Spectral analysis of N-body systems coupled to a bosonic field

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Abstract

We develop an extension of the abstract Mourre theory which consecutively is used to prove spectral properties of various systems coupled to a massless bosonic field. Our models include the spin–boson model and the standard model of quantum electrodynamics for a non–relativistic atom considered recently in [HS] and [BFS], respectively.

1 Introduction

It is well–known that Mourre's commutator method [M] is a powerful machinery for proving basic spectral properties of N–body Schrödinger operators H: Letting $\mathcal{F}(H), \sigma_{pp}(H)$ and $\sigma_{sc}(H)$ denote the set of thresholds (i.e. eigenvalues of sub–Hamiltonians), the set of eigenvalues and the singular continuous spectrum, respectively, these properties include

1) *$\mathcal{F}(H)$ is closed and countable.*

2) *The eigenvalues (counted with multiplicity) can only accumulate at $\mathcal{F}(H)$.*

3) *There is a limiting absorption principle away from $\mathcal{F}(H) \cup \sigma_{pp}(H)$. In particular $\sigma_{sc}(H) = \emptyset$.*

The purpose of the paper is to prove these or similar properties for various systems coupled weakly to a massless bosonic field. We discuss N–body as well as finite number of states systems motivated by the recent papers [BFS] and [HS], respectively. For the spin–boson Hamiltonians of [HS] the above properties should be replaced by

2)' *$\sigma_{pp}(H)$ is finite.*

3)' *There is a limiting absorption principle away from $\sigma_{pp}(H)$. In particular $\sigma_{sc}(H) = \emptyset$.*

To explain a basic idea of [HS] let us consider bosons with positive mass m which means that the 1–boson energy function is given by $\omega(k) = \sqrt{|k|^2 + m^2}$. We introduce $B = \frac{1}{2}(F \cdot p + p \cdot F)$; $F(k) = -\omega(k)|k|^{-2}k, p = -i\nabla_k$, as an operator on the 1–boson space $L^2\big(\mathbb{R}^\nu_k\big)$, and compute $i[\omega, B] = I$. Upon lifting to the symmetric Fock space $\mathcal{F}$ (by second quantization) this gives

$$i[d\Gamma(\omega), d\Gamma(B)] = N,$$

where N denotes the "number operator". Of course it is non–negative which in the applications of [HS] is used to establish a Mourre estimate for spin–boson Hamiltonians. To get the consequences $2)'$ and $3)'$ listed above the authors developed an extended abstract Mourre theory since their "conjugate operator" is not self–adjoint and consequently does not meet the requirements of [M]. (Notice that the above conjugate operators B and $A = d\Gamma(B)$ do not have self–adjoint realizations.)

It is noticed in [HS] that the massless case $m = 0$ is worse. As a matter the authors did not obtain results in this case (except for a certain compressed model in which N is bounded). The basic obstacle for $m = 0$ is that N is not bounded relatively to $H = d\Gamma(\omega)$, or more general

(1.1) $\quad (H+i)^{-1} i[H,A](H-i)^{-1}$ is unbounded.

In this paper we develop a Mourre theory that overcomes the above difficulty. We notice that all later refinements of Mourre's paper (see for example [PSS] and [BMP]) as a minimum requirement have boundedness of the first commutator in the above weak sense. In other words the property (1.1) does not fit into existing theories.

Of course our refinement of the Mourre theory involves new conditions. One of these is that the H–unbounded piece M of a decomposition $i[H,A] = M + G$ has that property that $i[H,M]$ is H–bounded. (Notice that in the above example the latter condition is trivial: $i[H,N] = 0$.)

Another complication is the one mentioned above: A might not be self–adjoint. The procedure of [HS] essentially consists in mimicking [M] using the semigroup generated by $-iA$ (assumed to exist). One ingredient of our approach is the introduction of a sequence of self–adjoint operators (A_n) which (by assumption) in some sense converges to A. Also each A_n is assumed to have the usual good commutator property $i[H,A_n]$ being H–bounded. Hence our procedure is facilitated by the basic technical results of [M] that are directly applicable for the approximating operators. In the example discussed above (1.1) and the lack of self–adjointness are due to the singularity at $k = 0$ of the vector field F, and therefore the approximating self–adjoint operators may naturally be constructed by suitably smoothing out the vector field.

Our examples concern Hamiltonians defined on a Hilbert space of the form $L^2(X) \otimes \mathcal{F}$. The spin–boson model corresponds to X being finite. For N–body systems X is a finite dimensional vector space.

In [BFS] the authors propose the conjugate operator $A = A_{el} \otimes I - I \otimes A_f$, where A_{el} is the generator of the dilations and A_f is the second quantized generator of the dilations lifted from the 1–photon space. The basic idea is to invoke the well–known Mourre estimate for "the N–body part" H_{el} of the Hamiltonian (given in terms of A_{el}) and then show local positivity away from the thresholds of H_{el} for small perturbations of

$$H_{el} \otimes I + I \otimes d\Gamma(\omega).$$

This procedure does not yield to the properties $1) - 3)$ listed above even for perturbations for which there is a natural notion of thresholds for the perturbed Hamiltonians. Explicitly the authors do not prove spectral properties in neighbourhoods of thresholds of H_{el}. Moreover to get spectral information in neighbourhoods of eigenvalues of H_{el} they need implicit conditions (cf. Fermi's Golden Rule) and an analyticity assumption on the perturbation.

In this paper we shall consider perturbations of the above tensor sum for which there is a natural notion of thresholds for the full Hamiltonian. We shall obtain all of

the properties $1)-3)$ for some classes of perturbations. Our basic idea is to consider the (non–self–adjoint) conjugate operators

$$A = CB_R \otimes I + I \otimes d\Gamma(B), \tag{1.2}$$

where C and R are adjustable positive parameters, B is given as in the beginning of this section, and B_R is somewhat similar to A_{el}. More precisely B_R is given in terms of a scaled version of the vector field invented by Graf [G]. We notice that by a previous result of the author [Sk, Theorem B2] this operator provides another Mourre estimate for "the N–body part" H_{el}. In [Sk] the latter estimate had obvious applications in studying propagation properties with locally singular potentials. In the present context the use of B_R is convenient not only for dealing with local singularities but for another reason too. (As a matter of fact we don't see how to obtain $1)-3)$ for non–trivial perturbations say by using (1.2) with B_R replaced by A_{el}.)

We emphasize that for all our examples we only have results for weak coupling. It is an open problem to go beyond this restriction for the classes of perturbations we consider. For the spin–boson Hamiltonians and our first model of an N–body system coupled to the bosonic field, the latter phrased as the electron–boson model, we give a very explicit bound on the perturbation (in fact similar to the one of [HS] obtained for positive mass). For the more complicated model, the so–called standard model of quantum electrodynamics for a non–relativistic atom, we do not derive an explicit bound. (In that case our methods would probably not yield realistic bounds.)

Although we shall not discuss it in this paper our limiting absorption principle may be used to show absence of embedded eigenvalues (in a weak coupling regime) in accordance to Fermi's Golden Rule and to treat related issues. This does not require the notion of resonance as defined in [BFS] or any other such notion. We notice that in the theory of N–body Schrödinger operators the analyticity assumption of [Si] was similarly removed in the treatment of instability of embedded eigenvalues of [AHS].

In Section 2 we formulate and derive the extended abstract Mourre theory. In Section 3 we discuss in detail the applications to the spin–boson model. We prove $2)'$ and $3)'$. In the next section we introduce "the electron–boson model" and do some preliminaries. As an easy corollary of the main result Proposition 4.4 we recover the Mourre estimate [Sk, Theorem B2]. In Section 5 we use the proposition to complete "the electron–boson model", that is to establish $1)-3)$. Finally in Section 6 we discuss the standard model of quantum electrodynamics for a non–relativistic atom. For a certain class of perturbations we obtain again the above properties. In that section we skip many details referring the reader to various procedures and preliminary results for the previously discussed models.

2 Abstract theory

For a (densely defined) operator T on a Hilbert space we denote by $\mathcal{D}(T)$ and $\rho(T)$ its domain and resolvent set, respectively.

Assumptions 2.1 *Let $H, M, A_n, n \in \mathbb{N}$, be self–adjoint operators on a Hilbert space $\mathcal{H}$. Suppose $M \geq \delta I$ for some positive number δ, and that there exists a closed symmetric operator A on $\mathcal{H}$ with the properties $-i \in \rho(A)$ and for some core $\mathcal{C}$ of A with $\mathcal{C} \subseteq \mathcal{D}(A_n)$ the identity $\lim_{n\to\infty} A_n\phi = A\phi$ holds for all $\phi \in \mathcal{C}$. Suppose*

1) (Compatibility) *The set $\mathcal{D}(H) \cap \mathcal{D}(M)$ is dense in $\mathcal{D}(H)$ as well as in $\mathcal{D}(M)$. The form $i[H, M]$ defined on this set extends to an $H-$ bounded operator $i[H, M]^0$, and*

$$\sup_{|t|<1} ||He^{itM}(H - i)^{-1}|| < \infty.$$

2) (First commutators) *For all $n \in \mathbb{N}$ the set $\mathcal{D}(H) \cap \mathcal{D}(A_n)$ is dense in $\mathcal{D}(H)$. The form $i[H, A_n]$ defined on this set extends to an $H-$ bounded operator H_n, and*

$$\sup_{|t|<1} ||He^{itA_n}(H - i)^{-1}|| < \infty.$$

Moreover there exists an $H-$ bounded (symmetric) operator G such that for all $\phi \in \mathcal{D}(H) \cap \mathcal{D}(M)$

$$\lim_{n\to\infty} H_n\phi = (M + G)\phi.$$

3) (Second commutators) *For all $n \in \mathbb{N}$ the set $\mathcal{D}(M) \cap \mathcal{D}(A_n)$ is dense in $\mathcal{D}(M)$. The form $i[M, A_n]$ defined on this set extends to an $M-$ bounded operator M_n, and*

$$\sup_{|t|<1} ||Me^{itA_n}M^{-1}|| < \infty.$$

Moreover there exists an $M-$ bounded operator M_∞ such that for all $\phi \in \mathcal{D}(M)$

$$\lim_{n\to\infty} M_n\phi = M_\infty\phi.$$

The form $i[G, A_n]$ defined on $\mathcal{D}(H) \cap \mathcal{D}(A_n)$ extends to an $H-$ bounded operator G_n. There exists an $H-$ bounded operator G_∞ such that for all $\phi \in \mathcal{D}(H)$

$$\lim_{n\to\infty} G_n\phi = G_\infty\phi.$$

4) (Positivity at E) *For a given $E \in \mathbb{R}$ there exist $\alpha > 0$, an open neighbourhood $\mathcal{U}$ of E and a compact operator K such that for all real–valued $f \in C_0^\infty(\mathcal{U})$ the form inequality*

$$M + f(H)Gf(H) \geq \alpha f(H)^2 - f(H)Kf(H) - (I - f(H))L(I - f(H))$$

holds on $\mathcal{D}(H) \cap \mathcal{D}\left(M^{\frac{1}{2}}\right)$ for some symmetric $H-$ bounded operator $L = L(f)$.

