ABSOLUTE CONTINUITY OF THE PERIODIC MAGNETIC SCHRÖDINGER OPERATOR

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ABSTRACT. We prove that the spectrum of the Schrödinger operator with periodic electric and magnetic potentials is absolutely continuous

1. INTRODUCTION

Consider on $L^2(\mathbb{R}^d)$, $d \geq 1$, a self-adjoint differential operator $\mathcal{A}(\mathbf{x}, \mathbf{D})$, $\mathbf{D} = -i\nabla$ with coefficients periodic with respect to a lattice Γ. The spectral analysis of this operator is based on the so-called Floquet decomposition (see for example [7] and [9]). This means that $\mathcal{A}(\mathbf{x}, \mathbf{D})$ can be represented as a direct integral of the operator family $\mathcal{A}_{\mathbf{k}} = \mathcal{A}(\mathbf{x}, \mathbf{D} + \mathbf{k})$ operating on functions defined on the torus $\mathbf{T} = \mathbb{R}^d/\Gamma$, in the *quasi-momentum* $\mathbf{k}$ over the unit cell $\tilde{\Omega}$ of the dual lattice. Under the assumption that each $\mathcal{A}_{\mathbf{k}}$ has a compact resolvent, the spectrum of the initial operator $\mathcal{A}(\mathbf{x}, \mathbf{D})$ is given by the union of *spectral bands*

$$I_l = \cup_{\mathbf{k} \in \tilde{\Omega}} E_l(\mathbf{k}),$$

the functions $E_l(\mathbf{k})$ being the discrete eigenvalues of $\mathcal{A}_{\mathbf{k}}$. If one of the functions E_l is constant in $\mathbf{k}$, then the associated band I_l degenerates into a point, which represents an eigenvalue of the operator $\mathcal{A}(\mathbf{x}, \mathbf{D})$. The question whether or not such a degeneration occurs for a given operator is very subtle. There is a limited number of operator classes for which one can give a definitive answer. In particular the only known results for scalar second order operators state the absence of such eigenvalues, whereas the fourth order operators may have them (see [7] for discussion).

The first result of this type belongs to L.E. Thomas [12], who established that the spectrum of the Schrödinger operator

$$\mathcal{A}(\mathbf{x}, \mathbf{D}) = -\Delta + V \tag{1.1}$$

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with a periodic electric potential $V = \bar{V}$ is absolutely continuous in the case $d = 3$ (see also [9] for generalisation to all $d \geq 2$). Later the Thomas' idea was used by L. Danilov [2] to prove the absolute continuity of the Dirac operator with a periodic electric potential. The key point of the Thomas' approach is to extend the operator $\mathcal{A}_{\mathbf{k}}$ into the complex plane in one of the components of the quasi-momentum, for example in k_1, and establish an estimate of the form

$$\|\mathcal{A}_{\mathbf{k}}^{-1}\| \leq C(\operatorname{Im} k_1) \tag{1.2}$$

with $C(t) \to 0$ as $t \to \infty$. Then due to the analyticity of $\mathcal{A}_{\mathbf{k}}$ in k_1 for the Hamiltonian (1.1) this estimate guarantees the absence of constant eigenvalues $E_l(\mathbf{k})$, which in its turn provides absolute continuity.

The Thomas' approach was applied by R. Hempel and I. Herbst in [5] (see also [6]) to the Schrödinger operator

$$H = (\mathbf{D} - \mathbf{a})^2 + V, \quad \mathbf{D} = -i\nabla, \tag{1.3}$$

with a periodic vector potential $\mathbf{a}$ and $V = 0$. It was found that the spectrum of H is absolutely continuous if the potential $\mathbf{a}$ is sufficiently small. Note that if the *magnetic field* $\mathbf{B} = \operatorname{curl} \mathbf{a}$ is periodic, then $\mathbf{a}$ may not be periodic, and then the spectrum may have eigenvalues of infinite multiplicity (see [3]).

Although R. Hempel and I. Herbst conjectured that absolute continuity must be true for all $\mathbf{a}$'s, the straightforward application of the Thomas' idea could not justify this conclusion, for the perturbation due to the magnetic field is of the first order. The latter obstacle was overcome in the recent paper [1] by M. Birman and T. Suslina in the case $d = 2$, who proved absolute continuity of H for all continuous $\mathbf{a}$'s. The special role in their approach was played by the $2D$-Pauli operator, which is known to admit a factorisation into two first order operators. This fact allows one to treat the perturbation due to the vector-potential $\mathbf{a}$ as a multiplicative one. This was the main reason why the Thomas' idea proved to be efficient. Although being very clear and elegant, the proof from [1] uses heavily the two-dimensionality of the problem and apparently cannot be generalised to higher dimensions.

In the present paper we prove that the magnetic periodic Schrödinger operator H is absolutely continuous in all dimensions (see Theorem 2.1 for the precise statement). The proof follows the Thomas' idea. However, as shows a counterexample constructed by T. Suslina [11] for $d = 3$, extending k_1 into the complex plane cannot provide the bound (1.2) for all vector potentials $\mathbf{a}$. Instead we investigate an extension of the operator $\mathcal{A}_{\mathbf{k}}$ to the complex plain with respect to the component of $\mathbf{k}$ along a specific direction $\boldsymbol{\alpha} \in \mathbb{R}^d$. The choice of $\boldsymbol{\alpha}$ depends on the vector-potential $\mathbf{a}$. A detailed plan of the proof is given in Sect.3 and we do not comment on it here. We only mention that our approach still contains some traces of the two-dimensional scheme from [1]. Namely, one of the most important observations is that the perturbation due to the magnetic potential can be represented as a multiplicative one, just as in [1]. However, instead of multiplication by a function one has to deal with a multiplication by a pseudo-differential operator.

Note that the Thomas' method is not the only one which can be used to tackle the absolute continuity problem. An alternative approach is described in [7]. Very roughly, its idea is to show that an eigefunction has to decay "extremely rapidly"and then, relying on a sort of unique continuation theorem at infinity, conclude that such a decay is impossible unless the eigenfunction is identically zero. In particular, this approach guarantees absolute continuity for divergence type second order periodic elliptic operators. On the other hand it does not allow one to handle the magnetic Hamiltonians.

The paper is organised as follows. The main result is stated in Sect. 2. Here we also reduce the problem to the estimate of the form (1.2) (see Theorem 2.2), reproducing the Thomas' argument. Sect. 3 contains an elaborate plan of the proof of Theorem 2.2 with precise statements of intermediate results. Their proofs are postponed until Sect. 5 - 7. Sect. 4 contains some results from the calculus of pseudo-differential operators on the torus.

We point out in conclusion that the successful solution of the problem for $d = 2$ in paper [1] has played a crucial role in author's decision to attack the absolute continuity problem for higher dimensions.

Notation. By bold lowercase letters we denote vectors in $\mathbb{R}^d$ and $\mathbb{Z}^d$, e.g. $\boldsymbol{\alpha} \in \mathbb{R}^d$, $\mathbf{m} \in \mathbb{Z}^d$. Bold uppercase letters are used for $d \times d$-matrices, e.g. $\mathbf{G}$. The only exception is made for letter $\mathbf{D}$ which is the standard notation for $-i\nabla$, i.e. $\mathbf{D} = (D_1, D_2, \dots, D_d)$, $D_j = -i\partial_j$. The notations $\mathbf{ab}$ and $\mathbf{aGb}$ stand for the scalar product in $\mathbb{R}^d$ and the quadratic form of the matrix $\mathbf{G}$ respectively.

When we use the notation $\mathbf{D}^{\mathbf{s}}$ we always assume that $\mathbf{s}$ is a vector from $\mathbb{Z}^d$ with non-negative components $s_1, s_2, \dots, s_d$ and then $\mathbf{D}^{\mathbf{s}} = D_1^{s_1} D_2^{s_2} \cdots D_d^{s_d}$.

2. Main result

2.1. **Main result.** We study in $L^2(\mathbb{R}^d)$, $d \geq 3$, the operator

$$H = (\mathbf{D} - \mathbf{a})\mathbf{G}(\mathbf{D} - \mathbf{a}) + V, \ \mathbf{D} = -i\nabla, \tag{2.1}$$

where $\mathbf{G} = \{g_{jl}\}$, $j, l = 1, 2, \dots, d$ is a symmetric positive-definite matrix with real-valued entries, $\mathbf{a} = (a_1, a_2, \dots, a_d)$ is a vector-function (magnetic vector-potential) with real-valued components and V is a real-valued function (electric potential). We suppose that $\mathbf{a}, V$ are both periodic. Obviously the operator H can be reduced by a suitable change of variables to (1.3). However it will be more convenient to deal with (2.1) for without loss of generality we may suppose that the vector-potential $\mathbf{a}$ and the electric potential V are periodic with respect to the cubic lattice $\Gamma = (2\pi\mathbb{Z})^d$. Denote

$$\Omega = [0, 2\pi)^d, \quad \mathbb{T} = \mathbb{R}^d/\Gamma, \qquad \tilde{\Omega} = [0, 1)^d.$$

Besides, it will be convenient to assume the following conditions:

$$\mathbf{DGa} = 0, \tag{2.2}$$

$$\int_\Omega \mathbf{a}(\mathbf{x}) d\mathbf{x} = 0. \tag{2.3}$$

The equality (2.2) is understood in the distributional sense. The requirements (2.2), (2.3) can be always satisfied by choosing an appropriate gauge. Under the condition (2.2) the operator H has the form

$$H = \mathbf{DGD} + X + V, \quad X = -2\mathbf{aGD} + \mathbf{aGa}.$$

Observe that the perturbation $X + V$ will be infinitesimally small with respect to the operator $\mathbf{DGD}$ if $\mathbf{a} \in L^\infty(\mathbb{R}^d)$, and $V \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ with $p = 2 (d = 2, 3)$ and $p > d/2 (d \geq 4)$. This guarantees self-adjointness of H on the domain $D(H) = D(\mathbf{DGD}) = H^2(\mathbb{R}^d)$. The above conditions are satisfied in the following Theorem which constitutes the main result of the paper:

Theorem 2.1. *Let $d \geq 3$. Let $\mathbf{a} \in C^{2d+3}(\mathbb{T})$ be a vector-potential, satisfying (2.2), (2.3), and let $V \in L^p_{loc}(\mathbb{R}^d)$, $p > d - 1$ be a function, both periodic with respect to the lattice $\Gamma = (2\pi\mathbb{Z})^d$. Then the spectrum of the operator H is absolutely continuous.*

2.2. **Floquet decomposition.** The proof of Theorem 2.1 will be based on the Floquet decomposition for the operator H in a direct integral. Let us introduce necessary notation. We identify the space $L^2(\mathbb{R}^d)$ with the direct integral

$$\mathfrak{H} = \int_{\tilde{\Omega}} L^2(\Omega) d\mathbf{k}.$$

The identification is implemented by the Gelfand transform

$$(Uf)(\mathbf{x}, \mathbf{k}) = e^{-i\mathbf{kx}} \sum_{\mathbf{m} \in \mathbb{Z}^d} e^{-i2\pi\mathbf{km}} f(\mathbf{x} + 2\pi\mathbf{m}),$$

which is initially defined on functions from the Schwarz class and extends by continuity to a unitary mapping from $L^2(\mathbb{R}^d)$ onto $\mathfrak{H}$. For every $\mathbf{k} \in \mathbb{R}^d$ we define on the periodic functions the operators

$$\begin{cases} H(\mathbf{k}) = & H_0(\mathbf{k}) + X(\mathbf{k}) + V, \\ H_0(\mathbf{k}) = & (\mathbf{D}+\mathbf{k})\mathbf{G}(\mathbf{D}+\mathbf{k}), \\ X(\mathbf{k}) = & -2\mathbf{a}\mathbf{G}(\mathbf{D}+\mathbf{k}) + \mathbf{a}\mathbf{G}\mathbf{a}. \end{cases} \tag{2.4}$$

