Transfer Operators for Coupled Analytic Maps

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Abstract

We consider analytic coupled map lattices over $\mathbf{Z}^d$ with exponentially decaying interaction. We introduce Banach spaces for the infinite-dimensional system that include measures with analytic, exponentially bounded finite-dimensional marginals. Using residue calculus and 'cluster expansion'-like techniques we define transfer operators on these Banach spaces. For these we get a unique probability measure that exhibits exponential decay of correlations.

0 Introduction

Coupled map lattices were introduced by K. Kaneko (cf. [12] for a review) as systems that are weak mixing wrt. spatio-temporal shifts. L.A. Bunimovich and Ya.G. Sinai proved in [6] (cf. also the remarks on that in [3]) the existence of an invariant measure and its exponential decay of correlations for a one-dimensional lattice of weakly coupled maps by constructing a Markov partition and relating the system to a two-dimensional spin system.
J. Bricmont and A. Kupiainen extend this result in [2] and [3, 4] to coupled circle maps over the $\mathbf{Z}^d$-lattice with analytic and Hölder-continuous weak interaction, respectively. They use a 'polymer' or 'cluster'-expansion for the

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Perron-Frobenius operator for the finite-dimensional subsystems over $\Lambda \subset \mathbf{Z}^d$ and write the nth iterate of this operator applied to the constant function 1 in terms of potentials for a $d+1$-dimensional spin system. Taking the limit as $n \to \infty$ and $\Lambda \to \mathbf{Z}^d$ they get existence and uniqueness (among measures with certain properties) of the invariant probability measure and exponential decay of correlations.

V. Baladi, M. Degli Esposti, S. Isola, E. Järvenpää and A. Kupiainen define in [1], for infinite-dimensional systems over the $\mathbf{Z}^d$ lattice, transfer operators on a Frechet space, and, for $d = 1$, on a Banach space; they study the spectral properties of these operators, viewing the coupled operator as a perturbation of the uncoupled one in the Banach case.

In [13] G. Keller and M. Künzle consider periodic or infinite one-dimensional lattices of weakly coupled maps of the unit interval. In particular they define transfer operators on the space BV of measures whose finite-dimensional marginals are of bounded variation and prove the existence of an invariant probability measure. For the infinite-dimensional system they further show that for a small perturbation of the uncoupled map any invariant measure in BV is close (in a specified sense) to the one they found.

Coupled map lattices with multi-dimensional local systems of hyperbolic type have been studied by Ya.B. Pesin and Ya.G. Sinai [16], M. Jiang [8, 9], M. Jiang and A. Mazel [10], M. Jiang and Ya.B. Pesin [11] and D.L. Volevich [17, 18].

More detailed surveys on coupled map lattices can be found in [5], [11] and [3].

In the above papers (except [1], [13]) the analysis has been done only for Banach spaces defined for finite subsets Λ of the lattice, and the (weak) limit of the invariant measure for $\Lambda \to \mathbf{Z}^d$ was taken afterwards.

Here we present a new point of view in which a natural Banach space and transfer operators are defined for the infinite lattice of weakly coupled analytic maps (Section 1). The space contains consistent families of analytic marginals over finite subsets of $\mathbf{Z}^d$. We take a weighted sup-norm so that the sup-norms of the marginals for the sub-systems over finitely many (say N) lattice points is bounded exponentially in N (Section 2). We identify an ample subset of this space with a set of *rca* measures (Section 4) that contains the unique invariant probability density (Section 2).

We derive exponential decay of correlations for this measure from (the proof of) the spectral properties of our transfer operators. (Sections 2, 7).

Our approach provides a natural setting for an analysis of the full $\mathbf{Z}^d$ Perron-Frobenius operator in terms of cluster expansions over finite subsets of the lattice. Using residue calculus we introduce an integral representation for the Perron-Frobenius operator for finite-dimensional sub-systems (Section 3) which yields a uniform control over the perturbation and also gives rise to an easy approach to stochastic perturbation (cf. [15]) which however we do not consider here.
Our 'cluster expansion' combinatorics (Section 5) uses ideas from [15] (cf. also [2]). Apart from the analysis of the one-dimensional operator, which is fairly standard and for which we refer to e.g. [2], the paper should be self-contained.

1 General Setting

We consider coupled map lattices in the following setting: The state space is $M = (S^1)^{\mathbf{Z}^d}$ where $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ is the unit circle in the complex plane and $d \in N$.
The map $S : M \to M$ is the composition $S = F \circ T^\epsilon$ of a coupling map T^ϵ depending on a (small) non-negative parameter ϵ and another parameter for the decay of interaction (cf. (1)) with an (uncoupled) map F that acts on each component of M separately. We make the following assumptions:

I $F(\mathbf{z}) = (f_p(z_p))_{p\in\mathbf{Z}^d}$ where $f_p : S^1 \to S^1$ are real analytic and expanding (i.e. $f_p' \geq \lambda_0 > 1$) maps that extend for some δ_1 holomorphically to the interior of an annulus $A_{\delta_1} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \mid -\delta_1 \leq \ln|z| \leq \delta_1\}$ and the family of Perron-Frobenius operators $\mathcal{L}_{f_p}$ for the indiviual systems satisfies uniformly a condition specified in Section (5.1).

We write $T^\epsilon : M \to M$ as $T^\epsilon(\mathbf{z}) = (T^\epsilon_p(\mathbf{z}))_{p\in\mathbf{Z}^d}$ and $T^\epsilon_p(\mathbf{z}) = z_p \exp[2\pi\imath\epsilon g_p(\mathbf{z})]$ with $g_p(\mathbf{z}) = \sum_{k=1}^\infty g_{p,k}(\mathbf{z})$. The functions $g_{p,k}$ is real valued on $(S^1)^{\mathbf{Z}^d}$ and depends only on those z_q with $||p-q|| \leq k$ (neighbours of distance at most k) where $||p|| \stackrel{\text{def}}{=} \sum_{l=1}^d |p_l|$.
We write $B_k(p) = \{q \in \mathbf{Z}^d \mid ||p-q|| \leq k\}$ and also denote by $g_{p,k}$ the function from the finite-dimensional torus $(S^1)^{B_k(p)}$ to $\mathbf{R}$.
We assume the following for the functions $g_{p,k}$:

II For all $p \in \mathbf{Z}^d$ and $k \geq 1$ the maps $g_{p,k}$ extend to a holomorphic map

$g_{p,k} : A_{\delta_1}^{B_k(p)} \to \mathbf{C}$ and

$$\| g_{p,k} \|_{A_{\delta_1}^{B_k(p)}} \leq c_1 \exp\left(-c_2 k^d\right) \tag{1}$$

with $c_1 > 0$ and c_2 bigger than a certain constant specified in (92).

The parameter c_1 is actually redundant as it is multiplied by ϵ in the definition of T_p^ϵ. We also have $\exp(-c_2 k^d) \leq \exp(-\xi)\exp(-\tilde{c}_2 k^d)$ for $\tilde{c}_2 = c_2 - \xi$, $\xi > 0$, i.e. for any ϵ we can make the interaction small only by taking c_2 large. But once we have chosen c_2 large enough to guarantee the convergence of the infinite sums in our analysis we can consider perturbations of the uncoupled map depending on the parameter ϵ only.
With the metric

$$d_\gamma(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} \sup_{p \in \mathbf{Z}^d} \gamma^{\|p\|} \|x_p - y_p\| \tag{2}$$

for $0 < \gamma < 1$ (M, d_γ) is a compact metric space. Its topology is the product topology on $(S^1)^{\mathbf{Z}^d}$. The Borel σ-algebra $\mathcal{B}$ on M is the same as the product σ-algebra. F and T^ϵ are continuous and measurable. Let $\mathcal{C}(M)$ denote the space of real-valued continuous functions on (M, d_γ) with the sup-norm and μ the Lebesgue (product) measure on M.
For $\Lambda_1 \subseteq \Lambda_2 \subseteq \mathbf{Z}^d$, with Λ_1 finite and an integrable function g on M depending only on the Λ_2-coordinates, we define the projection

$$(\pi_{\Lambda_1} g)(\mathbf{z}_{\Lambda_1}) \overset{\text{def}}{=} \int_{(S^1)^{\Lambda_2 \setminus \Lambda_1}} d\mu^{\Lambda_2 \setminus \Lambda_1}(\mathbf{z}_{\Lambda_2 \setminus \Lambda_1}) g(\mathbf{z}_{\Lambda_1} \vee \mathbf{z}_{\Lambda_2 \setminus \Lambda_1}) \tag{3}$$

2 Main Results

For finite $\Lambda \subset \mathbf{Z}^d$ let $H(A_\delta^\Lambda)$ be the space of continuous functions on the closed polyannulus A_δ^Λ that are holomorphic on its interior and write $\| \cdot \|_\Lambda$ for the sup-norm on $H(A_\delta^\Lambda)$. Let $\mathcal{F}$ be the set of all finite subsets (including $\emptyset$) of $\mathbf{Z}^d$. We denote by $\mathcal{H}$ the set of all consistent families $\phi = (\phi_\Lambda)_{\Lambda \in \mathcal{F}}$ of functions $\phi_\Lambda \in H(A_\delta^\Lambda)$. Consistency means $\pi_{\Lambda_1} \phi_{\Lambda_2} = \phi_{\Lambda_1}$ for $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$. We write $\mu(\phi) \overset{\text{def}}{=} \phi_\emptyset$.
We want to define a norm on a (sufficiently large) subspace of $\mathcal{H}$ that should at least contain 'product densities' like $h = (h_\Lambda)_{\Lambda \in \mathcal{F}}$ with $h_\Lambda(\mathbf{z}) = \prod_{p \in \Lambda} h_p(z_p)$, where $h_p \in H(A_\delta^{\{p\}})$ is the invariant probability density for the

single system over $\{p\}$ (cf. Section 5.1). As $\|h_p\|_{\{p\}} \leq c_h$ uniformly in p, the sup-norm $\|h_{\Lambda_1}\|_{\Lambda_1}$ does not grow faster than exponentially in $|\Lambda_1|$. Therefore we take a weighted sup-norm. For $0 < \vartheta < 1$ we define

$$\|\phi\|_\vartheta = \sup_{\Lambda \in \mathcal{F}} \vartheta^{|\Lambda|} \|\phi_\Lambda\|_\Lambda \tag{4}$$

and set $\mathcal{H}_\vartheta \overset{\text{def}}{=} \{\phi \in \mathcal{H} \mid \|\phi\|_\vartheta < \infty\}$. Then $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ is a Banach space. Analogously we define for $\Lambda \in \mathcal{F}$ the weighted norm on spaces $\mathcal{H}_{\Lambda,\vartheta}$ of consistent sub-families $(\phi_{\Lambda_1})_{\Lambda_1 \subseteq \Lambda}$:

$$\|\phi\|_{\Lambda,\vartheta} \overset{\text{def}}{=} \sup_{\Lambda_1 \subseteq \Lambda} \vartheta^{|\Lambda_1|} \|\phi_{\Lambda_1}\|_{\Lambda_1} \tag{5}$$

We get the same (topological) vector space as $\left(H(A_\delta^\Lambda), \|\cdot\|_\Lambda\right)$, but the constants for the estimates of the norms are unbounded as $|\Lambda|$ increases.
For given $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ and $N \in \mathbf{N}$ we have a map

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} : (\mathcal{H}_\vartheta, \|\cdot\|_\vartheta) \to (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}) \tag{6}$$

where $\mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}$ is the Perron-Frobenius operator for the finite-dimensional system over Λ_2 (cf. Section 3) with fixed boundary conditions (not included in the notation). The following definition of transfer operators for the infinite system does not depend on the choice of the boundary conditions.

Theorem 1 *For ϑ, ϵ sufficiently small, c_2, N sufficiently big and any $\Lambda_1 \in \mathcal{F}$:*

1. *The limit*

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F \circ T^\epsilon} \overset{\text{def}}{=} \lim_{\Lambda_2 \to \mathbf{Z}^d} \pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2} \tag{7}$$

$\in L\left((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta), (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta})\right)$ *exists and the family of these operators is uniformly (in Λ_1 and also in N) bounded. This defines operators $\mathcal{L}^N_{F \circ T^\epsilon}$ on $(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ by* $\left(\mathcal{L}^N_{F \circ T^\epsilon}\phi\right)_{\Lambda_1} \overset{\text{def}}{=} \pi_{\Lambda_1} \circ \mathcal{L}^N_{F \circ T^\epsilon}\phi$

For any $n \in \mathbf{N}$ we have $\mathcal{L}^n_{F \circ T^\epsilon} : \mathcal{H}_\vartheta \to \mathcal{H}_{\vartheta_n}$ with suitably chosen $0 < \vartheta_1 \leq \cdots \leq \vartheta_{N_0} = \vartheta_{N_0+1} = \cdots = \vartheta$.

In the case of finite-range interaction we can define a linear map $\mathcal{L}_{F \circ T^\epsilon}$ on $\mathcal{H}$ in the same way .

2. *There is a unique invariant probability measure* $\nu = (\nu_{\Lambda_1})_{\Lambda_1 \in \mathcal{F}} \in \mathcal{H}_\vartheta$. *In* $L(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta)$ *the sequence* $\left(\mathcal{L}^N_{F\circ T^\epsilon}\right)_{N \geq N_0}$ *converges exponentially fast:*

$$\left\|\mathcal{L}^N_{F\circ T^\epsilon} - \mu(\cdot)\nu\right\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta))} \leq c_3 \tilde{\eta}^N \tag{8}$$

for some $c_3 > 0$ *and* $0 < \tilde{\eta} < 1$.

For the invariant measure ν we have exponential decay of correlations for spatio-temporal shifts on the system:
Let $(e_1, \ldots, e_d)$ be a linearly-independent system of unit vectors in $\mathbf{Z}^d$. We define translations $\tau_{e_i}(p) \overset{\text{def}}{=} p + e_i$ for $p \in \mathbf{Z}^d$ and $(\tau_{e_i}(z))_p \overset{\text{def}}{=} z_{\tau_{e_i}(p)}$ for $z \in M$. In the following theorem we denote by τ (acting on M from the right) compositions $\tau = \tau_1 \circ \ldots \circ \tau_{m(\tau)}$ and by σ a composition of spatio-temporal shifts (on M): $\sigma = \sigma_1 \circ \ldots \circ \sigma_{m(\sigma)+m(\sigma)}$ with $\sigma_i \in \{S, \tau_{e_1}, \ldots, \tau_{e_d}\}$. We denote by $n(\sigma)$ the number of factors S and by $m(\sigma)$ the number of spatial translations in this product. For a translation-invariant system, i.e. $f_p = f$ and $g_p(\mathbf{z}) = g_{\tau^{-1}_{e_i}(p)}(\tau_{e_i}(z))$ for all $p \in \mathbf{Z}^d$, the time-shift S commutes with the translations.

Theorem 2 *For* ϑ,ϵ *and* c_2 *as in Theorem 1 there is a* $\kappa \in (0,1)$ *such that for all nonempty* $\Lambda_1, \Lambda_2 \in \mathcal{F}$ *with* $\Lambda_1 \cap \Lambda_2 = \emptyset$ *the following holds:*

1. *If* $g \in \mathcal{C}((S^1)^{\Lambda_1})$ *and* $f \in \mathcal{C}((S^1)^{\Lambda_2})$ *then* $\left|\int_M \nu d\mu\, gf - \left(\int_M \nu d\mu\, g\right)\left(\int_M \nu d\mu\, f\right)\right| \leq c_4 \vartheta^{-|\Lambda_1|-|\Lambda_2|} \|g\|_\infty \|f\|_\infty \kappa^{dist(\Lambda_1,\Lambda_2)}$
2. *If* $g \in \mathcal{C}((S^1)^{\Lambda_1})$ *and* $f \in \mathcal{H} \cap \mathcal{C}((S^1)^{\Lambda_2})$ *then*

$$\begin{aligned}\left|\int_M \nu d\mu\, g\circ\tau\circ S^n f - \left(\int_M \nu d\mu\, g\circ\tau\right)\left(\int_M \nu d\mu\, f\right)\right| \\ \leq \quad c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1|+|\Lambda_2|} \|g\|_\infty \|f\|_{\Lambda_2} \kappa^{m(\tau)} \tilde{\eta}^n\end{aligned} \tag{9}$$

   with suitable c_5 *and* $\tilde{\eta}$ *as in Theorem 1.*

3. *If the system is translation-invariant and* g, f *are as in (2.), then*

$$\begin{aligned}&\left|\int_M \nu d\mu \, g\circ\sigma f - \left(\int_M \nu d\mu \, g\right)\left(\int_M \nu d\mu \, f\right)\right| \\ &\leq \;\; c(\Lambda_1,\Lambda_2,\kappa) c_5^{|\Lambda_1|+|\Lambda_2|} \|g\|_\infty \|f\|_{\Lambda_2} \kappa^{m(\sigma)} \tilde{\eta}^{n(\sigma)}\end{aligned} \tag{10}$$

4. If $g, f \in \mathcal{C}(M)$ *then*

$$\lim_{\max\{m(\tau),n\}\to\infty} \left|\int_M \nu d\mu \, g\circ\tau\circ S^n f - \left(\int_M \nu d\mu \, g\circ\tau\right)\left(\int_M \nu d\mu \, f\right)\right| = 0. \tag{11}$$

5. If the system is translation-invariant and $g, f \in \mathcal{C}(M)$ *then*

$$\lim_{\max\{m(\sigma),n(\sigma)\}\to\infty} \int_M \nu d\mu \, g\circ\sigma \, f = \left(\int_M \nu d\mu \, g\right)\left(\int_M \nu d\mu \, f\right) \tag{12}$$

Remarks: 1)Statement (5.) means that for a translation-invariant system ν is mixing wrt. spatio-temporal shifts. According to (3.), the decay of correlations for observables g and h as specified in (2.) is exponentially fast.
2) We could choose the rate of decay κ first and then the other parameters.
3)The integration wrt. '$\nu d\mu$' will be defined in Section 4.
4) $c(\Lambda_1,\Lambda_2,\kappa)$ in (2.) and (3.) is a constant depending only on $\mathrm{dist}(\Lambda_1,\Lambda_2)$ and κ.

