L}\in\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,\eta)}\left\{\mathcal{F}(m^{\delta^*}_{\Delta_L},m_{\beta,\eta,\partial\Delta_L})\right\}} \tag{5.19}$$

It is not too difficult to check that there exists a constant $\Delta\mathcal{F} = \Delta\mathcal{F}(\beta,\theta)$, not depending on $\eta = \pm 1$ nor L, which is strictly positive if $\beta > 1$ and $\beta\theta$ satisfies (2.21), such that

$$\inf_{m^{\delta^*}_{\Delta_L}\in\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L,\eta)} \left\{\mathcal{F}(m^{\delta^*}_{\Delta_L}, m_{\beta,\eta,\partial\Delta_L})\right\} \ge \Delta\mathcal{F} \tag{5.20}$$

Now $\Xi(4\beta\theta,\Delta_L)$ can be estimated as before and this ends the proof of the Proposition 5.3. ■

By similar computations as in the proof of the Corollary 5.2, making the choice $z_2 = z_1$ it is easy to check that

Corollary 5.4 *There exists a constant $\tilde{c} = \tilde{c}(\beta,\theta)$ such that, if $\delta^*\log\frac{1}{\gamma} \downarrow 0$ when $\gamma \downarrow 0$, for all $\delta_2 > \delta^* > 0$, $\zeta_2 > 0$, for all $x > 0$, that satisfies*

$$\Delta\mathcal{F}(1-x) - \tilde{c}(\beta,\theta)\zeta_2 > 0 \tag{5.21}$$

for all intervals Δ_L of macroscopic length L that are included in an interval I that contains the origin, with $|I| \le \gamma^{-2}$ and satisfy

$$L \le L_2 \equiv \frac{\Delta\mathcal{F}(1-x) - \tilde{c}(\beta,\theta)\zeta_2}{c(\beta,\theta)(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}})} \tag{5.22}$$

if $\gamma = 2^{-n}$, with $I\!P$-probability one, for all but a finite number of indices n,

$$\mu_{\beta,\theta,\gamma}\left(\exists_{L:2\le|L|\le L_2}\,\exists_{\Delta_L\subset I}\ :\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L)\right) \le e^{-\frac{\beta x\Delta\mathcal{F}}{\gamma}} \tag{5.23}$$

The third family of events describes fluctuations around one equilibrium.

Let $\Delta_L=[\ell_1,\ell_2]$ with $\ell_i\in Z\!\!\!Z$ for $i=1,2$ be a macroscopic interval of length L, and $\delta_4>\delta_1>0$, $\zeta_4>\zeta_1>0$ be four real positive numbers, let us define for $\eta=+1$ or $\eta=-1$

$$
\begin{aligned}
&\mathcal{R}_{0,\eta}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L)\equiv\\
&\big\{\eta^{\delta_1,\zeta_1}(\ell_1)=\eta^{\delta_1,\zeta_1}(\ell_2)=\eta,\ \forall\ell\in(\ell_1,\ell_2),\ \eta^{\delta_1,\zeta_1}=0,\exists\tilde\ell_1,\tilde\ell_2,\ \tilde\ell_2-\tilde\ell_1=\tilde L\\
&\quad \ell_1<\tilde\ell_1<\tilde\ell_2\le\ell_2\ \ \eta^{\delta_4,\zeta_4}(\tilde\ell)=0\ \forall\tilde\ell:\ \tilde\ell_1\le\tilde\ell\le\tilde\ell_2\big\}
\end{aligned}
\tag{5.24}
$$

and $\mathcal{R}_{0}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L)\equiv\mathcal{R}_{0,+}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L)\cup\mathcal{R}_{0,-}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L)$.

Proposition 5.5 *Given $\beta>1$ and $\beta\theta$ that satisfies (2.21), $\delta_4>\delta_1>\delta^*$, $\zeta_4>\zeta_1>0$ and $z_3>0$ then there exists $\Omega_3=\Omega_3(\gamma,\delta^*,\delta_1,\delta_4,\zeta_1,\zeta_4,z_3,\Delta_L,\tilde L)$ such that on Ω_3*

$$
\begin{aligned}
&\mu_{\beta,\theta,\gamma}\left(\mathcal{R}_{0,\eta}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L)\right)\\
&\le e^{-\gamma^{-1}\left[c(\beta,\theta)(\zeta_4^3\delta_4\tilde L+\zeta_1^3\delta_1(L-\tilde L))-4\beta\zeta_1-2L(\delta^*+\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma})-2L\beta\theta\sqrt{\frac{\gamma}{\delta^*}}-\sqrt{L\gamma}4\beta\theta z_3\right]}
\end{aligned}
\tag{5.25}
$$

for some positive constants $c(\beta,\theta)$ and c and $P\left[\Omega_3\right]\ge1-e^{-\frac{z_3^2}{64}}$.
Proof: The proof is similar to the proofs of the Propositions 5.1 and 5.3. ■