The following result may be derived as in [M].

Lemma 2.2 *Suppose Assumptions 2.1 2) and 3). Let for* $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$ $A_n(\lambda) = \lambda^2(A_n + i\lambda)^{-1}$. *Then for* $|\lambda|$ *sufficiently large (possibly depending on* n*)* $A_n(\lambda)$ *preserves* $\mathcal{D}(H)$ *as well as* $\mathcal{D}(M)$. *Moreover*

1) *For all* $\phi \in \mathcal{D}(H)$

$$H_n\phi = \lim_{\lambda\to\infty} i[H, A_n(\lambda)]\phi,$$

$$G_n\phi = \lim_{\lambda\to\infty} i[G, A_n(\lambda)]\phi.$$

2) *For all* $\phi \in \mathcal{D}(M)$

$$M_n\phi = \lim_{\lambda\to\infty} i[M, A_n(\lambda)]\phi.$$

Corollary 2.3 *(The virial theorem) Suppose Assumptions 2.1 2) and 4) and (possibly replacing the second part of Assumption 2.1 2)) that for any* $\phi \in \mathcal{D}(H) \cap \mathcal{D}\left(M^{\frac{1}{2}}\right)$

$$\lim_{n\to\infty} \langle H_n\rangle_\phi = \langle M + G\rangle_\phi = ||M^{\frac{1}{2}}\phi||^2 + \langle G\rangle_\phi.$$

(Here $\langle T\rangle_\phi = \langle\phi, T\phi\rangle$*.) Suppose* $\{\phi_m\} \subseteq \mathcal{D}(H)$ *is a sequence of eigenstates,* $(H - E_m)\phi_m = 0$, $||\phi_m|| = 1$, *such that* $\phi_m \to 0$ *weakly and* $E_m \to E$. *Then there exists* $m_0 \in \mathbb{N}$ *such that* $\phi_m \notin \mathcal{D}\left(M^{\frac{1}{2}}\right)$ *for* $m \geq m_0$.

Proof We pick $f \in C_0^\infty(\mathcal{U})$ equal to one on a neighbourhood of E and compute for $\phi_m \in \mathcal{D}\left(M^{\frac{1}{2}}\right)$ the expectation

$$\langle M + G\rangle_{\phi_m} = \lim_{n\to\infty}\lim_{\lambda\to\infty} \langle i[H, A_n(\lambda)]\rangle_{\phi_m} = 0.$$

On the other hand if m is large enough

$$\langle M + G\rangle_{\phi_m} \geq \alpha - \langle K\rangle_{\phi_m} \geq \frac{\alpha}{2}$$

yielding a contradiction. □

Our main result is

Theorem 2.4 *(The limiting absorption principle) Suppose Assumptions 2.1 with* E *not being an eigenvalue of* H. *Then there exists a neighbourhood* $\mathcal{V}$ *of* E *such that*

1) $\sup_{Im z\neq 0, Re z\in\mathcal{V}} ||(A + i)^{-1\star}(H - z)^{-1}(A + i)^{-1}|| < \infty.$

2)

$$\sup_{Re z_1, Re z_2 \in \mathcal{V}, Im z_1 Im z_2 > 0, z_1 \neq z_2} \{|z_1 - z_2|^{-\frac{1}{3}}$$

$$||(A+i)^{-1\star}\Big\{(H-z_1)^{-1} - (H-z_2)^{-1}\Big\}(A+i)^{-1}||\} < \infty.$$

Remark 2.5 *The results of [M] are obtained from ours by letting $M = I$ and $A_n = A$. (Notice that Assumption 2.1 4) follows from the estimate for $M = 0$, cf. (3.16).)*

For simplicity of presentation we have not attempted to impose the weakest possible conditions under which our results may be obtained. Below we list some modified conditions under which Corollary 2.3 and Theorem 2.4 remain valid as can be justified actually by minor modifications of the proofs. Two of these conditions are reminiscences of [M] and [PSS] to which we refer for further details.

1) *Introducing the dual space $\mathcal{H}_{-2} = \mathcal{H}_2{}^\star, \mathcal{H}_2 = \mathcal{D}(H)$, the conditions of Assumptions 2.1 3) and 4), G_n and L being $H-$bounded, may be replaced by G_n and L being bounded as maps from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$. The condition on the existence of G_∞ may be replaced by $\sup_n ||(H+i)^{-1}G_n(H-i)^{-1}|| < \infty$.*

2) *The condition of Assumption 2.1 3) on the existence of M_∞ may be replaced by $\sup_n ||M^{-s}M_nM^{-s}|| < \infty$ for some $s < 1$.*

3) *The weight $(A+i)^{-1}$ of Theorem 2.4 1) may be replaced by $(A+i)^{-s} = e^{-\frac{s\pi}{2}i}\Gamma(s)^{-1}\int_0^\infty e^{itA}e^{-t}t^{s-1}dt$, $s > \frac{1}{2}$, where we use the notation e^{itA} for the semigroup generated by $-iA$.*

In order to prove Theorem 2.4 we need some preliminary results.

Lemma 2.6 *Suppose Assumption 2.1 1). Consider for $\epsilon \in \mathbb{R}\backslash\{0\}$ $H(\epsilon) = H - i\epsilon M$ with the domain $\mathcal{D} = \mathcal{D}(H) \cap \mathcal{D}(M)$. The adjoint operator is given by*

$$H(\epsilon)^\star = H(-\epsilon).$$

In particular $z \in \rho(H(\epsilon))$ for either $Imz + \delta\epsilon$ and ϵ both positive or both negative. (Here δ refers to the delta of Assumptions 2.1.) Moreover in these cases the resolvent $R_z^0(\epsilon) = (H(\epsilon) - z)^{-1}$ obeys the bound

$$||R_z^0(\epsilon)|| \leq |Imz + \delta\epsilon|^{-1}. \tag{2.1}$$

Proof Obviously $H(\epsilon)^\star \supseteq H(-\epsilon)$. Therefore for given $\psi \in \mathcal{D}\big(H(\epsilon)^\star\big)$ we just need to show that $\psi \in \mathcal{D}$. To that end we shall use that with $C = ||H(\epsilon)^\star\psi||$

$$|\langle H(\epsilon)\phi, \psi\rangle| \leq C||\phi|| \text{ for all } \phi \in \mathcal{D}. \tag{2.2}$$

For any non–real complex η we can compute (cf. [M])

$$M(H-\eta)^{-1}M^{-1} = (H-\eta)^{-1} - i(H-\eta)^{-1}i[H,M]^0(H-\eta)^{-1}M^{-1}. \tag{2.3}$$

Since obviously the right hand side is bounded we get that $\mathcal{D}$ is preserved by $(H-\eta)^{-1}$. Therefore we may replace ϕ in (2.2) by $(H-\bar{\eta})^{-1}\phi$. Commuting the resolvent by using (2.3) we thus obtain

$$|\Big\langle M\phi, (H-\eta)^{-1}\psi\Big\rangle| \leq C_{\psi,\eta}||\phi|| \text{ for all } \phi \in \mathcal{D}. \tag{2.4}$$

Since (by assumption) $\mathcal{D}$ is dense in $\mathcal{D}(M)$ we conclude that $(H-\eta)^{-1}\psi \in \mathcal{D}(M)$. By repeating the argument we obtain that $\phi_\kappa = D_\kappa\psi \in \mathcal{D}$ where for $\kappa > 0$

$$D_\kappa = H(\kappa H - i)^{-1}(\kappa H + i)^{-1} = \kappa^{-1}\Big(I + i(\kappa H - i)^{-1}\Big)(\kappa H + i)^{-1}.$$

Inserting in (2.2) yields

$$|\mathrm{Re}\langle H(\epsilon)\phi_\kappa, \psi\rangle| \leq C||\phi_\kappa|| \leq C||H(\kappa H + i)^{-1}\psi||. \tag{2.5}$$

On the other hand

$$|\mathrm{Re}\langle H(\epsilon)\phi_\kappa, \psi\rangle| \geq ||H(\kappa H + i)^{-1}\psi||^2 - |\epsilon\mathrm{Im}\langle M\phi_\kappa, \psi\rangle|. \tag{2.6}$$

To estimate the last term we notice that (by using a technique of [M] cf. Lemma 2.2)

$$\begin{aligned}
&\mathrm{Im}\langle M\phi_\kappa, \psi\rangle = \lim_{\lambda\to\infty}\frac{i}{2}\Big\langle\Big[Mi\lambda(M+i\lambda)^{-1}, D_\kappa\Big]\Big\rangle_\psi \\
&= \frac{1}{2}\big\langle R^+H^0R^+ + iR^-H^0R^-R^+ + iR^-R^+H^0R^+\big\rangle_\psi; \\
&R^+ = (\kappa H + i)^{-1}, R^- = (\kappa H - i)^{-1}, H^0 = i[H,M]^0.
\end{aligned}$$

In particular

$$\begin{aligned}
&|\epsilon\mathrm{Im}\langle M\phi_\kappa, \psi\rangle| \leq |\epsilon|\frac{3}{2}||\psi||\,||H^0(H-i)^{-1}||\,||(H-i)(\kappa H+i)^{-1}\psi|| \\
&\leq \frac{1}{2}||H(\kappa H+i)^{-1}\psi||^2 + C_1||\psi||^2,
\end{aligned} \tag{2.7}$$

where C_1 is independent of κ (and ψ).