3 Finite-dimensional Systems

We first consider 'finite-dimensional versions' of the maps F, T^ϵ etc. For a finite subset $\Lambda \in \mathbf{Z}^d$ and some fixed configuration $\mathbf{z}_{\Lambda^C} = (z_p)_{p\in\Lambda^C} \subset (S^1)^{\Lambda^C}$ on $\Lambda^C \stackrel{\text{def}}{=} \mathbf{Z}^d \setminus \Lambda$ we define $T^{\Lambda,\epsilon} : A^\Lambda_\delta \to \mathbf{C}^\Lambda$ by $(T^{\Lambda,\epsilon}(\mathbf{z}_\Lambda))_p \stackrel{\text{def}}{=} z_p \exp(2\pi\imath\epsilon g_p(\mathbf{z}_\Lambda \vee \mathbf{z}_{\Lambda^C}))$, where $\mathbf{z}_\Lambda \vee \mathbf{z}_{\Lambda^C} \in M$ agrees with $\mathbf{z}_\Lambda$ on its Λ-sites and with $\mathbf{z}_{\Lambda^C}$ on its Λ^C-sites.
We do not specify $\mathbf{z}_{\Lambda^C}$ in the notation of $T^{\Lambda,\epsilon}$. The restriction of F to A^Λ_δ is denoted by F^Λ.
With the following two propositions we ensure that for sufficiently small δ and ϵ (depending on δ but not on Λ or $\mathbf{z}_{\Lambda^C}$), $F^\Lambda \circ T^{\Lambda,\epsilon}$ maps A^Λ_δ to a bigger polyannulus (cf. [2]).

For $\Lambda \subset \mathbf{Z}^d$ we have the metric d_Λ on $(S^1)^\Lambda$ defined by

$$d_\Lambda(\mathbf{z},\mathbf{w}) \overset{\text{def}}{=} \sup\{|z_p - w_p| \mid p \in \Lambda\} \tag{13}$$

Proposition 1 *For all $c_7 \in (0,1)$, sufficiently small δ (depending on c_7) and ϵ (depending on c_7 and δ), and arbitrary $\Lambda \in \mathcal{F}$, $T^{\Lambda,\epsilon}$ maps A^Λ_δ biholomorphically onto its image and $T^{\Lambda,\epsilon}\left(A^\Lambda_\delta\right) \supset A^\Lambda_{c_7\delta}$, i.e. the image contains a sufficiently thick polyannulus. Also $T^{\Lambda,\epsilon}\left(\partial A^\Lambda_\delta\right) \cap A^\Lambda_{c_7\delta} = \emptyset$, i.e. the image of the boundary (the same as the boundary of the image) does not intersect the closed smaller polyannulus.*

Proposition 2 *Let the expanding maps $f_p : S^1 \to S^1$ satisfy condition* **I** *for some δ_1 and an expansion constant λ_0 and let $1 < \lambda < \lambda_0$. Then for all sufficiently small δ ($0 < \delta < \delta_0$) and all finite $\Lambda \subset \mathbf{Z}^d$ the map $F^\Lambda : A^\Lambda_\delta \to \mathbf{C}^\Lambda$ is locally biholomorphic, $A^\Lambda_{\lambda\delta} \subset F^\Lambda\left(A^\Lambda_\delta\right)$, i.e. the image contains a thicker polyannulus and furthermore all $\mathbf{z} \in A^\Lambda_{\lambda\delta}$ have the same number of preimages. We also have $A^\Lambda_{\lambda\delta} \cap F^\Lambda\left(\partial A^\Lambda_\delta\right) = \emptyset$.*

Combining Propositions 1 and 2 we have for fixed c_7 and (small) δ

$$F^\Lambda \circ T^{\Lambda,\epsilon}\left(A^\Lambda_\delta\right) \supset A^\Lambda_{c_7\lambda\delta} \tag{14}$$

and

$$F^\Lambda \circ T^{\Lambda,\epsilon}\left(\partial A^\Lambda_\delta\right) \cap A^\Lambda_{c_7\lambda\delta} = \emptyset \tag{15}$$

In particular, if we choose $c_7 > \frac{1}{\lambda}$ there is a disc of radius $(c_7\lambda - 1)\delta > 0$ around each point in A^Λ_δ that is entirely contained in $F^\Lambda \circ T^{\Lambda,\epsilon}\left(A^\Lambda_\delta\right)$. We will need this for Cauchy estimates. From now on we keep δ fixed.
In the next proposition we establish a special representation of the Perron-Frobenius operator for our finite system with $(S^1)^N = (S^1)^\Lambda$, $S^\epsilon = F^\Lambda \circ T^{\Lambda,\epsilon}$, ψ continuous (the proposition holds also for $\psi \in L^\infty(M)$) and ϕ continuous on the closed polyannulus $A^\Lambda_{\delta_1}$ and analytic in its interior.
First we give the definition of the Perron-Frobenius operator (cf. for example [14]).

Definition Let μ be a measure on a metric space M (with the Borel σ-algebra). The Perron-Frobenius operator $\mathcal{L}_S$ for a nonsingular measurable map $S : M \to M$ is defined via the equation

$$\int_M d\mu\,\psi\circ S\,\phi = \int_M d\mu\,\psi\,\mathcal{L}_S\phi \tag{16}$$

that, for given $\phi\in L^1(M)$, must hold for all $\psi\in L^\infty(M)$. The existence and uniqueness of $\mathcal{L}_S\phi\in L^1(M)$ is equivalent by the Radon-Nikodym Theorem to the absolute continuity (wrt. μ) of the measure associated to the functional $\psi\mapsto\int_M d\mu^N\psi\circ S\,\phi$, i.e. for all measurable $A\in M$, $\mu(A)=0$ implies $\mu(S^{-1}(A))=0$. This condition is called *nonsingularity* of S.

The normalized Lebesgue measure μ on S^1 is given by $d\mu(z)=\frac{dz}{2\pi\imath}\frac{1}{z}$ (this lifts wrt. the map $t\to e^{\imath t}$ to the normalized Lebesgue measure $\frac{dt}{2\pi}$ on $[0,2\pi)$) and the product measure μ^N on $(S^1)^N$ is given by

$$d\mu^N(\mathbf{z}) = \frac{d\mathbf{z}}{(2\pi\imath)^N}\frac{1}{\mathbf{z}} \overset{\text{def}}{=} \frac{dz_1}{2\pi\imath}\cdots\frac{dz_N}{2\pi\imath}\frac{1}{z_1}\cdots\frac{1}{z_N} \tag{17}$$

Proposition 3 *In the above setting the Perron-Frobenius-Operator can be written in the following way:*

$$\mathcal{L}_{S^\epsilon}\phi(w) = \int_{\Gamma^N}\frac{d\mathbf{z}}{(2\pi\imath)^N}\phi(z)\prod_{k=1}^{N}\left(\frac{1}{S_k^\epsilon(\mathbf{z})-w_k}\frac{S_k^\epsilon(\mathbf{z})}{z_k}\right) \tag{18}$$

where $\Gamma=\Gamma_+\cup\Gamma_-$ *is the positively-oriented boundary of* A_δ*.*

4 Further Remarks on the Infinite-Dimen/sional System

The subspace of functions that depend only on finitely many variables is dense in $(\mathcal{C}(M),\|\cdot\|_\infty)$, and each such function (say depending on $\mathbf{z}_\Lambda$ only) can be uniformly approximated by (the restriction of) functions in $\mathcal{H}(A_\delta^\Lambda)$. The dual space of $\mathcal{C}(M)$ is $rca(M)$ (see e.g. [7]), the space of bounded, regular, countably additive, real-valued set functions on $(M,\mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra. The norm on $rca(M)$ is the total variation. For given ϑ,Λ we consider rca measures with marginals $\phi_{\Lambda|(S^1)^\Lambda}$ over $(S^1)^\Lambda$ (restriction of ϕ_Λ to $(S^1)^\Lambda$) s.t. $\phi=(\phi_\Lambda)_{\Lambda\in\mathcal{F}}\in\mathcal{H}_\vartheta$. We remark that not every $\phi\in\mathcal{H}_\vartheta$ with real-valued marginals $\phi_{\Lambda|(S^1)^\Lambda}$ corresponds to an element in $rca(M)$ because its variation might not be bounded as $\int_\Lambda d\mu^\Lambda |\phi_\Lambda|$ might be unbounded in Λ. So we define for $\phi \in \mathcal{H}$

$$\|\phi\|_{var} \stackrel{\text{def}}{=} \lim_{\Lambda \to \mathbf{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda |\phi_\Lambda|. \tag{19}$$

We set $\mathcal{H}^{bv} \stackrel{\text{def}}{=} \{\phi \in \mathcal{H} : \|\phi\|_{var} < \infty\}$ and $\mathcal{H}^{bv}_\vartheta \stackrel{\text{def}}{=} \mathcal{H}^{bv} \cap \mathcal{H}_\vartheta$. In particular all real-analytic and non-negative $\phi \in \mathcal{H}$, i.e. $\phi_{\Lambda|(S^1)^\Lambda} \geq 0$ for all $\Lambda \in \mathcal{F}$, belong to this space.

We can view every $\phi \in \mathcal{H}^{bv}$ as an element of $rca(M)$: For $g \in \mathcal{C}(M)$ the net $(g_\Lambda)_{\Lambda \in \mathcal{F}}$ given by $g_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda(g)$ converges uniformly to g. We set

$$\phi(g) \stackrel{\text{def}}{=} \lim_{\Lambda \to \mathbf{Z}^d} \int_{(S^1)^\Lambda} d\mu^\Lambda g_\Lambda \phi_\Lambda \tag{20}$$

The limit exists because for $\Lambda_1 \subset \Lambda_2$

$$\begin{aligned} & \left| \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g_{\Lambda_1} \phi_{\Lambda_1} - \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} g_{\Lambda_2} \phi_{\Lambda_2} \right| \\ = & \left| \int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} (g_{\Lambda_1} - g_{\Lambda_2}) \phi_{\Lambda_2} \right| \\ \leq & \ \|g_{\Lambda_1} - g_{\Lambda_2}\|_{(S^1)^{\Lambda_2}} \|\phi\|_{var} \end{aligned} \tag{21}$$

gets arbitrarily small as $\Lambda_1 \to \mathbf{Z}^d$, i.e. the net has the Cauchy property.
We further see

$$\begin{aligned} \|\phi\|_{var} &= \sup_{\Lambda \in \mathcal{F}} \int_{(S^1)^\Lambda} d\mu^\Lambda |\phi_\Lambda| \\ &= \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \leq 1}} \int_{(S^1)^\Lambda} d\mu^\Lambda g\, \phi_\Lambda \\ &= \sup_{\substack{g \in \mathcal{C}(M) \\ \|g\|_\infty \leq 1}} |\phi(g)| \end{aligned} \tag{22}$$

so $\|\phi\|_{var}$ is in fact the total variation (the operator-norm, cf. [7]) of the corresponding linear functional on $\mathcal{C}(M)$.

Let $\mathcal{H}(\mathcal{F}) \stackrel{\text{def}}{=} \bigcup_{\Lambda \in \mathcal{F}} H(A^\Lambda_\delta)$, the subspace of functions depending on only finitely many variables. We define the product $g^1\phi \in \mathcal{H}_\vartheta$ of $g \in \mathcal{H}(A^{\Lambda_1}_\delta)$ and $\phi \in \mathcal{H}_\vartheta(\mathcal{F})$ by

$$(g^1\phi)_\Lambda \stackrel{\text{def}}{=} \pi_\Lambda(g^1\phi_{\Lambda_1\cup\Lambda}) \tag{23}$$

Lemma 1 *If $g^1 \in H(A_\delta^{\Lambda_1})$, $g^2 \in H(A_\delta^{\Lambda_2})$, $g \in \mathcal{C}(M)$ and $\phi \in \mathcal{H}_\vartheta$ the following holds*

1. *The product in (23) is well-defined and $\|g^1\phi\|_\vartheta \le \|g^1\|_{\Lambda_1}\vartheta^{-|\Lambda_1|}\|\phi\|_\vartheta$*
2. *$(g^1g^2)\phi = g^1(g^2\phi)$*
3. *g^2 is also an element of $\mathcal{H}_\vartheta$ and the product g^1g^2 as defined in (23) is the same as the usual product between functions on M.*
4. *$(g^1\phi)(g) = \phi(g^1g)$ where $(g^1\phi)$ and ϕ act as functionals.*
5. *$\mathcal{H}_\vartheta^{bv}$ is also a module over the ring $\mathcal{H}(\mathcal{F})$.*

5 Expansion of the Perron-Frobenius Operator

We split the integral kernel of the Perron-Frobenius operator for a finite-dimensional system. Recall $S_p : M \to (S^1)^{\{p\}}$, $S_p(\mathbf{z}) = f_p \circ T_p^\epsilon(\mathbf{z})$ with $T_p^\epsilon(\mathbf{z}) = z_p \exp\left(2\pi\imath\epsilon \sum_{k=1}^\infty g_{p,k}(\mathbf{z})\right) = z_p \prod_{k+1}^\infty \exp(2\pi\imath\epsilon g_{p,k}(\mathbf{z}))$.
If we consider only finite range interaction, say up to distance l, we have

$$T_{p,l}^\epsilon(\mathbf{z}) \stackrel{\text{def}}{=} z_p \exp(2\pi\imath\epsilon \sum_{k=1}^{l} g_{p,k}(\mathbf{z})) \tag{24}$$

For a finite-dimensional system (say on $(S^1)^{\Lambda_2}$) with fixed boundary conditions we have a special representation of $\mathcal{L}_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}$ in terms of the integral kernel (Proposition 3).

Proposition 4 *For the factors in the integral kernel in (18) we have the following splitting :*

$$\begin{aligned}\frac{1}{f_p \circ T_p^\epsilon(\mathbf{z}) - w_p}\frac{f_p \circ T_p^\epsilon(\mathbf{z})}{z_p} &= \frac{1}{f_p(z_p) - w_p}\frac{f_p(z_p)}{z_p} \\ &+ \frac{w_p}{z_p}\sum_{k=1}^{\infty}\frac{f_p \circ T_{p,k-1}^\epsilon(\mathbf{z}) - f_p \circ T_{p,k}^\epsilon(\mathbf{z})}{\left(f_p \circ T_{p,k-1}^\epsilon(\mathbf{z}) - w_p\right)\left(f_p \circ T_{p,k}^\epsilon(\mathbf{z}) - w_p\right)}\end{aligned} \tag{25}$$

The sum in the right hand side converges uniformly in $\mathbf{z} \in \Gamma^N$ *and* $w_p \in A_\delta$.