For each $\mathbf{k} \in \mathbb{R}^d$ the perturbation $H(\mathbf{k}) - H_0(\mathbf{0}) = H(\mathbf{k}) - \mathbf{DGD}$ is infinitesimally $H_0(\mathbf{0})$-bounded, so that the operators $H(\mathbf{k})$ are self-adjoint on the domain $D(H(\mathbf{k})) = D(H_0(\mathbf{0})) = H^2(\mathbb{T})$. Moreover, since $H_0(\mathbf{0})$ has a compact resolvent, so does $H(\mathbf{k})$. The family $H(\mathbf{k})$ represents the decomposition of H in the direct integral. Namely,

$$UHU^{-1} = \int_{\tilde{\Omega}} H(\mathbf{k}) d\mathbf{k}. \tag{2.5}$$

Let us single out a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}^d$ and set $\mathcal{L} = \{\mathbf{k} \in \mathbb{R}^d : \boldsymbol{\alpha}\mathbf{k} = 0\}$. Consider the family

$$\tilde{H}(k) = H\big(\boldsymbol{\alpha}(k - \boldsymbol{\mu}_\perp \mathbf{G}\boldsymbol{\alpha}) + \boldsymbol{\mu}_\perp\big),\ k \in \mathbb{R},\ \boldsymbol{\mu}_\perp \in \mathcal{L}. \tag{2.6}$$

The central role will be played by the operator obtained by extending this family to large complex values of k. Namely, set $k = \beta + i\rho$ with $\beta \in \mathbb{R}$ and $\rho > 0$ and define

$$\mathcal{H}(\rho) = \tilde{H}(\beta + i\rho).$$

We shall be interested in behaviour of $\mathcal{H}(\rho)$ in $\rho \to \infty$ uniformly with respect to $\boldsymbol{\mu}_\perp$ in a compact set. Let

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{||} + \boldsymbol{\mu}_\perp, \quad \boldsymbol{\mu}_{||} = \boldsymbol{\alpha}(\beta - \boldsymbol{\mu}_\perp \mathbf{G}\boldsymbol{\alpha}),\ \boldsymbol{\mu}_\perp \in \mathcal{L}. \tag{2.7}$$

We denote by $\mathcal{H}_{\mathbf{a}}(\rho)$ the operator $\mathcal{H}(\rho)$ with $V = 0$ and by $\mathcal{H}_0(\rho)$ – with $\mathbf{a} = 0, V = 0$. Vector $\boldsymbol{\mu}$ will play the role of a parameter and as a rule is omitted from notation. It will be convenient to normalise $\boldsymbol{\alpha}$ in such a way that $\boldsymbol{\alpha}\mathbf{G}\boldsymbol{\alpha} = 1$. Then it follows from (2.4) that

$$\begin{cases} \mathcal{H}_{\mathbf{a}}(\rho) = & \mathcal{H}_0(\rho) + \mathcal{X}(\rho), \\ \mathcal{H}_0(\rho) = & (\mathbf{D}+\boldsymbol{\mu})\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}) + 2i\rho\boldsymbol{\alpha}\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}) - \rho^2, \\ \mathcal{X}(\rho) = & -2\mathbf{a}\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}) - 2i\rho\boldsymbol{\alpha}\mathbf{G}\mathbf{a} + \mathbf{a}\mathbf{G}\mathbf{a}. \end{cases} \tag{2.8}$$

In the next subsection we shall show that Theorem 2.1 can be inferred from

Theorem 2.2. *Under the conditions of Theorem 2.1 there exist a number $\rho_0 = \rho_0(\mathbf{a})$ and a vector $\mathbf{n} = \mathbf{n}(\mathbf{a}) \in \mathbb{Z}^d$ such that for $\boldsymbol\alpha$ satisfying*

$$\boldsymbol\alpha\mathbf{G}\boldsymbol\alpha = 1, \quad \frac{\mathbf{G}\boldsymbol\alpha}{|\mathbf{G}\boldsymbol\alpha|} = \frac{\mathbf{n}}{|\mathbf{n}|}, \tag{2.9}$$

and $\beta = |\mathbf{G}\boldsymbol\alpha|(2|\mathbf{n}|)^{-1}$ one has

$$\|\mathcal{H}(\rho)u\| \ge c\rho\|u\|, \ \forall u \in C^\infty(\mathbb{T}), \tag{2.10}$$

for all $\rho \ge \rho_0$ with a constant c independent of ρ and $\boldsymbol\mu_\perp : |\boldsymbol\mu_\perp| \le C$.

Note that for the Schrödinger operator with $V = 0$ the results of [6] permit to deduce Theorem 2.1 if the lower bound (2.10) holds for some *fixed* value $\boldsymbol\mu_\perp$. We do not go into details.

2.3. **Reduction of the proof of Theorem 2.1 to Theorem 2.2.** In this subsection we merely reproduce the corresponding argument from the paper [1].

Since $D(\tilde H(k)) = D(\mathbf{DGD})$ does not depend on $\mathbf{k}$, the operators $\tilde H(k)$ in (2.6) form an analytic family of type A (see [8]) with respect to $k \in \mathbb{C}$. This family is self-adjoint in the sense that $\tilde H(k)^* = \tilde H(\bar k)$. Moreover, the resolvent of $(\tilde H(k) - z)^{-1}$ is compact for all $z \in \mathbb{C}$ where it exists. Therefore one can apply to $\tilde H(\mathbf{k})$ the results of the analytic perturbation theory. To do so we first introduce the following sets:

$$J = \{k \in \mathbb{R} : \boldsymbol\alpha\big(k - \boldsymbol\mu_\perp\mathbf{G}\boldsymbol\alpha\big) + \boldsymbol\mu_\perp \in \tilde\Omega \ \text{ for some } \ \boldsymbol\mu_\perp \in \mathcal{L}\},$$
$$\mathcal{K} = \{\boldsymbol\mu_\perp \in \mathcal{L} : \boldsymbol\alpha\big(k - \boldsymbol\mu_\perp\mathbf{G}\boldsymbol\alpha\big) + \boldsymbol\mu_\perp \in \tilde\Omega \ \text{ for some } \ k \in \mathbb{R}\},$$
$$J(\boldsymbol\mu_\perp) = \{k \in \mathbb{R} : \boldsymbol\alpha\big(k - \boldsymbol\mu_\perp\mathbf{G}\boldsymbol\alpha\big) + \boldsymbol\mu_\perp \in \tilde\Omega \ \}, \ \boldsymbol\mu_\perp \in \mathcal{K},$$

and note that

$$J = \bigcup_{\boldsymbol\mu_\perp \in \mathcal{K}} J(\boldsymbol\mu_\perp).$$

Consider $\tilde H(k)$ on a bounded interval $J' \supset \bar J$. In accordance with the analytic perturbation theory (see [8], Theorem VII.3.9) there exists a sequence of scalar functions $\lambda_l(k)$, $l \in \mathbb{N}$, and a sequence of vector-functions $\phi_l(\,\cdot\,, k)$ such that

1. The functions λ_l, ϕ_l are real-analytic on J';
2. For each $k \in J'$ and $l \in \mathbb{N}$
$$\tilde H(k)\phi_l(k) = \lambda_l(k)\phi_l(k);$$
3. The sequence $\phi_l(k)$ forms an orthonormal basis in $L^2(\Omega)$ for each $k \in J'$.

Theorem 2.1 results from the following statement, which serves as a key stone of the Thomas method:

(2.11) There are no constants among the functions $\lambda_l(k),\ k \in J'$.

To show why (2.11) implies Theorem 2.1 rewrite the direct integral (2.5) in the form

$$UHU^{-1} = \int_{\mathcal{K}} Z(\boldsymbol{\mu}_\perp) d\boldsymbol{\mu}_\perp, \text{ with } Z(\boldsymbol{\mu}_\perp) = \int_{J(\boldsymbol{\mu}_\perp)} \tilde{H}(k) dk.$$

Absolute continuity of $Z(\boldsymbol{\mu}_\perp)$ for each $\boldsymbol{\mu}_\perp \in \mathcal{K}$ follows from the properties 1 – 3 and (2.11) due to [9], Theorem XIII.86. This property in its turn ensures the absolute continuity of H by [9], Theorem XIII.85.

It remains to check that Theorem 2.2 implies (2.11). Suppose the converse, i.e. for some l there is an eigenvalue $\lambda_l(k) = \lambda = \mathrm{const}$ for all $k \in J'$. Then by [8], Theorems VII.1.10, VII.1.9 $\lambda_l(k) = \lambda$ for all $k \in \mathbb{C}$, which contradicts Theorem 2.2. This proves (2.11), and hence Theorem 2.1.

3. Plan of the proof of Theorem 2.2

This section provides a detailed plan of the proof of Theorem 2.2 with precise statements of intermediate results. Their proofs are given in subsequent sections. Each intermediate statement will be proved under minimal conditions on the objects involved. For instance, Theorem 3.1 below holds for any L^∞ magnetic vector potential $\mathbf{a}$ uniformly in $\boldsymbol{\alpha}$ satisfying the condition $\boldsymbol{\alpha}\mathbf{G}\boldsymbol{\alpha} = 1$.

The proof Theorem 2.2 splits naturally into two parts: first we prove it for $V = 0$, i.e. for the operator $\mathcal{H}_{\mathbf{a}}(\rho)$, and then incorporate the electric potential V. The first part is much more complicated and requires new methods compared to [12], whereas the second part uses basically the same argument as in [12]. Due to this circumstance the most of the paper is concerned with the proof of the bound (2.10) for $V = 0$.

Introduce the matrix $\mathbf{F} = \sqrt{\mathbf{G}}$ and denote by f_- and f_+ the minimal and the maximal eigenvalues of $\mathbf{F}$ respectively. Due to the condition $\boldsymbol{\alpha}\mathbf{G}\boldsymbol{\alpha} = 1$ one has $|\mathbf{F}\boldsymbol{\alpha}| = 1$. Define the unit vector

$$\mathbf{e} = \mathbf{F}\boldsymbol{\alpha},$$

which often will be more convenient to use than $\boldsymbol{\alpha}$.

3.1. **Partition of $L^2(\Omega)$.** To establish the lower bound (2.10) we need to decompose the space $L^2(\Omega)$ into two mutually orthogonal subspaces. Introduce the Fourier transform:

$$\hat{u}(\mathbf{m}) = (\mathcal{F}u)(\mathbf{m}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_\Omega e^{-i\mathbf{m}\mathbf{x}} u(\mathbf{x}) d\mathbf{x},$$

which defines a unitary mapping from $L^2(\Omega)$ onto $\ell^2(\mathbb{Z}^d)$. Let $\omega > 0$ be a number. Define the following subsets of the dual lattice $\mathbb{Z}^d$:

$$\begin{aligned} M = M(\rho,\omega) = \{\mathbf{m} \in \mathbb{Z}^d : |\mathbf{e}\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| \le \omega,& \\ \big||\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho\big| \le \omega\},& \\ N = N(\rho,\omega) = \mathbb{Z}^d \setminus M(\rho,\omega).& \end{aligned}$$

Throughout the rest of the paper the parameter ω is a constant which will be chosen to be sufficiently large. In all the estimates below the constants C, c with or without indices are independent of ρ and ω. Besides, here and everywhere below the dependence on the vector $\boldsymbol{\mu}$ is not reflected in the notation, since $\boldsymbol{\mu}$ will be varying in a compact set of $\mathbb{R}^d$ and unless otherwise stated the constants in all the estimates will be independent of $\boldsymbol{\mu}$.

Note that the symbol

$$h(\mathbf{m},\rho) = (\mathbf{m}+\boldsymbol{\mu})\mathbf{G}(\mathbf{m}+\boldsymbol{\mu}) + 2i\rho\boldsymbol{\alpha}\mathbf{G}(\mathbf{m}+\boldsymbol{\mu}) - \rho^2. \tag{3.1}$$

of the operator $\mathcal{H}_0(\rho)$ from (2.8) obeys the lower bound

$$|h(\mathbf{m};\rho)| \ge \omega\rho, \ \forall \mathbf{m} \in \mathfrak{N}(\rho,\omega). \tag{3.2}$$

Let $P(\mathbf{m};\rho,\omega), Q(\mathbf{m};\rho,\omega)$ be the characteristic functions of the sets M, N respectively. Introduce also smooth cut-off functions supported near the shell $|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho| \le \omega$. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\zeta(t) = 1$ for $|t| \le 1$, $\zeta(t) = 0$ for $|t| \ge 2$ and $\zeta(t) \le 1$ for all t. Then for any $\omega > 0$ and $j = 1, 2$ define

$$\zeta_j(\mathbf{m};\rho,\omega) = \zeta\big((|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho)(j\omega)^{-1}\big), \ \mathbf{m} \in \mathbb{Z}^d. \tag{3.3}$$

Now define the subspaces of $L^2(\Omega)$ associated with the sets M, N:

$$\begin{aligned} \mathfrak{M} = \mathfrak{M}(\rho,\omega) = \{u \in L^2(\Omega) : \hat{u}(\mathbf{m}) = 0, \ \forall \mathbf{m} \notin M(\rho,\omega)\}, \\ \mathfrak{N} = \mathfrak{N}(\rho,\omega) = \{u \in L^2(\Omega) : \hat{u}(\mathbf{m}) = 0, \ \forall \mathbf{m} \notin N(\rho,\omega)\}. \end{aligned}$$

It is clear that the operators $P = P(\rho,\omega) = P(\mathbf{D};\rho,\omega)$ and $Q = Q(\rho,\omega) = Q(\mathbf{D};\rho,\omega)$ are the orthogonal projections on $\mathfrak{M}(\rho,\omega)$ and $\mathfrak{N}(\rho,\omega)$ respectively. Using (3.2) we shall prove that the operator $H_0(\rho)$ dominates the perturbation $\mathcal{X}(\rho)$ on the subspace $\mathfrak{N}(\rho,\omega)$ which almost immediately leads to a bound of the form (2.10) for the operator $\mathcal{H}_{\mathbf{a}}Q$.