Combining (2.5), (2.6) and (2.7) yields (by subtraction)

$$\tfrac{1}{2}||H(\kappa H+i)^{-1}\psi||^2 - C_1||\psi||^2 \leq ||H(\epsilon)^\star\psi||\,||H(\kappa H+i)^{-1}\psi||, \tag{2.8}$$

from which we obtain the bound

$$||H(\kappa H+i)^{-1}\psi|| \leq C_\psi, \tag{2.9}$$

the constant C_ψ being independent of κ. Letting $\kappa \to 0$ we obtain that $\psi \in \mathcal{D}(H)$. This fact combined with (2.2) implies that also $\psi \in \mathcal{D}(M)$.

The second part of the lemma follows from the first part just proven and the fact that the numerical range of $H(\epsilon)$ is a subset of $\epsilon\{\eta | \operatorname{Im}\eta \leq -\delta\}$. □

Lemma 2.7 *Suppose the assumption of Lemma 2.6 and that $f \in C_0^\infty(\mathbb{R})$ is equal to one on a neighbourhood of E. Then there exist constants $C, \epsilon_0 > 0$ and a neighbourhood $\mathcal{V}$ of E such that*

$$||(H-i)(I-f(H))R_z^0(\epsilon)|| \leq C$$

provided $|\epsilon| \leq \epsilon_0$, $\epsilon Imz > 0$ and $Rez \in \mathcal{V}$.

Proof We pick an almost analytic extension $\tilde{f} \in C_0^\infty(\mathbb{C})$ of f so that we can represent

$$f(H) = \frac{1}{\pi}\int_{\mathbb{C}} \left(\bar{\partial}\tilde{f}\right)(\eta)(H-\eta)^{-1}dudv,\ \eta = u+iv. \tag{2.10}$$

In conjunction with (2.3) we obtain that

$$Mf(H)M^{-1} \text{ and } [f(H), M] \text{ are bounded.} \tag{2.11}$$

In particular for any $\phi \in \mathcal{H}$

$$\psi = (I-f(H))R_z^0(\epsilon)\phi \in \mathcal{D}{=}\mathcal{D}(H) \cap \mathcal{D}(M).$$

We commute

$$(H(\epsilon)-z)\psi = (I-f(H))\phi - \epsilon i[f(H), M]R_z^0(\epsilon)\phi,$$

yielding together with (2.11) and Lemma 2.6

$$\begin{aligned} &||(H(\epsilon)-z)\psi|| \leq ||(I-f(H))\phi|| + |\epsilon|C_1||R_z^0(\epsilon)\phi|| \\ &\leq C_2||\phi|| \end{aligned} \tag{2.12}$$

for constants C_1 and C_2 independent of ϵ and z.

We can also compute

$$||(H(\epsilon)-z)\psi||^2 = ||(H-\operatorname{Re}z)\psi||^2 + ||(\epsilon M + \operatorname{Im}z)\psi||^2 - \epsilon\langle i[H,M]\rangle_\psi. \tag{2.13}$$

Clearly the last term on the right hand may be estimated by

$$|\epsilon\langle i[H,M]\rangle_\psi| \leq |\epsilon|C_3\,||(H-i)\psi||^2. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) gives the bound

$$||(H-\operatorname{Re}z)\psi||^2 + ||(\epsilon M + \operatorname{Im}z)\psi||^2 \leq C_2^2||\phi||^2 + |\epsilon|C_3||(H-i)\psi||^2. \tag{2.15}$$

Clearly we can find a neighbourhood $\mathcal{V}$ of E and a positive number κ such that for all $t \in \mathbb{R}$ and $s \in \mathcal{V}$

$$\Big\{(t-s)^2 - 2\kappa|t-i|^2\Big\}|1-f(t)|^2 \geq 0. \tag{2.16}$$

For $|\epsilon|C_3 \leq \kappa$ and Rez $(= s) \in \mathcal{V}$ (2.15) and (2.16) give the bound (by subtraction)

$$||(H-i)\psi||^2 \leq \kappa^{-1}C_2^2||\phi||^2. \tag{2.17}$$

□

In the rest of this section we impose Assumptions 2.1 with E not being an eigenvalue of H. We pick a real–valued $f \in C_0^\infty(\mathbb{R})$ equal to one on a neighbourhood of E such that the form inequality

$$M + f(H)Gf(H) \geq \tfrac{\alpha}{2}f(H)^2 - (I-f(H))L(I-f(H)) \tag{2.18}$$

holds on $\mathcal{D}(H)\cap\mathcal{D}\Big(M^{\frac{1}{2}}\Big)$. We would like to prove Lemmas 2.6 and 2.7 for the perturbed operator $H(\epsilon) - i\epsilon f(H)Gf(H)$. Introducing the notation

$$R_z(\epsilon) = (H - i\epsilon(M + f(H)Gf(H)) - z)^{-1}$$

for its resolvent we have

Lemma 2.8 *There exist constants $C, \epsilon_0 > 0$ and a neighbourhood $\mathcal{V}$ of E such that*

$$||R_z(\epsilon)|| \leq \tfrac{C}{|\epsilon|}, \tag{2.19}$$

$$||(H-i)(I-f(H))R_z(\epsilon)|| \leq C \tag{2.20}$$

provided $|\epsilon| \leq \epsilon_0$, ϵIm$z > 0$ and Re$z \in \mathcal{V}$.

Proof It is not obvious that the resolvent exists for all z in question. Clearly a perturbation argument based on (2.1) gives the existence for large values of $|\mathrm{Im}z|$. Below we shall prove (2.19) in a domain of the desired form assuming that the resolvent exists. Then by a simple connectedness argument it follows that it exists in the whole domain.

So suppose z is given such that $R_z(\epsilon)$ exists. Then from

$$R_z(\epsilon) = R_z^0(\epsilon)(I + i\epsilon f(H)Gf(H)R_z(\epsilon))$$

and Lemma 2.7 we obtain (with ϵ_0 and $\mathcal{V}$ given as in Lemma 2.7)

$$||(H-i)(I-f(H))R_z(\epsilon)|| \leq C_1(1 + |\epsilon|\,||R_z(\epsilon)||). \tag{2.21}$$

By (2.18) and Lemma 2.6

$$R_{\bar{z}}(-\epsilon)f(H)^2R_z(\epsilon) \leq \tfrac{2}{\alpha}\Big(\tfrac{\mathrm{Im}R_z(\epsilon)}{\epsilon} + R_{\bar{z}}(-\epsilon)(I-f(H))L(I-f(H))R_z(\epsilon)\Big), \tag{2.22}$$

yielding

$$||f(H)R_z(\epsilon)||^2 \leq C_2\Big(\tfrac{||R_z(\epsilon)||}{|\epsilon|} + ||(H-i)(I-f(H))R_z(\epsilon)||^2\Big). \tag{2.23}$$

Combining (2.21) and (2.23) we obtain

$$\begin{aligned} ||R_z(\epsilon)||^2 &\leq 2\big(||(I-f(H))R_z(\epsilon)||^2 + ||f(H)R_z(\epsilon)||^2\big) \\ &\leq 2C_1^2(1+|\epsilon|\,||R_z(\epsilon)||)^2 + 2C_2\left(\frac{||R_z(\epsilon)||}{|\epsilon|} + C_1^2(1+|\epsilon|\,||R_z(\epsilon)||)^2\right). \end{aligned} \tag{2.24}$$

We may assume that $2C_1^2(1+C_2)\epsilon_0^2 < 1$. Then (by subtraction) (2.24) implies the bound (2.19). Upon combining with (2.21) we get (2.20). □

With $R_z(\epsilon)$ given as in Lemma 2.8 we introduce the notation

$$F_z(\epsilon) = (A+i)^{-1\star}R_z(\epsilon)(A+i)^{-1}.$$

Lemma 2.9 *In addition to the bounds of Lemma 2.8 we have (with a possibly larger constant C)*

$$||R_z(\epsilon)(A+i)^{-1}|| \leq |\epsilon|^{-\frac{1}{2}}C(1+||F_z(\epsilon)||)^{\frac{1}{2}}, \tag{2.25}$$

$$||\epsilon M R_z(\epsilon)(A+i)^{-1}|| \leq C(1+||F_z(\epsilon)||)^{\frac{1}{2}} \tag{2.26}$$

for all ϵ and z given as in the lemma.

Proof For any $\phi \in \mathcal{H}$ we put $\psi = R_z(\epsilon)(A+i)^{-1}\phi$. The expectation of (2.22) in the state $(A+i)^{-1}\phi$ gives

$$||f(H)\psi||^2 \leq \frac{2}{\alpha}\left(\frac{||F_z(\epsilon)||}{|\epsilon|}||\phi||^2 + C_1||(H-i)(I-f(H))\psi||^2\right).$$

Upon combining with Lemma 2.8 we thus obtain

$$\begin{aligned} ||\psi||^2 &\leq 2||f(H)\psi||^2 + 2||(I-f(H))\psi||^2 \\ &\leq \left(\frac{4}{\alpha}\frac{||F_z(\epsilon)||}{|\epsilon|} + C_2\right)||\phi||^2, \end{aligned} \tag{2.27}$$

which clearly gives (2.25).

As for (2.26) we compute

$$(A+i)^{-1}\phi = (H(\epsilon)-z)\psi - i\epsilon f(H)Gf(H)\psi,$$

which in conjunction with (2.13), (2.14) and Lemma 2.8 leads to

$$||(A+i)^{-1}\phi||^2 \geq \frac{1}{2}||(H(\epsilon)-z)\psi||^2 - \epsilon^2 C_3||\psi||^2$$

$$\geq \frac{1}{2}||(\epsilon M + \mathrm{Im}z)\psi||^2 - \big(|\epsilon|C_4 + \epsilon^2 C_3\big)||(H-i)\psi||^2$$

$$\geq \frac{1}{2}||\epsilon M\psi||^2 - 2\big(|\epsilon|C_4 + \epsilon^2 C_3\big)||(H-i)f(H)||^2||\psi||^2 - C_5||\phi||^2.$$

So by (2.27)

$$||\epsilon M\psi||^2 \leq C_6\big(|\epsilon|\,||\psi||^2 + ||\phi||^2\big)$$

$$\leq C_6\left(\frac{4}{\alpha}||F_z(\epsilon)|| + |\epsilon|C_2 + 1\right)||\phi||^2,$$

which clearly gives (2.26). □

Proof of Theorem 2.4 We shall use Lemmas 2.8 and 2.9 to prove the differential inequality

$$||\tfrac{d}{d\epsilon}F_z(\epsilon)|| \leq |\epsilon|^{-\frac{1}{2}}C(1 + ||F_z(\epsilon)||). \tag{2.28}$$

In conjunction with (2.19) this will give Theorem 2.4 1) by three integrations (iterations) with respect to ϵ using the fact that for any $\phi \in \mathcal{H}$

$$\lim_{\epsilon\to 0} R_z(\epsilon)\phi = (H-z)^{-1}\phi,$$

which in turn is readily obtained from Assumption 2.1 1) and (2.1).