5.1 Unperturbed Operator

The first summand in (25) is just the one which appears in the uncoupled system (i.e. $T^{\epsilon=0} = \mathrm{id}$) and in this case each lattice site can be considered separately. We denote by $\mathcal{L}_{f_p}$ the restriction of the Perron-Frobenius operator to the Banach space of functions on S^1 that extend continuously on the closed annulus A_δ and holomorphically on the interior A_δ. $\|\cdot\|_{A_\delta}$ denotes the supremum over A_δ. The operator

$$\mathcal{L}_{f_p} : (\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta}) \to (\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta})$$

has 1 as simple eigenvalue and the rest of its spectrum is contained in a disc around 0 of radius strictly smaller than 1. It splits into

$$\mathcal{L}_{f_p} = Q_p + R_p \tag{26}$$

with

$$R_p Q_p = Q_p R_p = 0 \tag{27}$$

and

$$\left\| R_p^n \right\|_{L(\mathcal{H}(A_\delta), \|\cdot\|_{A_\delta})} \leq c_r \eta^n \tag{28}$$

with $c_r > 0$, $0 < \eta < 1$. For proofs of these statements see e.g. [2].
Q_p is the projection onto the one-dimensional eigenspace spanned by $h_p \in \mathcal{H}(A_\delta)$, whose restriction to S^1 is positive and has integral $\int_{S^1} d\mu \, h_p = 1$.
We assume in condition **I** regarding the family $(f_p)_{p \in \mathbf{Z}^d}$ that $\|h_p\|_{A_\delta} \leq c_h$ and that the exponential bound in (28) holds uniformly in p. This is the case for example if the f_p are uniformly close to each other as is shown using analytic perturbation theory.
$\mathcal{L}_{f_p}$ preserves the integral and so does Q_p because of (27) and (28). Γ_+ is homologous to S^1. So we can write Q_p as

$$\begin{aligned} Q_p g(w) &= h_p(w) \int_{S^1} d\mu \, g && (29) \\ &= h_p(w) \int_{\Gamma_+} \frac{dz}{2\pi\imath} \frac{1}{z} g(z) && (30) \\ &= \int_{\Gamma} \frac{dz}{2\pi\imath} \frac{1}{z} h_p(w,z) g(z) && (31) \end{aligned}$$

where we have used that g is holomorphic in A_δ and defined:

$$h_p(w_p, z_p) \overset{\text{def}}{=} \begin{cases} h_p(w_p) & \text{for} \quad z_p \in \Gamma_+ \\ 0 & \text{for} \quad z_p \in \Gamma_- \end{cases} \tag{32}$$

The idempotency $Q_p^2 = Q_p$ reads in the integral representation

$$\int_\Gamma \frac{dz^2}{2\pi\imath} \frac{1}{z^2} \int_\Gamma \frac{dz^1}{2\pi\imath} \frac{1}{z^1} h_p(w, z^2) h_p(z^2, z^1) g(z^1) = \int_\Gamma \frac{dz^1}{2\pi\imath} \frac{1}{z^1} h_p(w, z^1) g(z^1) \tag{33}$$

According to Proposition 3 the operator R_p can be written

$$R_p g(w) = \int_\Gamma \frac{dz}{2\pi\imath} \frac{1}{z} r_p(w, z) g(z) \tag{34}$$

with

$$r_p(w, z) = \frac{1}{f_p(z) - w} \frac{f_p(z)}{z} - h_p(w, z). \tag{35}$$

Then equation (27) reads in the integral representation

$$\begin{aligned} \int_\Gamma \frac{dz_p^2}{2\pi\imath} \frac{1}{z_p^2} \int_{S^1} \frac{dz_p^1}{2\pi\imath} \frac{1}{z_p^1} r_p(w_p, z_p^2) h_p(z_p^2) g(z^1) &= 0, & (36) \\ \int_{S^1} \frac{dz_p^2}{2\pi\imath} \frac{1}{z_p^2} \int_\Gamma \frac{dz_p^1}{2\pi\imath} \frac{1}{z_p^1} r_p(z_p^2, z_p^1) g(z^1) &= 0 & (37) \end{aligned}$$

5.2 Perturbed Operator

In view of (25) we set

$$\beta_{p,k}(w_p, \mathbf{z}) \overset{\text{def}}{=} \frac{w_p}{z_p} \frac{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})}{(f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p)(f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p)} \quad . \tag{38}$$

This corresponds to the difference between the operators for systems with interaction of finite-range of order k and $k-1$, respectively. We have the estimate

$$\begin{aligned} & |\beta_{p,k}(w_p, \mathbf{z})| & (39) \\ \leq & \left| \frac{w_p}{z_p} \right| |f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p|^{-1} |f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p|^{-1} \end{aligned}$$

$$
\begin{aligned}
& \times|f_p \circ T^{\epsilon}_{p,k-1}(\mathbf{z}) - f_p \circ T^{\epsilon}_{p,k}(\mathbf{z})| \\
\leq\ & \frac{1+\delta}{1-\delta}|c_7\lambda - 1|^{-1}|c_7\lambda - 1|^{-1}\|f'_p\|_{\{p\}} c\epsilon \exp(-c_2 k^d) \\
\leq\ & c_8 \epsilon \exp(-c_2 k^d)
\end{aligned}
$$

uniformly in $p \in \mathbf{Z}^d$, $w_p \in A_\delta$, $\mathbf{z} \in M$.

5.3 Time N Step

Now we want to estimate the norm of (6) or equivalently that of

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} : (\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}) \to (\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}) \tag{40}$$

$$
\begin{aligned}
\mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2},\epsilon}\phi(\mathbf{z}^0) \qquad & = \textstyle\int_{\Gamma^{\Lambda_2}} \frac{d\mathbf{z}^{-1}}{(2\pi\imath)^{|\Lambda_2|}} \cdots \int_{\Gamma^{\Lambda}_2} \frac{d\mathbf{z}^{-N}}{(2\pi\imath)^{|\Lambda_2|}} \prod_{t=-N}^{-1} \prod_{p \in \Lambda_2} \\
& \times \Big((h_p(z_p^{t+1}, z_p^t) + r_p(z_p^{t+1}, z_p^t) + \textstyle\sum_{k=1}^{\infty} \beta_{p,k}(z_p^{t+1}, \mathbf{z}^t) \Big)
\end{aligned} \tag{41}
$$

Distributing the product we get infinitely many summands. In each factor there is for each $-N \leq m \leq -1$, $p \in \Lambda_2$ a choice between h_p, r_p and $\beta_{p,k}$ ($1 \leq k < \infty$) and we can interpret such a choice graphically as a *configuration* as follows (cf. [2, 15]):
On $\Lambda_2 \times \{-N, \ldots, 0\}$ we represent

- $h_p\left(z_p^{t+1}, z_p^t\right)$ by an *h-line* from (p,t) to $(p,t+1)$
- $r_p\left(z_p^{t+1}, z_p^t\right)$ by an *r-line* from (p,t) to $(p,t+1)$

Figure 1: h-line and r-line

- $\beta_{p,k}\left(z_p^{t+1},\mathbf{z}^t\right)$ by a *k-triangle* (actually rather a cone or pyramid but in our pictures for $d=1$ it is a triangle) with apex $(p,t+1)$ and base points (q,t) with $\|p-q\| \leq k$. (So some of the base points might not lie in $\Lambda_2 \times \{-N,\ldots,-1\}$ but all the apices lie in $\Lambda_2 \times \{-N+1,\ldots,0\}$.)

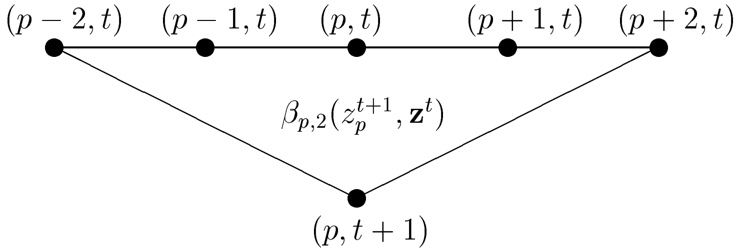

Figure 2: 2-triangle

Note that if $v(k) \overset{\text{def}}{=} |B_k(0)|$ denotes the number of base points of a k-triangle, we have the estimate $v(k) \leq (3k)^d$.
Each choice corresponds to a configuration and for each configuration $\mathcal{C}$ we have an operator $\mathcal{L}_{\mathcal{C}}$. So we can write

$$\mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}} = \sum_{\mathcal{C}} \mathcal{L}_{\mathcal{C}} \tag{42}$$

Some of these summands are zero namely if

- a factor $h_p\left(z_p^{t+2},z_p^{t+1}\right) r_p\left(z_p^{t+1},z_p^t\right)$ or $r_p\left(z_p^{t+2},z_p^{t+1}\right) h_p\left(z_p^{t+1},z_p^t\right)$ appears, but no factor $\beta_{q,k}\left(z_q^{t+2},\mathbf{z}^{t+1}\right)$ with $\|p-q\| \leq k$ (i.e. an h-line follows or is followed by an r-line and at their common endpoint no triangle is attached with any of its basepoints). This follows since, by Fubini's Theorem, one can first perform the $dz_p^{t+1}dz_p^t$-integration and get zero by (36) or (37). (Note that no other terms depend on z_p^{t+1} and the remaining factors and integrations (up to time t+1) correspond to the function $g(z^1)$ in (36) or (37).)
- if a term $h_p\left(z_p^{t+2},z_p^{t+1}\right)\beta_{p,k}\left(z_p^{t+1},\mathbf{z}^t\right)$ appears but no $\beta_{q,l}\left(z_q^{t+2},\mathbf{z}^{t+1}\right)$ with $\|p-q\| \leq l$ (i.e. a triangle is followed by an h-line and at their common endpoint (the apex of the triangle) no other triangle is attached with any of its basepoints).

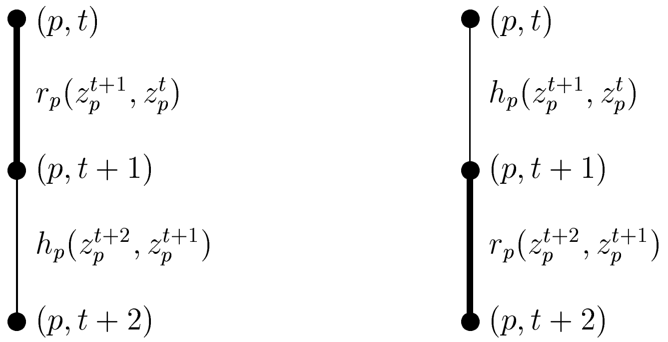

Figure 3: Consecutive r-line and h-line

$$\begin{aligned}\beta_{p,k}\left(w_p, \mathbf{z}\right) &= \frac{w_p}{z_p} \frac{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})}{(f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p)(f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p)} \\ &= \frac{w_p}{z_p}\left[\frac{1}{f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p} - \frac{1}{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p}\right]\end{aligned} \tag{43}$$

By the Residue Theorem:

$$\int_{S^1} \frac{dw_p}{2\pi\imath} \frac{1}{w_p} \beta_{p,k}\left(w_p, \mathbf{z}\right) = 0 \tag{44}$$

because the poles at $w_p = f_p \circ T^\epsilon_{p,k}(\mathbf{z})$ and $w_p = f_p \circ T^\epsilon_{p,k-1}(\mathbf{z})$ (with $\mathbf{z} \in \Gamma^N$, in particular $z_p \in \Gamma_+$ or Γ_-) both lie either outside Γ_+ or inside Γ_- as f_p is expanding and $T^\epsilon_{p,k}$ close to $T^\epsilon_{p,k-1}$ and the two summands have residue $\frac{-1}{z_p}$ and $\frac{1}{z_p}$, respectively.

This identity is a consequence of the fact that $\beta_{p,k}$ is the kernel of a difference between two transfer operators (for the systems with interaction of range k and $k-1$) both preserving the integral. So the range of this operator consists of functions with integral zero and these are annihilated by the operator corresponding to h_p.

Furthermore we note that in

$$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} = \sum_{\mathcal{C}} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \tag{45}$$

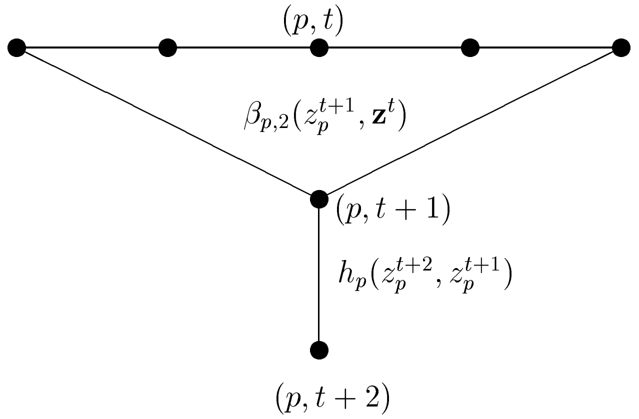

Figure 4: Combination 2-triangle and h-line

we get $\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}}=0$ unless $\mathcal{C}$ ends with h-lines in all points of $(\Lambda_2\setminus\Lambda_1)\times\{0\}$ because of (37), (44) and the fact that π_{Λ_1} means integration over $(S^1)^{\Lambda_2\setminus\Lambda_1}$. So we just have to sum over non-zero configurations that end (at time 0) with r-lines or triangles at most in $\Lambda_1\times\{0\}$. Let $\mathcal{C}$ be a configuration with exactly n_r r-lines and $n_{\beta,k}$ k-triangles for $0\le k<\infty$ (so the set of triangles is given by $n_\beta\overset{\text{def}}{=}(n_{\beta,1},n_{\beta,2},\ldots)$ with $|n_\beta|\overset{\text{def}}{=}\sum_{k=1}^{\infty}n_{\beta,k}<\infty$).

We have to find an upper bound for the norm of each $\mathcal{L}_{\mathcal{C}}$. We do so by collecting r- and h-lines into chains and estimating the contributions of integrating the factors corresponding to these parts of the configuration.

Definition A sequence of lines from (p,t) to $(p,t+1)$, …, $(p,t+k-1)$ to $(p,t+k)$ with $p\in\Lambda_2$ and $-N\le t\le t+k\le 0$ such that to the points $(p,t+1)\ldots(p,t+k-1)$ no triangles are attached is called an *h-chain of length k*. If such an h-chain is not contained in a longer one it is called a *maximal h-chain*. Then (p,t) and $(p,t+k)$ are denoted its *endpoints*. The definitions of *r-chain* etc. are analogous. Furthermore let $\tilde\Lambda_{\mathcal{C}}$ be the set of points $p\in\Lambda_2$ that appear as the $\mathbf{Z}^d$-coordinate of a base point (p,t) of a triangle in $\mathcal{C}$ and $\Lambda_{\mathcal{C}}$ the set of those points $p\in\mathbf{Z}^d$ that appear as the $\mathbf{Z}^d$-coordinate of an apex (p,t) that does not lie above any other triangle. Λ_r is the set of $r\in\mathbf{Z}^d\setminus\tilde\Lambda_{\mathcal{C}}$ that appear as the $\mathbf{Z}^d$ coordinate of an r-line (this implies that there is an r-chain from time $-N$ to time 0). $\Lambda(\mathcal{C})\overset{\text{def}}{=}\tilde\Lambda_{\mathcal{C}}\cup\Lambda_r$.
In Figure 5 there are for example maximal r-chains from $(1,-3)$ to $(1,0)$ or from $(2,-3)$ to $(2,-2)$. $\Lambda_2=\{1,\ldots,8\}$, $\tilde\Lambda_{\mathcal{C}}=\{2,\ldots,7\}$, $\Lambda_{\mathcal{C}}=\{4\}$ and $\Lambda_r=\{1\}$.

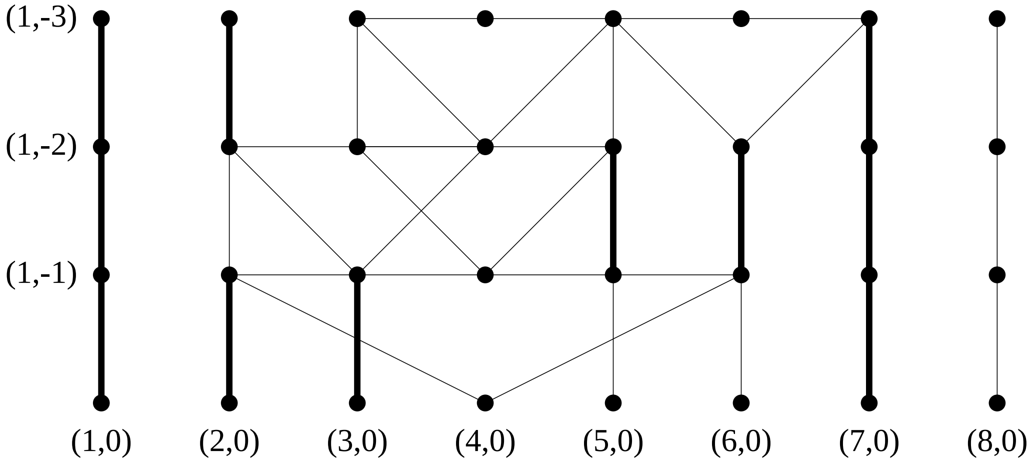

Figure 5: Example for a configuration

As each k-triangle has $v(k) \leq (3k)^d$ base points we have

$$\tilde{\Lambda}_{\mathcal{C}} \leq \sum_{k=1}^{\infty} (3k)^d n_{\beta,k} \tag{46}$$

To get the estimate for (40) we proceed in the following order:

1. We integrate in $\left|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi\left(\mathbf{z}^0_{\Lambda_1}\right)\right|$ over all dz^t_p for which a factor $r_p(z^{t+1}_p, z^t_p)$ appears. For each maximal r-chain of length l we get according to (28) a factor not greater than $c_r \eta^l$.