An immediate consequence of this result is the
Corollary 5.6 *Given $\beta>1$ and $\beta\theta$ that satisfies (2.21),there exists two constants $\tilde c_i=\tilde c_i(\beta,\theta)$ for $i=1,2$ such that if $\delta^*\log\frac{1}{\gamma}\downarrow0$ when $\gamma\downarrow0$, for all $\delta_4>\delta_1>\delta^*$, $\zeta_4>\zeta_1>0$ that satisfy*

$$
\delta_4\zeta_4^3\ge\delta_1\zeta_1^3\ge\tilde c_1\left(\sqrt{\frac{\gamma}{\delta^*}}\vee\gamma\log\frac{1}{\gamma}\right)
\tag{5.26}
$$

and $\delta_4\zeta_4^3\ge\tilde c_1\zeta_1$, for all $1>x>0$, for all intervals Δ_L of macroscopic length L that are included in an interval I that contains the origin, with $|I|\le\gamma^{-2}$ if $\gamma=2^{-n}$, with $I\!\!P$-probability one, for all but a finite number of indices n, for all $\tilde L\ge1$,

$$
\mu_{\beta,\theta,\gamma}\left(\exists_{L:2\le L\le|I|}\exists_{\Delta_L\subset I}:\ \mathcal{R}_{0}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L)\right)\le e^{-\frac{\tilde c_2(\beta,\theta)\tilde Lx\zeta_4^3\delta_4}{\gamma}}
\tag{5.27}
$$

Therefore if we denote

$$
\mathcal{O}_0^{\delta_1,\zeta_1}(I)\equiv\bigcup_{R:\ R_1\le R\le|I|}\ \bigcup_{\Delta_R\subset I}\mathcal{O}_0^{\delta_1,\zeta_1}(\Delta_R)
\tag{5.28}
$$

$$
\mathcal{W}^{\delta_2,\zeta_2}(I)\equiv\bigcup_{L:\ 2\le L\le L_2}\ \bigcup_{\Delta_L\subset I}\mathcal{W}^{\delta_2,\zeta_2}(\Delta_L)
\tag{5.29}
$$

and

$$\mathcal{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}(I) \equiv \bigcup_{L:\, 2\le L\le |I|} \bigcup_{\Delta_L\subset I} \bigcup_{\tilde L:\, 1\le \tilde L\le L} \mathcal{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\Delta_L,\tilde L) \tag{5.30}$$

then, for an appropriate choice of various parameters, δ_i,ζ_i for $i:\ 1\le i\le 4$, as a consequence of the Corollaries 5.2, 5.4 and 5.6, all the previous sets have a Gibbs-Probability that goes to zero, $I\!P$-almost surely. It is convenient to make the choices $\delta_1=\delta_2$, $\zeta_2=\zeta_1$, $\zeta_4>\zeta_1$ and $\delta_4\ge\delta_1$. We note that $\eta^{\delta_1,\zeta_1}(\ell)=\eta$ implies $\eta^{\delta_4,\zeta_4}(\ell)=\eta$. Therefore on the complementary of the unions of the previous sets we can only have runs of length at most R_1 of $\eta^{\delta_1,\zeta_1}=0$ followed by runs of length at least L_2 of equilibrium $\eta^{\delta_4,\zeta_4}(\ell)=\eta$. Namely blocks $\eta^{\delta_1,\zeta_1}(\ell)=0$ between adjacent blocks of the same equilibrium can be only $\eta^{\delta_4,\zeta_4}=\eta$, since (5.27).

The next step is to prove that the length of the previous run of $\eta^{\delta_4,\zeta_4}=\eta$ which is at least L_2 is in fact bounded from below by a much larger quantity. We define, see (5.14), for $\eta\in\{+1,-1\}$, $\ell_1<\tilde\ell_1<\tilde\ell_2<\ell_2$ with $2\le\tilde\ell_1-\ell_1\le R_1$, $\ell_2-\tilde\ell_2\le R_1$,

$$\begin{aligned}&\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2)\equiv\\ &\quad\big\{\eta^{\delta_4,\zeta_4}(\ell_1)=\eta^{\delta_4,\zeta_4}(\ell_2)=\eta,\ \eta^{\delta_4,\zeta_4}(\tilde\ell_1-1)=\eta^{\delta_4,\zeta_4}(\tilde\ell_1)=-\eta,\\ &\eta^{\delta_4,\zeta_4}(\ell)=-\eta\ \forall\ell:\tilde\ell_1+1\le\ell\le\tilde\ell_2-1,\ \eta^{\delta_4,\zeta_4}(\tilde\ell_2)=\eta^{\delta_4,\zeta_4}(\tilde\ell_2+1)=-\eta,\ \eta^{\delta_4,\zeta_4}(\ell_2)=\eta\big\}\end{aligned} \tag{5.31}$$

In the following proposition we will show that uniformly in the choices of $\tilde\ell_1$, $\tilde\ell_2$, ℓ_1 and ℓ_2 in a fixed interval $\mathcal{J}$ of suitable length, this set of events has small probability.