To prove (2.28) we compute (for $\epsilon \neq 0$)

$$\tfrac{d}{d\epsilon}F_z(\epsilon) = (A+i)^{-1\star}R_z(\epsilon)i(M + f(H)Gf(H))R_z(\epsilon)(A+i)^{-1}. \tag{2.29}$$

The middle term is rewritten as

$$M + f(H)Gf(H) = (M+G) - (I-f)Gf - G(I-f). \tag{2.30}$$

Upon substituting (2.30) into the right hand side of (2.29) we obtain three terms. The second and third terms are bounded by

$$|\epsilon|^{-\frac{1}{2}}C(1 + ||F_z(\epsilon)||)^{\frac{1}{2}}\Big(\leq |\epsilon|^{-\frac{1}{2}}C(1 + ||F_z(\epsilon)||)\Big) \tag{2.31}$$

by (2.20) and (2.25).

It remains to bound the operator

$$B_z(\epsilon) = (A+i)^{-1\star}R_z(\epsilon)i(M+G)R_z(\epsilon)(A+i)^{-1}. \tag{2.32}$$

For that we rewrite $\langle \phi_1, B_z(\epsilon)\phi_2\rangle$ for $\phi_1, \phi_2 \in \mathcal{C}_1 = (A+i)\mathcal{C}$ as follows.

We insert (cf. Assumption 2.1 2) and Lemma 2.2 1))

$$
\begin{aligned}
M + G &= \lim_{n\to\infty}\lim_{\lambda\to\infty} i[H, A_n(\lambda)] = T_1 + \cdots + T_5;\\
T_1 &= \lim_{n\to\infty}\lim_{\lambda\to\infty} i[H - i\epsilon(M + fGf), A_n(\lambda)],\\
T_2 &= -\epsilon \lim_{n\to\infty}\lim_{\lambda\to\infty} [M, A_n(\lambda)],\\
T_3 &= -\epsilon \lim_{n\to\infty}\lim_{\lambda\to\infty} [f, A_n(\lambda)]Gf,\\
T_4 &= -\epsilon \lim_{n\to\infty}\lim_{\lambda\to\infty} f[G, A_n(\lambda)]f,\\
T_5 &= -\epsilon \lim_{n\to\infty}\lim_{\lambda\to\infty} fG[f, A_n(\lambda)].
\end{aligned}
\tag{2.33}
$$

The contribution from T_1 is given by

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{\lambda\to\infty} \Big\langle \phi_1, \Big\{(A+i)^{-1\star} i\Big[R_z(\epsilon), \lambda A_n(A_n + i\lambda)^{-1}\Big](A+i)^{-1}\Big\}\phi_2\Big\rangle\\
&= \Big\langle R_{\bar z}(-\epsilon)(A+i)^{-1}\phi_1, A(A+i)^{-1}\phi_2\Big\rangle - \Big\langle A(A+i)^{-1}\phi_1, R_z(\epsilon)(A+i)^{-1}\phi_2\Big\rangle.
\end{aligned}
\tag{2.34}
$$

Here we used that for $\phi \in \mathcal{C}_1$

$$
\lim_{n\to\infty}\lim_{\lambda\to\infty} i\lambda A_n(A_n + i\lambda)^{-1}(A+i)^{-1}\phi = \lim_{n\to\infty} A_n(A+i)^{-1}\phi = A(A+i)^{-1}\phi.
$$

Since $\mathcal{C}_1$ is dense in $\mathcal{H}$ we obtain from (2.34), the Cauchy Schwarz inequality and (2.25) that the contribution from T_1 to (2.32) is bounded by the constant (2.31).

We are left with bounding the contributions from $T_2, \cdots, T_5$ to (2.32). For that we compute as forms on $\mathcal{D}(M)$ using Assumptions 2.1, Lemmas 2.2 and 2.6, (2.10) and (2.3)

$$
\begin{aligned}
T_2 &= i\epsilon M_\infty,\\
T_3 &= -i\epsilon\frac{1}{\pi}\int_{\mathbb{C}} \Big(\bar\partial\tilde f\Big)(\eta)(H-\eta)^{-1}(M+G)(H-\eta)^{-1}dudvGf,\\
T_4 &= i\epsilon f G_\infty f,\\
T_5 &= -i\epsilon fG\frac{1}{\pi}\int_{\mathbb{C}} \Big(\bar\partial\tilde f\Big)(\eta)(H-\eta)^{-1}(M+G)(H-\eta)^{-1}dudv.
\end{aligned}
\tag{2.35}
$$

By (2.3) the terms T_3 and T_5 are of the form

$$
T_3 = \epsilon M B \text{ and } T_5 = -\epsilon B^\star M;\ B \text{ bounded.}
$$

Hence by (2.25), (2.26) and the Cauchy Schwarz inequality all terms of (2.35) contribute to (2.32) by operators with the bounding constant

$$
|\epsilon|^{-\frac{1}{2}} C(1 + ||F_z(\epsilon)||).
$$

We have proved (2.28).

The second statement Theorem 2.4 2) may be proved using (2.25) and (2.28) (without the last factor) exactly as in [PSS]. □

3 The spin–boson model

Let $\mathcal{F}$ denote the symmetric Fock space over $L^2(\mathbb{R}^\nu)$, $\nu \in \mathbb{N}$. Let $m \in \mathbb{N}$. On $\mathcal{H} = \mathbb{C}^m \otimes \mathcal{F} = \int\limits_X \oplus \mathcal{F} dx$, $X = \{1, \cdots, m\}$, we consider a Hamiltonian of the form

$$H = S \otimes I + I \otimes H_f + V,\ V = \int\limits_X \oplus \{a(\lambda_x) + a^\star(\lambda_x)\} dx, \tag{3.1}$$

where S is self–adjoint on $\mathbb{C}^m$, H_f is the second quantization $d\Gamma(w)$ on $\mathcal{F}$ of the operator of multiplication by the function $\omega(k) = |k|$ on $L^2(\mathbb{R}^\nu)$, and a and $a^\star$ are the operator of annihilation and creation, respectively, of a function $\lambda_x(\cdot) \in L^2_\omega := L^2\Big(\mathbb{R}^\nu, \Big(1 + \omega(k)^{-1}\Big) dk\Big)$. We use the notation $||\lambda_x||_\omega$ for the corresponding weighted $L^2 -$ norm.

It is well–known (cf. [BFS] and [HS]) that for any ϕ in the form domain of $I \otimes H_f$

$$|| \int\limits_X \oplus a(\lambda_x)\, dx \phi ||,\ || \int\limits_X \oplus a^\star(\lambda_x) dx \phi || \leq \sup_{x \in X} ||\lambda_x||_\omega \big\langle I \otimes H_f + I \big\rangle_\phi^{\frac{1}{2}}. \tag{3.2}$$

Hence by the Kato–Rellich theorem [RS, Theorem X.12] the Hamiltonian (3.1) with domain $\mathcal{D}(H) = \mathbb{C}^m \otimes \mathcal{D}\big(H_f\big)$ is self–adjoint.

We shall impose the following stronger condition. Let for $i, j \in \{0, 1, 2\}, i + j \leq 2$, $x \in X$

$$\lambda_x^{(i,j)}(k) = |k|^{-j} \left(\frac{\partial}{\partial |k|}\right)^i \lambda_x(k)$$

considered as distributions on $C_0^\infty(\mathbb{R}^\nu \backslash \{0\})$. We demand

$$\lambda_x^{(i,j)}(\cdot) \in L^2_\omega. \tag{3.3}$$

We can now verify Assumptions 2.1 1)–3). Up to this point we have only specified H. Consider the semigroup $U(t)$ acting on $L^2(\mathbb{R}^\nu)$ given by

$$U(t)\phi(k) = F(|k| > t) \left(1 - \frac{t}{|k|}\right)^{\frac{\nu - 1}{2}} \phi\left((|k| - t)\frac{k}{|k|}\right),\ t \geq 0.$$

It is generated by $-iB$ (that is $U(t) = e^{itB}$) where B is the closure on $C_0^\infty(\mathbb{R}^\nu \backslash \{0\})$ of

$$B = -\frac{1}{2}\left(\frac{k}{|k|} \cdot p + p \cdot \frac{k}{|k|}\right),\ p = -i\left(\frac{\partial}{\partial k_1}, \cdots, \frac{\partial}{\partial k_\nu}\right).$$

The A of Assumptions 2.1 is given by $A = I \otimes d\Gamma(B)$ where $-id\Gamma(B)$ is the generator of the second quantized semigroup $\Gamma(U(t))$. Generating a contraction semigroup

$-1 \in \rho(-iA)$. Consequently $-i \in \rho(A)$. We specify (the last product consisting of all finite symmetric linear combinations of products of C_0^∞–functions)

$$\mathcal{C} = \bigcup_{p\in\mathbb{N}} \bigoplus_{q=0}^{p} \mathbb{C}^m \otimes \left(C_0^\infty(\mathbb{R}^\nu\backslash\{0\})^{\otimes_s q}\right) (\subseteq \mathcal{H}). \tag{3.4}$$

It follows immediately from [RS, Theorem X.49] that $\mathcal{C}$ is a core of A.

By the formal computations on $\mathcal{F}$

$$\begin{aligned} &[d\Gamma(B), a^\star(\lambda_x)] = a^\star(B\lambda_x),\\ &[d\Gamma(B), a(\lambda_x)] = -a(B\lambda_x), \end{aligned} \tag{3.5}$$

we can (on a formal level) understand (3.3) as conditions guaranteeing H–boundedness of $i[V, A]$ and $i[i[V, A], A]$. As for the commutator with $I \otimes H_f$ we have (still formally)

$$\begin{aligned} &i\big[I \otimes H_f, A\big] = I \otimes i[d\Gamma(\omega), d\Gamma(B)]\\ &= I \otimes d\Gamma(i[\omega, B]) = I \otimes d\Gamma(I) = I \otimes N, \end{aligned} \tag{3.6}$$

where N is the "number operator".