2. For each maximal h-chain starting at (p,t) and ending at $(p, t+l)$ we perform the integration

$$\int_\Gamma \frac{dz^{t+l-1}_p}{2\pi\imath} \cdots \int_\Gamma \frac{dz^{t+1}_p}{2\pi\imath} h_p(z^{t+l}_p, z^{t+l-1}_p) \cdots h_p(z^{t+1}_p, z^t_p) = h_p(z^{t+l}_p) \tag{47}$$

3. We perform the integration corresponding to π_{Λ_1}

$$\prod_{p \in \Lambda_2 \setminus \Lambda_1} \int_{S^1} \frac{dz^0_p}{2\pi\imath} h_p(z^0_p) = 1 \tag{48}$$

4. In the remaining integral we estimate uniformly $|\beta_{p,k}(z_p^{t+1}, \mathbf{z}^t)|$ by (39) and each (from step 2 and 3 remaining) factor $h_p(z_p^t)$ by $\|h_p\|_{A_\delta} \leq c_h$ and $|\phi(\mathbf{z}^{-N})|$ by $\|\phi_{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r}\|_{A_\delta^{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r}}$ (cf. remark below).

Remark For all points $q \notin \tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r$ we must have h-chains in $\mathcal{C}$ from $(q, -N)$ to $(q, 0)$. Therefore we have

$$\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}(\mathbf{z}^0_{\Lambda_1}) = \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r}(\mathbf{z}^0_{\Lambda_1}) \tag{49}$$

where on the righthandside we use the same notation '$\mathcal{L}_{\mathcal{C}}$' for the operator on $H_{A_\delta^{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r}, \vartheta}$.

So if $\tilde{n}_r$ denotes the number of maximal r-chains and $\tilde{n}_h$ the number of maximal h-chains having spatial coordinates in $\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_1$ (for otherwise they are 'integrated away' giving a factor of 1) we get

$$\begin{aligned} & \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi\|_{\Lambda_1} \\ \leq \; & (c_1 \epsilon)^{|n_\beta|} \exp\left(-c_2 \sum_{k=1}^{\infty} k^d n_{\beta,k} \right) c_h^{\tilde{n}_h} c_r^{\tilde{n}_r} \eta^{n_r} \|\phi_{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r}\|_{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r} \end{aligned} \tag{50}$$

with

$$\begin{aligned} \|\phi_{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r}\|_{\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r} & \leq \vartheta^{-|\Lambda_r| - \sum_{k=1}^{\infty} (3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2, \vartheta} \\ & \leq \vartheta^{-|\Lambda_r|} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2, \vartheta} \end{aligned} \tag{51}$$

for all $\Lambda_2 \in \mathcal{F}$.

6 Operators for the Infinite-Dimensional System

Estimates (50) and (51) bound the particular summands in an expansion like (41). We see that triangles and maximal r-chains in a configuration $\mathcal{C}$ lead to small factors on the right hand side of (50). (A maximal r-chain consisting of n r-lines contributes a factor $c_r\eta^n$. The factor c_r is greater than 1 in general. But either it will be compensated for by a small factor due to a triangle e.g. as in (91) or n will be large, cf. e.g. (95)). This motivates the following definition of the length of a configuration. The length gives rise to a lower bound for the number of triangles or r-lines, i.e. a long configuration will lead to a small contribution in the total sum in (41).

Definition The *length*, length($\mathcal{C}$), of a configuration $\mathcal{C}$ (that we got in an expansion like (42)) is the maximal difference $0-t$ such that there are points (p,t) and $(q,0)$ being end-points of r-lines or base points or apices of triangles. (Note that if there are any triangles or r-lines, there is also a triangle or r-line ending at $\Lambda \times \{0\}$.) If there are no triangles or r-lines in $\mathcal{C}$ its length is zero. We identify two non-zero configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ if they agree in their triangles and r-lines (but might have different t_0, Λ_2). Then for a configuration $\mathcal{C}$ length($\mathcal{C}$), $L(\mathcal{C})$, $\tilde{\Lambda}_{\mathcal{C}}$, $\Lambda(\mathcal{C})$ (as in the definition on page 17) and the operator $\pi_\Lambda \circ \mathcal{L}_{\mathcal{C}} \in L((\mathcal{H}(A_\delta^{\Lambda(\mathcal{C})}), \|\cdot\|_{\Lambda(\mathcal{C})}), (\mathcal{H}(A_\delta^{\Lambda}), \|\cdot\|_\Lambda))$ are still well-defined.

For $\Lambda_1 \in \mathcal{F}$ we define $E(\Lambda_1)$ as the set of all non-zero configurations $\mathcal{C}$ in some $\Lambda_2 \times \{-t_0, \ldots, 0\}$ with $\Lambda_1 \subset \Lambda_2 \in \mathcal{F}$, $t_0 \in N$ and $t_0 >$ length($\mathcal{C}$), and that end at time 0 with triangles or r-lines at most in $\Lambda_1 \times \{0\}$.

Further we define $E_N(\Lambda_1)$ as the set of non-zero configurations $\mathcal{C}$ in $\Lambda_2 \times \{-N, \ldots, 0\}$ with $\Lambda_1 \subset \Lambda_2 \in \mathcal{F}$ and $\Lambda_{\mathcal{C}} \subseteq \Lambda_2$.

We define

$$\nu_\Lambda \overset{\text{def}}{=} \sum_{\mathcal{C} \in E(\Lambda)} \pi_\Lambda \circ \mathcal{L}_{\mathcal{C}} h_{\Lambda(\mathcal{C})} \tag{52}$$

The convergence of this infinite sum and other properties of ν are proved in the following proposition additional to Theorem 1.

Proposition 5 *Let ϑ, ϵ, c_2, N and Λ_1 as in Theorem 1.*

1. $$\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon} = \sum_{\mathcal{C} \in E_N(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \tag{53}$$

2. $$\left\| \mathcal{L}^N_{F\circ T^\epsilon} - \mathcal{L}^{N+1}_{F\circ T^\epsilon} \right\|_{L((\mathcal{H}_\vartheta, \|\cdot\|_\vartheta))} \leq c_9 \tilde{\eta}^N \tag{54}$$

3. *$\nu = (\nu_{\Lambda_1})_{\Lambda_1 \in \mathcal{F}} \in \mathcal{H}^{bv}_\vartheta$, $\mu(\nu) = 1$ and $\nu \geq 0$ and ν satisfies (8) in Theorem 1.*

4. *For* $N_1, N_2 \in N$ *the operator* $\mathcal{L}^{N_2}_{F\circ T^\epsilon}$ *is defined on* $\mathcal{L}^{N_1}_{F\circ T^\epsilon}(\mathcal{H}_\vartheta) \subset \mathcal{H}_{\vartheta_{N_1}}$ *and maps this space to* $\mathcal{H}_{\vartheta_{N_1+N_2}}$ *and*

$$\mathcal{L}^{N_2}_{F\circ T^\epsilon} \circ \mathcal{L}^{N_1}_{F\circ T^\epsilon} = \mathcal{L}^{N_1+N_2}_{F\circ T^\epsilon} \tag{55}$$

5. ν *is the unique* $\mathcal{L}_{F\circ T^\epsilon}$*-invariant element in* $\mathcal{H}_\vartheta$ *with* $\mu(\nu) = 1$

6. *For* $g \in \mathcal{C}(M)$ *and* $\phi \in \mathcal{H}^{bv}_\vartheta$

$$\int_M d\mu\, g \circ S\, \phi = \int_M d\mu\, g \mathcal{L}_{F\circ T^\epsilon}\phi \tag{56}$$

and in particular

$$\mu(\phi) = \mu(\mathcal{L}_{F\circ T^\epsilon}\phi) \tag{57}$$

For finite-range interaction all this also holds for $\phi \in \mathcal{H}^{bv}$*.*

7. $\mathcal{L}_{F\circ T^\epsilon}$ *is non-negative, i.e.* $\phi \geq 0$ *implies* $\mathcal{L}_{F\circ T^\epsilon}\phi \geq 0$*.*

8. *For* $\phi \in \mathcal{H}^{bv}_\vartheta$ *we have the estimate* $\|\mathcal{L}_{F\circ T^\epsilon}\phi\|_{var} \leq \|\phi\|_{var}$*. For finite-range interaction all this also holds for* $\phi \in \mathcal{H}^{bv}$*.*

7 Decay of Correlations

We have found the unique $\nu \in \mathcal{H}_\vartheta$ with $\mu(\nu) = 1$. This corresponds to a non-negative measure on $(M, \mathcal{B})$ whose marginal on $(S^1)^\Lambda$ wrt. μ^Λ is given by (the restriction of) ν_Λ. We need the following proposition about exponential decay of correlations for ν for the proof of Theorem 2.

Theorem 3 *For sufficiently small* ϑ *and* ϵ*, big* c_2*, finite disjoint* Λ_1, Λ_2 *and* $f \in H(A^{\Lambda_2}_\delta)$ *there are a* $\kappa \in (0,1)$ *and a* $\tilde{\vartheta} \in (0,1)$ *such that*

1. $\|\nu_{\Lambda_1\cup\Lambda_2} - \nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2,\vartheta} \leq c_{10}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}$
2. $\|\pi_{\Lambda_1}(f\nu) - \nu(f)\nu_{\Lambda_1}\|_{\Lambda_1,\vartheta} \leq c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}$
3. $\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon}(f\nu) - \nu(f)\nu_{\Lambda_1}\|_{\Lambda_1,\tilde{\vartheta}} \leq c_{12}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}\tilde{\eta}^N$
   for every $N \geq 0$.

Remark As in Theorem 2 we can choose the rate of decay κ first and then the other parameters.

8 Proofs

Proof of Proposition 1 We have a Cauchy estimate for the partial derivatives of the functions $g_{p,k} : A_\delta^{B_k(p)} \to \mathbf{C}$ on a smaller polyannulus. Let $q \in B_k(p)$, Then

$$\begin{aligned}
\left\| \frac{\partial}{\partial z_q} g_{p,k} \right\|_{A_\delta^{B_k(p)}} &\leq \frac{1}{|e^{\delta} - e^{\delta_1}|} c_1 \exp(-c_2 k^d) \qquad (58) \\
&= c_{13} \exp(-c_2 k^d) \qquad (59)
\end{aligned}$$

Also note that $\frac{\partial}{\partial z_q} g_{p,k} = 0$ for $q \notin B_k(p)$. Therefore

$$\begin{aligned}
\left\| \frac{\partial}{\partial z_q} g_p \right\|_{A_\delta^{\mathbf{Z}^d}} &= \left\| \frac{\partial}{\partial z_q} \sum_{k=\|p-q\|}^{\infty} g_{p,k} \right\|_{A_\delta^{\mathbf{Z}^d}} \qquad (60) \\
&\leq c_{13} \sum_{k=\|p-q\|}^{\infty} \exp(-c_2 k^d) \\
&\leq c_{13} \frac{1}{1 - \exp(-c_2)} \exp\left(-c_2 \|p-q\|^d\right) \\
&= c_{14} \exp\left(-c_2 \|p-q\|^d\right)
\end{aligned}$$

Now we consider everything in the lift given by $pr : \mathbf{C}_\delta^\Lambda \to A_\delta^\Lambda$, $(\tilde{z}_p)_{p\in\Lambda} \mapsto \left(e^{\imath \tilde{z}_p}\right)_{p\in\Lambda}$, where $\mathbf{C}_\delta \overset{\text{def}}{=} \{w \in \mathbf{C} \mid \Im w \in [-\delta, \delta]\}$.

Then we have $\left(\widetilde{T^{\Lambda,\epsilon}}(\tilde{\mathbf{z}})\right)_p = \tilde{z}_p + 2\pi\epsilon \tilde{g}_p(\tilde{\mathbf{z}})$. The function $\tilde{g}_p(\tilde{\mathbf{z}}) = g_p(pr(\tilde{\mathbf{z}}))$ satisfies the same estimate (1) with a different constant $\tilde{c_1}$ for $\delta < \delta_1$ sufficiently small since pr and its partial derivatives are uniformly bounded on $\mathbf{C}_\delta^\Lambda$.

Then we have

$$\left| D\left(\widetilde{T^{\Lambda,\epsilon}}(\tilde{\mathbf{z}})\right)_{p,q} - \delta_{p,q} \right| \leq 2\pi\epsilon \tilde{c_1} \exp\left(-c_{13} \|p-q\|^d\right)$$

In particular the row sum norm (the operator-norm induced by the l^∞-norm on $\mathbf{C}^\Lambda$) of $\left(D\widetilde{T^{\Lambda,\epsilon}} - \mathrm{id}\right)$ is smaller than 1 for $\epsilon < \epsilon_0$ small enough. According

to Lemma 2 (cf. below and noting that $\mathbf{C}_\delta$ is convex), $\widetilde{T^{\Lambda,\epsilon}}$ is a biholomorphic map onto its image and so is $T^{\Lambda,\epsilon}$.
Now fix $\delta < \delta_0$ according to the first part of the proof. If $\mathbf{z} \in \partial A_\delta^\Lambda$ we have $z_p \in \partial A_\delta$ for at least one $p \in \Lambda$. From the formula $z_p' \overset{\text{def}}{=} T_p^{\Lambda,\epsilon}(\mathbf{z}) = z_p \exp\left(2\pi\imath\epsilon g_p(\mathbf{z})\right)$ and the assumption that g_p is uniformly bounded on A_{δ_1} we see

$$\left|\ln|z_p'|\right| \geq \delta - c_{16}\epsilon > c_{17}\delta \tag{61}$$

for $\epsilon \leq \epsilon_0 < \frac{1-c_{17}}{c_{16}}\delta$.
Now assume $\emptyset \neq A_{c\delta} \setminus T^{\Lambda,\epsilon}(A_\delta) \ni \mathbf{z}$. Let s be the line-segment between $\mathbf{z}$ and its nearest point $\mathbf{w}$ on $\left(S^1\right)^\Lambda$ (wrt. the metric d_Λ). For each point $\mathbf{y}$ on s the inequality $\ln d_\Lambda(\mathbf{w},\mathbf{y}) \leq \ln d_\Lambda(\mathbf{w},\mathbf{z}) \leq c_{17}\delta$ holds.
In particular there is a $\mathbf{y} \in T^{\Lambda,\epsilon}\left(\partial A_\delta^\Lambda\right)$ on s with $|y_p| \leq c_{17}\delta$ for all $p \in \Lambda$, but this contradicts the estimate (61) above. □

Lemma 2 *If $T : U \to \mathbf{C}^n$ is a holmorphic map on a convex set $U \subset \mathbf{C}^n$ and satisfies the estimate $\|DT(z) - id\| \leq c_{18} < 1$ then T is biholomorphic onto its image (in this lemma the chosen norm on $\mathbf{C}^n$ and the corresponding operator norm are both denoted by $\|\cdot\|$).*

Proof T is locally biholomorphic by the Inverse Function Theorem. So we only have to show injectivity. Let $z^0, z^1 \in U$ with $T\left(z^0\right) = T\left(z^1\right)$ and $\gamma : [0,1] \to U$, $\gamma(t) = z^o + t(z^1 - z^0)$. Then

$$\begin{aligned}
\left\|z^1 - z^0\right\| &= \left\|T\left(z^1\right) + z^1 - T\left(z^0\right) - z^0\right\| \\
&= \|T \circ \gamma(1) + \gamma(1) - T \circ \gamma(0) + \gamma(0)\| \\
&= \left\|\int_0^1 \left(DT\left(\gamma(t)\right) - \mathrm{id}\right)\left(z^1 - z^0\right) dt\right\| \\
&\leq \left\|z^1 - z^0\right\| \int_0^1 \|DT\left(\gamma(t)\right) - \mathrm{id}\| \, dt \\
&\leq \left\|z^1 - z^0\right\| c_{18}
\end{aligned} \tag{62}$$

which implies $z^1 = z^0$. □

Proof of Proposition 2 As F acts on each coordinate separately by an f_p we have (in view on the chosen metric (13)) to show the statement just for the map f (we drop the index p), i.e. the case Λ containing just one element.