On the contrary, estimating $\mathcal{H}_{\mathbf{a}}P$ is far from being easy for the operators $\mathcal{H}_0(\rho)$ and $\mathcal{X}(\rho)$ are of the same order of magnitude on $\mathfrak{M}(\rho,\omega)$. To overcome this difficulty we prove first of all that on the subspace $\mathfrak{M}(\rho,\omega)$ the operator $\mathcal{H}_{\mathbf{a}}(\rho)$ reduces to the first order operator

$$T_{\mathbf{a}} = T_{\mathbf{a}}(\rho) = (\mathbf{D} + \boldsymbol{\mu} - \mathbf{a})\mathbf{F}(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{D})) - \rho, \tag{3.4}$$

where

$$\boldsymbol{\nu}(\mathbf{m}) = \frac{\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})}{|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|}. \tag{3.5}$$

More precisely, the following theorem holds:

Theorem 3.1. *Let $\mathbf{a} \in L^\infty(\Omega)$ and the operator $T_{\mathbf{a}}(\rho)$ be as defined above. Then*

$$\|\big(\mathcal{H}_{\mathbf{a}}(\rho) - 2\rho T_{\mathbf{a}}(\rho)\big)\zeta_1(\mathbf{D};\rho,\omega)\| \le (2\omega + f_+\|\mathbf{a}\|_{L^\infty})^2, \tag{3.6}$$

for all $\mathbf{e} \in \mathbb{S}^{d-1}, \omega > 0$ and $\rho > 0$.

See subsection 5.4 for proof.

Theorem 3.1 reduces the problem to estimating from below the operator $T_{\mathbf{a}}$. Observe that it is very easy to estimate the unperturbed operator $T_0(\rho)$. Indeed, representing $\boldsymbol{\mu}$ in the form (2.7), for any vector $\boldsymbol{\alpha}$ satisfying (2.9), we have

$$\begin{aligned} \min_{\mathbf{m}\in\mathbb{Z}^d} |(\mathbf{m}+\boldsymbol{\mu})\mathbf{F}\mathbf{e}| &= \min_{\mathbf{m}\in\mathbb{Z}^d} |(\mathbf{m}+\boldsymbol{\mu}_{\|}+\boldsymbol{\mu}_{\perp})\mathbf{G}\boldsymbol{\alpha}| \\ &\ge |\beta|, \text{ if } \ |\beta| \le |\mathbf{G}\boldsymbol{\alpha}|(2|\mathbf{n}|)^{-1}. \end{aligned} \tag{3.7}$$

Since $|\mathbf{G}\boldsymbol{\alpha}| = |\mathbf{F}\mathbf{e}| \ge f_-$, this implies that

$$\|T_0(\rho)u\| \ge f_-(2|\mathbf{n}|)^{-1}\|u\|, \ \beta = |\mathbf{G}\boldsymbol{\alpha}|(2|\mathbf{n}|)^{-1}, \ \ \forall u \in C^\infty(\mathbb{T}), \tag{3.8}$$

uniformly in $\rho \ge 0$. To reduce $T_{\mathbf{a}}(\rho)$ to $T_0(\rho)$ we shall find a similarity transformation ("gauge transformation") W such that the operator $WT_{\mathbf{a}}(\rho)W^{-1}$ would coincide with $T_0(\rho)$ up to some insignificant errors. Below we explain how to find such a transformation.

3.2. **Gauge transformation.** We shall find a boundedly invertible operator W such that $WT_{\mathbf{a}}(\rho)W^{-1}$ is close to $T_0(\rho)$ in some sense. We begin with constructing such a transformation for the operator obtained from (3.4) by replacing $\boldsymbol{\nu}(\mathbf{D})$ with a fixed unit vector $\boldsymbol{\gamma} \in \mathbb{S}^{d-1}$:

$$T^{(\boldsymbol{\gamma})}_{\mathbf{a}} = T^{(\boldsymbol{\gamma})}_{\mathbf{a}}(\rho) = (\mathbf{D} + \boldsymbol{\mu} - \mathbf{a})\mathbf{F}(i\mathbf{e} + \boldsymbol{\gamma}) - \rho,$$

A reduction to $T_0^{(\gamma)}$ would be secured if we find a scalar potential for $T_{\mathbf{a}}^{(\boldsymbol\gamma)}$, that is a periodic complex-valued smooth function $\psi = \psi_{\boldsymbol\gamma}$ such that

$$(ie + \gamma)\mathbf{F}\mathbf{D}\psi = -(ie + \gamma)\mathbf{F}\mathbf{a}. \tag{3.9}$$

Indeed, if such a function exists then

$$e^{\psi} T_{\mathbf{a}}^{(\boldsymbol\gamma)} = T_0^{(\gamma)} e^{\psi}. \tag{3.10}$$

However, to find ψ we need to make two assumptions:

1. The vector-potential $\mathbf{a}$ is a trigonometric polynomial, i.e. for some $L \in \mathbb{N}$ and $K_L = \{\mathbf{m} \in \mathbb{Z}^d : 0 < |\mathbf{m}| \le L\}$ it can be represented as follows:

$$\mathbf{a}(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \sum_{0<|\mathbf{m}|\le L} \hat{\mathbf{a}}(\mathbf{m}) e^{i\mathbf{m}\mathbf{x}}. \tag{3.11}$$

2. The vector $\mathbf{e}$ is chosen in such a way that

$$\mathbf{e}\mathbf{F}\mathbf{m} \neq 0 \quad \text{for all } \mathbf{m} \in K_L. \tag{3.12}$$

In Lemma 3.7 we construct such an $\mathbf{e}$ explicitly.

Lemma 3.2. *Let* $\mathbf{a}$ *be a trigonometric polynomial of the form* (3.11). *Suppose that* $\mathbf{e} \in \mathbb{S}^{d-1}$ *obeys* (3.12). *Then for any* $\boldsymbol\gamma \in \mathbb{S}^{d-1}$ *there exists a function* $\psi = \psi_{\boldsymbol\gamma} \in C^\infty(\mathbb{T})$ *satisfying the equation* (3.9). *If one requires*

$$\hat\psi(\mathbf{m}) = 0, \ \ \mathbf{m} \notin K_L, \tag{3.13}$$

then ψ *is defined uniquely and satisfies the upper bounds*

$$\sup_{\mathbf{x},\boldsymbol\gamma} |\mathbf{D}^{\mathbf{s}} \psi_{\boldsymbol\gamma}(\mathbf{x})| \le C_{\mathbf{s}}, \tag{3.14}$$

and

$$\sup_{\mathbf{x}} |\mathbf{D}^{\mathbf{s}}(\psi_{\boldsymbol\gamma} - \psi_{\boldsymbol\eta})| \le C_{\mathbf{s}} |\boldsymbol\gamma - \boldsymbol\eta|, \tag{3.15}$$

with constants $C_{\mathbf{s}}$ *independent of* $\boldsymbol\gamma, \boldsymbol\eta$, *that depend only on* $\mathbf{a}, \mathbf{e}$.

See subsection 5.1 for proof.

Note that this Lemma does not require that $\boldsymbol\alpha$ satisfies (2.9).

In contrast to $T_{\mathbf{a}}^{(\boldsymbol\gamma)}$ the required gauge transformation for the operator (3.4) will be performed by a pseudo-differential operator (PDO) on the torus $\mathbb{T}$ (see (4.1) for definition). Assuming as above that $\mathbf{a}$ is a trigonometric polynomial (3.11) and $\mathbf{e}$ satisfies (3.12), let $\psi_{\boldsymbol\gamma} = \psi_{\boldsymbol\gamma}(\mathbf{x}; \mathbf{a})$ be the function constructed in Lemma 3.2 with $\boldsymbol\gamma = \boldsymbol\nu(\mathbf{m})$, $\mathbf{m} \in \mathbb{Z}^d$. Denote

$$\psi(\mathbf{x}, \mathbf{m}; \mathbf{a}; \rho, \omega) = \psi_{\boldsymbol\gamma}(\mathbf{x}; \mathbf{a}) \zeta_2(\mathbf{m}; \rho, \omega), \tag{3.16}$$

with $\rho, \omega > 0$. To avoid cumbersome notation, when this does not cause any confusion, we omit some or all of the parameters on which ψ depends and write simply $\psi(\mathbf{x}, \mathbf{m})$ or just ψ. The role of the multiplication by e^{ψ} in (3.10) will be now played by the PDO with the symbol $b = \exp\psi$:

$$
\begin{aligned}
(3.17)\quad (Wu)(\mathbf{x}) &= (W(\mathbf{a}; \rho, \omega)u)(\mathbf{x}) \\
&= (2\pi)^{-\frac{d}{2}} \sum_{\mathbf{m}} \exp\big(\psi(\mathbf{x}, \mathbf{m}; \mathbf{a}; \rho, \omega)\big)\hat{u}(\mathbf{m})e^{i\mathbf{m}\mathbf{x}}, \ \forall u \in L^2(\Omega).
\end{aligned}
$$

The properties of W are summarised in

Lemma 3.3. *Let* (3.11), (3.12) *be fulfilled and let the operator* $W = W(\mathbf{a}; \rho, \omega), \rho > 0, \omega > 0$ *be as defined above. Then* $\|W\| \le C$. *Moreover,*

1. *The operator* $W(\mathbf{a}; \rho, \omega)$ *maps* $C^\infty(\mathbb{T})$ *into itself;*
2. *For any* $v \in C^{2d+3}(\mathbb{T})$

$$
(3.18)\quad \|[W(\mathbf{a}; \rho, \omega), v](\mathbf{D} + \boldsymbol{\mu})^{\mathbf{s}}\| \le C(\rho + \omega + 1)^s (1 + \omega)^{-1}, \qquad \rho \ge 6\omega > 0, \ s = |\mathbf{s}| \le 1;
$$

3. *For a sufficiently large* $\omega_0 = \omega_0(\mathbf{a}, \mathbf{e})$ *and* $\rho \ge 6\omega, \omega \ge \omega_0$ *the operator* $W(\mathbf{a}; \rho, \omega)$ *is invertible and*
$$\|W(\mathbf{a}, \rho, \omega)^{-1}\| \le C.$$

The constants in the above estimates depend only on $\mathbf{a}$ *and* $\mathbf{e}$.

See subsection 5.2 for proof.

We do not write down the precise relation between $WT_{\mathbf{a}}W^{-1}$ and T_0, since it is more convenient to apply the gauge transformation directly to the operator $\mathcal{H}_{\mathbf{a}}$, before its reduction to $T_{\mathbf{a}}$, i.e. before applying Theorem 3.1. It is done in the next subsection.

As H. Isozaki and P. Kuchment pointed out to the author, a transformation similar to W was used in the papers [4] and [10] in a different context.

3.3. **Application of W to $\mathcal{H}_{\mathbf{a}}$.** To begin with we remove the condition that $\mathbf{a}$ should be a trigonometric polynomial. However, to make use of the operator W constructed above, we split $\mathbf{a}$ into the sum of a trigonometric polynomial $\mathbf{a}'$ of the form (3.11) and a remainder $\mathbf{a}''$:

$$
(3.19)\qquad \begin{cases}
\mathbf{a} = & \mathbf{a}' + \mathbf{a}'', \\
\mathbf{a}'(\mathbf{x}) = & (2\pi)^{-\frac{d}{2}} \sum_{0 < |\mathbf{m}| \le L} \hat{\mathbf{a}}(\mathbf{m}) e^{i\mathbf{m}\mathbf{x}}, \\
\mathbf{a}''(\mathbf{x}) = & (2\pi)^{-\frac{d}{2}} \sum_{|\mathbf{m}| > L} \hat{\mathbf{a}}(\mathbf{m}) e^{i\mathbf{m}\mathbf{x}}.
\end{cases}
$$

Note that $\hat{\mathbf{a}}(\mathbf{0}) = 0$ due to (2.3).