Motivated by the above computations we define with P_Ω given as the projection in $\mathcal{F}$ onto the vacuum

$$\begin{aligned} &M = I \otimes (N + P_\Omega)\ (\geq I),\\ &A_n = I \otimes d\Gamma(B_n);\\ &B_n = \frac{1}{2}(F_n \cdot p + p \cdot F_n),\ F_n(k) = \frac{-k}{\sqrt{|k|^2 + n^{-2}}}. \end{aligned} \tag{3.7}$$

The flow generated by the smooth vector field F_n is given by

$$\frac{d}{dt}\theta_n(k,t) = F_n(\theta_n(k,t)),\ \theta_n(k,0) = k,$$

and satisfies the bound

$$\sup_{|t|<1} \sup_{k\in\mathbb{R}^\nu\backslash\{0\}} \tfrac{|\theta_n(k,t)|}{|k|} \leq e^n. \tag{3.8}$$

To verify Assumption 2.1 2) we notice that the formula

$$\big(e^{itB_n}\phi\big)(k) = (\det \theta_n'(k,t))^{\frac{1}{2}}\phi(\theta_n(k,t)) \tag{3.9}$$

combined with (3.8) gives the bound

$$\begin{aligned} &\sup_{|t|<1} ||H_f e^{itd\Gamma(B_n)}\big(H_f - i\big)^{-1}||\\ &\leq \sup_{|t|<1} \sup_{p\in\mathbb{N}} \sup_{k_1,\cdots,k_p\in\mathbb{R}^\nu} |\sum_{q=1}^{p} \omega(\theta_n(k_q,-t))\left(\sum_{q=1}^{p}\omega(k_q) - i\right)^{-1}| \leq e^n. \end{aligned}$$

Clearly the same bound holds upon extending from $\mathcal{F}$ to $\mathcal{H} = \mathbb{C}^m \otimes \mathcal{F}$. Hence (cf. (3.2)) the bound of Assumption 2.1 2) holds. To verify that $i[H, A_n]$ extends to a H–bounded operator we notice that $\mathcal{C}$ given by (3.4) is invariant under e^{itA_n} (cf. (3.8) and (3.9)). By a result of Mourre [M, Proposition II.1] we therefore only need to show that $i[H, A_n]$ defined as a form on $\mathcal{C}$ extends to a H–bounded operator. But in this sense we obtain from (3.5) and (3.6) that

$$
\begin{aligned}
&i[H, A_n] = I \otimes d\Gamma(\omega_n) + V_n^{(1)};\\
&\omega_n(k) = \frac{|k|}{\sqrt{|k|^2 + n^{-2}}},\ V_n^{(1)} = \int_X \oplus \left\{ a\left(\lambda_{n,x}^{(1)}\right) + a^\star\left(\lambda_{n,x}^{(1)}\right)\right\} dx,\\
&\lambda_{n,x}^{(1)} = -iB_n\lambda_x = \frac{|k|}{\sqrt{|k|^2 + n^{-2}}}\left(\lambda_x^{(1,0)} + \frac{\frac{\nu}{n^2} + (\nu-1)|k|^2}{2(|k|^2 + n^{-2})}\lambda_x^{(0,1)}\right).
\end{aligned}
\tag{3.10}
$$

Since $\omega_n \leq n\omega$ and $\lambda_{n,x}^{(1)} \in L^2_\omega$ both terms are H–bounded.

We readily obtain the last part of Assumption 2.1 2) by using (3.2), (3.10) and the Lebesgue dominated convergence theorem. Here we put

$$
\begin{aligned}
&G = \int_X \oplus \left\{ a\left(\lambda_x^{(1)}\right) + a^\star\left(\lambda_x^{(1)}\right)\right\} dx - I \otimes P_\Omega;\\
&\lambda_x^{(1)} = \lambda_x^{(1,0)} + \frac{\nu - 1}{2}\lambda_x^{(0,1)}.
\end{aligned}
$$

Clearly for all $\phi \in \mathcal{C}$ $\lim_{n\to\infty} A_n\phi = A\phi$.

As for Assumption 2.1 1) we notice that $\mathcal{C}$ is also a core of M and that it is invariant under e^{itM}. The bound of Assumption 2.1 1) is trivial. So by the same argument as above we only need to compute $i[H, M]$ as a form on $\mathcal{C}$:

$$
i[H, M] = -\int_X \oplus \left\{ \bar{P}_\Omega a(i\lambda_x) + a^\star(i\lambda_x)\bar{P}_\Omega\right\} dx;\ \bar{P}_\Omega = I - P_\Omega,
\tag{3.11}
$$

which obviously is H–bounded.

As for Assumption 2.1 3) we compute by techniques familiar by now

$$M_n = M_\infty = 0,$$

$$G_n = V_n^{(2)} = \int_X \oplus \left\{ a\left(\lambda_{n,x}^{(2)}\right) + a^\star\left(\lambda_{n,x}^{(2)}\right)\right\} dx;$$

$$\lambda_{n,x}^{(2)} = -iB_n\lambda_x^{(1)} = \sum_{i=0}^{2} c_{n,i}\lambda_x^{(i,2-i)},$$

$$G_\infty = V^{(2)} = \int_X \oplus \left\{ a\left(\lambda_x^{(2)}\right) + a^\star\left(\lambda_x^{(2)}\right)\right\} dx;$$

$$\lambda_x^{(2)} = \sum_{i=0}^{2} c_i\lambda_x^{(i,2-i)},$$

for smooth coefficient functions obeying

$$c_i(k) = \lim_{n\to\infty} c_{n,i}(k), \quad \sup_{\substack{i=0,1,2\\ n\in\mathbb{N}, k\in\mathbb{R}^\nu}} |c_{n,i}(k)| < \infty.$$

Finally we shall verify Assumption 2.1 4) for all $E \in \mathbb{R}$ assuming

$$\beta := \sup_{x\in X} \int_{\mathbb{R}^\nu} |\lambda_x^{(1)}(k)|^2 dk < 1. \tag{3.12}$$

For that we introduce the unitary operator (cf. [HS])

$$T = \exp\left(\int_X \oplus \left\{ a\left(\lambda_x^{(1)}\right) - a^\star\left(\lambda_x^{(1)}\right)\right\} dx \right). \tag{3.13}$$

By the proof of [RS, Theorem X.41] $\mathcal{C}$ constitutes a set of analytic vectors for the exponent. This fact justifies a small computation yielding to:

$$\begin{aligned} &I\otimes N + \int_X \oplus \left\{ a\left(\lambda_x^{(1)}\right) + a^\star\left(\lambda_x^{(1)}\right)\right\} dx \\ &= T\left(I\otimes N - \int_X \oplus dx \int_{\mathbb{R}^\nu} |\lambda_x^{(1)}(k)|^2 dk \right) T^{-1} \qquad (3.14) \\ &\geq T(I\otimes N - \beta)T^{-1}, \end{aligned}$$

as a form on $\mathcal{C}$.

By a simple density argument (3.14) extends to $\mathcal{D}\left(M^{\frac{1}{2}}\right)$ and therefore in particular to $\mathcal{D}(H) \cap \mathcal{D}\left(M^{\frac{1}{2}}\right)$.

For any real–valued $f \in C_0^\infty(\mathbb{R})$ we decompose the form

$$
\begin{aligned}
&M + f(H)Gf(H) = P - L + f(H)Lf(H) - f(H)Kf(H);\\
&P = I \otimes N + \int_X \oplus \left\{ a\Big(\lambda_x^{(1)}\Big) + a^\star\Big(\lambda_x^{(1)}\Big) \right\} dx + TI \otimes P_\Omega T^{-1},\\
&L = \int_X \oplus \left\{ a\Big(\lambda_x^{(1)}\Big) + a^\star\Big(\lambda_x^{(1)}\Big) \right\} dx + TI \otimes P_\Omega T^{-1} - I \otimes P_\Omega,\\
&K = TI \otimes P_\Omega T^{-1}.
\end{aligned}
\tag{3.15}
$$

Clearly K is compact. By (3.12) and (3.14) $P \geq (1-\beta) =: 2\alpha > 0$. We decompose

$$2\alpha I = 2\alpha f(H)^2 + 2\alpha\Big(I - f(H)^2\Big)$$

and write

$$
\begin{aligned}
&2\alpha - L + f(H)(L - 2\alpha)f(H)\\
&= -(I - f(H))(L - 2\alpha)(I - f(H))\\
&+ (I - f(H))B^\star f(H) + f(H)B(I - f(H)),
\end{aligned}
$$

where $B = B(f)$ is bounded. Since the last two terms on the right hand side may be estimated for any $\epsilon > 0$ as

$$
\begin{aligned}
&(I - f(H))B^\star f(H) + f(H)B(I - f(H))\\
&\geq -\epsilon^2 f(H)BB^\star f(H) - \epsilon^{-2}(I - f(H))^2
\end{aligned}
\tag{3.16}
$$

we obtain the inequality of Assumption 2.1 4) by choosing ϵ such that $\epsilon^2 ||B||^2 \leq \alpha$.

We have verified Assumptions 2.1. The additional assumption of Corollary 2.3 can be verified by (3.10) and the argument after (3.10). In fact we can prove the following stronger result.

Theorem 3.1 *Suppose (3.3) and (3.12). Let $\mathcal{H}_{pp}$ be the space of eigenstates of H. Then $\mathcal{H}_{pp} \subseteq \mathcal{D}\Big(M^{\frac{1}{2}}\Big)$,*

$$\dim \mathcal{H}_{pp} \leq m(1-\beta)^{-1}, \tag{3.17}$$

and the bounds of Theorem 2.4 hold for any E not being an eigenvalue of H. In particular $\mathcal{H}_{sc}$, the singular continuous subspace of H, is empty.