Consider the lift $\mathbf{R}_\delta \times \mathbf{R} \ni (r,\phi) \mapsto re^{\imath\phi}$ where $\mathbf{R}_\delta \overset{\text{def}}{=} [1-\ln\delta, 1+\ln\delta]$. This defines (modulo $(0,2\pi)$) a $(0,2\pi)$-periodic map $\tilde{f} = \left(\tilde{f}_r, \tilde{f}_\phi\right)$ via $f\left(re^{\imath\phi}\right) = \tilde{f}_r(r,\phi)e^{\imath\tilde{f}_\phi(r,\phi)}$. On $\{1\}\times\mathbf{R}$ one has $\frac{\partial}{\partial r}\tilde{f}_r \geq \lambda_0$ and so because of periodicity and a compactness argument, $\frac{\partial}{\partial r}\tilde{f}_r \geq \lambda$ on a thin $(0<\delta<\delta_0$ small) strip $\mathbf{R}_\delta \times \mathbf{R}$. It follows similarly, as in the proof of Proposition 1, that $\tilde{f}(\mathbf{R}_\delta\times\mathbf{R}) \supset \mathbf{R}_{\lambda\delta}\times\mathbf{R}$, $\tilde{f}$ is diffeomorphic onto its image and each point in $\mathbf{R}_\delta\times\mathbf{R}$ has the same number of preimages (which is equal to $\left(\tilde{f}(1,2\pi)-\tilde{f}(1,0)\right)/2\pi$). From this the claim about f follows. □

Proof of Proposition 3 We substitute the expression (18) into the right-hand side of equation (16) and get

$$\int_{T^N}\frac{d\mathbf{w}}{(2\pi\imath)^N}\frac{1}{\mathbf{w}}\psi(\mathbf{w})\int_{\Gamma^n}\frac{d\mathbf{z}}{(2\pi\imath)^N}\phi(\mathbf{z})\prod_{k=1}^{N}\left(\frac{1}{S_k^\epsilon(\mathbf{z})-w_k}\frac{S_k^\epsilon(\mathbf{z})}{z_k}\right) \tag{63}$$

As (16) is linear in ψ we can assume (by using a continuous partition of unity) that ψ vanishes outside a small set $K\subset T^N$ having distinct preimages under S^t (for all $0\leq t\leq\epsilon$) contained in $K_\alpha = K_{\alpha_1}\times\cdots\times K_{\alpha_N}$ such that each K_α is contained in a polydisc $D_\alpha = D_{\alpha_1}\times\cdots\times D_{\alpha_N}$. These are mutually disjoint and $S_\alpha^t \overset{\text{def}}{=} S^t_{|K_\alpha}$ is biholomorphic onto K (for all $0\leq t\leq\epsilon$). (To make this more precise we note that for $t=0$ the map S^0 is the product of maps f_i $(1\leq i\leq N)$ and each f_i gives rise to an M_i-fold covering map of A_δ. So locally we can index the disjoint preimages of K under S^0 by $\alpha=(\alpha_1,\ldots,\alpha_N)$ where $1\leq\alpha_i\leq M_i$. If we take the set K small enough this is still true under small $(0\leq t\leq\epsilon)$ perturbations.)

For given $\mathbf{w}\in K$, index α as above, $k\in\{1,\ldots,N\}$ and fixed $z_l\in K_{\alpha_l}$ $(l\neq k)$ the function $z_k\mapsto(S^\epsilon_{\alpha,k}(z_1,\cdots,z_k,\cdots,z_N)-w_k)^{-1}$ has exactly one simple pole in D_{α_k} and is holomorphic in A_δ^Λ away from this pole. Therefore we get the same if we just integrate around these poles.

$$=\int_K\frac{d\mathbf{w}}{(2\pi\imath)^N}\frac{1}{\mathbf{w}}\psi(\mathbf{w})\sum_\alpha\left(\prod_{k=1}^N\int_{\partial D_{\alpha_k}}\frac{dz_k}{2\pi\imath}\right)\phi(\mathbf{z})\prod_{k=1}^N\frac{S^\epsilon_{\alpha,k}(\mathbf{z})}{z_k}\prod_{k=1}^N\frac{1}{S^\epsilon_{\alpha,k}(\mathbf{z})-w_k} \tag{64}$$

For each α we can write each of the inner integrals as an integral of a differential form over the distinguished boundary $b_0D_\alpha \overset{\text{def}}{=} \partial D_{\alpha_1}\times\ldots\times\partial D_{\alpha_N}$, parameterized by $[0,1)^N\ni t\mapsto(e^{2\pi\imath t_1},\ldots,e^{2\pi\imath t_N})$, whence

$$\int_{b_0 D_\alpha} \phi(\mathbf{z}) \prod_{k=1}^{N} \frac{S^\epsilon_{\alpha,k}(\mathbf{z})}{z_k} \prod_{k=1}^{N} \frac{1}{S^\epsilon_{\alpha,k}(\mathbf{z}) - w_k} dz_1 \wedge \ldots \wedge dz_N \tag{65}$$

We want to split the singular factor into a product of single poles in each variable. So we apply the transformation $\mathbf{u} = S_\epsilon(\mathbf{z}) \overset{\text{def}}{=} S^\epsilon_\alpha(\mathbf{z})$.

$$\int_{S_\epsilon(b_0 D_\alpha)} \phi \circ S_\epsilon^{-1}(\mathbf{u}) \prod_{k=1}^{N} \frac{u_k}{(S_\epsilon^{-1}(\mathbf{u}))_k} \det(S_\epsilon^{-1})'(\mathbf{u}) \prod_{k=1}^{N} \frac{1}{u_k - w_k} du_1 \wedge \ldots \wedge du_N \tag{66}$$

where $(S_\epsilon^{-1})'$ is the complex derivative and so is holomorphic in $\mathbf{u}$. To apply Cauchy's formula we have to integrate over a product of cycles (each lying in $\mathbf{C}$). The map $t \mapsto S_t \overset{\text{def}}{=} S^t_\alpha$ is a homotopy between S_ϵ and the product map S_0 and avoids singularities of the integrand in (66) since for ϵ small enough the set $\{S_t(b_0 D_\alpha) \mid 0 \leq t \leq \epsilon\}$ has positive distance (uniformly in Λ) from the set of singularities $\bigcup_{k=1}^{N} \{u \in D_\alpha : u_k = w_k\}$. $S_0(b_0 D_\alpha) = S_{0,1}(\partial D_{\alpha_1}) \times \ldots \times S_{0,N}(\partial D_{\alpha_N})$ is a product of cycles and hence a cycle. The differential n-form in (66) is a cocycle because its coefficient is holomorphic. So we get by Stokes' theorem

$$= \int_{S_0(b_0 D_\alpha)} \phi \circ S_\epsilon^{-1}(\mathbf{u}) \prod_{k=1}^{N} \frac{u_k}{(S_\epsilon^{-1}(\mathbf{u}))_k} \det(S_\epsilon^{-1})'(\mathbf{u}) \prod_{k=1}^{N} \frac{1}{u_k - w_k} du_1 \wedge \ldots \wedge du_N \tag{67}$$

and by Cauchy's formula

$$= \phi \circ S_\epsilon^{-1}(\mathbf{w}) \prod_{k=1}^{N} \frac{w_k}{(S_\epsilon^{-1}(\mathbf{w}))_k} \frac{1}{\det(S_\epsilon'(S_\epsilon^{-1}(\mathbf{w})))} \tag{68}$$

So (64) is equal to

$$\sum_\alpha \int_K \frac{d\mathbf{w}}{(2\pi\imath)^N} \frac{1}{\mathbf{w}} \psi(\mathbf{w}) \phi \circ (S^\epsilon_\alpha)^{-1}(\mathbf{w}) \frac{1}{\det(S^\epsilon)'((S^\epsilon_\alpha)^{-1}(\mathbf{w}))} \prod_{k=1}^{N} \frac{w_k}{((S^\epsilon_\alpha)^{-1}(\mathbf{w}))_k} \tag{69}$$

For each index α, the map S^ϵ_α gives rise to a coordinate transformation $\mathbf{u} = (S^\epsilon_\alpha)^{-1}(\mathbf{w})$.

$$= \sum_\alpha \int_{K_\alpha} \frac{d\mathbf{u}}{(2\pi\imath)^N} \frac{1}{\mathbf{u}} \psi \circ S^\epsilon_\alpha(\mathbf{u}) \phi(\mathbf{u}) \tag{70}$$

As $\psi \circ F = 0$ outside $\bigcup_\alpha K_\alpha$ and the K_α are mutually disjoint this equals

$$\begin{aligned} &= \int_{(S^1)^N} \frac{d\mathbf{u}}{(2\pi\imath)^N} \frac{1}{\mathbf{u}} \psi \circ S(\mathbf{u}) \phi(\mathbf{u}) && (71) \\ &= \int_{(S^1)^N} d\mu^N \psi \circ S \, \phi && (72) \end{aligned}$$

as was to be shown. $\square$

Proof of Lemma 1 Consistency follows from

$$\begin{aligned} \pi_{\Lambda_3}(g^1\phi)_{\Lambda_4} &= \pi_{\Lambda_3} \circ \pi_{\Lambda_4}(g^1\phi_{\Lambda\cup\Lambda_4}) && (73) \\ &= \pi_{\Lambda_3}(g^1\phi_{\Lambda\cup\Lambda_4}) \\ &= \pi_{\Lambda_3}(g^1\phi_{\Lambda\cup\Lambda_3}) \\ &= (g^1\phi)_{\Lambda_3} \end{aligned}$$

for all $\Lambda_3 \subset \Lambda_4 \in \mathcal{F}$.
As g^1 depends only on the Λ_1-coordinates we have

$$\begin{aligned} \|(g^1\phi)_{\Lambda_1\cup\Lambda}\|_{\Lambda_1\cup\Lambda} &= \|g^1\phi_{\Lambda_1\cup\Lambda}\|_{\Lambda_1\cup\Lambda} && (74) \\ &\leq \|g^1\|_{\Lambda_1} \|\phi_{\Lambda_1\cup\Lambda}\|_{\Lambda_1\cup\Lambda} \\ &\leq \|g^1\|_{\Lambda_1} \vartheta^{-|\Lambda_1|-|\Lambda|} \|\phi\|_\vartheta \end{aligned}$$

and so

$$\vartheta^{|\Lambda|} \|g^1\phi)_\Lambda\|_\Lambda \leq \|g^1\|_{\Lambda_1} \vartheta^{-|\Lambda_1|} \|\phi\|_\vartheta \tag{75}$$

and

$$\|g\phi\|_\vartheta \leq \|g^1\|_{\Lambda_1} \vartheta^{-|\Lambda_1|} \|\phi\|_\vartheta \tag{76}$$

For Λ_1 fixed the product is continuous in both factors.
(2.) follows from

$$\begin{aligned} ((g^1g^2)\phi)_\Lambda &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}\phi_{\Lambda\cup\Lambda_1\cup\Lambda_2}) \\ &= \pi_\Lambda(g^1_{\Lambda_1}\pi_{\Lambda\cup\Lambda_1}(g^2_{\Lambda_2}\phi_{\Lambda\cup\Lambda_1\cup\Lambda_2})) \\ &= \pi_\Lambda(g^1_{\Lambda_1}\pi_{\Lambda\cup\Lambda_1}(g^2\phi)) \\ &= (g^1(g^2\phi))_\Lambda \end{aligned} \tag{77}$$

To see (3.) we note that the projection of the product of g^1 and g^2 is

$$\pi_\Lambda(g^1g^2) = \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}) \tag{78}$$

and the product in the sense of (23) has Λ-marginal

$$\begin{aligned} \pi_\Lambda(g^1g^2) &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda\cup\Lambda_2}) \\ &= \pi_\Lambda(g^1_{\Lambda_1}g^2_{\Lambda_2}) \end{aligned} \tag{79}$$

as g^2 does not depend on $\Lambda \setminus \Lambda_2$-coordinates.
If $\Lambda_1 \subseteq \Lambda_2$ then

$$\begin{aligned} g_{\Lambda_2}(g^1\phi)_{\Lambda_2} &= g_{\Lambda_2}g^1\phi_{\Lambda_2} \\ &= (g^1g)_{\Lambda_2}\phi_{\Lambda_2} \end{aligned} \tag{80}$$

and so (4.) follows from

$$\begin{aligned} (g^1\phi)(g) &= \lim_{\Lambda_2\to\mathbf{Z}^d}\int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} g_{\Lambda_2}(g^1\phi)_{\Lambda_2} \\ &= \lim_{\Lambda_2\to\mathbf{Z}^d}\int_{(S^1)^{\Lambda_2}} d\mu^{\Lambda_2} (g^1g)_{\Lambda_2}\phi_{\Lambda_2} \\ &= \phi(g^1g) \end{aligned} \tag{81}$$

$$\begin{aligned} \|g^1\phi\|_{var} &= \lim_{\Lambda\to\mathbf{Z}^d}\int_{(S^1)^{\Lambda}} d\mu^{\Lambda} |(g^1\phi)_\Lambda| \\ &= \lim_{\substack{\Lambda\to\mathbf{Z}^d \\ \Lambda\supset\Lambda_1}}\int_{(S^1)^{\Lambda}} d\mu^{\Lambda} |g^1||\phi_\Lambda| \\ &\leq \|g^1\|_{\Lambda_1}\|\phi\|_{var} \end{aligned} \tag{82}$$

□

Proof of Proposition 4 We get recursively

$$
\begin{aligned}
&\frac{1}{f_p \circ T^\epsilon_{p,l}(\mathbf{z}) - w_p} \frac{f_p \circ T^\epsilon_{p,l}(\mathbf{z})}{z_p} \qquad (83)\\
&= \frac{1}{f_p \circ T^\epsilon_{p,l-1}(\mathbf{z}) - w_p} \frac{f_p \circ T^\epsilon_{p,l-1}(\mathbf{z})}{z_p} \\
&\quad + \frac{w_p}{z_p} \frac{f_p \circ T^\epsilon_{p,l-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,l}(\mathbf{z})}{(f_p \circ T^\epsilon_{p,l-1}(\mathbf{z}) - w_p)(f_p \circ T^\epsilon_{p,l}(\mathbf{z}) - w_p)} \\
&= \frac{1}{f_p(z_p) - w_p} \frac{f_p(\mathbf{z})}{z_p} + \frac{w_p}{z_p} \sum_{k=1}^{l} \frac{f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - f_p \circ T^\epsilon_{p,k}(\mathbf{z})}{(f_p \circ T^\epsilon_{p,k-1}(\mathbf{z}) - w_p)(f_p \circ T^\epsilon_{p,k}(\mathbf{z}) - w_p)}
\end{aligned}
$$

The estimate (39) yields uniform convergence of this sum as $l \to \infty$. So we get (25). □

In (50) we estimate the norm of the operator corresponding to one particular configuration in terms of its different kinds of lines and triangles. Now we have to bound sums over all such configurations as they arise in expansions for the full operators. For this we use our analysis and some combinatorics at the same time. The idea is that a configuration of a given length must have at least a certain number of triangles and r-chains that lead to small factors in the estimates. In fact some special r-chains could not be replaced by h-chains in the configuration as we would get the zero operator.

Definition A maximal r-chain going from an apex downwards to the next base or bottom point is called an *a-r-chain.* (If the apex coincides with a base or bottom point the a-r-chain has length zero.)
The *a-r-length* of a configuration $\mathcal{C}$ is the sum of the lenghts of all its a-r-chains plus the number of its triangles, i.e. if $\mathcal{C}$ has $|n_\beta|$ triangles with corresponding a-r-chains of length $l_1, \ldots, l_{|n_\beta|}$ then

$$
\begin{aligned}
\text{a-r-length}(\mathcal{C}) &\overset{\text{def}}{=} l_1 + \cdots + l_{|n_\beta|} + |n_\beta| \qquad (84)\\
&= (l_1 + 1) + \cdots + (l_{|n_\beta|} + 1)
\end{aligned}
$$

(In particular a-r-length$(\mathcal{C}) \geq |n_\beta|$.)

We call a maximal r-chain going from a base point (p, t) of a triangle to $(p, -N)$ (such that $(p, -N)$ is not a base point of another triangle) a *u-r-chain* (upwards going r-chain), a maximal r-chain going downwards from a basepoint a *d-r-chain* (*d-h-chains* are defined analogously), and a maximal r-chain going from a bottom point $(p, 0)$ to $(p, -N)$ an *l-r-chain* (long r-chain). The configuration in Figure 5 has length 3, a-r-length 6, only one a-r-chain of positive length from $(6, -2)$ to $(6, -1)$, only one u-r-chain of positive length from $(3, -3)$ to $(3, -2)$, and only one l-r-chain from $(1, -3)$ to $(1, 0)$.

We prepare the proofs of Theorem 1 and Proposition 5 in the following technical proposition that provides the basic analysis and combinatorics for all other proofs.