Assuming that $\mathbf{e}$ obeys (3.12) denote $W = W(\rho,\omega) = W(\mathbf{a}';\rho,\omega)$ and introduce the operator

$$\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) = \mathcal{H}_{\mathbf{a}}(\rho) + 2\rho\mathbf{a}'\mathbf{F}(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{D}))\zeta_2(\mathbf{D};\rho,\omega) \tag{3.20}$$

with $\boldsymbol{\nu}(\mathbf{m})$ as in (3.5). Sometimes it will be useful for us to rewrite this operator in the form

$$\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) = \mathcal{H}_{\mathbf{a}}(\rho) - 2\rho\,\mathrm{Op}(g) \tag{3.21}$$

where $\mathrm{Op}(g)$ is the PDO with the symbol

$$\begin{aligned} g(\mathbf{x},\mathbf{m}) = g(\mathbf{x},\mathbf{m};\rho,\omega) &= (i\mathbf{e} + \boldsymbol{\nu}(\mathbf{m}))\mathbf{F}\mathbf{D}\psi(\mathbf{x},\mathbf{m};\rho,\omega) \\ &= -(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{m}))\mathbf{F}\mathbf{a}'(\mathbf{x})\zeta_2(\mathbf{m};\rho,\omega). \end{aligned} \tag{3.22}$$

The operator $W(\rho,\omega)$ "intertwines" $\mathcal{H}_{\mathbf{a}}(\rho)$ and $\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)$:

Theorem 3.4. *Let* $\mathbf{a} \in C^{2d+3}(\mathbb{T})$, $\mathbf{a}', \mathbf{a}''$ *be as in* (3.19) *and* $\mathbf{e}$ *obey* (3.12). *Then for any* $\omega > 0, \rho \ge 18\omega$ *one has*

$$\| W(\mathbf{a}';\rho,\omega)\mathcal{H}_{\mathbf{a}}(\rho) - \tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)W(\mathbf{a}';\rho,\omega)\| \le C(\omega + \rho\omega^{-1}) \tag{3.23}$$

with a constant C *depending only on* $\mathbf{a}, L$ *and* $\mathbf{e}$.

See subsection 5.3 for proof.

Once the transformation W is constructed, the condition (3.12) becomes unnecessary. In particular, we do not need it to describe the properties of the operator $\tilde{\mathcal{H}}_{\mathbf{a}}$. Instead of Theorem 3.1 we now have

Theorem 3.5. *Let the operators* $\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)$, $T_{\mathbf{a}}(\rho)$ *be as defined in* (3.20) *and* (3.4) *respectively. If* $\mathbf{a} \in L^\infty(\Omega)$ *then*

$$\begin{aligned} &\| \big(\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) - 2\rho T_0(\rho)\big)\zeta_1(\mathbf{D};\rho,\omega)\| \\ &\qquad\qquad \le 4\rho f_+ \|\mathbf{a}''\|_{L^\infty} + (2\omega + f_+\|\mathbf{a}\|_{L^\infty})^2. \end{aligned} \tag{3.24}$$

If in addition $\mathbf{a} \in C^{2d+3}(\mathbb{T})$ *then*

$$\begin{aligned} &\| \zeta_1(\mathbf{D};\rho,\omega)\big(\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) - 2\rho T_0(\rho)\big)\| \\ &\qquad \le 4\rho f_+ \|\mathbf{a}''\|_{L^\infty} + (2\omega + f_+\|\mathbf{a}\|_{L^\infty})^2 \\ &\qquad + C(\rho+1)(\omega+1)^{-1}\big(|||\mathbf{a}|||_{2d+3}^2 + |||\mathbf{a}|||_{2d+3} + |||\mathbf{a}'|||_{2d+3}\big), \end{aligned} \tag{3.25}$$

for all $\omega > 0$ *and* $\rho > 0$ *uniformly in* $\mathbf{e} \in \mathbb{S}^{d-1}$.

See subsection 5.4 for proof.

3.4. **Proof of Theorem 2.2 for $V = 0$.** First we establish a lower bound for the operator $\tilde{\mathcal{H}}_{\mathbf{a}}(\rho, \omega)$. To this end we will choose the number L in (3.19) so that the bound

$$\max_{\mathbf{x}} |\mathbf{a}''(\mathbf{x})| \le \frac{f_-}{32 f_+ |\mathbf{n}|}. \tag{3.26}$$

is satisfied. Now Theorem 3.5 and the estimate (3.8) enable us to prove the following

Theorem 3.6. *Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{a})$ and $\boldsymbol{\mu}$ be as in (2.9) and (2.7) respectively. Let L be such that (3.26) is fulfilled. Suppose that $\mathbf{a} \in C^{2d+3}(\mathbb{T})$ and set $\beta = |\mathbf{G}\boldsymbol{\alpha}||2\mathbf{n}|^{-1}$. Then there exist two numbers $\omega_0 = \omega_0(\mathbf{a})$, $\rho_0 = \rho_0(\mathbf{a}, \omega)$ such that one has*

$$\|\tilde{\mathcal{H}}_{\mathbf{a}}(\rho, \omega) u\| \ge f_- \frac{\rho}{8|\mathbf{n}|} \|u\|, \ \forall u \in C^\infty(\mathbb{T}) \tag{3.27}$$

for all $\omega \ge \omega_0, \rho \ge \rho_0$ uniformly in $\boldsymbol{\mu}_\perp : |\boldsymbol{\mu}_\perp| \le C$.

See subsection 6.2 for proof.

This theorem implies Theorem 2.2 for $V = 0$. To check this we must first convince ourselves that under the conditions of Theorem 2.2 one can find a vector $\mathbf{e} = \mathbf{F}\boldsymbol{\alpha}$ and a number L such that both (3.12) and (3.26) are fulfilled.

Lemma 3.7. *Let us fix $L \in \mathbb{N}$. Let $\boldsymbol{\alpha}$ be a vector defined by (2.9) with*

$$n_j = (rL)^{j-1}, \ r \in \mathbb{N}, \ j = 1, 2, \dots, d. \tag{3.28}$$

Then for $r > d$ one has $\boldsymbol{\alpha}\mathbf{G}\mathbf{m} \ne 0$ for all $\mathbf{m} \in K_L$.

The next lemma establishes the existence of a suitable L:

Lemma 3.8. *Let $\mathbf{a} \in H^s(\mathbb{T})$ with some $s > 3d/2 - 1$ and let $\boldsymbol{\alpha}$ be defined in (2.9) with $\mathbf{n}$ as in (3.28). Then there exists a number $L \in \mathbb{N}$ such that the vector-potential $\mathbf{a}''$ defined in (3.19) obeys the bound (3.26).*

For proof of Lemmas 3.7 and 3.8 see subsection 6.3.

Now to deduce Theorem 2.2 for $V = 0$ notice that in view of Lemma 3.3 we have the two-sided estimate $c\|u\| \le \|Wu\| \le C\|u\|$ uniform in $\rho \ge 6\omega, \omega \ge \omega_0$. Thus (3.27) ensures that for any $u \in C^\infty(\mathbb{T})$ one has

$$\begin{aligned} \|\mathcal{H}_{\mathbf{a}}(\rho) u\| \ge c' \|W \mathcal{H}_{\mathbf{a}}(\rho) u\| &\ge c' \big[\|\tilde{\mathcal{H}}_{\mathbf{a}}(\rho, \omega) W u\| - C(\omega + \rho\omega^{-1}) \|u\| \big] \\ &\ge c' \big[f_- \rho (8|\mathbf{n}|)^{-1} \|Wu\| - C(\omega + \rho\omega^{-1}) \|u\| \big] \\ &\ge c'(c'' - C\omega\rho^{-1} - C\omega^{-1}) \rho \|u\|. \end{aligned}$$

Hence for $\omega \geq 3C/c''$, $\rho \geq 3C\omega/c''$ the estimate

$$\|\mathcal{H}_{\mathbf{a}}(\rho)u\| \geq c\rho\|u\| \tag{3.29}$$

follows.

3.5. **Proof of Theorem 2.2.** To incorporate the electric potential V we establish the following estimate:

Lemma 3.9. *Let $\boldsymbol{\alpha}$ be as defined in (2.9) and $\boldsymbol{\mu}$ be of the form (2.7) with $\beta = |\mathbf{G}\boldsymbol{\alpha}||2\mathbf{n}|^{-1}$. Suppose that $V \in L^p(\Omega)$ where $p \geq 2$ if $d = 2, 3$ and $p > d/2$ if $d \geq 4$. Then*

$$\|Vu\| \leq C\rho^{-1+(d-1)p^{-1}}\big(\|\mathcal{H}_{\mathbf{a}}(\rho)u\| + \rho\|u\|\big), \ C = C(\mathbf{a}, \mathbf{n}), \tag{3.30}$$

uniformly in $\boldsymbol{\mu}_\perp : |\boldsymbol{\mu}_\perp| \leq C$ for all $\rho \geq \rho_0$ with a sufficiently big ρ_0.

See subsection 7.2 for proof.

Suppose now that the conditions of Theorem 2.2 are fulfilled. Since $p > d - 1$, according to (3.30), for sufficiently large $\rho_0 = \rho_0(\epsilon)$ we have

$$\|Vu\| \leq \epsilon\big(\|\mathcal{H}_{\mathbf{a}}(\rho)u\| + \rho\|u\|\big), \ \forall \epsilon > 0, \ \rho \geq \rho_0.$$

Thus by (3.29)

$$\begin{aligned}\|\mathcal{H}(\rho)u\| &\geq \|\mathcal{H}_{\mathbf{a}}(\rho)u\| - \|Vu\| \\ &\geq (1-\epsilon)\|\mathcal{H}_{\mathbf{a}}(\rho)u\| - \epsilon\rho\|u\| \geq \big(c(1-\epsilon) - \epsilon\big)\rho\|u\|.\end{aligned}$$

For sufficiently small ϵ we obtain the required result.

4. Pseudodifferential operators on the torus

4.1. In this section we obtain some elementary results from the calculus of pseudo-differential operators on the torus $\mathbb{T}$, which will be used in subsequent sections.

Let $b = b(\mathbf{x}, \mathbf{m})$ be a complex-valued function on $\mathbb{T} \otimes \mathbb{Z}^d$ such that $b(\,\cdot\,, \mathbf{m}) \in C^p(\mathbb{T})$, $p \geq 1$ and introduce the operator

$$\big(\mathrm{Op}(b)u\big)(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} \sum_{\mathbf{m}} b(\mathbf{x}, \mathbf{m})\hat{u}(\mathbf{m})e^{i\mathbf{m}\mathbf{x}}, \ u \in L^2(\Omega). \tag{4.1}$$

We shall be interested in symbols satisfying the condition

$$\sup_{\mathbf{x},\mathbf{m}} |\mathbf{D}^{\mathbf{s}} b(\mathbf{x}, \mathbf{m})| \leq C_{\mathbf{s}}, \ |\mathbf{s}| \leq p. \tag{4.2}$$

For such symbols introduce the norms:

$$|\!|\!| b |\!|\!|_p = \max_{|\mathbf{s}| \leq p} \sup_{\mathbf{x},\mathbf{m}} |\mathbf{D}^{\mathbf{s}} b(\mathbf{x}, \mathbf{m})|.$$

We start with proving a very elementary test of boundedness:

Lemma 4.1. *If $b \in C^{d+1}(\mathbb{T})$ obeys* (4.2)*, then the operator* $\operatorname{Op}(b)$ *is bounded on $L^2(\Omega)$ and*

$$\|\operatorname{Op}(b)\| \le C |||b|||_{d+1},$$

where the constant C depends only on d.