Proof For any eigenstate ϕ of H the expectation $\langle H_n \rangle_\phi$ vanishes, cf. the proof of Corollary 2.3. Hence we obtain from (3.10) and the Lebesgue monotone and dominated convergence theorems that $\langle M + G \rangle_\phi$ vanishes (by letting $n \to \infty$). In particular $\phi \in \mathcal{D}\Big(M^{\frac{1}{2}}\Big)$. Moreover, cf. the above verification of Assumption 2.1 4),

$$\left\langle I \otimes N - \int_X \oplus dx \int_{\mathbb{R}^\nu} |\lambda_x^{(1)}(k)|^2 dk \right\rangle_{T^{-1}\phi} = 0.$$

Estimating yields

$$\langle I(1-\beta) - I \otimes P_\Omega \rangle_{T^{-1}\phi} \leq 0,$$

which in turn (by summing over an orthonormal basis in $\mathcal{H}_{pp}$) gives (3.17). □

4 The electron–boson model, preliminaries

We shall now study Hamiltonians given by (3.1) with $\mathcal{H}$ replaced by $\mathcal{H} = L^2(X) \otimes \mathcal{F} = \int\limits_X \oplus \mathcal{F} dx$, where X is a finite dimensional real vector space with an inner product. The operator S is now replaced by the (minus) Laplace operator on $L^2(X)$, that is $S = -\triangle$ and H_f is given exactly as before while finally V is replaced by

$$\begin{aligned} V &= \sum_{b \in \mathcal{B}} V_b, \\ V_b &= \int\limits_{X^b} \oplus \left\{ a\left(\lambda^b_{x^b}\right) + a^\star\left(\lambda^b_{x^b}\right) + v^b\left(x^b\right) I \right\} dx^b. \end{aligned} \tag{4.1}$$

Here $\mathcal{B}$ is a finite index set for a collection of subspaces $X^b \subset X$ on each of which the corresponding "pair potential", V_b, has the specified form. If $\lambda^b = 0$ for all b then $H_{el} := -\triangle + V$ acts on the factor $L^2(X)$ and is usually called a generalized Schrödinger operator.

In this section we shall require

$$\begin{aligned} &\lambda^b = \lambda^b_{(\cdot)}(\cdot) \in L^\infty\left(X^b, L^2_\omega\right), \\ &||\partial_y^\alpha \left(|y|^\alpha \lambda^b_y(\cdot) \right)||_\omega \to 0 \text{ for } |y| \to \infty;\ |\alpha| \leq 1, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &v^b\left(-\triangle^b + 1\right)^{-1} \text{ compact},\ v^b \text{ real-valued}, \\ &|\partial_y^\alpha \left(|y|^{|\alpha|} v^b(y) \right)| \to 0 \text{ for } |y| \to \infty;\ |\alpha| \leq 1. \end{aligned} \tag{4.3}$$

The above derivatives are only required to exist outside a compact subset of X^b and may possibly there be defined as distributional derivatives. (See the beginning of Section 5 for more details.)

By (3.2), (4.2), (4.3) (the latter conditions only used with $|\alpha| = 0$) and the Kato–Rellich theorem we infer that H is essentially self–adjoint on $\mathcal{D}(\triangle) \otimes \mathcal{D}\left(H_f\right)$.

We now introduce various notation well–known for generalized Schrödinger operators. We may assume that the family $\{X_b | b \in \mathcal{B}\}$, defined by writing X_b for the orthogonal complement of X^b in X, is closed with respect to intersection and that there exist $b_{\min}, b_{\max} \in \mathcal{B}$ such that $X^{b_{\min}} = 0$ and $X^{b_{\max}} = X$. The set $\mathcal{B}$ is ordered by writing $b_1 \subset b_2$ if $X^{b_1} \subset X^{b_2}$. We introduce projections on X by putting

$\Pi^b x = x^b$ and $\Pi_b x = x_b$ for $x = x^b \oplus x_b \in X = X^b \oplus X_b$. Similar notation $p = p^b \oplus p_b$ will be used for components of the momentum operator $p = -i\nabla = (p_1, \cdots, p_{\dim X})$. We define on $\mathcal{H}^b = L^2\big(X^b\big) \otimes \mathcal{F} = \int\limits_{X^b} \oplus \mathcal{F} dx$ the sub–Hamiltonian

$$H^b = \big(p^b\big)^2 \otimes I + I \otimes H_f + V^b;\ V^b = \sum_{c \subset b} V_c. \tag{4.4}$$

For $b = b_{\min}$ the first term on the right hand side of (4.4) is interpreted as zero, that is $H^{b_{\min}} = H_f + V^{b_{\min}}$ on $\mathcal{F}$. We also define (on $\mathcal{H} = L^2(X_b) \otimes \mathcal{H}^b$)

$$H_b = (p_b)^2 \otimes I + I \otimes H^b, \tag{4.5}$$

and more general for $b \subset c$ (on $\mathcal{H}^c = L^2\big(X_b^c\big) \otimes \mathcal{H}^b$)

$$H_b^c = (p_b^c)^2 \otimes I + I \otimes H^b.$$

As for the usual generalized Schrödinger operators the latter Hamiltonians are useful for conducting induction arguments.

We define for $b \neq b_{\min}$ the set of thresholds as the set of eigenvalues of sub-Hamiltonians, that is

$$\mathcal{F}^b = \bigcup_{c \overset{\subset}{\neq} b} \sigma_{pp}(H^c). \tag{4.6}$$

Let $d^b : \mathbb{R} \to \mathbb{R}$ be given by

$$d^b(E) = \begin{cases} \inf\limits_{\tau \in \mathcal{F}^b(E)} (E - \tau),\ \mathcal{F}^b(E) := \mathcal{F}^b \cap (-\infty, E] \neq \emptyset, \\ 1,\ \mathcal{F}^b(E) = \emptyset. \end{cases} \tag{4.7}$$

We abbreviate $d^{b_{\max}}(E) = d(E)$ and $\mathcal{F}^{b_{\max}} = \mathcal{F}(H)$ (assuming here $b_{\max} \neq b_{\min}$).

Similarly we define for $c \subset b$ ($b = b_{\min}$ not excluded) with $\mathcal{P}_c^b = \cup_{c \subset d \subset b} \sigma_{pp}\big(H^d\big)$ and some constant $C \geq 0$

$$d_c^b(E) = \begin{cases} \inf\limits_{\tau \in \mathcal{P}_c^b(E)} (E - \tau),\ \mathcal{P}_c^b(E) := \mathcal{P}_c^b \cap (-\infty, E] \neq \emptyset, \\ C,\ \mathcal{P}_c^b(E) = \emptyset. \end{cases} \tag{4.8}$$

Clearly for any $\epsilon > 0$

$$d_c^b(E + \epsilon) \leq d_c^b(E) + \epsilon, \tag{4.9}$$

and similarly for the function defined in (4.7).

For $R \in \mathbb{R}$ the notation $F(\cdot < R)$ stands for the characteristic function of the interval $(-\infty, R)$. Let $F(\cdot \geq R) = 1 - F(\cdot < R)$.

For $\delta > 0$ the notation η_δ stands for any function $\eta \in C_0^\infty(\mathbb{R})$ obeying $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $|t| \leq \delta$ and $\eta(t) = 0$ for $|t| > 2\delta$.

We shall prove the following result expressed in terms of the definition (4.8).

Lemma 4.1 *Let* $b, c \in \mathcal{B}$,$c \subset b$, $E_0 \in \mathbb{R}, \epsilon > 0$, K_f *be compact on* $\mathcal{F}$ *and finally* $h^c \in L^\infty(X^c)$ *be compactly supported. Then*

$$\sup_{E \leq E_0} ||B_c^b(\delta, E, \epsilon)|| \to 0 \textit{ for } \delta \to 0;$$

$$B_c^b(\delta, E, \epsilon) = \eta_\delta\Big(H^b - E\Big)\Big(B^b \otimes K_f\Big),$$

$$B^b = h^c(x^c)F\bigg(\Big(p_c^b\Big)^2 < d_c^b(E+\epsilon) - 2\epsilon\bigg).$$

(For $b = b_{\min}$ B^b *is the number* $h^c(0)F\big(0 < d_c^b(E+\epsilon) - 2\epsilon\big)$*.)*

Proof We consider the statement of the lemma as a collection of statements labelled by $b \in \mathcal{B}$ and holding for all other quantities (including the constant C of (4.8)). Then we shall proceed by induction using the ordering of $\mathcal{B}$. Thus the first step will be to verify the statement for $b = b_{\min}$. For that we may proceed more general by showing it for $c = b$ in which case we may actually assume that $b = b_{\max}$ (since $b_{\min} = b_{\max}$ is not excluded at this point). To do that we notice that for $\delta < \frac{1}{2}$ and $E \leq E_0$

$$\begin{aligned} &B_c(\delta, E, \epsilon) = \eta_\delta(H - E)B_1 K B_2; \\ &B_1 = F(H < E_0 + 1)\big(\big(p^2+1\big) \otimes I\big), \\ &K = \Big(\big(p^2+1\big)^{-1} h^c\Big) \otimes K_f, \\ &B_2 = B_2(E, \epsilon) = F\Big(\big(p_c\big)^2 < d_c(E+\epsilon) - 2\epsilon\Big) \otimes I. \end{aligned} \tag{4.10}$$

Here we have omitted the notation $b = b_{\max}$ at various places while we have kept the notation c for a latter purpose. (If $b = b_{\min}$ we may interpret $B_1 = I$ and $K = h^c(0)K_f$.) Clearly B_1 is bounded and K is compact. If $E \in \sigma_{pp}(H)$ then $B_2 = 0$. On the other hand if $E \notin \sigma_{pp}(H)$ then by the compactness $||B_c(\delta, E, \epsilon)|| \to 0$ for $\delta \to 0$. To obtain this convergence uniformly with respect to $E \leq E_0$ we proceed by the way of contradiction. So suppose for some $\kappa > 0$ that

$$||B_c(\delta_n, E_n, \epsilon)|| > \kappa \tag{4.11}$$

for sequences $\delta_n \to 0$ and $E_n \to E$.

We notice that (4.9) implies the bound

$$d_c(E_n + \epsilon) - 2\epsilon \leq d_c\Big(E + \frac{\epsilon}{2}\Big) - \epsilon;\ |E - E_n| \leq \frac{\epsilon}{2},$$

yielding for large enough n that

$$B_c(\delta_n, E_n, \epsilon) = \eta_{\delta_n}(H - E_n)B_c\big(\delta, E, \tfrac{\epsilon}{2}\big)B_2(E_n, \epsilon). \tag{4.12}$$

Here $\delta > 0$ may be chosen arbitrarily. Choosing it so small that $||B_c\big(\delta, E, \frac{\epsilon}{2}\big)|| \leq \kappa$ we obviously have a contradiction to (4.11).