Proposition 5.7 *Given $\beta>1$, $0<x<1$, $p>1$, $\hat c>0$, $\rho>0$, if $\theta\le\frac{x^2\Delta\mathcal{F}}{48\sqrt{\hat c(p+1+2\rho)}}$ then there exist $\gamma_0>0$ and $c_0>0$ such that for $\gamma\le\gamma_0$, if $\zeta_4 g_2(1/\zeta_4)\le\frac{x^2\Delta\mathcal{F}}{96}(1\wedge\frac{\beta}{\sqrt{\hat c(p+1+2\rho)}})$, for all $\delta_4>\delta^*=c_0\gamma\log\log\frac{1}{\gamma}$, for all intervals $I=[\ell_1,\ell_2]$ such that $|I|\le\hat c(\gamma\log\log\frac{1}{\gamma})^{-1}$, and for any $I\subset\mathcal{J}$, $|\mathcal{J}|=\tilde c\gamma^{-1}(\log 1/\gamma)^p$ for some positive constant $\tilde c$, on a set $\Omega_4=\Omega_4(\mathcal{J},\beta,\theta,\gamma)$ that satisfies*

$$I\!P[\Omega_4]\ge 1-\frac{2\tilde c}{\hat c}(\log\frac{1}{\gamma})^{p+\rho}e^{-(\log\log\frac{1}{\gamma})(p+2\rho+1)} \tag{5.32}$$

we have, uniformly on all intervals $[\tilde\ell_1,\tilde\ell_2]\subset I$ and uniformly on $I\subset\mathcal{J}$,

$$\mu_{\beta,\theta,\gamma}\left(\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2)\right)\le\exp\left[-\frac{\beta}{\gamma}x(1-x)\Delta\mathcal{F}\right] \tag{5.33}$$

for $\eta=\pm1$

Proof:

The first step is to restrict ourself to a finite volume Gibbs measure. Since $\eta^{\delta_4,\zeta_4}(\ell_1) = \eta^{\delta_4,\zeta_4}(\ell_2) = \eta$, we get

$$\mu_{\beta,\theta,\gamma}\left(\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2)\right) \le e^{4\beta\frac{\zeta_4}{\gamma}}\mu_{\beta,\theta,\gamma}\left(\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2) \mid \Sigma_{\partial\Delta_L}\right)(0) \tag{5.34}$$

Using the fact that $\eta^{\delta_4,\zeta_4}(\tilde\ell_1) = \eta^{\delta_4,\zeta_4}(\tilde\ell_1 - 1)$ and $\eta^{\delta_4,\zeta_4}(\tilde\ell_2+1) = \eta^{\delta_4,\zeta_4}(\tilde\ell_2)$ we can also decouple the interval $[\tilde\ell_1 - 1, \tilde\ell_2 + 1]$ from the interval $[\ell_1,\ell_2]$. This will produce three adjacent intervals. We associate, the interaction between the first and the second interval to the first one, and the interaction between the second and the third interval to the third one. This will give, up to a factor $e^{8\beta\frac{\zeta_4}{\gamma}}$, a product of three terms each one being localized on one of the three intervals. We make a rough estimate for the terms corresponding to the first and the third interval. Applying an argument similar to the one given in the Corollary 5.4, we get that, with a $I\!P$-probability 1, uniformly with respect to all intervals $[\tilde\ell_1,\tilde\ell_2]$ included in an interval $\mathcal{J}$ containing the origin, with $|\mathcal{J}| \le \frac{1}{\gamma^2}$,

$$\begin{aligned}\mu_{\beta,\theta,\gamma}\left(\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2) \mid \Sigma_{\partial\Delta_L}\right)(0) &\le e^{12\beta\frac{\zeta_4}{\gamma}}e^{-\frac{\beta x\Delta\mathcal{F}}{\gamma}} \\ &\times \mu_{\beta,\theta,\gamma,\tilde\ell_1,\tilde\ell_2}\left(\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\tilde\ell_1,\tilde\ell_2)\right)(0,0)\end{aligned} \tag{5.35}$$

where $\mu_{\beta,\theta,\gamma,\tilde\ell_1,\tilde\ell_2}(.)(0,0)$ is the Gibbs measure in the macroscopic volume $[\tilde\ell_1,\tilde\ell_2]$ with empty boundary conditions, see (2.4), with $\tilde\sigma = 0$.