Proposition 6 *For sufficiently small ϑ, ϵ and big c_2 and N we have for all $\Lambda_1 \subseteq \Lambda_2 \in \mathcal{F}$ the following bound for the terms in the expansion of (45) for $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}$ with constants c_{19}, c_{20}:*

1. $$\sum_{\mathcal{C}:\mathrm{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}}\|_{L\left(\left(\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}\right), \left(\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}\right)\right)} \leq c_{19}\tilde{\eta}^N \tag{85}$$

   with $\tilde{\eta} \overset{\mathrm{def}}{=} \sqrt{\eta} < 1$

2. $$\left\|\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}\right\|_{L\left(\left(\mathcal{H}_{\Lambda_2,\vartheta}, \|\cdot\|_{\Lambda_2,\vartheta}\right), \left(\mathcal{H}_{\Lambda_1,\vartheta}, \|\cdot\|_{\Lambda_1,\vartheta}\right)\right)} \leq c_{20} \tag{86}$$

Proof

1. We fix $0 \leq K \leq |\Lambda_1|$ and $\Lambda_3 \subseteq \Lambda_1$ with $|\Lambda_3| = K$ (so there are $\binom{|\Lambda_1|}{K}$ possible choices for Λ_3) and want to estimate the number of configurations $\mathcal{C}$ such that $\Lambda_{\mathcal{C}} = \Lambda_3$. So let us consider such a configuration. We call the triangles whose apex lies at, or whose a-r-chain ends in, $\Lambda_3 \times \{0\}$, *root triangles*. We can assign to $\mathcal{C}$ a graph as follows: We start with a star graph with a star point labelled (0) and K leaves, labelled $(0, 1), \ldots, (0, K)$. These leaves are in bijection with $\Lambda_3 \times \{0\}$. Now we add successively for each l-triangle in $\mathcal{C}$ a small *l-tree* (a star graph with one star point and $v(l)$ leaves) to the graph and label the

new vertices: If an l-triangle lies with its apex or ends with its a-r-chain on a basepoint of another triangle (for that we have already assigned a small tree) or on a point in $\Lambda_3 \times \{0\}$ (this point is labelled say $s = (s_1, \dots, s_n)$) we add a small l-tree to the graph by identifying its star point with s and label the $v(l)$ new leaves in the graph $(s_1, \dots, s_n, 1)$,...,$(s_1, \dots, s_n, v(l))$. Since, for example, an apex could coincide with more than one other triangle's basepoint we introduce a linear order on the set of tuples (and so on the set of vertices of the labelled graph):

We say $s = (s_1, \dots, s_n) \prec t = (t_1, \dots, t_m)$ if $n < m$ and $s_i = t_i$ for $1 \leq i \leq n$ or if $s_i = t_i$ $(1 \leq i \leq k)$ and $s_k < t_k$ for some k.

In our successive assignment of triangles to small trees we always choose the next triangle such that the corresponding small tree is attached to the smallest (wrt. $\prec$) labelled leaf in the graph. This also defines a unique choice of the triangle and the leaf where we attach the small tree. So every $\mathcal{C}$ is completely determined by its corresponding labelled graph and the length of its a-r-chains. Note that it is not the case that for every graph together with a choice of lengths for the particular a-r-chains there was a corresponding configuration, but at least we have found an injection between these two data.

For the configuration in Figure 5, for example, we get the following labelled graph:

If $n_{\beta,k}$ is the total number of k-triangles, the number of such corresponding sets of graphs is not greater than $4^K \prod_{k=1}^{\infty} c_{21}^{k^d n_{\beta,k}}$ (by Lemma 3, see below). As mentioned above we have for each of the $|n_\beta|$ a-r-chains a length $0 \leq l_i < \infty$. The a-r-length is

$$L = (l_1 + 1) + \dots + (l_{|n_\beta|} + 1). \tag{87}$$

So $L \geq |n_\beta|$. For a given n_β with $|n_\beta| \geq 1$ and $L \geq 1$ there are $\binom{L-1}{|n_\beta|-1}$ different choices of $(l_1, \dots, l_{|n_\beta|})$ that satisfy (87). For $|n_\beta| = 0$ we have $L = 0$ and the (unique) configuration without triangles or r-lines. So in any case the number of choices is bounded from above by $\binom{L}{|n_\beta|}$. The integration over these $|n_\beta|$ a-r-chains leads to a factor $c_r^{|n_\beta|} \eta^L$ in our estimates (cf. (50)).

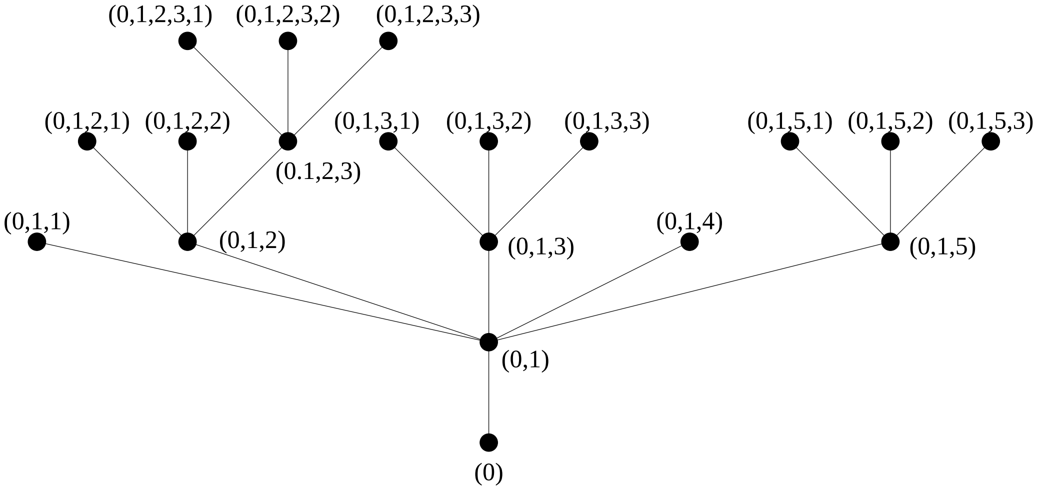

Figure 6: The labelled graph for the configuration in Figure 5

2. There are choices between d-r-chains and d-h-chains in the configuration.

   There are not more than $\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}$ base points for which we can choose between a d-h-chain (giving factor c_h in our estimates) and a d-r-chain (giving factor at most $c_r\eta$). So the total sum over these combinations is bounded from above by

   $$(c_h + c_r\eta)^{\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}} \le \prod_{k=1}^{\infty}\left(\exp(c_{22}k^d)\right)^{n_{\beta,k}}$$

3. There are choices between u-r-chains and u-h-chains in the configuration.

   There are not more than $\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}$ basepoints. To each of them we can attach either a u-h-chain, giving a factor c_h, or a u-r-chain, giving a factor $c_r\eta^{\max\{0,N-L\}}$, because if $N - L > 0$, such a u-r-chain cannot have length smaller than $N - L$, for otherwise it would not end in $\Lambda_2 \times \{-N\}$. If $N - L > 0$ there must be at least one u-r-chain, so we get in total a factor not greater than

   $$(c_h + c_r)^{\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}}\eta^{\max\{0,N-L\}} = \prod_{k=1}^{\infty}\left(\exp(c_{23}k^d)\right)^{n_{\beta,k}}\eta^{\max\{0,N-L\}} \tag{88}$$

4. There are only choices left between l-h-chains and l-r-chains in $(\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}) \times \{-N, \ldots, 0\}$, giving factor c_h or $c_r\eta^N$ respectively. Let l $(0 \le l \le |\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}| \le |\Lambda_1| - K)$ denote the number of l-r-chains in such a choice. For given l there are $\binom{|\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}|}{l} \le \binom{|\Lambda_1|-K}{l}$ different subsets Λ_r of $\Lambda_1 \setminus \tilde{\Lambda}_{\mathcal{C}}$ of cardinality l (that corresponds to a particular choice of exactly l l-r-chains.) The configuration $\mathcal{C}$ is determined by all the choices mentioned up to now.

In the configuration $\mathcal{C}$ there are h-chains at points with $\mathbf{Z}^d$-coordinate in $\Lambda_1 \setminus (\tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r)$. The operator $\mathcal{L}_{\mathcal{C}}$ acts on ϕ_{Λ_2} by integration over these coordinates. So for the uniform estimate of $\mathcal{L}_{\mathcal{C}}\phi_{\tilde{\Lambda}}$ we will use (51).
Therefore we have the estimate

$$
\begin{aligned}
& \vartheta^{|\Lambda_1|} \sum_{\mathcal{C}:\mathrm{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1} && (89)\\
\le\; & \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K \le |n_\beta| < \infty}} 4^K \prod_{k=1}^{\infty} \Big(\exp(c_{21}k^d)\Big)^{n_{\beta,k}} (c_1\epsilon)^{|n_\beta|} && (90)\\
& \times \prod_{k=1}^{\infty} \Big(\exp(-c_2k^d)\Big)^{n_{\beta,k}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} c_r^{|n_\beta|}\eta^L \prod_{k=1}^{\infty} \Big(\exp(c_{22}k^d)\Big)^{n_{\beta,k}} \\
& \times \prod_{k=1}^{\infty} \Big(\exp(c_{23}k^d)\Big)^{n_{\beta,k}} \eta^{\max\{0,N-L\}} \sum_{l=0}^{|\Lambda_1|-K} \binom{|\Lambda_1|-K}{l} (c_r\eta^N)^l \\
& \times c_h^{|\Lambda_1|-K-l}\vartheta^{-l} \prod_{k=1}^{\infty} \vartheta^{-(3k)^d n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta} \\
\le\; & \vartheta^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K \le |n_\beta| < \infty}} 4^K (c_1\epsilon c_r)^{|n_\beta|} && (91)\\
& \times \prod_{k=1}^{\infty} \exp((c_{21} - c_2 + c_{22} + c_{23} - 3^d \ln\vartheta)k^d)^{n_{\beta,k}} \\
& \times \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \eta^{\max\{N,L\}} (\vartheta^{-1}c_r\eta^N + c_h)^{|\Lambda_1|-K} \|\phi\|_{\Lambda_2,\vartheta}.
\end{aligned}
$$

We set $\epsilon_1 \stackrel{\mathrm{def}}{=} 4\epsilon c_1 c_r$ and $\epsilon_2 \stackrel{\mathrm{def}}{=} \sqrt{\epsilon_1}$. Then we have $\epsilon_1^{|n_\beta|} \le \epsilon_2^K \epsilon_2^{|n_\beta|}$. We set

$\tilde{c_2} \overset{\text{def}}{=} c_2 - c_{21} - c_{22} - c_{23} + 3^d \ln\vartheta$. Then $\tilde{c_2} > 0$ if

$$c_2 > c_{21} + c_{22} + c_{23} - 3^d \ln\vartheta \tag{92}$$

(We assume this condition on the decay of the coupling.) Further we split $\eta^{\max\{N,L\}} \le \tilde{\eta}^L \tilde{\eta}^N$ with $\tilde{\eta} = \sqrt{\eta}$. Then (91) can be bounded as follows:

$$\begin{aligned}
\le\; & \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} (c_r\eta^N + \vartheta c_h)^{|\Lambda_1|-K} \epsilon_2^K \sum_{\substack{n_\beta \\ K \le |n_\beta| < \infty}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} \tilde{\eta}^L \epsilon_2^{|n_\beta|} \qquad (93) \\
& \times \prod_{k=1}^{\infty} \Big(\exp(-\tilde{c_2} k^d)\Big)^{n_{\beta,k}} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N \\
\le\; & (c_r\eta^N + \vartheta c_h + \epsilon_2)^{|\Lambda_1|} \sum_{L=0}^{\infty} \sum_{n=0}^{L} \binom{L}{n} \tilde{\eta}^L \epsilon_2^n \sum_{\substack{n_\beta \\ |n_\beta| = n}} \prod_{k=1}^{\infty} \Big(\exp(-\tilde{c_2} k^d)\Big)^{n_{\beta,k}} \\
& \times \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N
\end{aligned}$$

We have

$$\sum_{\substack{n_\beta \\ |n_\beta| = n}} \prod_{k=1}^{\infty} \Big(\exp(-\tilde{c_2} k^d)\Big)^{n_{\beta,k}} \le \prod_{k=1}^{\infty} \sum_{n_{\beta,k}=0}^{\infty} \Big(\exp(-\tilde{c_2} k^d)\Big)^{n_{\beta,k}} \tag{94}$$

and the last infinite product converges (to c_{24} say) since for k sufficiently large $\exp(-\tilde{c_2} k^d) < \frac{1}{2}$ and $\sum_{n_{\beta,k}=0}^{\infty} \Big(\exp(-\tilde{c_2} k^d)\Big)^{n_{\beta,k}} \le 1 + 2\exp(-\tilde{c_2} k^d)$ and $\sum_{k=0}^{\infty} \exp(-\tilde{c_2} k^d) < \infty$ (Recall $\prod_{k=1}^{\infty}(1 + u_k)$ *convergent* $\Leftarrow \sum_{k=1}^{\infty} |u_k| < \infty$.)

$$\begin{aligned}
\le\; & (\epsilon_2 + c_r\tilde{\eta}^N + c_h\vartheta)^{|\Lambda_1|} c_{24} \sum_{L=0}^{\infty} (\epsilon_2 + \tilde{\eta})^L \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N \\
\le\; & (\epsilon_2 + c_r\tilde{\eta}^N + c_h\vartheta)^{|\Lambda_1|} \frac{1}{1 - \epsilon_2 - \tilde{\eta}} c_{24} \|\phi\|_{\Lambda_2,\vartheta} \tilde{\eta}^N \qquad (95) \\
\le\; & c_{19} \tilde{\eta}^N \|\phi\|_{\Lambda_2,\vartheta} \qquad (96)
\end{aligned}$$

for ϑ and ϵ sufficiently small and N sufficiently large. This also holds for $\Lambda \subset \Lambda_1$. So (1.) is proved.
If $\mathcal{C}$ is a non-zero configuration of length $0 \le m < N$ in the expansion of $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}}$ it has no l-r-chains. So this time we have $l(\mathcal{C}) = 0$. Using the splitting $\eta^L \le \tilde{\eta}^L \tilde{\eta}^m$ we get in a similar way

$$
\begin{align}
&\vartheta^{|\Lambda_1|} \sum_{\substack{\mathcal{C}:\mathrm{length}(\mathcal{C})=m,\\ l(\mathcal{C})=0}} \|\pi_{\Lambda_1}\circ \mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1} \tag{97}\\
\le\ & \vartheta^{|\Lambda_1|}\sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K}\sum_{\substack{n_\beta\\ K\le|n_\beta|<\infty}} 4^K\prod_{k=1}^\infty\Big(\exp(c_{21}k^d)\Big)^{n_{\beta,k}} \nonumber\\
& \times(c_1\epsilon)^{|n_\beta|}\prod_{k=1}^\infty\Big(\exp(-c_2k^d)\Big)^{n_{\beta,k}}\sum_{L=|n_\beta|}^\infty\binom{L}{|n_\beta|}c_r^{|n_\beta|}\tilde{\eta}^L\prod_{k=1}^\infty\Big(\exp(c_{22}k^d)\Big)^{n_{\beta,k}} \nonumber\\
& \times\prod_{k=1}^\infty\Big(\exp(c_{23}k^d)\Big)^{n_{\beta,k}} c_h^{|\Lambda_1|-K}\prod_{k=1}^\infty \vartheta^{-(3k)^d n_{\beta,k}}\|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N \nonumber\\
\le\ & \sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K}(c_h\vartheta)^{|\Lambda_1|-K}\sum_{\substack{n_\beta\\ K\le|n_\beta|<\infty}}(c_1\epsilon 4c_r)^{|n_\beta|}\prod_{k=1}^\infty\Big(\exp(-\tilde{c_2}k^d)\Big)^{n_{\beta,k}} \nonumber\\
& \times\sum_{L=|n_\beta|}^\infty\binom{L}{|n_\beta|}\tilde{\eta}^L\tilde{\eta}^m\|\phi\|_{\Lambda_2,\vartheta} \nonumber\\
\le\ & \sum_{K=0}^{|\Lambda_1|}\binom{|\Lambda_1|}{K}(c_h\vartheta)^{|\Lambda_1|-K}\epsilon_2^K\sum_{\substack{n_\beta\\ K\le|n_\beta|<\infty}}\sum_{L=|n_\beta|}^\infty\binom{L}{|n_\beta|}\tilde{\eta}^L\epsilon_2^{|n_\beta|} \nonumber\\
& \times\prod_{k=1}^\infty\Big(\exp(-\tilde{c_2}k^d)\Big)^{n_{\beta,k}}\tilde{\eta}^m\|\phi\|_{\Lambda_2,\vartheta} \nonumber\\
\le\ & (\epsilon_2+c_h\vartheta)^{|\Lambda_1|}\frac{1}{1-\epsilon_2-\tilde{\eta}}c_{25}\tilde{\eta}^m\|\phi\|_{\Lambda_2,\vartheta} \nonumber\\
\le\ & c_{26}\tilde{\eta}^m\|\phi\|_{\Lambda_2,\vartheta} \tag{98}
\end{align}
$$

Again this also holds for $\Lambda\subset\Lambda_1$ and so

$$
\vartheta^{|\Lambda_1|} \sum_{\substack{\mathcal{C}:\mathrm{length}(\mathcal{C})=m,\\ l(\mathcal{C})=0}} \|\pi_{\Lambda_1}\circ \mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1,\vartheta}\le c_{26}\|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^m \tag{99}
$$

Therefore

$$
\left\|\pi_{\Lambda_1}\circ\mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}\right\|_{L((\mathcal{H}_{\Lambda_2,\vartheta},\|\cdot\|_{\Lambda_2,\vartheta}),(\mathcal{H}_{\Lambda_1,\vartheta}\|\cdot\|_{\Lambda_1,\vartheta}))} \ \le\ \sum_{m=0}^N c_{26}\tilde{\eta}^m \tag{100}
$$

$$
\begin{aligned}
&\leq \sum_{m=0}^{\infty} c_{26}\tilde{\eta}^m \\
&\leq c_{20}
\end{aligned}
$$

□

Lemma 3 *1. The number of labelled tree graphs with exactly n edges is smaller than 2^{2n}*

2. The number of labelled tree graphs corresponding to configurations that have exactly $n_{\beta,k}$ k-triangles ($|n_{\beta,k}| < \infty$) and end (at time 0) in $\Lambda_3 \times \{0\}$ and not in any smaller set is bounded from above by $4^{|\Lambda_3|} \prod_{k=1}^{\infty} c_{21}^{k^d n_{\beta,k}}$ with $c_{21} = 4^{3^d}$.