Proof. One checks directly that

$$\|\operatorname{Op}(b)u\|^2 = \sum_{\mathbf{m},\mathbf{n}} g(\mathbf{m},\mathbf{n})\hat{u}(\mathbf{m})\overline{\hat{u}(\mathbf{n})},$$

$$g(\mathbf{m},\mathbf{n}) = (2\pi)^{-d} \int_\Omega b(\mathbf{x},\mathbf{m})\overline{b(\mathbf{x},\mathbf{n})}e^{i(\mathbf{m}-\mathbf{n})\mathbf{x}}d\mathbf{x}.$$

Integrating by parts we see that

$$|g(\mathbf{m},\mathbf{n})| \le C |||b|||_{d+1}^2 (1 + |\mathbf{m}-\mathbf{n}|)^{-(d+1)}.$$

Therefore the coefficient by $\|u\|^2$ in the r.h.s. of the estimate

$$\|\operatorname{Op}(b)u\|^2 \le \left[\sup_{\mathbf{n}} \sum_{\mathbf{m}} |g(\mathbf{m},\mathbf{n})|\right]^{\frac{1}{2}} \left[\sup_{\mathbf{m}} \sum_{\mathbf{n}} |g(\mathbf{m},\mathbf{n})|\right]^{\frac{1}{2}} \|u\|^2,$$

is finite and satisfies the required estimate. □

From now on we assume that all the symbols we consider, satisfy (4.2).

For the next Lemma we shall need one more characteristics of the symbol b. Namely, for a symbol $b \in C^p(\mathbb{T})$ obeying the condition

$$|\mathbf{D}^{\mathbf{s}}\big(b(\mathbf{x},\mathbf{m}) - b(\mathbf{x},\mathbf{n})\big)| \le C_{\mathbf{s}}|\mathbf{m}-\mathbf{n}|^{\varkappa},\ \varkappa \ge 0,\ \forall \mathbf{m},\mathbf{n} \in \mathbb{Z}^d, \tag{4.3}$$

define the semi-norm

$$\langle b\rangle_{p,\varkappa} = \max_{|\mathbf{s}|\le p}\ \sup_{\mathbf{x},\mathbf{m},\mathbf{n}} \frac{|\mathbf{D}^{\mathbf{s}}\big(b(\mathbf{x},\mathbf{m}) - b(\mathbf{x},\mathbf{n})\big)|}{|\mathbf{m}-\mathbf{n}|^{\varkappa}}.$$

Lemma 4.2. *Suppose that $b_1 \in C^p(\mathbb{T})$, $p > d$ and $b_2 \in C^q(\mathbb{T})$, $q = p + d + t$ with some $t > 0$. Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ be subsets of $\mathbb{Z}^d$ such that* $\operatorname{dist}\{\Lambda_1,\ \Lambda_2\} \ge r \ge 0$. *Suppose also that*

$$b_1(\mathbf{x},\mathbf{m}) = 0,\ \forall \mathbf{m} \notin \Lambda_1, \quad b_2(\mathbf{x},\mathbf{m}) = 0,\ \forall \mathbf{m} \notin \Lambda_2. \tag{4.4}$$

Then the following is true:

1. *If b_1 obeys the condition* (4.3) *for some $\varkappa \in [0, t)$, then*

$$\|\operatorname{Op}(b_1)\operatorname{Op}(b_2) - \operatorname{Op}(b_1 b_2)\| \le C\langle b_1\rangle_{p,\varkappa}\ |||b_2|||_q (1 + r)^{-t+\varkappa}, \tag{4.5}$$

with a constant C depending only on p, d, t.

2. *If one of the sets* Λ_1 *or* Λ_2 *is contained in the ball* $\Lambda(R) = \{\mathbf{m} : |\mathbf{m}| \le R\}$, $R > 0$, *then*

$$\|\mathbf{D}^{\mathbf{s}} \operatorname{Op}(b_1) \operatorname{Op}(b_2)\| \le C |||b_1|||_{p+s} \ |||b_2|||_q (R^s + r^s)(1+r)^{-t}, \tag{4.6}$$

and

$$\| \operatorname{Op}(b_1) \operatorname{Op}(b_2) \mathbf{D}^{\mathbf{s}}\| \le C |||b_1|||_{p} \ |||b_2|||_q (R^s + r^s)(1+r)^{-t}, \tag{4.7}$$

for any $\mathbf{s} : s = |\mathbf{s}| < t$, *with a constant* C *depending only on* p, d, t.

Proof. Calculate directly:

$$\begin{aligned}
&(2\pi)^d (\operatorname{Op}(b_1) \operatorname{Op}(b_2) u)(\mathbf{y}) \\
&\qquad = \sum_{\mathbf{n}} b_1(\mathbf{y}, \mathbf{n}) e^{i\mathbf{n}\mathbf{y}} \sum_{\mathbf{m}} \hat{u}(\mathbf{m}) \int_\Omega e^{-i(\mathbf{n}-\mathbf{m})\mathbf{x}} b_2(\mathbf{x}, \mathbf{m}) d\mathbf{x} \\
&\qquad = \sum_{\mathbf{m}} \hat{u}(\mathbf{m}) e^{i\mathbf{m}\mathbf{y}} \sum_{\mathbf{n}} \int_\Omega e^{-i(\mathbf{n}-\mathbf{m})(\mathbf{x}-\mathbf{y})} b_1(\mathbf{y}, \mathbf{n}) b_2(\mathbf{x}, \mathbf{m}) d\mathbf{x}.
\end{aligned} \tag{4.8}$$

To prove (4.5) split this expression in two:

$$(2\pi)^d \Gamma_1 u(\mathbf{y}) = \sum_{\mathbf{m}} \hat{u}(\mathbf{m}) e^{i\mathbf{m}\mathbf{y}} \sum_{\mathbf{n}} \int_\Omega e^{-i(\mathbf{n}-\mathbf{m})(\mathbf{x}-\mathbf{y})} b_1(\mathbf{y}, \mathbf{m}) b_2(\mathbf{x}, \mathbf{m}) d\mathbf{x}$$

$$(2\pi)^d \Gamma_2 u(\mathbf{y}) = \sum_{\mathbf{m}} g(\mathbf{y}, \mathbf{m}) \hat{u}(\mathbf{m}) e^{i\mathbf{m}\mathbf{y}},$$

$$g(\mathbf{y}, \mathbf{m}) = \sum_{\mathbf{n}} \big(b_1(\mathbf{y}, \mathbf{n}) - b_1(\mathbf{y}, \mathbf{m})\big) K(\mathbf{y}, \mathbf{m}, \mathbf{n});$$

$$K(\mathbf{y}, \mathbf{m}, \mathbf{n}) = \int_\Omega \exp\big[-i(\mathbf{n}-\mathbf{m})(\mathbf{x}-\mathbf{y})\big] b_2(\mathbf{x}, \mathbf{m}) d\mathbf{x}.$$

By the Poisson summation formula

$$\sum_{\mathbf{n}} e^{-i\mathbf{n}(\mathbf{x}-\mathbf{y})} = (2\pi)^{\frac{d}{2}} \delta(\mathbf{x}-\mathbf{y}),$$

so that $\Gamma_1 = \operatorname{Op}(b_1 b_2)$. To handle Γ_2 we integrate by parts to conclude that for any $\mathbf{l}$

$$|\mathbf{D}^{\mathbf{l}} K(\mathbf{y}, \mathbf{m}, \mathbf{n})| \le C(1 + |\mathbf{n}-\mathbf{m}|)^{|\mathbf{l}|-q} |||b_2|||_q, \tag{4.9}$$

and hence

$$|||g|||_p \le C \langle b_1 \rangle_{p,\varkappa} |||b_2|||_q \sup_{\mathbf{m} \in \Lambda_2} \sum_{\mathbf{n} : |\mathbf{n}-\mathbf{m}| \ge r} (1 + |\mathbf{m}-\mathbf{n}|)^{p-q+\varkappa}.$$

For $q = p + d + t$ and $t > \varkappa$, this produces the bound (4.5).

As far as the inequalities (4.6) and (4.7) are concerned, we shall prove only (4.6). Applying $\mathbf{D}^{\mathbf{s}}$ to (4.8) directly, we conclude that the function $\mathbf{D}^{\mathbf{s}}\operatorname{Op}(b_1)\operatorname{Op}(b_2)u(\mathbf{x})$ is a linear combination of terms having the form

$$\sum_{\mathbf{n}} \mathbf{n}^{\mathbf{s}_2}\mathbf{D}^{\mathbf{s}_1} b_1(\mathbf{y},\mathbf{n})e^{i\mathbf{n}\mathbf{y}} \sum_{\mathbf{m}} \hat{u}(\mathbf{m}) \int_{\Omega} e^{-i(\mathbf{n}-\mathbf{m})\mathbf{x}} b_2(\mathbf{x},\mathbf{m})d\mathbf{x}$$
$$= \sum_{\mathbf{m}} \tilde{g}(\mathbf{y},\mathbf{m})\hat{u}(\mathbf{m})e^{i\mathbf{m}\mathbf{y}},$$

with $\mathbf{s}_1+\mathbf{s}_2=\mathbf{s}$ and

$$\tilde{g}(\mathbf{y},\mathbf{m}) = \sum_{\mathbf{n}} \mathbf{n}^{\mathbf{s}_2}\mathbf{D}^{\mathbf{s}_1} b_1(\mathbf{y},\mathbf{n})K(\mathbf{y},\mathbf{m},\mathbf{n}).$$

In view of (4.9) and (4.4)

$$|||\tilde{g}|||_p \le C|||b_1|||_{p+s}\ |||b_2|||_q \sup_{\mathbf{m}\in\Lambda_2} \sum_{\mathbf{n}\in\Lambda_1} |\mathbf{n}|^s(1+|\mathbf{n}-\mathbf{m}|)^{p-q}.$$

If $\Lambda_1\subset\Lambda$, then $|\mathbf{n}|^s \le R^s$ and thus (4.6) follows. If $\Lambda_2\subset\Lambda$, then $|\mathbf{n}|^s \le C(|\mathbf{n}-\mathbf{m}|^s+R^s)$ which again leads to (4.6). □

This Lemma has the following simple but useful consequences:

Corollary 4.3. *Let $b\in C^q(\mathbb{T})$ be a symbol, obeying* (4.2), (4.3) *and let $q>2d+1+\varkappa$. Suppose that*

$$|b(\mathbf{x},\mathbf{m})| \ge c > 0, \forall \mathbf{x}\in\Omega,\ \ \mathbf{m}\in\mathbb{Z}^d.$$

Then there exists a number $\delta_0>0$, which depends only on $d,\varkappa,q$, such that under the condition

$$\langle b\rangle_{d+1,\varkappa}|||b^{-1}|||_q + \langle b^{-1}\rangle_{d+1,\varkappa}|||b|||_q \le \delta_0$$

the operator $T=\operatorname{Op}(b)$ is boundedly invertible, T^{-1} is defined on $L^2(\Omega)$ and

$$\|T\| \le C|||b|||_{d+1},\quad \|T^{-1}\| \le C|||b^{-1}|||_{d+1}, \tag{4.10}$$

with a constant C depending only on δ_0 and the dimension d.

Proof. Denote $T^\dagger=\operatorname{Op}(b^{-1})$. Notice that $\operatorname{Op}(bb^{-1})=I$ and hence according to Lemma 4.2

$$\|TT^\dagger - I\| \le C\langle b\rangle_{d+1,\varkappa}|||b^{-1}|||_q \le C\delta_0,$$
$$\|T^\dagger T - I\| \le C\langle b^{-1}\rangle_{d+1,\varkappa}|||b|||_q \le C\delta_0.$$

Thus for sufficiently small δ_0 the operators T and $T^\dagger$ are both invertible and have bounded inverse operators, defined on $L^2(\Omega)$:

$$T^{-1} = T^\dagger\big(TT^\dagger\big)^{-1},\ \ (T^\dagger)^{-1} = T\big(T^\dagger T\big)^{-1}.$$

Consequently

$$\|T^{-1}\| \le (1 - C\delta_0)^{-1}\|T^\dagger\|, \quad \|(T^\dagger)^{-1}\| \le (1 - C\delta_0)^{-1}\|T\|.$$

Now the estimates (4.10) follow from Lemma 4.1. $\square$

Corollary 4.4. *Let* $b_1, b_2 \in C^q(\mathbb{T})$, $q = 2d + 1 + t$, $t > 0$, *be two symbols satisfying* (4.2) *and* (4.3) *and with* $\varkappa < t$. *Then*

$$\|\big[\operatorname{Op}(b_1), \operatorname{Op}(b_2)\big]\| \le C\big[\langle b_1\rangle_{d+1,\varkappa} |||b_2|||_q + \langle b_2\rangle_{d+1,\varkappa} |||b_1|||_q\big], \tag{4.11}$$

with a constant C *depending only on* d.