It remains to estimate the last term. Recalling (4.3), (4.4) and (4.5) it is easy to see that if we denote $\tilde I_{12} \equiv [\tilde\ell_1,\tilde\ell_2]$ we have

$$\mu_{\beta,\theta,\gamma,\tilde\ell_1,\tilde\ell_2}\left(\tilde{\mathcal{W}}_\eta^{\delta_4,\zeta_4}(\tilde\ell_1,\tilde\ell_2)\right)(0,0) \le e^{\beta\gamma^{-1}[\delta^*(\tilde\ell_2-\tilde\ell_1)]}\frac{Z_{-\eta,\delta_4,\zeta_4}(\tilde I_{12})}{Z_{\eta,\delta_4,\zeta_4}(\tilde I_{12})} \tag{5.36}$$

where the last term is similar to the one defined in (4.5), with $\mathcal{R}(\tau) = \mathcal{R}^{\delta_4,\zeta_4}(\tilde\ell_1,\tilde\ell_2,\tau)$. Writing in a similar way as we did in (4.10), with self explaining notations, we have

$$\frac{Z_{-\eta,\delta_4,\zeta_4}(\tilde I_{12})}{Z_{\eta,\delta_4,\zeta_4}(\tilde I_{12})} = e^{\Delta\mathcal{G}(m^{\delta^*}_{\beta,\tilde I_{12}},\epsilon)}\frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde I_{12})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde I_{12})} \tag{5.37}$$

Using the estimate (4.27) we get

$$I\!P\left[\log\frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde I_{12})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde I_{12})} \ge \frac{\epsilon}{\gamma}\right] \le \exp\left(-\frac{\epsilon^2}{212\gamma|\tilde I_{12}|\beta\theta\zeta_4 g_2(1/\zeta_4)}\right) \tag{5.38}$$

To get a result which is true *uniformly* with respect to all subintervals $\tilde{I}_{12}$ of I, and for any I in a given interval $\mathcal{J}$ of length $\tilde{c}(\gamma)^{-1}(\log 1/\gamma)^p$ containing the origin, we need a modification of the Ottaviani inequality [29] that takes into account that we do not have sum of random variables *i.e.* an additive process but merely an approximate additive process.

To simplify notations, given an interval $\tilde{I} \subset I$, let us call $Y(\tilde{I}) \equiv \log \frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde{I})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde{I})}$.

Lemma 5.8 *For any given interval* I

$$I\!P\left[\max_{\tilde{I}_{1,2}\subset I}\left|Y(\tilde{I}_{12})\right| \geq \beta\frac{4\epsilon+12\zeta_4}{\gamma}\right] \leq \frac{I\!P\left[|Y(I)| \geq \beta\frac{\epsilon}{\gamma}\right]}{\inf_{\tilde{I}_{12}\subset I} I\!P\left[\left|Y(\tilde{I}_{12})\right| \leq \beta\frac{\epsilon}{\gamma}\right]} \tag{5.39}$$

Proof: Recall that $[\ell_1,\ell_2] \equiv I$ and intervals $\tilde{I}_{12} = [\tilde{\ell}_1,\tilde{\ell}_2]$. Using the fact that for all $\tilde{I}_{12} \subset I$, $|Y(\tilde{I}_{12})| \leq |Y([\ell_1,\tilde{\ell}_1])| + |Y([\ell_1,\tilde{\ell}_2])| + \beta\frac{4\zeta_4}{\gamma}$, we get $|Y(\tilde{I}_{12})| \leq 2\max_{\ell_1\leq\tilde{\ell}\leq\ell_2}|Y([\ell_1,\tilde{\ell}])| + \beta\frac{4\zeta_4}{\gamma}$. Therefore

$$I\!P\left[\max_{\tilde{I}_{1,2}\subset I}\left|Y(\tilde{I}_{12})\right| \geq \beta\frac{4\epsilon+12\zeta_4}{\gamma}\right] \leq I\!P\left[\max_{\ell_1\leq\tilde{\ell}\leq\ell_2}\left|Y([\ell_1,\tilde{\ell}])\right| \geq \beta\frac{2\epsilon+4\zeta_4}{\gamma}\right] \tag{5.40}$$

Let $\tau = \inf\left\{t \geq \ell_1\ ;\ |Y([\ell_1,t])| \geq \beta\frac{2\epsilon+4\zeta_4}{\gamma}\right\}$, $\inf(\emptyset) = \infty$. Since, for all $k \in [\ell_1,\ell_2]$, $|Y(I)| \geq |Y([\ell_1,k])| - |Y([k+1,\ell_2])| - \beta\frac{4\zeta_4}{\gamma}$, we have

$$\left\{\tau = k\right\} \cap \left\{|Y([k+1,\ell_2])| \leq \beta\frac{\epsilon}{\gamma}\right\} \subset \left\{|Y(I)| \geq \beta\frac{\epsilon}{\gamma}\right\} \tag{5.41}$$