Next we conduct the induction step assuming again $b = b_{\max}$, so that we can now use the statement of the lemma for all $b \neq b_{\max}$. Suppose $c \neq b_{\max}$. Since the last argument above involving (4.12) did not use the property $c = b_{\max}$, it suffices to show convergence for fixed E. For that purpose we pick a family $\{j_b\}, b \neq b_{\max}$, of functions on X each one being smooth and homogeneous of degree 0 outside a compact set. We assume that

$$0 \leq j_b \leq 1, \ \sum_b j_b = 1$$

and for any $b' \not\subset b$

$$|x| j_b(x) \leq C|x^{b'}| j_b(x); \ C = C(b, b') < \infty. \tag{4.13}$$

We insert this partition

$$\eta_\delta(H - E)B \otimes K_f = \sum_b \eta_\delta(H - E) j_b B \otimes K_f. \tag{4.14}$$

Clearly we may assume that the constant C entering in the definition of B through (4.8) is large. Assuming $C \geq E_0 + \epsilon - \inf \mathcal{P}_{b_{\min}}^{b_{\max}}$ it is readily seen that

$$d_{c''}^{b''}(E') \leq d_{c'}^{b'}(E'); \ c'' \subset c' \subset b' \subset b'', E' \leq E_0 + \epsilon. \tag{4.15}$$

In particular

$$F_c = F_{b_{\max}} F_c; \ F_{c'} = F_{c'}\Big((p_{c'})^2 < d_{c'}(E + \epsilon) - 2\epsilon\Big),$$

and thus we can write for b with $c \not\subset b$

$$\begin{aligned}
&\eta_\delta(H - E) j_b B \otimes K_f = \eta_\delta(H - E) B_1 K B_2 B_3; \\
&B_1 = F(H < E_0 + 1)\big(\big(p^2 + 1\big) \otimes I\big), \\
&K = \Big(\big(p^2 + 1\big)^{-1} j_b h^c\Big) \otimes K_f, \\
&B_2 = F_{b_{\max}} \otimes I, \ B_3 = F_c \otimes I.
\end{aligned}$$

By (4.13) K is compact and hence we recognize the form (4.10) treated above.

Thus we only need to bound those terms with $c \subset b$. To do that we split

$$\begin{aligned}
&\eta_\delta(H - E) j_b = \eta_\delta(H - E) j_b \eta_{\delta'}(H_b - E) + \eta_\delta(H - E) T\big(\delta'\big); \\
&T\big(\delta'\big) = \eta_{\delta'}(H - E) j_b - j_b \eta_{\delta'}(H_b - E), \ \delta' \geq 2\delta.
\end{aligned} \tag{4.16}$$

We may write

$$\eta_\delta(H-E)T\big(\delta'\big)B\otimes K_f=\eta_\delta(H-E)KB_2B_3;$$
$$K=T\big(\delta'\big)\big(h^c\otimes K_f\big),$$
$$B_2=F_{b_{\max}}\otimes I,\ B_3=F_c\otimes I.$$

By a computation using (2.10) and (4.13) we see that

$$||T\big(\delta'\big)(F(|x|\geq R)\otimes I)||\to 0 \text{ for } R\to\infty,$$

implying that K is compact. Hence for fixed $\delta'>0$ we can again proceed as above to treat the contribution from the second term on the right hand side of (4.16). It remains to show that the contribution from the first term on the right hand side of (4.16) can be estimated arbitrarily small by choosing $\delta'>0$ small enough. For that purpose we use (4.5) to reduce to a statement for the sub-Hamiltonian H^b. We decompose $(p_c)^2=\big(p_c^b\big)^2+(p_b)^2$ and estimate (cf. (4.9) and (4.15))

$$d_c(E+\epsilon)-(p_b)^2\leq d_c\Big(E-(p_b)^2+\epsilon\Big)\leq d_c^b\Big(E-(p_b)^2+\epsilon\Big), \tag{4.17}$$

yielding

$$\begin{aligned}&||\eta_{\delta'}(H_b-E)B\otimes K_f||\leq||\eta_{\delta'}\Big(H^b-\Big(E-(p_b)^2\Big)\Big)\cdots\\&\Big(h^c(x^c)F\Big(\Big(p_c^b\Big)^2<d_c^b\Big(E-(p_b)^2+\epsilon\Big)-2\epsilon\Big)\Big)\otimes K_f||\\&\leq\sup_{E'\leq E_0}||\eta_{\delta'}\Big(H^b-E'\Big)\Big(h^c(x^c)F\Big(\Big(p_c^b\Big)^2<d_c^b\big(E'+\epsilon\big)-2\epsilon\Big)\Big)\otimes K_f||.\end{aligned}\tag{4.18}$$

Clearly by the induction hypothesis the right hand side of (4.18) $\to 0$ for $\delta'\to 0$ completing the proof. □

In the rest of this section $b_{\min}\neq b_{\max}$.

We can now prove a similar result expressed in terms of the definition (4.7).

Corollary 4.2 *Let $c\in\mathcal{B}$, $E\in\mathbb{R},\epsilon>0$, K_f be compact on $\mathcal{F}$ and finally $h^c\in L^\infty(X^c)$ be compactly supported. Then for any given $\varepsilon>0$ there exists for all small enough $\delta>0$ a compact operator $K=K(\varepsilon,\delta)$ on $\mathcal{H}$ such that*

$$||B_c-K||\leq\varepsilon;$$
$$B_c=\eta_\delta(H-E)\big(B\otimes K_f\big),\ B=h^c(x^c)F\Big((p_c)^2<d(E+\epsilon)-2\epsilon\Big).$$

Proof We use the partition of unity in the proof of Lemma 4.1 to obtain (4.14) (with the present definition of B). The terms with $c\not\subset b$ are compact. As for the terms

with $c \subset b$ we use (4.16) and the discussion following (4.16) to conclude that we only need to find a $\delta' > 0$ such that

$$\sum_{c \subset b \neq b_{\max}} ||\eta_{\delta'}(H_b - E)B \otimes K_f|| \leq \varepsilon. \tag{4.19}$$

For that we pick $C = \max(1, E + \epsilon - \inf \mathcal{F}(H))$ to be used as input in the definition (4.8) for any b and c such that $c \subset b \neq b_{\max}$. Then it is readily seen (cf. (4.17)) that

$$d(E+\epsilon) - (p_b)^2 \leq d\Big(E - (p_b)^2 + \epsilon\Big) \leq d_c^b\Big(E - (p_b)^2 + \epsilon\Big),$$

which implies the bound (4.18). Using Lemma 4.1 to the right hand side of (4.18) we clearly obtain (4.19) for some small $\delta' > 0$. □

In order to apply Corollary 4.2 we introduce a certain vector field on X invented by Graf [G]. We have enlisted a collection of its characteristics in the following lemma. These may be derived by mimicking [G] or a simplified procedure due to Derezinski [D]. The property 5) was derived in [Sk, Appendix A].

Lemma 4.3 *There exist on X a smooth vector field ω with symmetric derivative $\omega_\star$ and a partition of unity $\{\tilde{q}_b\}$ indexed by $b \in \mathcal{B}$ and consisting of smooth functions, $0 \leq \tilde{q}_b \leq 1$, such that for some positive constants r_1 and r_2*

1) $\omega_\star(x) \geq \sum_b \Pi_b \tilde{q}_b(x)$.

2) $\omega^b(x) = 0$ *if* $|x^b| < r_1$.

3) $|x^c| > r_1$ *on supp*$(\tilde{q}_b)$ *if* $c \not\subset b$.

4) $|x^b| < r_2$ *on supp*$(\tilde{q}_b)$.

5) *For all multiindicies α and $n \in \mathbb{N} \cup \{0\}$ there exists $C \in \mathbb{R}$:*
$|\partial_x^\alpha \tilde{q}_b(x)| + |\partial_x^\alpha (x \cdot \nabla)^n (\omega(x) - x)| \leq C;\ x \in X$.

We state some consequences. Combining 3) and 4) yields to

$$\tilde{q}_c(x)\tilde{q}_b(kx) = 0 \text{ if } c \not\subset b;\ k = \tfrac{r_1}{r_2}. \tag{4.21}$$

We define for any $b \in \mathcal{B}$

$$q_b(x) = \tilde{q}_b(kx)\left(\sum_c \tilde{q}_c(kx)^2\right)^{-\frac{1}{2}}.$$

Then by 1) and (4.21)

$$\omega_\star(x) \geq \sum_b \Pi_b q_b(x)^2. \tag{4.22}$$

Finally using 2) and the fact that ω is a gradient field we obtain

$$\omega(x) = \omega(x_b) \text{ if } |x^b| < r_1. \tag{4.23}$$

We shall now consider the operator B_R on $L^2(X)$ given by

$$B_R = \tfrac{1}{2}\big(R\omega\big(\tfrac{x}{R}\big)\cdot p + p\cdot\omega\big(\tfrac{x}{R}\big)R\big);\ R > 0. \tag{4.24}$$

More precisely we shall consider the formal commutator

$$\begin{aligned} &i[H, B_R\otimes I] := \Big(2p\omega_\star\Big(\frac{x}{R}\Big)p - \frac{1}{2R^2}(\triangle(\nabla\cdot\omega))\Big(\frac{x}{R}\Big)\Big)\otimes I + \sum_{b\in\mathcal{B}} V_b'; \\ &V_b' = -R\omega\Big(\frac{x}{R}\Big)^b\cdot\int\limits_{X^b}\oplus\Big\{a\Big(\nabla^b\lambda^b_{x^b}\Big) + a^\star\Big(\nabla^b\lambda^b_{x^b}\Big) + \Big(\nabla^b v^b\Big)\Big(x^b\Big)I\Big\}dx^b. \end{aligned} \tag{4.25}$$

Here R should be chosen so large that the right hand side makes sense as a symmetric operator on $\mathcal{D}(H)$. Notice that this is possible due to our conditions (4.2) and (4.3), (3.2) and Lemma 4.3 2) and 5). In fact we observe that

$$||V_b'(H-i)^{-1}|| \to 0 \text{ for } R\to\infty. \tag{4.26}$$

To state the main result of this section we notice (cf. [RS, Theorem X.41]) that any potential $\tilde{V} = \sum\limits_{b\in\mathcal{B}}\tilde{V}_b$ of the form (4.1) with $\tilde{v}^b = 0$ and $\tilde{\lambda}^b$ obeying (4.2) is essentially self–adjoint on

$$\mathcal{C} = \bigcup_{p\in\mathbb{N}}\bigoplus_{q=0}^{p} C_0^\infty(X)\otimes\Big(C_0^\infty(\mathbb{R}^\nu\backslash\{0\})^{\otimes_s q}\Big)\ (\subseteq\mathcal{H}). \tag{4.27}$$

In the next section $\mathcal{C}$ will play a similar role as the space defined by (3.4) did in Section 3.

