A REVERSED POINCARE INEQUALITY FOR MONOTONE FUNCTIONS

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Abstract. We derive a family of sharp reversed Poincaré inequalities for positive monotone functions defined on a finite interval of the real line. The method we introduce here to derive our main result can be used to derive more general inequalities.

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The purpose of this article is to derive a new family of sharp inequalities for positive monotone functions defined on a finite interval of the real line. For simplicity we will consider intervals of the form $(0,b)$ with $0 < b < \infty$. By an appropriate translation our result can be extended to intervals of the form (a,b) with $-\infty < a < b < \infty$.

The inequalities we present here are embodied in the following

THEOREM 1. *Let g be a nonnegative nondecreasing function defined on $(0,b)$. Assume $g \in C^1(0,b)$, and denote its derivative by g'. Let $q > 2$. Then,*

$$\frac{\left(\int_0^b (g'(u))^{1/q}\, du\right)^q}{\int_0^b g(u) u\, du} \le \frac{b^{q-3}}{2^{q-2}} B\left(\frac{1}{2}, \frac{q-2}{q-1}\right)^{q-1}, \tag{1}$$

where B denotes the Beta function, i.e., $B(n,m) \equiv \Gamma(n)\Gamma(m)/\Gamma(n+m)$. In particular, for $q = 3$ we have

$$\frac{\left(\int_0^b (g'(u))^{1/3}\, du\right)^3}{\int_0^b g(u) u\, du} \le \frac{\pi^2}{2}. \tag{2}$$

Equality is obtained in (1) for a unique function g, up to a multiplicative constant.

Proof. We will assume $\int_0^b ug(u)\, du < \infty$ since otherwise the inequality is trivial. Now, since g is increasing and nonnegative we have $0 \le g(u)(b^2-u^2) < 2\int_u^b \tilde{u} g\tilde{u})\, d\tilde{u} < \infty$ and therefore $\lim_{u\to b} g(u)(b-u)^{q/(q-1)} = 0$.

Let $n = q-1 > 1$, and consider the following two point boundary value problem,

$$-\frac{d}{dx}\left(\left|\frac{du}{dx}\right|^{n-2}\frac{du}{dx}\right) = \lambda u \qquad \text{on } (0,1), \tag{3}$$

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with $u'(0) = 0$ and $u(1) = 0$. Denote $u(0) = b$. For every constant $b > 0$ this two point boundary value problem has a unique positive solution if and only if

$$\lambda = \lambda(b) = \frac{n-1}{2n}(\frac{b}{2})^{n-2} B(1/2, 1-1/n)^n \tag{4}$$

(see the Lemma below). This positive solution is at least $C^1(0,1)$ and it is monotonic decreasing, i.e., $u'(x) < 0$. Henceforth, for a given $b > 0$ we fix $\lambda = \lambda(b)$, given by (4). Let $p = -du/dx > 0$. Eqn. (3) is equivalent to the system of first order equations

$$\begin{aligned} u' &= -p \\ p' &= \frac{\lambda u}{(n-1)p^{n-2}} \end{aligned} \tag{5}$$

The functions $u(x)$, $p(x)$ corresponding to a solution of (5) trace out a trajectory in the (u,p)–plane (usually called the phase plane). Such a trajectory has slope

$$\frac{dp}{du} = -\frac{\lambda u}{(n-1)p^{n-1}}, \tag{6}$$

at any point where $p \neq 0$. Now, let $g \in C^1(0,b)$ with $g' \geq 0$ and $g \geq 0$ but otherwise arbitrary. Multiplying (6) by $np^{n-1}g(u)$ and integrating in u from 0 to b we get,

$$\frac{n}{n-1}\lambda \int_0^b ug(u)\,du = -n\int_0^b p^{n-1}\frac{dp}{du}g(u)\,du \geq \int_0^b p^n g'(u)\,du, \tag{7}$$

where the last inequality follows by integrating by parts, using $g(0) \geq 0$, and the fact that $\lim_{u\to b} g(u)(b-u)^{q/(q-1)} = 0$. Note that in the neighborhood of $u = b$, $p(u) \approx (b-u)^{q/(q-1)}$ (see the Lemma below). Now, notice that $1 = \int_0^1 dx = \int_0^b du/p(u)$. Thus, for any positive constant K we can write the right side of (7) as

$$I = \int_0^b p^n g'(u)\,du = \int_0^b (p^n g'(u) + \frac{K}{p})\,du - K. \tag{8}$$

Since $K > 0$ and $g' \geq 0$,

$$p^n g'(u) + \frac{K}{p} \geq \frac{(n+1)}{n}(ng'(u))^{1/(n+1)} K^{n/(n+1)}. \tag{9}$$