Therefore, making a partition over the possible values of τ and using independence, we get

$$I\!P\left[|Y(I)| \geq \beta\frac{\epsilon}{\gamma}\right] \geq \inf_{\ell_1\leq k\leq\ell_2} I\!P\left[|Y([k+1,\ell_2])| \leq \beta\frac{\epsilon}{\gamma}\right]\sum_{k=\ell_1}^{\ell_2} I\!P\left[\tau = k\right] \tag{5.42}$$

Using the definition of τ, we get (5.39). ■

We assume without loss of generality, that $\mathcal{J}$ is centered at the origin and that $|I| = \hat{c}\frac{1}{\gamma\log\log\frac{1}{\gamma}}$, for a given $\hat{c}$. We make a block decomposition of the interval $\mathcal{J}$ into blocks of length $\hat{c}(2\gamma\log\log\frac{1}{\gamma})^{-1}$, that is $\mathcal{J} = \cup_{-j_1\leq j\leq j_1}\widehat{I}_j$ with $2j_1+1 = \left[\frac{2\tilde{c}}{\hat{c}}(\log 1/\gamma)^p\log\log\frac{1}{\gamma}\right]$. Note that any interval I we consider is included in the union of three consecutive intervals $\widehat{I}_{[j,j+2]} \equiv \widehat{I}_j \cup \widehat{I}_{j+1} \cup \widehat{I}_{j+2}$ for some $-j_1 \leq j \leq j_1 - 2$. Therefore we get, denoting $\max^*_{I\subset\mathcal{J}}$

the maximum over the intervals I such that $|I| = \hat{c}(\gamma \log\log \frac{1}{\gamma})^{-1}$ that are in $\mathcal{J}$, for all $\epsilon > 0$, setting $\tilde{\epsilon} = 4\epsilon + 12\zeta_4$, we have

$$I\!P\left[\max_{I\subset\mathcal{J}}{}^{*} \max_{\tilde{I}_{12}\subset I} |Y(\tilde{I}_{12})| \geq \beta\frac{\tilde{\epsilon}}{\gamma}\right] \leq \frac{2\tilde{c}(\log 1/\gamma)^{p}\log\log\frac{1}{\gamma}}{\hat{c}} I\!P\left[\max_{\tilde{I}_{12}\subset \widehat{I}_{[0,2]}} |Y(\tilde{I}_{12})| \geq \beta\frac{\tilde{\epsilon}}{\gamma}\right] \tag{5.43}$$

Using (5.38) and (5.39) we have

$$I\!P\left[\max_{I\subset\mathcal{J}}{}^{*} \max_{\tilde{I}_{12}\subset I} \left|\log\frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde{I}_{12})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde{I}_{12})}\right| \geq \beta\frac{\tilde{\epsilon}}{\gamma}\right] \leq \frac{2\tilde{c}(\log\frac{1}{\gamma})^{p+\rho}}{\hat{c}} \frac{e^{-u\log\log\frac{1}{\gamma}}}{1-e^{-u\log\log\frac{1}{\gamma}}} \tag{5.44}$$

where $u \equiv \frac{\tilde{\epsilon}^2\beta^2}{212\hat{c}\beta\theta\zeta_4 g_2(1/\zeta_4)}$ and $\rho > 0$ is small as we want. We assume for the moment that the various parameters are chosen such that $u \geq p + 1 + 2\rho$. Using the first Borel-Cantelli lemma, recalling that $\gamma = 2^{-n}$, we get that with a $I\!P$-probability 1, for all but a finite number of indices n

$$\max_{I\subset\mathcal{J}}{}^{*} \max_{\tilde{I}_{12}\subset I} \frac{Z_{-\eta,0,\delta_4,\zeta_4}(\tilde{I}_{12})}{Z_{\eta,0,\delta_4,\zeta_4}(\tilde{I}_{12})} \leq e^{\beta\frac{4\epsilon+12\zeta_4}{\gamma}} \tag{5.45}$$

It remains to estimate the first term in the right hand side of (5.37). We have $\Delta\mathcal{G}(m^{\delta^*}_{\beta,\tilde{I}_{12}}) = -\eta\sum_{x\in\mathcal{C}_{\delta^*}(\tilde{I}_{12})} X(x)$ where

$$X(x) \equiv -2\beta\theta\lambda(x)|D(x)|\left[m^{\delta^*}_{\beta,1} + m^{\delta^*}_{\beta,2} + \Xi(x,\beta\theta,\alpha)\right] - \lambda(x)\log\frac{\Psi_{\beta\theta,\alpha(x),m^{\delta^*}_{\beta,2}}\Psi_{0,0,m^{\delta^*}_{\beta,1}}}{\Psi_{\beta\theta,\alpha(x),m^{\delta^*}_{\beta,1}}\Psi_{0,0,m^{\delta^*}_{\beta,2}}} \tag{5.46}$$

with $\Xi(x,\beta\theta,\alpha) \equiv [\widehat{\varphi}(m^{\delta^*}_{\beta,1},\lambda(x)\beta\theta,\alpha) - \widehat{\varphi}(m^{\delta^*}_{\beta,2},\lambda(x)\beta\theta,\alpha)]$. Therefore we need to estimate from above the probability of