Proof 1.) For every labelled tree graph in question we can define a path starting and ending at the root point (0) and running through each edge exactly twice in the following way. From a (labelled) vertex $t = (t_1, \ldots, t_n)$ we go to the to the next greater (wrt. $\prec$) vertex where we haven't yet been (*going up*), or if this is not possible (i.e. t is a leaf or we have already been at all vertices $(t_1, \ldots, t_{n+1})$) back to $(t_1, \ldots, t_{n-1})$ (*going down*). So we return to (0) after $2n$ steps. We encode the path in a word $(a_1, \ldots, a_{2n})$ with $a_i = 1$ if we go up in the ith step and $a_i = 0$ otherwise. Obviously the labelled graph is uniquely determined by its word. (Note that not every word of length $2n$ with symbols "0" and "1" corresponds to such a labelled graph. But the map between these two data is injective.) As there are 2^n words of length $2n$ with at most two different symbols this is also an upper bound for the number of graphs in question.
To see (2.) we note that the number of edges in such a tree graph is not greater than $K + \sum_{k=1}^{\infty} (3k)^d n_{\beta,k}$. □

Proof of Theorem 1 The difference between $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_2} \circ T^{\Lambda_2,\epsilon}} \circ \pi_{\Lambda_2}$ and $\pi_{\Lambda_1} \circ \mathcal{L}^N_{F^{\Lambda_3} \circ T^{\Lambda_3,\epsilon}} \circ \pi_{\Lambda_3}$ for $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \in \mathcal{F}$ is due to the summands involving configurations that do not lie completely (with all its triangles) in $\Lambda_2 \times \{0, -1, \ldots\}$. For those we have the lower bound for the spatial extension of the set of triangles:

$$
\begin{aligned}
b(\mathcal{C}) &\overset{\text{def}}{=} \sum_{k=1}^{\infty} k n_{\beta,k} \qquad (101) \\
&\geq \operatorname{dist}(\Lambda_1, \Lambda_2^C)
\end{aligned}
$$

As the analysis in the proof of Proposition 6 shows we have in the estimate for each such configuration a factor

$$\begin{aligned}
&\prod_{k=1}^{\infty}\left(\exp(-\tilde{c_2}k^d)\right)^{n_{\beta,k}} \\
&\leq \prod_{k=1}^{\infty}\left[\exp(-(\tilde{c_2}-\xi)k^d)\right]^{n_{\beta,k}}\prod_{k=1}^{\infty}\left(\exp(-\xi k n_{\beta,k})\right) \\
&\leq \prod_{k=1}^{\infty}\left[\exp(-(\tilde{c_2}-\xi)k^d)\right]^{n_{\beta,k}}\exp\left(-\xi\mathrm{dist}(\Lambda_1,\Lambda_2^C)\right)
\end{aligned} \tag{102}$$

If we take $\xi > 0$ small enough we can take out a factor $\exp\left(-\xi\mathrm{dist}(\Lambda_1,\Lambda_2^C)\right)$ and do the analysis with the remaining factor as before since $\tilde{c_2}-\xi > 0$. So we get

$$\begin{aligned}
&\|\pi_{\Lambda_1}\circ\mathcal{L}^N_{F^{\Lambda_2}\circ T^{\Lambda_2,\epsilon}}\circ\pi_{\Lambda_2}-\pi_{\Lambda_1}\circ\mathcal{L}^N_{F^{\Lambda_3}\circ T^{\Lambda_3,\epsilon}}\circ\pi_{\Lambda_3}\|_{L\left((\mathcal{H}_\vartheta,\|\cdot\|_\vartheta),\left(\mathcal{H}_{\Lambda_1,\vartheta},\|\cdot\|_{\Lambda_1,\vartheta}\right)\right)} \\
&\qquad\leq c_{27}\exp\left(-\xi\mathrm{dist}(\Lambda_1,\Lambda_2^C)\right)
\end{aligned} \tag{103}$$

with some constant c_{27} and the limit in (7) exists. The second statement in (1.) follows from (89) and (97) with ϑ replaced by a sufficiently small $\tilde{\vartheta}$. For example, (89) becomes

$$\begin{aligned}
&\tilde{\vartheta}^{|\Lambda_1|}\sum_{\mathcal{C}:\mathrm{length}(\mathcal{C})=N}\|\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}}\phi_{\Lambda_2}\|_{\Lambda_1} \\
&\leq c_{28}(\epsilon_2+c_r\tilde{\eta}^n\frac{\tilde{\vartheta}}{\vartheta}+c_h\tilde{\vartheta})^{|\Lambda_1|}\|\phi\|_{\Lambda_2,\vartheta}\tilde{\eta}^N
\end{aligned} \tag{104}$$

and the term in brackets is smaller than 1 if $\tilde{\vartheta}$ and $\frac{\tilde{\vartheta}}{\vartheta}$ are small enough. The statement for systems with finite-range interaction follows from the fact that in that case all limits are already attained for some sufficiently large $\Lambda_2 \in \mathcal{F}$ and that all considered sums are finite.
(2.) follows from (3.) and (5.) of Proposition 5. □

Proof of Proposition 5 With the same argument as in the proof of (1.) in Theorem 1 we see that the right-hand side term in (53) differs from the operator in (45) only in summands for $\mathcal{C}$ with $b(\mathcal{C}) \geq \mathrm{dist}(\Lambda_1, \Lambda_2^C)$. So the difference is bounded by $c_{29} \exp\left(-\xi \mathrm{dist}(\Lambda_1, \Lambda_2^C)\right)$ for some $c_{29} > 0$.

In order to prove (2.) we first observe that configurations $\mathcal{C} \in E_N(\Lambda_1)$ of length $\leq N-1$ extend canonically to $\mathcal{C}' \in E_{N+1}(\Lambda_1)$ with $\mathcal{L}_{\mathcal{C}} = \mathcal{L}_{\mathcal{C}'}$ because there are only h-lines in the step from time $-N$ to $-N+1$. So we can extend $\mathcal{C}$ to $\mathcal{C}'$ on $\Lambda_2 \times \{-N-1, \ldots, 0\}$ (where Λ_2 is so big that $\Lambda_2 \times \{-N-1, \ldots, 0\}$ contains all triangles of $\mathcal{C}$) by adding h-lines from $(p, -N-1)$ to $(p, -N)$ for all $p \in \Lambda_2$ and obviously $\mathcal{L}_{\mathcal{C}} = \mathcal{L}_{\mathcal{C}'}$.

Note that a configuration $\mathcal{C}'$ in $\Lambda_2 \times \{-N-1, 0\}$ of length $\leq N-1$ is the extension in the above sense of a (uniquely defined) $\mathcal{C}$.

So in the difference (54), all terms $\mathcal{L}_{\mathcal{C}}$ with length $\leq N-1$ are cancelled. Using (1.) of Proposition 6, (99) and (1.) of this proposition we get for all $\Lambda_1 \in \mathcal{F}$

$$\begin{aligned} \left\| \left(\pi_{\Lambda_1} \circ \mathcal{L}^N_{F\circ T^\epsilon} - \pi_{\Lambda_1} \circ \mathcal{L}^{N+1}_{F\circ T^\epsilon}\right) \phi \right\|_{\Lambda_1, \vartheta} &\leq \left(c_{19}\tilde{\eta}^N + c_{20}\tilde{\eta}^N + c_{19}\tilde{\eta}^{N+1}\right) \|\phi\|_\vartheta \\ &\leq c_{30}\tilde{\eta}^N \|\phi\|_\vartheta \qquad (105) \end{aligned}$$

with c_{30} independent of Λ_1. This proves (2.)

Recall that by Theorem 1 the operators $\mathcal{L}^N_{F\circ T^\epsilon} \in L\left(\mathcal{H}_\vartheta, \|\cdot\|_\vartheta\right)$ are well defined for $N \geq N_0$ and, by part (2.), give rise to a Cauchy sequence. With the same argument we see that the infinite sum in the definition of ν_Λ (cf. (52)) converges and $\nu \in \mathcal{H}_\vartheta$. $\nu \geq 0$ and so $\nu \in \mathcal{H}^{bv}$ will follow from (7.).

The difference in (8) is only due to configurations of length $\geq N$ and can therefore be estimated (as before in (105)) by $c\tilde{\eta}^N$, which proves (8).

For $\Lambda_1 \in \mathcal{F}$,

$$\begin{aligned} &\pi_{\Lambda_1} \circ \mathcal{L}^{N_2}_{F\circ T^\epsilon} \circ \mathcal{L}^{N_1}_{F\circ T^\epsilon} \phi \qquad (106) \\ &= \sum_{\mathcal{C}_2 \in E_{N_2}(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_2}\left(\mathcal{L}^{N_1}_{F\circ T^\epsilon}\phi\right) \\ &= \sum_{\mathcal{C}_2 \in E_{N_2}(\Lambda_1)} \left(\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_2} \circ \sum_{\mathcal{C}_1 \in E_{N_2}(\Lambda(\mathcal{C}_2))} \pi_{\Lambda(\mathcal{C}_2)} \circ \mathcal{L}_{\mathcal{C}_1} \phi_{\Lambda(\mathcal{C}_1)}\right) \\ &= \sum_{\substack{\mathcal{C}_2 \in E_{N_2}(\Lambda_1) \\ \mathcal{C}_1 \in E_{N_2}(\Lambda(\mathcal{C}_2))}} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_2 \circ \mathcal{C}_1} \phi_{\Lambda(\mathcal{C}_1)} \\ &= \sum_{\mathcal{C}_3 \in E_{N_1+N_2}(\Lambda_1)} \pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_3} \phi_{\Lambda(\mathcal{C}_3)} \end{aligned}$$

$$= \quad \pi_{\Lambda_1} \circ \mathcal{L}_{F\circ T^\epsilon}^{N_1+N_2} \phi.$$

Note that we sum over infinitely many $\mathcal{C}_1, \mathcal{C}_2$. A priori, the distribution is only valid for finite partial sums. In terms of configurations we 'put $\mathcal{C}_1$ on $\mathcal{C}_2$' to get $\mathcal{C}_3 = \mathcal{C}_2 \circ \mathcal{C}_1$ (which might be a zero configuration) and in fact such a splitting exists and is unique for every non-zero $\mathcal{C}_3$. So the net of finite partial sums over $\mathcal{C}_3$ we get converges to the infinite expansion (53) of the right-hand side of (55) and (4.) is proved. We have by (3.) $\lim_{N\to\infty} \mathcal{L}_{F\circ T^\epsilon}^N h = \mu(h) = \nu$ and so by (4.) $\mathcal{L}_{F\circ T^\epsilon}\nu = \nu$ and also $\mu(\nu) = 1$, by (6.) which we will show below. For any $\phi \in \mathcal{H}_\vartheta$ with $\mathcal{L}_{F\circ T^\epsilon}\phi = \phi$ and $\mu(\phi) = 1$ we have

$$\phi = \lim_{N\to\infty} \mathcal{L}_{F\circ T^\epsilon}^N \phi = \mu(\phi)\nu = \nu \tag{107}$$

so (5.) is proved.
To prove (6.), we consider first the special case $g \in \mathcal{C}((S^1)^\Lambda)$.

$$\begin{aligned} \int_M d\mu\, g \circ S\, \phi &= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_M d\mu\, g \circ S_{\Lambda_1} \phi \\ &= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g \circ S_{\Lambda_1} \phi_{\Lambda_1} \\ &= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g\, \mathcal{L}_{F^{\Lambda_1} \circ T^{\Lambda_1,\epsilon}} \phi_{\Lambda_1} \\ &= \lim_{\Lambda_1 \to \mathbf{Z}^d} \int_{(S^1)^{\Lambda}} d\mu^{\Lambda} g\, \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1} \circ T^{\Lambda_1,\epsilon}} \pi_{\Lambda_1} \phi \\ &= \int_M d\mu\, g \mathcal{L}_{F\circ T^\epsilon} \phi \end{aligned} \tag{108}$$

So (6.) is true for $g \in \mathcal{C}((S^1)^\Lambda)$. By assumption ϕ and so $\mathcal{L}_{F\circ T^\epsilon}\phi$ are in $\mathcal{H}^{bv}$ (by part (8.) for whose proof we just use the above special case of (6.)), i.e. they correspond to continuous linear functionals. For any $g \in \mathcal{C}(M)$ the net $(g_\Lambda)_{\Lambda\in\mathcal{F}}$ converges uniformly to g as $\Lambda \to \mathbf{Z}^d$, as does $(g_\Lambda \circ S)_{\Lambda\in\mathcal{F}}$ to $g \circ S$. So by continuity of the integral operators the equality also holds for g. The special case (6.) follows from taking $g \equiv 1$. For finite-range interaction the limits in (108) are already attained for sufficiently large $\Lambda_1 \in \mathcal{F}$ and all the computations work with $\phi \in \mathcal{H}^{bv}$.
We have, by definition, $(\mathcal{L}_{F\circ T^\epsilon}\phi)_\Lambda \overset{\text{def}}{=} \lim_{\Lambda_1\to\mathbf{Z}^d} \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}} \phi_{\Lambda_1}$. If that was negative somewhere there would be a $\Lambda_1 \in \mathcal{F}$ with $\pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1}\circ T^{\Lambda_1,\epsilon}} \phi_{\Lambda_1}$ having negative values and we could find a non-negative $g \in \mathcal{C}((S^1)^\Lambda)$ such that

$$\int_{(S^1)^\Lambda} d\mu^\Lambda g\, \pi_\Lambda \circ \mathcal{L}_{F^{\Lambda_1} \circ T^{\Lambda_1,\epsilon}} \phi_{\Lambda_1} < 0 \tag{109}$$

But by (6.) the integral equals

$$\int_{(S^1)^{\Lambda_1}} d\mu^{\Lambda_1} g \circ S\, \phi_{\Lambda_1} \geq 0 \tag{110}$$

So $\mathcal{L}_{F \circ T^\epsilon}$ is non-negative. Finally (8.) follows from

$$\begin{aligned} \|\mathcal{L}_{F\circ T^\epsilon}\phi\|_{var} &= \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \leq 1}} \int_M d\mu\, g\, \mathcal{L}_{F\circ T^\epsilon}\phi \qquad (111) \\ &= \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \leq 1}} \int_M d\mu\, g \circ S\, \phi \\ &\leq \sup_{\Lambda \in \mathcal{F}} \sup_{\substack{g \in \mathcal{C}((S^1)^\Lambda) \\ \|g\|_\infty \leq 1}} \|g\|_\infty \|\phi\|_{var} \\ &= \|\phi\|_{var} \end{aligned}$$

□

Proof of Theorem 3

$$\begin{aligned} \nu_{\Lambda_1 \cup \Lambda_2} &= \sum_{\mathcal{C} \in E(\Lambda_1 \cup \Lambda_2)} \pi_{\Lambda_1 \cup \Lambda_2} \circ \mathcal{L}_{\mathcal{C}} h \qquad (112) \\ &= \sum_{\substack{\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \\ b(\mathcal{C}) \leq \frac{1}{2}\text{dist}(\Lambda_1, \Lambda_2)}} (\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}_1} h)(\pi_{\Lambda_2} \circ \mathcal{L}_{\mathcal{C}_2} h) \\ &+ \sum_{\substack{\mathcal{C} \\ b(\mathcal{C}) > \frac{1}{2}\text{dist}(\Lambda_1, \Lambda_2)}} \pi_{\Lambda_1 \cup \Lambda_2} \circ \mathcal{L}_{\mathcal{C}} h \end{aligned}$$