Suppose in addition that b_1, b_2 *satisfy* (4.4) *and that either* Λ_1 *or* Λ_2 *is contained in the ball* $\Lambda(R) = \{\mathbf{m} : |\mathbf{m}| \le R\}$, $R > 0$. *Then*

$$\begin{aligned}\|\big[\operatorname{Op}(b_1), \operatorname{Op}(b_2)\big]\mathbf{D}^{\mathbf{s}}\| \le C\big[\langle b_1\rangle_{d+1,\varkappa} |||b_2|||_q + \langle b_2\rangle_{d+1,\varkappa} |||b_1|||_q\big] R^s \\ + \big[|||b_1|||_{d+1} |||b_2|||_q + |||b_2|||_{d+1} |||b_1|||_q\big] R^s(1+R)^{-t},\end{aligned} \tag{4.12}$$

for any $\mathbf{s}, |\mathbf{s}| < t$.

Proof. Inequality (4.11) results directly from (4.5) with $\Lambda_1 = \Lambda_2 = \mathbb{Z}^d$ and $r = 0$.

Let $\chi = \chi(\mathbf{m})$ be the characteristic function of the ball $\Lambda(2R)$. Represent the operator in the l.h.s. of (4.12) as follows:

$$\begin{aligned}&[\operatorname{Op}(b_1), \operatorname{Op}(b_2)]\chi(\mathbf{D})\mathbf{D}^{\mathbf{s}} \\ &\quad + \operatorname{Op}(b_1)\operatorname{Op}(b_2)(1 - \chi(\mathbf{D}))\mathbf{D}^{\mathbf{s}} - \operatorname{Op}(b_2)\operatorname{Op}(b_1)(1 - \chi(\mathbf{D}))\mathbf{D}^{\mathbf{s}}.\end{aligned}$$

In view of (4.11) the first summand obeys (4.12). Since b_1 and b_2 enter (4.12) symmetrically, it suffices to prove this inequality assuming that $\Lambda_1 \subset \Lambda(R)$. Then the third operator in the above sum turns into zero. To establish (4.12) for the second operator rewrite it in the form

$$\operatorname{Op}(b_1)\operatorname{Op}(\tilde b_2)\mathbf{D}^{\mathbf{s}}, \ \ \tilde b_2(\mathbf{x}, \mathbf{m}) = b_2(\mathbf{x}, \mathbf{m})(1 - \chi(\mathbf{m})).$$

Now Lemma 4.2 ensures that its norm is bounded by $|||b_1|||_{d+1} |||b_2|||_q R^s(1+R)^{-t}$, as required. $\square$

4.2. Here we establish some useful properties of the functions $\zeta_j(\mathbf{m}; \rho, \omega)$ and $\boldsymbol{\nu}(\mathbf{m})$ defined in (3.3) and (3.5), viewed as symbols.

The function $\zeta_j(\mathbf{m}; \rho, \omega)$, $j = 1, 2$, satisfies (4.3) for any p and $\varkappa = 1$, and

$$\langle \zeta_j(\,\cdot\,; \rho, \omega)\rangle_{p,1} \le C\omega^{-1}, \ \forall p \ge 0, \ \rho, \omega > 0. \tag{4.13}$$

Since $\zeta_j(\,\cdot\,;\rho,\omega)$ is supported in the ball $\Lambda\big(f_-^{-1}(\rho+2j\omega+|\boldsymbol{\mu}|)\big)$, Corollary 4.4 with $t=2$ and Lemma 4.2 provide the bound

$$\begin{aligned}(4.14)\quad \|[\zeta_j(\mathbf{D};\rho,\omega),v]\mathbf{D}^{\mathbf{s}}\| \le C(\rho+\omega+1)^s(1+\omega)^{-1}|\!|\!|v|\!|\!|_{2d+3},\\ \forall v\in C^{2d+3}(\mathbb{T}),\quad \rho,\omega>0,\ s=|\mathbf{s}|\le 1.\end{aligned}$$

For any $\mathbf{m}\in\mathbb{Z}^d$ and a fixed $\boldsymbol{\mu}\in\mathbb{R}^d\setminus\mathbb{Z}^d$ define

$$\boldsymbol{\eta}(\mathbf{m};\rho,\omega)=\zeta_2(\mathbf{m};\rho,\omega)\boldsymbol{\nu}(\mathbf{m}). \tag{4.15}$$

Taking into account (4.13) and the fact that

$$|\boldsymbol{\nu}(\mathbf{m})-\boldsymbol{\nu}(\mathbf{n})|\le \frac{2f_+}{f_-}\frac{|\mathbf{m}-\mathbf{n}|}{\max\{|\mathbf{m}+\boldsymbol{\mu}|,|\mathbf{n}+\boldsymbol{\mu}|\}}, \tag{4.16}$$

we see that

$$\langle\boldsymbol{\eta}(\,\cdot\,;\rho,\omega)\rangle_{p,1}\le C\omega^{-1},\ \forall p\ge 0,\ \rho\ge 6\omega>0. \tag{4.17}$$

5. Gauge transformation. Proof of Theorem 3.5

5.1. **Scalar potential for the operator $T_{\mathbf{a}}^{(\gamma)}$.** In this subsection we find a gauge function ψ which guarantees the validity of (3.10).

Proof of Lemma 3.2. Passing to the Fourier transform, we obtain from (3.9)

$$(i\mathbf{e}+\boldsymbol{\gamma})\mathbf{F}\mathbf{m}\hat{\psi}(\mathbf{m})=-(i\mathbf{e}+\boldsymbol{\gamma})\mathbf{F}\hat{\mathbf{a}}(\mathbf{m}).$$

In view of (3.12) this equation (uniquely) defines the Fourier coefficients $\hat{\psi}(\mathbf{m})$ for all $\mathbf{m}\in K_L$:

$$\hat{\psi}(\mathbf{m})=-\frac{(i\mathbf{e}+\boldsymbol{\gamma})\mathbf{F}\hat{\mathbf{a}}(\mathbf{m})}{i\mathbf{e}\mathbf{F}\mathbf{m}+\boldsymbol{\gamma}\mathbf{F}\mathbf{m}} \tag{5.1}$$

Extending $\hat{\psi}$ to all $\mathbf{m}\in\mathbb{Z}^d$ in accordance with (3.13), we guarantee the uniqueness of ψ.

The estimates (3.14), (3.15) follow directly from (5.1) if one takes into account that

$$|\mathbf{e}\mathbf{F}\mathbf{m}|\ge c=c(\mathbf{e},L),\ \forall\mathbf{m}\in K_L.$$

Let us prove, for instance, the estimate (3.14). According to (5.1) and (3.13) $|\mathbf{m}|^s|\hat{\psi}(\mathbf{m})|\le 2L^{1+s}c^{-2}|\hat{\mathbf{a}}(\mathbf{m})|$, $\forall s\ge 0$, so that

$$(2\pi)^{\frac{d}{2}}|D^{\mathbf{s}}\psi(\mathbf{x})|\le\left|\sum_{|\mathbf{m}|\le L}|\mathbf{m}|^s\hat{\psi}(\mathbf{m})e^{i\mathbf{m}\mathbf{x}}\right|\le c^{-2}L^{1+s}\sum_{|\mathbf{m}|\le L}|\hat{\mathbf{a}}(\mathbf{m})|,\ |\mathbf{s}|\le s.$$

which leads to (3.14). $\square$

5.2. **Gauge transformation for $T_{\mathbf{a}}(\rho)$: trigonometric polynomials.** Let the function $\psi(\mathbf{a};\mathbf{x},\mathbf{m})$ and the operator $W(\mathbf{a};\rho,\omega)$ be defined in (3.16) and (3.17) respectively. We point out some estimates resulting from (3.14), (3.15), (4.13) and (4.16):

$$\begin{cases} |||\psi|||_p + |||e^{\pm\psi}|||_p \le C_p, \ \rho,\omega > 0, \\ \langle\psi\rangle_{p,1} + \langle e^{\pm\psi}\rangle_{p,1} \le C_p\omega^{-1}, \ \rho \ge 6\omega > 0, \end{cases} \qquad \forall p \ge 0. \tag{5.2}$$

Here and below the constants C are independent of ρ,ω and $\boldsymbol{\mu} : |\boldsymbol{\mu}| \le C$, but may depend on $\mathbf{a}$. We are now able to give

Proof of Lemma 3.3. The estimate $\|W\| \le C$ follows from Lemma 4.1. Statement 1 results from the fact that $\psi(\,\cdot\,,\mathbf{m};\rho,\omega) \in C^\infty(\mathbb{T})$ and the estimates (3.14). Statement 3 is a direct consequence of Corollary 4.3 and (5.2).

To prove statement 2 write $[W,v] = -[I-W,v]$ and recall that the symbol $1-e^{\psi}$ of the operator $I-W$ is supported inside the ball $\Lambda\big(f_-^{-1}(\rho+4\omega+|\boldsymbol{\mu}|)\big)$. Hence by (5.2) and the second part of Corollary 4.4 with $t=2$ the estimate (3.18) holds. □

5.3. **General a. Proof of Theorem 3.4.** Let us split $\mathbf{a}$ into the sum $\mathbf{a}'$ and $\mathbf{a}''$ as in (3.19) and assume that $\boldsymbol{\alpha}$ obeys (3.12).

Step 1. Let $W = W(\mathbf{a}';\rho,\omega)$ and $\mathcal{H}_0 = \mathcal{H}_0(\rho)$. Compute first the commutator $[\mathcal{H}_0, W]$. Note that

$$\begin{aligned}[(\mathbf{D}+\boldsymbol{\mu})\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}), W] &= (\mathbf{D}+\boldsymbol{\mu})\mathbf{G}[\mathbf{D},W] + [\mathbf{D},W]\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}) \\ &= 2(\mathbf{D}+\boldsymbol{\mu})\mathbf{G}[\mathbf{D},W] - [\mathbf{D},\mathbf{G}[\mathbf{D},W]].\end{aligned}$$

Thus

$$[\mathcal{H}_0, W] = 2(\mathbf{D}+\boldsymbol{\mu})\mathbf{G}[\mathbf{D},W] - [\mathbf{D},\mathbf{G}[\mathbf{D},W]] + 2i\rho\boldsymbol{\alpha}\mathbf{G}[\mathbf{D},W].$$

It is easy to see that

$$[\mathcal{X}, W] = -2\mathbf{a}\mathbf{G}[\mathbf{D},W] - 2[\mathbf{a},W]\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}) - 2i\rho\boldsymbol{\alpha}\mathbf{G}[\mathbf{a},W] + [\mathbf{a}\mathbf{G}\mathbf{a},W],$$

so that

$$\begin{aligned} &[\mathcal{H}_{\mathbf{a}}, W] = \mathcal{C}_1 + \mathcal{C}_2; \\ &\qquad \mathcal{C}_1 = 2(\mathbf{D}+\boldsymbol{\mu})\mathbf{G}[\mathbf{D},W] + 2i\rho\boldsymbol{\alpha}\mathbf{G}[\mathbf{D},W], \\ &\qquad \mathcal{C}_2 = \ -2[\mathbf{a},W]\mathbf{G}(\mathbf{D}+\boldsymbol{\mu}) - 2i\rho\boldsymbol{\alpha}\mathbf{G}[\mathbf{a},W] \\ &\qquad\qquad + [\mathbf{a}\mathbf{G}\mathbf{a},W] - 2\mathbf{a}\mathbf{G}[\mathbf{D},W] - [\mathbf{D},\mathbf{G}[\mathbf{D},W]]. \end{aligned} \tag{5.3}$$

Step 2. Let g be the symbol from (3.22). We shall show that

$$\|\mathcal{C}_1 - 2\rho\,\mathrm{Op}(g)\,\mathrm{Op}(e^{\psi})\| \le C(\omega + \rho\omega^{-1}), \tag{5.4}$$

for any $\omega > 0$ and $\rho \ge 18\omega$. First of all check that

$$\|(\mathbf{D}+\boldsymbol{\mu})\mathbf{G}[\mathbf{D},W] - (\mathbf{D}+\boldsymbol{\mu})\zeta_2(\mathbf{D};\rho,3\omega)\mathbf{G}[\mathbf{D},W]\| \\ \le C_n(\omega+\rho+1)(1+\omega)^{-n},\ \forall n\ge 2, \tag{5.5}$$

for all $\omega, \rho > 0$. To this end consider the operator in the l.h.s. as the product of the three operators $\mathbf{G}(\mathbf{D}+\boldsymbol{\mu})$, $(1-\zeta_2(\mathbf{D};\rho,3\omega))$ and $[\mathbf{D},W]$. Notice that the operator $[\mathbf{D},W]$ is a PDO with the symbol $\mathbf{D}\psi(\mathbf{x},\mathbf{m})\exp\psi(\mathbf{x},\mathbf{m})$, which is supported inside the set $M(\rho,4\omega)\subset \Lambda\big(f_-^{-1}(\rho+4\omega+|\boldsymbol{\mu}|)\big)$. On the other hand, the symbol $1-\zeta_2(\mathbf{m};\rho,3\omega)$ is supported outside $M(\rho,6\omega)$. For the distance between these two sets is bounded below by $2f_+^{-1}\omega$, in view (4.6) the bound (5.5) is valid.