$$\mathcal{A} \equiv \left\{\max_{I\subset\mathcal{J}}{}^{*} \max_{\tilde{I}_{12}\subset I} \left|\gamma\sum_{x\in\mathcal{C}_{\delta^*}(\tilde{I}_{12})} X(x)\right| \geq s\right\} \tag{5.47}$$

for $s > 0$. For our purpose it is enough to prove (5.47) for $s \leq s_0$, for a given s_0. This will be done in two steps that are similar to the proof of (5.44). First we give an estimate for a fixed $\tilde{I}_{12}$ and then we make a block decomposition of $\mathcal{J}$ into blocks of length $\hat{c}(2\gamma\log\log\frac{1}{\gamma})^{-1}$. Arguing as before we apply the usual Ottaviani Inequality. All of this is standard and it is just an adaptation of the proof of the upper bound in the Law of the Iterated Logarithm given by De Acosta [1]. It follows from the exponential Markov inequality and independence that, for all $\lambda \geq 0$,

$$I\!P\left[\gamma\sum_{x\in\mathcal{C}_{\delta^*}(\tilde{I}_{12})} X(x) \geq s\right] \leq e^{-st} \prod_{x\in\mathcal{C}_{\delta^*}(\tilde{I}_{12})} I\!E\left[e^{t\gamma X(x)}\right] \tag{5.48}$$

To estimate the previous Laplace transform, we use $e^x \le 1 + x + \frac{x^2}{2} e^{|x|}$, $\forall x \in I\!R$. Using the fact that $I\!E(X) = 0$, we get

$$I\!E\left[e^{t\gamma X(x)}\right] \le 1 + (t\gamma)^2 \frac{I\!E[X^2(x)]}{2} e^{t\gamma \|X(x)\|_\infty} \tag{5.49}$$

Using the Proposition 3.4, if γ is small enough, and how small depends on $\beta\theta$ to absorb the last term in (3.54), we have for some positive constant c, $\|X(x)\|_\infty \le 4\beta\theta \frac{\delta^*}{\gamma}(1 + c\beta\theta)$. On the other hand it is easy to check that, calling $I\!E[|D(x)|^2] = V^2(x) = \frac{\delta^*}{\gamma}$, we have also for some positive constant c, if γ is small enough $I\!E[X^2(x)] \le 16(\beta\theta)^2(1 + c\beta\theta)^2 \frac{\delta^*}{\gamma}$. Using $1 + x \le e^x$ $\forall x \in I\!R$ and $|\mathcal{C}_{\delta^*}(\tilde{I}_{12})| = \frac{|\tilde{I}_{12}|}{\delta^*}$, we get easily

$$\prod_{x \in \mathcal{C}_{\delta^*}(\tilde{I}_{12})} I\!E\left[e^{tX(x)}\right] \le \exp\left[\gamma 8 (t\beta\theta)^2 (1 + c\beta\theta) |\tilde{I}_{12}| e^{t\delta^* 4\beta\theta(1+c\beta\theta)}\right] \tag{5.50}$$

The choice of t depends on $|\tilde{I}_{12}|$. If $\gamma|\tilde{I}_{12}| \ge \frac{\delta^*}{g_3(\gamma)}$ with $\lim_{\gamma\downarrow 0} g_3(\gamma) = 0$ as slowly as we want, we choose $t = \frac{s}{16\gamma|\tilde{I}_{12}|(\beta\theta)^2(1+c\beta\theta)}$. If $\gamma|\tilde{I}_{12}| \le \frac{\delta^*}{g_3(\gamma)}$, we choose $t = \frac{s \log\log\frac{1}{\gamma}}{32\hat{c}(\beta\theta)^2(1+c\beta\theta)} s$. Assuming that $g_3(\gamma)$ is such that $\gamma(\log\log\frac{1}{\gamma})^2 \le (g_3(\gamma))^2$, in both the cases, we get

$$I\!P\left[\left|\gamma \sum_{x \in \mathcal{C}_{\delta^*}(\tilde{I}_{12})} X(x)\right| \ge \beta s\right] \le 2\exp\left[-\frac{s^2 \log\log\frac{1}{\gamma}(1 - 2s_0 c g_3(\gamma))}{32\hat{c}(\theta)^2(1 + c\beta\theta)}\right] \tag{5.51}$$

for $s \le s_0$ and for some constant c [1]. To get uniformity with respect to all subintervals that are in I, we write simply

$$\max_{\tilde{I}_{12} \subset I} \left|\gamma \sum_{x \in \mathcal{C}_{\delta^*}(\tilde{I}_{12})} X(x)\right| \le 2 \max_{\ell_1 \le \ell \le \ell_2} \left|\gamma \sum_{x=\ell_1}^{\ell} X(x)\right| \tag{5.52}$$