The inequality is obtained by minimizing the left side with respect to p in $(0,\infty)$. Equality is attained in (9) if and only if

$$np^{n+1}\frac{dg}{du} = K. \tag{10}$$

¿From (8) and (9) we get

$$I \geq \frac{(n+1)}{n}\left(\int_0^b (ng'(u))^{1/(n+1)}\,du\right) K^{n/(n+1)} - K. \tag{11}$$

Choosing $K = n(\int_0^b g'(u)^{1/(n+1)}\,du)^{n+1}$, we obtain

$$I \geq \left(\int_0^b g'(u)^{1/(n+1)}\,du\right)^{n+1}. \tag{12}$$

Finally from (7) and (12) we get

$$\frac{n}{n-1}\lambda \int_0^b ug(u)\,du \geq \left(\int_0^b g'(u)^{1/(n+1)}\,du\right)^{n+1}. \tag{13}$$

Then (1) follows from (4) and (13). To conclude with the proof of the theorem we need to discuss the case of equality. Equality in (1) is attained if and only if the function g satisfies Eqn. (10), and $g(0) = 0$. The function $p(u)$ is given by $p(u) = (n\lambda(b^2 - u^2)/(2(n-1)))^{1/n}$, which follows from Eqn. (6) and $p(b) = 0$. Eqn. (10) has a unique solution satisfying the initial condition $g(0) = 0$, which is given by

$$\hat{g}(u) = \frac{K}{n}\int_0^u \left(\frac{\lambda n}{n-1}\right)^{(n+1)/n} (b^2 - u^2)^{-(n+1)/n}\,du \tag{14}$$

This $\hat{g}$ is unique up to the multiplicative constant K, it is increasing and positive for $u \in (0, b)$. Clearly $\hat{g} \in C^1(0, b)$. As $u \to b$, $\hat{g}(u) \approx (b-u)^{-1/n}$ and therefore $\int u\hat{g}(u)\,du < \infty$ if and only if $n > 1$, i.e., $q > 2$. □

Notice that starting from a two point boundary value problem different from (3), and using the same sort of techniques, one can prove other type of inequalities for monotone functions.

The existence and uniqueness of a positive solution to the two point boundary value problem (3), with $u'(0) = u(1) = 0$, can be proven in several ways (see, e.g., [4]). We state the results needed here in the following

LEMMA 2. *For the two point boundary value problem given by (3), for $n > 1$, with $u'(0) = u(1) = 0$ and $u(0) = b > 0$ there exists a unique positive solution if and only if $\lambda = \lambda(b)$ is given by (4). Moreover u is decreasing and*

$$u'(x) = -\left[\left(\frac{n\lambda(b)}{2(n-1)}\right)(b^2 - u^2)\right]^{1/n}. \tag{15}$$

Proof. By an elementary computation we have that for any $n > 1$,

$$\int_0^b \frac{du}{(b^2-u^2)^{1/n}} = b^{(n-2)/n}\int_0^{\pi/2} (\cos\theta)^{(n-2)/n}\,d\theta = b^{(n-2)/n}\frac{1}{2}B(\frac{1}{2}, \frac{n-1}{n}). \tag{16}$$

Now, for $\lambda(b)$ given by (4), define implicitly the function $u : [0, 1] \to [0, b]$ by

$$\int_{u(x)}^b \frac{dv}{(b^2-v^2)^{1/n}} = \left[\frac{\lambda(b)n}{2(n-1)}\right]^{1/n} x. \tag{17}$$

It follows from (17) that $u(0) = 0$ and that

$$u'(x) = -\left[\left(\frac{\lambda(b)n}{2(n-1)}\right)(b^2 - u^2)\right]^{1/n} \leq 0. \tag{18}$$

Now, from (4), (16), and (17) we have $u(1) = 0$. Also, from (18) and $u(0) = b$ we have $u'(0) = 0$. Finally, it follows directly from (18) that u satisfies (3). From (16)

and a standard bootstrap argument one can show that $u \in C^1(0,1)$. In fact one can show that u is real analytic away from the origin.

The uniqueness of the positive solution to the two point boundary value problem (3) follows from a result of M. Otani [4] by setting his exponent $q = 2$. See also [5], for extensions of this eigenvalue problem to higher dimensions and for references on this problem. □

Phase–space techniques similar to the ones we have used here to prove (1) have been used previously by us. In fact, we have followed the same type of ideas to prove a variational characterization for the speed of propagating fronts of a certain class of nonlinear diffusion equations (e.g., the Fisher–Kolmogorov equation) [1] [2]. We have also employed similar methods to find a variational characterization for the principal eigenvalue associated to nonlinear, one–dimensional, Hamiltonian systems [3].

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