Now, by definitions (3.3), (3.5) and (4.15)

$$\|\mathbf{F}(\mathbf{D}+\boldsymbol{\mu})\zeta_2(\mathbf{D};\rho,3\omega) - \rho\boldsymbol{\eta}(\mathbf{D};\rho,3\omega)\| \\ \le \max_{\mathbf{m}} \big||\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho\big|\zeta_2(\mathbf{m};\rho,3\omega) \le 12\omega,$$

and hence

$$\begin{aligned}\|\mathcal{C}_1 - 2\rho(\boldsymbol{\eta}(\mathbf{D};\rho,3\omega)+i\mathbf{e})\mathbf{F}\operatorname{Op}(\mathbf{D}\psi e^{\psi})\| &\le \\ \le 2\|(\mathbf{D}+\boldsymbol{\mu})\mathbf{G}(1-\zeta_2(\mathbf{D};\rho,3\omega))[\mathbf{D},W]\| & \\ + 2\|\big[\mathbf{F}(\mathbf{D}+\boldsymbol{\mu})\zeta_2(\mathbf{D};\rho,3\omega) - \rho\boldsymbol{\eta}(\mathbf{D};\rho,3\omega)\big]\mathbf{F}[\mathbf{D},W]\| & \\ \le 2C_n(\rho+\omega+1)(1+\omega)^{-n} + C\omega,\ \forall\omega,\rho>0. &\end{aligned}$$

Observe that by (5.2) and Lemma 4.2

$$\|\operatorname{Op}(\mathbf{D}\psi e^{\psi}) - \operatorname{Op}(\mathbf{D}\psi)W\| \le C(1+\omega)^{-1},\ \rho\ge 6\omega>0,$$

and by (3.22), (4.17) and Lemma 4.2

$$\|\big(\boldsymbol{\eta}(\mathbf{D};\rho,3\omega)+i\mathbf{e}\big)\operatorname{Op}(\mathbf{D}\psi) - \operatorname{Op}(g)\| \le C(1+\omega)^{-1},\ \rho\ge 18\omega>0.$$

Now (5.4) follows.

Step 3. Finally, we prove that

$$\|\mathcal{C}_2\| \le C\rho\omega^{-1},$$

for any $\omega > 0$ and $\rho \ge 6\omega$. Indeed, the operators $[\mathbf{D},W]$ and $[\mathbf{D},\mathbf{G}[\mathbf{D},W]]$ are PDO's with the symbols $\mathbf{D}\psi e^{\psi}$ and $(\mathbf{D}\mathbf{G}\mathbf{D}\psi + \mathbf{D}\psi\mathbf{G}\mathbf{D}\psi)e^{\psi}$ respectively. Therefore by Lemma 4.1 the last three operators in the definition (5.3) are bounded uniformly in ρ,ω.

The first two operators in (5.3) are bounded by $C\rho\omega^{-1}$ in view of (3.18).

Combining steps 1, 2 and 3 we arrive at the bound

$$\|W\mathcal{H}_{\mathbf{a}} - \mathcal{H}_{\mathbf{a}}W + 2\rho\operatorname{Op}(g)W\| \le C(\omega+\rho\omega^{-1}).$$

Using the definition (3.21), we obtain (3.23). The proof of Theorem 3.4 is completed.

5.4. **Reduction to $T_0(\rho)$: proof of Theorem 3.5.** We begin with proving Theorem 3.1.

Proof of Theorem 3.1. Denote

$$S = \mathcal{H}_{\mathbf{a}}(\rho)\zeta_1(\mathbf{D};\rho,\omega).$$

Let $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{m})$ be the unit vector defined in (3.5). Let us look at the symbol (3.1). Write

$$\begin{aligned}(\mathbf{m}+\boldsymbol{\mu})\mathbf{G}(\mathbf{m}+\boldsymbol{\mu}) - \rho^2 \\ = |\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|^2 - \rho^2 = (|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho)(|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| + \rho).\end{aligned}$$

Therefore, taking into account the estimate

$$||\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho| \le 2\omega,\ \ \mathbf{m}\in M(\rho, 2\omega),$$

we have

$$|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|^2 - \rho^2 - 2\rho(|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| - \rho) \le 4\omega^2.$$

Since $|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})| = \boldsymbol{\nu}(\mathbf{m})\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})$, by (3.1) for $\mathbf{m}\in M(\rho,2\omega)$ we have

$$h(\mathbf{m},\rho) = h^{(0)}(\mathbf{m},\rho) + h^{(1)}(\mathbf{m},\rho),$$

where

$$h^{(0)}(\mathbf{m},\rho) = 2\rho\big[i\mathbf{e}\mathbf{F}(\mathbf{m}+\boldsymbol{\mu}) + \boldsymbol{\nu}\mathbf{F}(\mathbf{m}+\boldsymbol{\mu}) - \rho\big],\ \ \mathbf{e} = \mathbf{F}\boldsymbol{\alpha},$$

and the error $h^{(1)}(\mathbf{m},\rho)$ obeys the estimate $|h^{(1)}(\mathbf{m},\rho)| \le 4\omega^2$. Similarly, the operator $\mathcal{X}(\rho)$ from (2.8) can be decomposed as follows:

$$\mathcal{X}(\rho) = \mathcal{X}^{(0)}(\rho) + \mathcal{X}^{(1)}(\rho),$$

with

$$\begin{aligned}\mathcal{X}^{(0)}(\rho) &= -2\rho\mathbf{a}\mathbf{F}\boldsymbol{\nu}(\mathbf{D}) - 2i\rho\mathbf{e}\mathbf{F}\mathbf{a},\\ \mathcal{X}^{(1)}(\rho) &= -2\mathbf{a}\mathbf{F}\big(\mathbf{F}(\mathbf{D}+\boldsymbol{\mu}) - \boldsymbol{\rho}(\mathbf{D})\big) + \mathbf{a}\mathbf{G}\mathbf{a},\ \ \boldsymbol{\rho}(\mathbf{m}) = \rho\boldsymbol{\nu}(\mathbf{m}).\end{aligned}$$

Clearly $\|\mathcal{X}^{(1)}(\rho)\zeta_1(\mathbf{D};\rho,\omega)\| \le 4f_+\omega\|\mathbf{a}\|_{L^\infty} + f_+^2\|\mathbf{a}\|^2_{L^\infty}$. Therefore the operator S can be represented in the form $S = S^{(0)} + S^{(1)}$, with the operator $S^{(1)} = \big(\mathrm{Op}(h^{(1)}) + \mathcal{X}^{(1)}\big)\zeta(\mathbf{D};\rho,\omega)$, which satisfies the bound

$$\|S^{(1)}\| \le (2\omega + f_+\|\mathbf{a}\|_{L^\infty})^2, \tag{5.6}$$

and

$$S^{(0)} = 2\rho\big[i(\mathbf{D}+\boldsymbol{\mu}-\mathbf{a})\mathbf{F}\mathbf{e} + (\mathbf{D}+\boldsymbol{\mu}-\mathbf{a})\mathbf{F}\boldsymbol{\nu}(\mathbf{D}) - \rho\big]\zeta_1(\mathbf{D};\rho,\omega).$$

Comparing $S^{(0)}$ with (3.4), we arrive at the representation

$$S^{(0)} = 2\rho T_{\mathbf{a}}(\rho)\zeta_1(\mathbf{D};\rho,\omega).$$

Now this equality and (5.6) lead to (3.6). □

Proof of Theorem 3.5. We begin with proving (3.24). Observe that by (3.4)

$$(5.7)\quad \big(T_{\mathbf{a}}(\rho) - T_0(\rho) + \mathbf{a}'\mathbf{F}(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{D}))\big)\zeta_1(\mathbf{D};\rho,\omega) \\ = -\mathbf{a}''\mathbf{F}(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{D}))\zeta_1(\mathbf{D};\rho,\omega)$$

for any $\rho, \omega > 0$. Thus by (3.20)

$$\|\big(\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) - 2\rho T_0(\rho)\big)\zeta_1(\mathbf{D};\rho,\omega)\| \\ \le \|\big(\mathcal{H}_{\mathbf{a}}(\rho) - 2\rho T_{\mathbf{a}}(\rho)\big)\zeta_1(\mathbf{D};\rho,\omega)\| + 4\rho f_+\|\mathbf{a}''\|_{L^\infty}.$$

Now (3.6) ensures (3.24).

To prove (3.25) notice that $\zeta_1 = \zeta_1(\mathbf{D};\rho,\omega)$ commutes with $\mathcal{H}_0(\rho)$ and $T_0(\rho)$, so that by (2.8), (3.20)

$$[\zeta_1, \tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) - 2\rho T_0(\rho)] = [\zeta_1, \mathcal{X}(\rho)] + 2\rho[\zeta_1, \mathbf{a}']\mathbf{F}\big(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{D})\big)\zeta_2(\mathbf{D};\rho,\omega) \\ = -2[\zeta_1, \mathbf{a}]\mathbf{G}(\mathbf{D} + \boldsymbol{\mu}) - 2i\rho\boldsymbol{\alpha}\mathbf{G}[\zeta_1, \mathbf{a}] + [\zeta_1, \mathbf{aGa}] \\ + 2\rho[\zeta_1, \mathbf{a}']\mathbf{F}\big(i\mathbf{e} + \boldsymbol{\nu}(\mathbf{D})\big)\zeta_2(\mathbf{D};\rho,\omega).$$

The terms in the r.h.s. are bounded by

$$C(\rho+1)(\omega+1)^{-1}\big(|||\mathbf{a}|||_{2d+3}^2 + |||\mathbf{a}|||_{2d+3} + |||\mathbf{a}'|||_{2d+3}\big)$$

by virtue of (4.14). Along with (3.24) this estimate leads to (3.25). □

6. Lower bounds: proof of Theorem 3.6

6.1. **Lower bounds on the subspaces $\mathfrak{M}$ and $\mathfrak{N}$.** The proof of Theorem 3.6 relies on the following two lemmas:

Lemma 6.1. *Let $\boldsymbol{\alpha}$ be as in (2.9), $\mathbf{a} \in L^\infty(\Omega)$ and $\mathbf{a}''$ obey (3.26). Let $\boldsymbol{\mu}$ be as in (2.7) with $\beta = |\mathbf{G}\boldsymbol{\alpha}||2\mathbf{n}|^{-1}$. Then there exist positive numbers $\omega_0 = \omega_0(\mathbf{a})$, $\rho_0 = \rho_0(\mathbf{a},\omega)$ such that*

$$(6.1)\qquad \|\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)P(\rho,\omega)u\| \ge f_- \frac{\rho}{2|\mathbf{n}|}\|P(\rho,\omega)u\|,\ \forall u \in C^\infty(\mathbb{T}),$$

for all $\omega \ge \omega_0$, $\rho \ge \rho_0$, uniformly in $\boldsymbol{\mu}_\perp : |\boldsymbol{\mu}_\perp| \le C$.

Proof. For $P(\rho,\omega) = P(\rho,\omega)\zeta_1(\mathbf{D};\rho,\omega)$, we have for $u = P(\rho,\omega)u$

$$\|\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)u\| \ge 2\rho\|T_0(\rho)u\| - \|\big(\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega) - 2\rho T_0(\rho)\big)\zeta_1(\mathbf{D};\rho,\omega)\|\|u\|.$$

Now (6.1) results from (3.8), (3.26) and (3.24). □

Lemma 6.2. *There exists a number $\omega_0 = \omega_0(\mathbf{a})$ such that for all $\omega \geq \omega_0, \rho \geq 4$ one has*

$$\|\tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)Q(\rho,\omega)u\| \geq \frac{\omega\rho}{2}\|Q(\rho,\omega)u\|,\ \forall u \in C^\infty(\mathbb{T}), \tag{6.2}$$

for all $\mathbf{e} \in \mathbb{S}^{d-1}$.