Therefore, using the Ottaviani inequality

$$I\!P\left[\max_{\ell_1 \le \ell \le \ell_2} \left|\gamma \sum_{x=\ell_1}^{\ell} X(x)\right| \ge 2\beta s\right] \le \frac{I\!P\left[\left|\gamma \sum_{x=\ell_1}^{\ell_2} X(x)\right| \ge \beta s\right]}{\inf_{\ell_1 \le \ell \le \ell_2} I\!P\left[\left|\gamma \sum_{x=\ell_1}^{\ell} X(x)\right| \le \beta s\right]} \tag{5.53}$$

[1] Remark that given $s_0 > 0$ it is always possible to find $\gamma_0 > 0$ such that for $\gamma \le \gamma_0$, the quantity $(1 - 2s_0 c g_3(\gamma))$ is strictly positive.

we get, setting $\tilde{u}=\frac{(1-2s_0cg_3(\gamma))}{32\hat{c}(\theta)^2(1+c\beta\theta)}$ by an argument similar to the one that gives (5.43),

$$
I\!P\left[\max_{I\subset\mathcal{J}}{}^{*}\max_{\tilde{I}_{12}\subset I}\left|\gamma\sum_{x\in\mathcal{C}_{\delta^*}(I)}X(x)\right|\geq 2\beta s\right]\leq\frac{4\tilde{c}(\log\frac{1}{\gamma})^{p+\rho}}{\hat{c}}e^{-s^2\tilde{u}\log\log\frac{1}{\gamma}} \tag{5.54}
$$

We then collect (5.34), (5.36), (5.37), (5.45) obtaining

$$
\mu_{\beta,\theta,\gamma}\left(\mathcal{W}_{\eta}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\tilde{\ell}_1,\tilde{\ell}_2,\ell_2)\right)\leq\exp\left[-\frac{\beta}{\gamma}\left(x\Delta\mathcal{F}-24\zeta_4-4\epsilon-4s-\delta^*|I|\right)\right] \tag{5.55}
$$

We make the following choices $s\leq s_0=\frac{x^2\Delta\mathcal{F}}{16}$, $\epsilon=\frac{1}{16}x^2\Delta\mathcal{F}$, $c_0=\frac{x^2\Delta\mathcal{F}}{4\hat{c}}$, $\zeta_4\leq\frac{x^2\Delta\mathcal{F}}{96}$ this will give us (5.33). We take γ_0 such that $(1-2s_0cg_3(\gamma_0))=\frac{1}{2}$. To be able to satisfy $s^2\tilde{u}\geq p+2\rho+1$ *and* $s\leq s_0$, we impose $\theta\leq\frac{x^2\Delta\mathcal{F}}{48\sqrt{\hat{c}(p+2\rho+1)}}$ and we can take $s=16\theta\sqrt{\hat{c}(p+2\rho+1)}$. Recalling that we need also $u\geq p+2\rho+1$, we impose that ζ_4 is such that $\zeta_4g_2(1/\zeta_4)\leq\frac{x^2\beta\Delta\mathcal{F}}{72\sqrt{\hat{c}(p+2\rho+1)}}$, that is with the condition above we assume $\zeta_4g_2(1/\zeta_4)\leq\frac{x^2\Delta\mathcal{F}}{96}\left[1\wedge\frac{\beta}{\sqrt{\hat{c}(p+2\rho+1)}}\right]$ and we get (5.32). This ends the proof of the Proposition 5.7. ∎

Proof of Theorem 2.4:

We prove that the complementary of the set $\mathcal{R}^{\delta_4,\zeta_4}(\ell_1,\ell_2,R_1)\bigcup\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2)$ has Gibbs-probability that goes to zero as $e^{-\frac{c_4(\beta.\theta)\delta_4\zeta_4^3}{\gamma}}$. We decompose

$$
\mathcal{A}\equiv\left(\mathcal{R}^{\delta_4,\zeta_4}(\ell_1,\ell_2,R_1,+)\right)^c\bigcap\left(\mathcal{R}^{\delta_4,\zeta_4}(\ell_1,\ell_2,R_1,-)\right)^c=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_4 \tag{5.56}
$$

where, see (5.30), $\mathcal{A}_1\equiv\mathcal{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([\ell_1+2R_1,\ell_2-2R_1])$ and, see (5.29), $\mathcal{A}_2\equiv\mathcal{W}^{\delta_4,\zeta_4}([\ell_1+2R_1,\ell_2-2R_1])$, While $\mathcal{A}_1$ and $\mathcal{A}_2$ refer to the behaviour of the profiles in the bulk of the interval, $\mathcal{A}_3$ and $\mathcal{A}_4$ consider the behaviour of the profiles in a region close to the boundaries. Namely for a given $\eta\in\{-1,+1\}$, we can be in $\left(\mathcal{R}^{\delta_4,\zeta_4}(\ell_1,\ell_2,R_1,\eta)\right)^c$ just because we have $\eta^{\delta_4,\zeta_4}(\ell_1+2R_1)\neq\eta$ or $\eta^{\delta_4,\zeta_4}(\ell_2-2R_1)\neq\eta$. Let us define