In estimating the second summand we note that if we sum in formula (89) and (97) just over $\mathcal{C}$ with $b(\mathcal{C}) \geq \frac{1}{2}\text{dist}(\Lambda_1, \Lambda_2)$ we can take out from $\prod_{k=1}^\infty \left(\exp(-\tilde{c_2} k^d)\right)^{n_{\beta,k}}$ a factor $\exp(-\xi \frac{1}{2})\text{dist}(\Lambda_1, \Lambda_2)$ (like in the proof of Proposition 5). The rest of the analysis is as in the proof of Proposition 6. We can take ξ such that $\exp(-\xi \frac{1}{2}) = \kappa$ if c_2 is sufficiently large and get

$$\| \sum_{\substack{\mathcal{C} \\ b(\mathcal{C}) > \frac{1}{2}\text{dist}(\Lambda_1, \Lambda_2)}} \pi_{\Lambda_1 \cup \Lambda_2} \circ \mathcal{L}_{\mathcal{C}} h \|_{\Lambda_1 \cup \Lambda_2} \tag{113}$$

$$\begin{aligned} &\leq \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} c_{31} \|h\|_\vartheta \vartheta^{-|\Lambda_1|-|\Lambda_2|} \\ &\leq c_{32} \vartheta^{-|\Lambda_1|-|\Lambda_2|} \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} \end{aligned} \tag{114}$$

We write for the first summand in (112)

$$\begin{aligned} &\sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})\leq\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) \\ &\quad = \nu_{\Lambda_1}\nu_{\Lambda_2} - \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h) \end{aligned} \tag{115}$$

and estimate in a similar way

$$\Big\| \sum_{\substack{\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2 \\ b(\mathcal{C})>\frac{1}{2}\mathrm{dist}(\Lambda_1,\Lambda_2)}} (\pi_{\Lambda_1}\circ\mathcal{L}_{\mathcal{C}_1}h)(\pi_{\Lambda_2}\circ\mathcal{L}_{\mathcal{C}_2}h)\Big\|_{\Lambda_1\cup\Lambda_2} \leq c_{33}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} \tag{116}$$

(115) and (116) also hold for all $\Lambda_1' \subseteq \Lambda_1$, $\Lambda_2' \subseteq \Lambda_2$ and (1.) follows.

$$\begin{aligned} \pi_{\Lambda_1}(f\nu) &= \pi_{\Lambda_1}(f\nu_{\Lambda_1\cup\Lambda_2}) \\ &= \pi_{\Lambda_1}(f\nu_{\Lambda_1}\nu_{\Lambda_2} + f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2})) \\ &= \nu(f)\nu_{\Lambda_1} + \pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2})) \end{aligned} \tag{117}$$

and using $\|\pi_{\Lambda_1}\|_\infty = 1$ we get

$$\|\pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\|_{\Lambda_1} \leq \|f\|_{\Lambda_2}\|\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}\|_{\Lambda_1\cup\Lambda_2} \tag{118}$$

and so by (1.)

$$\|\pi_{\Lambda_1}(f(\nu_{\Lambda_1}\nu_{\Lambda_2} - \nu_{\Lambda_1\cup\Lambda_2}))\|_{\Lambda_1} \leq c_{16}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} \tag{119}$$

This holds for all $\Lambda_1' \subset \Lambda_1$, so (2.) is proved.
We set $\phi = f\nu - \nu(f)\nu$. So $\pi_{\Lambda_1}\circ\mathcal{L}^N_{F\circ T^\epsilon}(f\nu) - \nu(f)\nu_{\Lambda_1} = \pi_{\Lambda_1}\circ\mathcal{L}^N_{F\circ T^\epsilon}\phi$. We estimate the $\|\cdot\|_{\Lambda_1,\tilde{\vartheta}}$-norm of the last term as in the proof of Proposition 6, but this time using the finer estimates from (2.)

$$\begin{aligned} \|\phi_{\Lambda(\mathcal{C})}\|_{\Lambda(\mathcal{C})} &\leq \vartheta^{-|\Lambda(\mathcal{C})|}c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\kappa^{\mathrm{dist}(\Lambda(\mathcal{C}),\Lambda_2)} \\ &\leq c_{11}\vartheta^{-|\Lambda_2|}\|f\|_{\Lambda_2}\vartheta^{-|\Lambda_r(\mathcal{C})|-\sum_{k=1}^\infty(3k)^d n_{\beta,k}}\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^\infty kn_{\beta,k}} \end{aligned} \tag{120}$$

where as before $\Lambda(\mathcal{C}) \overset{\text{def}}{=} \tilde{\Lambda}_{\mathcal{C}} \cup \Lambda_r$.
So we get analogously to formulae (89) and (90):

$$
\begin{aligned}
&\tilde{\vartheta}^{|\Lambda_1|} \sum_{\mathcal{C}:\mathrm{length}(\mathcal{C})=N} \|\pi_{\Lambda_1} \circ \mathcal{L}_{\mathcal{C}} \phi_{\Lambda_2}\|_{\Lambda_1} \qquad (121)\\
\leq\ & \tilde{\vartheta}^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq |n_\beta|<\infty}} 4^K \prod_{k=1}^{\infty} \Big(\exp(c_{21}k^d)\Big)^{n_{\beta,k}} (c_1\epsilon)^{|n_\beta|} \\
& \times \prod_{k=1}^{\infty} \Big(\exp(-c_2 k^d)\Big)^{n_{\beta,k}} \sum_{L=|n_\beta|}^{\infty} \binom{L}{|n_\beta|} c_r^{|n_\beta|} \eta^L \prod_{k=1}^{\infty} \Big(\exp(c_{22}k^d)\Big)^{n_{\beta,k}} \\
& \times \prod_{k=1}^{\infty} \Big(\exp(c_{23}k^d)\Big)^{n_{\beta,k}} \eta^{\max\{0,N-L\}} \sum_{l=0}^{|\Lambda_1|-K} \binom{|\Lambda_1|-K}{l} (c_r\eta^N)^l c_h^{|\Lambda_1|-K-l} \\
& \times c_{11}\vartheta^{-|\Lambda_2|} \|f\|_{\Lambda_2} \vartheta^{-l-\sum_{k=1}^{\infty}(3k)^d n_{\beta,k}(\mathcal{C})} \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)-\sum_{k=1}^{\infty} k n_{\beta,k}} \\
\leq\ & c_{11}\tilde{\vartheta}^{|\Lambda_1|} \sum_{K=0}^{|\Lambda_1|} \binom{|\Lambda_1|}{K} \sum_{\substack{n_\beta \\ K\leq |n_\beta|<\infty}} 4^K (c_1\epsilon c_r)^{|n_\beta|} \\
& \times \prod_{k=1}^{\infty} \Big(\exp((c_{21} - c_2 + c_{22} + c_{23} - 3^d \ln\vartheta - \ln\kappa)k^d)\Big)^{n_{\beta,k}} \\
& \times \sum_{L=n_\beta}^{\infty} \eta^{\max\{L,N\}} (\vartheta^{-1}c_r\eta^N + c_h)^{|\Lambda_1|-K} \vartheta^{-|\Lambda_2|} \|f\|_{|\Lambda_2|} \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}
\end{aligned}
$$

This time we set $\tilde{c_2} = c_2 - c_{21} - c_{22} - c_{23} + 3^d \ln\vartheta + \ln\kappa$ and with the same analysis as from (90) to (96) we get:

$$
\leq c_{34}(\epsilon_2 + c_r\tilde{\eta}^N \frac{\tilde{\vartheta}}{\vartheta} + c_h\tilde{\vartheta})^{|\Lambda_1|} \|\nu\|_\vartheta \vartheta^{-|\Lambda_2|} \|f\|_{\Lambda_2} \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} \tilde{\eta}^N. \qquad (122)
$$

For sufficiently small ϵ_2 and $\tilde{\vartheta}$ the term in brackets is smaller than one. Note that there is no condition on N. So we get the same estimates for all $n \geq 0$ and these also hold for $\Lambda \subset \Lambda_1$. So in analogy with (54) we get

$$
\left\|\mathcal{L}_{F\circ T^\epsilon}^N \phi - \mathcal{L}_{F\circ T^\epsilon}^{N+1} \phi\right\|_{\Lambda_1} \leq c_{35}\vartheta^{-|\Lambda_2|} \|f\|_{\Lambda_2} \kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)} \tilde{\eta}^N \qquad (123)
$$

and as $\mu(\phi) = 0$ we conclude (3.) $\square$

Proof of Theorem 2

$$\begin{aligned}
&\left|\int_M \nu d\mu\, gf - \left(\int_M \nu d\mu\, g\right)\left(\int_M \nu d\mu\, f\right)\right| \\
\leq\ & \left|\int_{(S^1)^{\Lambda_1\cup\Lambda_2}} d\mu^{\Lambda_1\cup\Lambda_2}(\nu_{\Lambda_1\cup\Lambda_2}-\nu_{\Lambda_1}\nu_{\Lambda_2})gf\right| \\
\leq\ & \|\nu_{\Lambda_1\cup\Lambda_2}-\nu_{\Lambda_1}\nu_{\Lambda_2}\|_{\Lambda_1\cup\Lambda_2}\|g\|_\infty\|f\|_\infty \\
\leq\ & c_{10}\vartheta^{-|\Lambda_1|-|\Lambda_2|}\|g\|_\infty\|f\|_\infty\kappa^{\mathrm{dist}(\Lambda_1,\Lambda_2)}
\end{aligned} \tag{124}$$

so (1.) is proved.

$$\begin{aligned}
&\left|\int_M \nu d\mu\, g\circ\tau\circ S^n f - \left(\int_M \nu d\mu\, g\circ\tau\right)\left(\int_M \nu d\mu\, f\right)\right| \\
=\ & \left|\int_M d\mu\, g\circ\tau\left(\pi_{\tau^{-1}(\Lambda_1)}\circ\mathcal{L}^n_{F\circ T^\epsilon}(f\nu)-\nu(f)\nu_{\tau^{-1}(\Lambda_1)}\right)\right| \\
\leq\ & c_{12}c_5^{|\Lambda_1|+|\Lambda_2|}\|f\|_{\Lambda_2}\|g\|_\infty\kappa^{\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)}\tilde{\eta}^n
\end{aligned} \tag{125}$$

Here we have used (3.) of Theorem 3 and set $c_5 \overset{\text{def}}{=} \tilde{\vartheta}^{-1}$. From

$$\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2) \geq m(\tau)-\mathrm{diam}(\Lambda_1,\Lambda_2) \tag{126}$$

follows

$$\kappa^{\mathrm{dist}(\tau^{-1}(\Lambda_1),\Lambda_2)} \leq c(\Lambda_1,\Lambda_2,\kappa)\kappa^{m(\tau)} \tag{127}$$

where $c(\Lambda_1,\Lambda_2,\kappa)$ depends only on Λ_1, Λ_2 and κ. If τ and S commute, (3.) follows from (2.).

We prove (4.) by approximating g and f by functions and for that we can apply estimate (2.). For a given $\gamma > 0$ we choose $\Lambda_1 \in \mathcal{F}$ so large that $\|g-g_{\Lambda_1}\|_\infty \leq \gamma$. Further there exists an $\tilde{f}_{\Lambda_2} \in \mathcal{H}(A_\delta^{\Lambda_2})$ with $\|f-\tilde{f}_{\Lambda_2}\|_\infty \leq \gamma$ (sup-norm on $(S^1)^{\mathbf{Z}^d}$). So

$$\begin{aligned}
&\left|\int_M \nu d\mu\, g\circ\tau\circ S^n f - \left(\int_M \nu d\mu\, g\circ\tau\right)\left(\int_M \nu d\mu\, f\right)\right| \\
\leq\ & \left|\int_M \nu d\mu\,(g-g_{\Lambda_1})\circ\tau\circ S^n f\right| + \left|\int_M \nu d\mu\, g_{\Lambda_1}\circ\tau\circ S^n\,(\tilde{f}_{\Lambda_2}-f)\right| \\
&+\left|\int_M \nu d\mu\, g_{\Lambda_1}\circ\tau\circ S^n\tilde{f}_{\Lambda_2} - \left(\int_M \nu d\mu\, g_{\Lambda_1}\circ\tau\right)\left(\int_M \nu d\mu\,\tilde{f}_{\Lambda_2}\right)\right|
\end{aligned} \tag{128}$$

$$
\begin{aligned}
& +\left|\left(\int_M \nu d\mu\, g_{\Lambda_1} \circ \tau\right)\left(\int_M \nu d\mu\, (f - \tilde{f}_{\Lambda_2})\right)\right| \\
& +\left|\left(\int_M \nu d\mu\, (g - g_{\Lambda_1}) \circ \tau\right)\left(\int_M \nu d\mu\, f\right)\right| \\
\leq\ & \|g - g_{\Lambda_1}\|_\infty \|f\|_\infty + \|g_{\Lambda_1}\|_\infty \|f - \tilde{f}_{\Lambda_2}\|_\infty \\
& + c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1| + |\Lambda_2|} \|g_{\Lambda_1}\|_\infty \|\tilde{f}_{\Lambda_2}\|_{\Lambda_2} \tilde{\eta}^{n(\sigma)} \kappa^{m(\sigma)} \\
& + \|g_{\Lambda_1}\|_\infty \|f - f_{\Lambda_2}\|_\infty + \|g - g_{\Lambda_1}\|_\infty \|f_{\Lambda_2}\|_\infty \\
\leq\ & (2\|f\|_\infty + 2\|g\|_\infty + 3\gamma)\, \gamma \\
& + c(\Lambda_1, \Lambda_2, \kappa) c_5^{|\Lambda_1| + |\Lambda_2|} (\|g\|_\infty + \gamma) \|\tilde{f}_{\Lambda_2}\|_{\Lambda_2} \tilde{\eta}^{n(\sigma)} \kappa^{m(\sigma)}
\end{aligned}
$$

and this gets arbitrarily small as we first choose γ, then Λ_1, Λ_2 and f_{Λ_2} and finally $\max\{m(\sigma), n(\sigma)\}$.
(5.) follows from (4.) and the commutation of the τ_{e_i} with S. □

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References

[1] V. Baladi, M. Degli Esposti, S. Isola, E. Järvenpää, A. Kupiainen: The spectrum of weakly coupled map lattices, preprint

[2] J. Bricmont and A. Kupiainen: Coupled Analytic Maps, *Nonlinearity* 8, pp.379-396, 1995

[3] J. Bricmont and A. Kupiainen: High Temperature Expansions and Dynamical Systems, *Comm. Math. Phys.* 178, pp.73-732

[4] J. Bricmont and A. Kupiainen: Infinite dimensional SRB-measures, preprint

[5] L.A. Bunimovich: Coupled map lattices: One step forward and two steps back, *Physica D 86*, 1995, pp. 248-255

[6] L.A. Bunimovich, Y.G. Sinai: Space-time chaos in coupled map lattices, *Nonlinearity 1*, pp. 491-516, 1988

[7] N. Dunford and J.T. Schwartz: Linear Operattors Part I, *Interscience*, New York, 1958

[8] M. Jiang:Equilibrium states for lattice models of hyperolic type *Nonlinearity 8*, no.5, 1994, pp.631-659

[9] M. Jiang: Ergodic Properties of Coupled Map Lattices of Hyperbolic Type, *Penns. State University Dissertation*, 1995

[10] M. Jiang and A. Mazel: uniqueness of Gibbs States and exponential Decay of Correlation for some Lattice models, *Journal of Statistical Physics* 82, no. 3-4, 1995

[11] M. Jiang and Ya.B. Pesin: Equilibrium Measures for Coupled Map Lattices: Existence, Uniqueness and Finite-Dimensional Approximations, Preprint

[12] K. Kaneko (ed.): Theory and Applications of Coupled map Lattices, *J. Wiley*, 1993

[13] G. Keller and M. Künzle: Transfer Operators for Coupled Map Lattices, *Ergodic Theory & Dynamical Systems* Vol. 12, pp.297-318, 1992

[14] A. Lasota and M.C. Mackey: Chaos, Fractals and Noise, *Springer*, 1994

[15] C. Maes and A. Van Moffaert: Stochastic Stability of Weakly Coupled Lattice Maps, preprint

[16] Ya.B. Pesin and Ya.G. Sinai: Space-time chaos in chains of weakly interacting hyperbolic mappings, *Advances in Soviet Mathematics* Vol.3, 1991, pp.165-198

[17] D.L. Volevich: The Sinai-Bowen-Ruelle Measure for a Multidimensional Lattice of interacting Hyperbolic Mappings, *Russ. Akad. Dokl. Math.* 47, 1993, pp.117-121

[18] D.L. Volevich: Construction of an analogue of Bowen-Ruell-Sinai measure for a multidimensional lattice of interacting hyperbolic mappings *Russ. Akad. Math. Sbornik* Vol.79, 1994, pp.347-363