Before we proceed with the proof of this Lemma we establish a relative bound for the perturbation $\mathcal{X}(\rho)$ which will be also useful in Section 7. Set $A = \max_{\mathbf{x}} |\mathbf{a}(\mathbf{x})|$. By (2.8)

$$\begin{aligned}\|\mathcal{X}(\rho)u\| &\leq 2\|\mathbf{aG}(\mathbf{D}+\boldsymbol{\mu})u\| + 2\rho\|\mathbf{Fa}u\| + \|\mathbf{aGa}u\|\\ &\leq 2Af_+\big(\|\mathbf{F}(\mathbf{D}+\boldsymbol{\mu})u\| + (f_+A+\rho)\|u\|\big)\\ &\leq 2Af_+\Big[(2Af_+\rho)^{-1}\||\mathbf{F}(\mathbf{D}+\boldsymbol{\mu})|^2u\| + (f_+A + f_+A\rho + \rho)\|u\|\Big].\end{aligned}$$

Using the fact that

$$\begin{aligned}\||\mathbf{F}(\mathbf{D}+\boldsymbol{\mu})|^2u\| &\leq \|\big[|\mathbf{F}(\mathbf{D}+\boldsymbol{\mu})|^2 + 2i\rho\mathbf{eF}(\mathbf{D}+\boldsymbol{\mu})\big]u\|\\ &\leq \|\mathcal{H}_0(\rho)u\| + \rho^2\|u\|,\end{aligned}$$

we obtain from here that

$$\|\mathcal{X}(\rho)u\| \leq \rho^{-1}\|\mathcal{H}_0(\rho)u\| + C(\rho+1)\|u\|, \tag{6.3}$$

with some constat C depending on $\mathbf{a}$ and f_+.

Proof of Lemma 6.2. Since the second operator in the r.h.s. of (3.20) is bounded by $C\rho, C = C(\mathbf{a})$, uniformly in ω, it suffices to establish the bound of the form (6.2) for the operator $\mathcal{H}_{\mathbf{a}}(\rho)$ only. Precisely,

$$\|\mathcal{H}_{\mathbf{a}}(\rho)Qu\| \geq \frac{3\omega\rho}{4}\|Qu\| - C\rho\|Qu\|. \tag{6.4}$$

It follows from (2.8) and (6.3) that for $\rho \geq 4$ one has

$$\|\mathcal{H}_{\mathbf{a}}(\rho)u\| \geq 3/4\|\mathcal{H}_0(\rho)u\| - C\rho\|u\|.$$

Now (6.4) follows from (3.2). □

6.2. **Proof of Theorem 3.6.** Let $P = P(\rho,\omega)$, $Q = Q(\rho,\omega)$ and $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{\mathbf{a}}(\rho,\omega)$. Notice that

$$\|\tilde{\mathcal{H}}u\|^2 = \|\tilde{\mathcal{H}}Pu\|^2 + \|\tilde{\mathcal{H}}Qu\|^2 + 2\,\mathrm{Re}\langle\tilde{\mathcal{H}}Pu, \tilde{\mathcal{H}}Qu\rangle.$$

For any $\epsilon > 0$ we have

$$\begin{aligned}2\,\mathrm{Re}\langle\tilde{\mathcal{H}}Pu, \tilde{\mathcal{H}}Qu\rangle \leq\ &\epsilon(\|\tilde{\mathcal{H}}Pu\|^2 + \|\tilde{\mathcal{H}}Qu\|^2)\\ &+ \epsilon^{-1}(\|Q\tilde{\mathcal{H}}Pu\|^2 + \|P\tilde{\mathcal{H}}Qu\|^2).\end{aligned}$$

Thus for any $\epsilon > 0$

$$\|\tilde{\mathcal{H}}u\|^2 \geq (1-\epsilon)(\|\tilde{\mathcal{H}}Pu\|^2 + \|\tilde{\mathcal{H}}Qu\|^2) - \epsilon^{-1}(\|Q\tilde{\mathcal{H}}Pu\|^2 + \|P\tilde{\mathcal{H}}Qu\|^2). \tag{6.5}$$

Since $P = \zeta_1(\mathbf{D};\rho,\omega)P$ and $QT_0(\rho)P = 0$, from (3.24) and (3.26) we obtain

$$\|Q\tilde{\mathcal{H}}Pu\| \leq \big[f_-\rho(8|\mathbf{n}|)^{-1} + C(\omega^2+1)\big]\|Pu\|, \ C = C(\mathbf{a},\mathbf{F}).$$

For sufficiently large $\rho_0 = \rho_0(\omega,\mathbf{a})$ and $\rho \geq \rho_0$ this quantity is bounded by

$$\frac{3f_-\rho}{16|\mathbf{n}|}\|Pu\|.$$

Analogously with the help of (3.25) we prove that

$$\|P\tilde{\mathcal{H}}Qu\| \leq \frac{3f_-\rho}{16|\mathbf{n}|}\|Qu\|.$$

Furthermore from (6.2) and (6.1), we obtain that

$$\|\tilde{\mathcal{H}}Pu\|^2 + \|\tilde{\mathcal{H}}Qu\|^2 \geq \left(\frac{f_-\rho}{2|\mathbf{n}|}\right)^2\|u\|^2$$

for $\omega \geq \omega_0$ with a sufficiently large ω_0. Therefore (6.5) leads to the estimate

$$\begin{aligned}\|\tilde{\mathcal{H}}u\|^2 &\geq (1-\epsilon)\left(\frac{f_-\rho}{2|\mathbf{n}|}\right)^2\|u\|^2 - \epsilon^{-1}\left(\frac{3f_-\rho}{16|\mathbf{n}|}\right)^2\|u\|^2 \\ &= \left(\frac{f_-\rho}{2|\mathbf{n}|}\right)^2\big[(1-\epsilon) - \epsilon^{-1}(3/8)^2\big]\|u\|^2.\end{aligned}$$

Set $\epsilon = 1/2$ and notice that the expression in the brackets is bounded from below by $1/16$. Clearly, this implies (3.27).

6.3. **Proof of Lemmas 3.7 and 3.8.**

Proof of Lemma 3.7. For $l \leq d$ it follows from the equation

$$\sum_{j=1}^{l}(\mathbf{G}\boldsymbol{\alpha})_j m_j = 0, \ m \in K_L,$$

that

$$|m_l| \leq (rL)^{-l+1}\sum_{j=1}^{l-1}|m_j|(rL)^{j-1} \leq (rL)^{-1}\sum_{j=1}^{l-1}|m_j| \leq (rL)^{-1}dL = dr^{-1}.$$

Since $r > d$, we have $m_l = 0$. Starting with $l = d$ and subsequently repeating the argument for $l = d-1, d-2, ..., 1$ we arrive at $m_j = 0, j \geq 0$. □

Proof of Lemma 3.8. A straightforward calculation shows that

$$\begin{aligned}(2\pi)^{\frac{d}{2}}|\mathbf{a}''(\mathbf{x})| &\leq \sum_{|\mathbf{m}|>L} |\hat{\mathbf{a}}(\mathbf{m})| \\ &\leq \left[\sum_{\mathbf{m}} (1+|m|)^{2s}|\hat{\mathbf{a}}(\mathbf{m})|^2\right]^{\frac{1}{2}} \left[\sum_{|\mathbf{m}|>L} (1+|\mathbf{m}|)^{-2s}\right]^{\frac{1}{2}} \\ &\leq C(1+L)^{-s+\frac{d}{2}}\|\mathbf{a}\|_{H^s}.\end{aligned}$$

Since $|\mathbf{n}| \leq CL^{d-1}$ and $s > 3d/2 - 1$, it is now clear that for sufficiently large L the required bound is satisfied. □

7. Proof of Theorem 2.2

7.1. **Relative estimates.** So far we have been studying the operator $\mathcal{H}_{\mathbf{a}}(\rho)$ without any electric field. Now we are going to include V in our considerations. To begin with we establish a Sobolev type inequality for periodic functions. The crucial role is played by the Hausdorff-Young inequality (see [13]):

Proposition 7.1. *Let $u = u(\mathbf{x})$ be a function defined on the cube $\Omega = [0, 2\pi)^d$. Suppose that the sequence of its Fourier coefficients $\hat{u} = \{\hat{u}(\mathbf{m})\}$, $\mathbf{m} \in \mathbb{Z}^d$ belongs to l^q with some $q \in [1, 2]$. Then*

$$\|u\|_{L^p} \leq (2\pi)^{d(1/2-1/q)}\|\hat{u}\|_{l^q}, \ p^{-1} + q^{-1} = 1. \tag{7.1}$$

Let $\mathcal{H}_0(\rho)$ be defined in (2.8). We also assume that $\boldsymbol{\alpha}$ is defined as in (2.9), $\boldsymbol{\mu}$ has the form (2.7) and $\beta = |\mathbf{G}\boldsymbol{\alpha}|(2|\mathbf{n}|)^{-1}$. All the constants below are independent of $\rho, \boldsymbol{\mu}_\perp : |\boldsymbol{\mu}_\perp| \leq C$, but may depend on the functions $V, \mathbf{a}$ and the vector $\mathbf{n}$ in the definition of $\boldsymbol{\alpha}$.

Lemma 7.2. *Let $\boldsymbol{\alpha}, \boldsymbol{\mu}$ be as specified above. Suppose that $V \in L^p(\Omega)$ where $p \geq 2$ if $d = 2, 3$ and $p > d/2$ if $d \geq 4$. Then*

$$\|Vu\| \leq C\rho^{-1+(d-1)p^{-1}}\|\mathcal{H}_0(\rho)u\|, u \in C^\infty(\mathbb{T}), \tag{7.2}$$

for all $\rho \geq 1$.

Proof. By Hölder's inequality and Proposition 7.1, for any $q \geq 2$ such that $p^{-1} + q^{-1} = 2^{-1}$ one has

$$\|Vu\|^2 \leq \|V\|^2_{L^p}\|u\|^2_{L^q} \leq C_q\|V\|^2_{L^p}\|\hat{u}\|^2_{l^t}, \ t^{-1} = 1 - q^{-1} = p^{-1} + 2^{-1}.$$

Let $h(\mathbf{m},\rho)$ be as defined in (3.1). According to (3.7) the symbol $h(\mathbf{m},\rho)$ in (3.1) does not vanish, so one can write

$$\sum_{\mathbf{m}} |\hat{u}(\mathbf{m})|^t = \sum_{\mathbf{m}} |\hat{u}(\mathbf{m})|^t |h(\mathbf{m},\rho)|^t |h(\mathbf{m},\rho)|^{-t}$$
$$\le \left[\sum_{\mathbf{m}} |\hat{u}(\mathbf{m})|^2 |h(\mathbf{m},\rho)|^2\right]^{\frac{t}{2}} \left[\sum_{\mathbf{m}} |h(\mathbf{m},\rho)|^{-\frac{2t}{2-t}}\right]^{1-\frac{t}{2}}.$$

To estimate the sum in the second factor split it into two parts: $\Sigma_1+\Sigma_2$ with

$$\Sigma_1 = \sum_{\mathbf{m}\in N(\rho,1)} \left||\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|^2 + 2i\rho\mathbf{e}\mathbf{F}(\mathbf{m}+\boldsymbol{\mu}) - \rho^2\right|^{-\frac{2t}{2-t}},$$
$$\Sigma_2 = \sum_{\mathbf{m}\in M(\rho,1)} \left|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|^2 + 2i\rho\mathbf{e}\mathbf{F}(\mathbf{m}+\boldsymbol{\mu}) - \rho^2\right|^{-\frac{2t}{2-t}}.$$

Estimate Σ_2 using (3.7) and ignoring the term $|\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|^2-\rho^2$:

$$\Sigma_2 \le C\rho^{-\frac{2t}{2-t}} \sum_{\mathbf{m}:||\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})|-\rho|<1} 1 \le C\rho^{-\frac{2t}{2-t}+d-1}.$$

To handle Σ_1 we neglect $2i\rho\mathbf{e}\mathbf{F}(\mathbf{m}+\boldsymbol{\mu})$ and estimate Σ_1 by the integral

$$\int_{|z-\rho|\ge 1} \frac{z^{d-1}}{(z^2-\rho^2)^{\frac{2t}{2-t}}} dz.$$

This integral converges since $4t(2-t)^{-1} = 2p > d$. A routine computation shows that Σ_1 obeys the same bound as Σ_2. Combining the above estimates, we obtain

$$\|Vu\|^2 \le C\rho^{-2+(d-1)(2/t-1)} \|\mathcal{H}_0(\rho)u\|^2.$$

For $t^{-1} = p^{-1} + 2^{-1}$, this leads to (7.2). $\square$

7.2. **Proof of Lemma 3.9.** It suffices to refer to (6.3) and Lemma 7.2.

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