$$
\mathcal{A}_3^{\eta}(\ell)\equiv\left\{m^{\delta^*}:\eta^{\delta_4,\zeta_4}(\ell)\neq\eta\right\} \tag{5.57}
$$

and

$$
\mathcal{A}_3=\bigcup_{\eta,\eta'\in\{-1,+1\}^2}\mathcal{A}_3^{\eta}(\ell_1+2R_1)\cup\mathcal{A}_3^{\eta'}(\ell_2-2R_1) \tag{5.58}
$$

Suppose that a profile is in $\mathcal{A}_3^{\eta}(\ell_1+2R_1)$ then we can have four alternatives.

The block ℓ_1+2R_1 has $\eta^{\delta_4,\zeta_4}(\ell_1+2R_1)=0$ or $\eta^{\delta_4,\zeta_4}(\ell_1+2R_1)=-\eta$ and it is sandwiched at a distance smaller than $2R_1$ by two blocks with the same η's or with different η's. In this last case the profiles are fronts.

It is easy to see that

$$\begin{aligned}\mathcal{A}_3\cap\left(\mathcal{O}_0^{\delta_1,\zeta_1}([\ell_1+R_1,\ell_2-R_1])\right)^c\cap\mathcal{A}\subset\quad&\mathcal{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\\ \bigcup\mathcal{W}^{\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\bigcup&\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\end{aligned}\tag{5.59}$$

It remains to consider what is left in $\mathcal{A}$. The presence of $\mathcal{A}_4$ comes from the fact that in the definition of $\mathcal{A}_1$, there are four parameters $\delta_4,\zeta_4,\delta_1,\zeta_1$ and since $\delta_4\zeta_4^3\geq\tilde{c}_1(\beta,\theta)\zeta_1$ we can have blocks such that $\eta^{\delta_1,\zeta_1}=0$ but $\eta^{\delta_4,\zeta_4}=1$. Let us define

$$\mathcal{A}_4^\eta(\ell)\equiv\left\{m^{\delta^*}:\eta^{\delta_4,\zeta_4}(\ell)=\eta,\eta^{\delta_1,\zeta_1}(\ell)=0\right\}\tag{5.60}$$

and

$$\mathcal{A}_4=\bigcup_{\eta,\eta'\in\{-1,+1\}^2}\mathcal{A}_4^\eta(\ell_1+2R_1)\cup\mathcal{A}_4^{\eta'}(\ell_2-2R_1)\tag{5.61}$$

Arguing as before we get

$$\begin{aligned}&\mathcal{A}_4\cap\left(\mathcal{O}_0^{\delta_1,\zeta_1}([\ell_1+R_1,\ell_2-R_1])\right)^c\cap\mathcal{A}\subset\\ &\quad\mathcal{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\bigcup\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\end{aligned}\tag{5.62}$$

It is now clear that we have

$$\begin{aligned}&\mathcal{A}\cap\left(\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2)\right)^c\cap\left(\mathcal{O}_0^{\delta_1,\zeta_1}([\ell_1+R_1,\ell_2-R_1])\right)^c\subset\\ &\quad\mathcal{R}_0^{\delta_1,\zeta_1,\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\bigcup\mathcal{W}^{\delta_4,\zeta_4}([\ell_1+R_1,\ell_2-R_1])\end{aligned}\tag{5.63}$$

and (2.35) follows immediatly from the Corollaries 5.2, 5.4, 5.6 and the Proposition 5.7. ■

Proof of Theorem 2.5:

Taking into account (2.35), we must check that for $\ell_1\leq\ell_2\leq\ell_3$ that belongs to $\mathcal{J}$, an event of the form

$$\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_1,\ell_2,\eta)\cap\mathcal{V}^{\delta_1,\zeta_1,\delta_4,\zeta_4}(\ell_2,\ell_3,\eta)\tag{5.64}$$

with $\ell_2-\ell_1\leq\ell_{\hat{c}}(\gamma)$ and $\ell_3-\ell_2\leq\ell_{\hat{c}}(\gamma)$ has small Gibbs-probability and moreover that this is true with a very high $I\!P$-probability, uniformly for $\ell_1\leq\ell_2\leq\ell_3$ in $\mathcal{J}$. But it is immediate to see that those events are controlled by the Proposition 5.7.

¿From the Theorem 2.3, denoting by $c_2 = \left[\frac{c(x,\rho,\gamma)}{(\beta\theta)^2(m_{\beta,1}+m_{\beta,2})^2}\right]$, see (2.29), we end the proof of the Theorem 2.5. ■

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