Proposition 4.4 *Let $\tilde{V}$ be given as above, $E\in\mathbb{R}$ and $\epsilon>0$. Then there exists $R_0>0$ so that for any $R\geq R_0$ there exist an open neighbourhood $\mathcal{U}=\mathcal{U}(R)$ of E and a compact operator $K=K(R)$ on $\mathcal{H}$ such that with $T=\exp\Big(i\tilde{V}\Big)$*

$$\begin{aligned} &f(H)i[H,B_R\otimes I]f(H) \\ &\geq f(H)\big\{2(d(E+\epsilon)-3\epsilon)TI\otimes P_\Omega T^{-1} - K - \epsilon I\big\}f(H) \end{aligned} \tag{4.28}$$

for all real–valued $f\in C_0^\infty(\mathcal{U})$.

Proof Let in the following the notation $o(1)$ stands for an R–depending bounded operator obeying $||o(1)||\to 0$ for $R\to\infty$. Operators or forms on the form $o(1)(H-i)$ or $(H+i)o(1)(H-i)$ are denoted by $o_H(1)$ and $o_{HH}(1)$, respectively. By (4.22), Lemma 4.3 5) and (4.26)

$$i[H,B_R\otimes I]\geq 2\sum_b Q_b(R)\otimes I + o_H(1);\ Q_b(R) = q_b\big(\tfrac{x}{R}\big)(p_b)^2 q_b\big(\tfrac{x}{R}\big), \tag{4.29}$$

as a form inequality on $\mathcal{D}(H)$.

For each of the terms $Q_b(R)\otimes I$ we may write $(p_b)^2=\sum\limits_{j=1}^{\dim X_b} p_j^2$, $(p_1,\cdots,p_{\dim X_b})$ being the moment operator with respect to some orthonormal basis in X_b, and then estimate

$$\begin{aligned} Q_b(R)\otimes I &\geq \sum_{j=1}^{\dim X_b} P_j^\star\, T\, I\otimes P_\Omega T^{-1} P_j \\ &= T\sum_{j=1}^{\dim X_b} P_j^\star I\otimes P_\Omega P_j T^{-1}+o_{HH}(1) \\ &= TQ_b(R)\otimes P_\Omega T^{-1}+o_{HH}(1); \\ P_j &= \Big(p_j q_b\Big(\frac{x}{R}\Big)\Big)\otimes I. \end{aligned} \tag{4.30}$$

The commutations of the middle step require justification: We introduce $T(t)=\exp\Big(it\tilde{V}\Big)$; $-1\leq t\leq 1$, and obtain by a differentiation on the core $\mathcal{C}\subset\mathcal{D}(H)$

$$TP_j\,T^{-1}-P_j=\sum_c \int\limits_0^1 T(t) i\Big[\tilde{V}_c,P_j\Big]T(-t)dt. \tag{4.31}$$

Here by assumption

$$\tilde{V}_c=\int\limits_{X^c}\oplus\Big\{a\Big(\tilde{\lambda}^c_{x^c}\Big)+a^\star\Big(\tilde{\lambda}^c_{x^c}\Big)\Big\}dx^c, \tag{4.32}$$

where $\tilde{\lambda}^b$ satisfies (4.2). We compute

$$i\Big[\tilde{V}_c,P_j\Big]=-\int\limits_{X^c}\oplus\Big\{a\Big(\partial_j\tilde{\lambda}^c_{x^c}\Big)+a^\star\Big(\partial_j\tilde{\lambda}^c_{x^c}\Big)\Big\}q_b\big(\tfrac{x}{R}\big)dx^c. \tag{4.33}$$

Clearly this operator is zero for $c\subset b$ and in general of the form $o_H(1)$ (cf. Lemma 4.3 3)). By iterating the formula (4.31) we obtain

$$\begin{aligned} TP_j\,T^{-1}-P_j-\sum_{c\not\subset b} i\Big[\tilde{V}_c,P_j\Big] &= \sum_d\sum_{c\not\subset b}\int\limits_0^1 dt\int\limits_0^t ds\, T(s) i^2\Big[\tilde{V}_d,\Big[\tilde{V}_c,P_j\Big]\Big]T(-s) \\ &= \sum_d\sum_{c\not\subset b}\frac{1}{2}\, f_{c,d}\otimes I, \end{aligned} \tag{4.34}$$

where by the canonical commutation relations and (4.33) we have computed

$$i^2\Big[\tilde{V}_d,\Big[\tilde{V}_c,P_j\Big]\Big]=\left(-2q_b\big(\tfrac{x}{R}\big)\operatorname{Im}\int\limits_{\mathbb{R}^\nu}\tilde{\lambda}^d_{x^d}(k)\overline{\partial_j\tilde{\lambda}^c_{x^c}(k)}dk\right)\otimes I=:f_{c,d}(x)\otimes I. \tag{4.38}$$

Since $f_{c,d} \otimes I = o(1)$ we can write (4.34) on the form

$$T^{-1}P_j = P_j T^{-1} + o_H(1) \tag{4.35}$$

as an operator on $\mathcal{C}$ and thus (by density) on $\mathcal{D}(H)$.

Clearly we can justify the middle step of (4.30) by using (4.35) and the adjoint formula.

Using

$$Q_b(R) \geq (d(E+\epsilon) - 3\epsilon) q_b\Big(\frac{x}{R}\Big) F\Big((p_b)^2 \geq d(E+\epsilon) - 3\epsilon\Big) q_b\Big(\frac{x}{R}\Big)$$

to the right hand side of (4.30) and substituting in (4.29) yields to

$$\begin{aligned} &i[H, B_R \otimes I] \geq 2(d(E+\epsilon) - 3\epsilon) T I \otimes P_\Omega T^{-1} \\ &- 2d(E+\epsilon) \sum_b T P_b(R) \otimes P_\Omega T^{-1} + o_{HH}(1); \\ &P_b(R) = q_b\Big(\frac{x}{R}\Big) F\Big((p_b)^2 < d(E+\epsilon) - 3\epsilon\Big) q_b\Big(\frac{x}{R}\Big). \end{aligned} \tag{4.36}$$

For each of the terms $TP_b(R) \otimes P_\Omega T^{-1}$ we shall now use the following analogue of (4.31).

$$\begin{aligned} &TP_b(R) \otimes P_\Omega T^{-1} - P_b(R) \otimes P_\Omega \\ &= \sum_{c_1} \int_0^1 T(t_1) i a d^1_{c_1}(P_b(R) \otimes P_\Omega) T(-t_1) dt_1, \end{aligned} \tag{4.37}$$

utilizing here and below the notation

$$\begin{aligned} &ad^m_{c_1,\cdots,c_m}(B) = \Big[\tilde{V}_{c_m}, ad^{m-1}_{c_1,\cdots,c_{m-1}}(B)\Big], \\ &ad^0(B) = B;\ c_1, \cdots, c_m \in \mathcal{B}. \end{aligned}$$

The commutator in (4.37) is given by

$$\begin{aligned} &ad^1_{c_1}(P_b(R) \otimes P_\Omega) = B^{(1)}_{c_1} + P_b(R) \otimes I a d^1_{c_1}(I \otimes P_\Omega); \\ &B^{(m)}_{c_1,\cdots,c_m} = \Big[\tilde{V}_{c_m}, P_b(R) \otimes I\Big] ad^{m-1}_{c_1,\cdots,c_{m-1}}(I \otimes P_\Omega). \end{aligned} \tag{4.38}$$

For $c_m \subset b$ $B^{(m)}_{c_1,\cdots,c_m} = 0$. In general $B^{(m)}_{c_1,\cdots,c_m} = o(1)$ (cf. (4.42) below and Lemma 4.3 3)). To treat the contribution from the second term on the right hand side of (4.38) to (4.37) we iterate by inserting

$$\begin{aligned} &T(t_1)P_b(R) \otimes I i\, ad^1_{c_1}(I \otimes P_\Omega) T(-t_1) - P_b(R) \otimes I i\, ad^1_{c_1}(I \otimes P_\Omega) \\ &= \sum_{c_2} \int_0^{t_1} T(t_2)(P_b(R) \otimes I) i^2\, ad^2_{c_1,c_2}(I \otimes P_\Omega) T(-t_2) dt_2 + o(1), \end{aligned} \tag{4.39}$$

where we used that $B^{(2)}_{c_1,c_2} = o(1)$. By iterating the above procedure n ($\in \mathbb{N}$) times we obtain the formula

$$
\begin{aligned}
&TP_b(R) \otimes P_\Omega T^{-1} = T_n + R_n + o(1);\\
&T_n = \sum_{m=0}^{n-1} \sum_{c_1,\cdots,c_m} (m!)^{-1} P_b(R) \otimes Ii^m \, ad^m_{c_1,\cdots,c_m}(I \otimes P_\Omega),\\
&R_n = \sum_{c_1,\cdots,c_n} \int_0^1 dt_1 \cdots \int_0^{t_{n-1}} dt_n \, T(t_n) P_b(R) \otimes Ii^n \, ad^n_{c_1,\cdots,c_n}(I \otimes P_\Omega) T(-t_n).
\end{aligned}
\tag{4.40}
$$

We claim that

$$
||R_n|| \to 0 \text{ for } n \to \infty, \tag{4.41}
$$

uniformly with respect to R.

To show (4.41) we notice the bound (cf. the proof of [RS, Theorem X.41])

$$
||ad^n_{c_1,\cdots,c_n}(I \otimes P_\Omega)|| \leq 4^n \sqrt{n!} \sup_{c,x^c} \left(\int_{\mathbb{R}^\nu} |\tilde{\lambda}^c_{x^c}(k)|^2 dk \right)^{\frac{n}{2}}, \tag{4.42}
$$

which upon computing the integrals implies

$$
||R_n|| \leq C^n (n!)^{-\frac{1}{2}};\ C = 4(\#\mathcal{B}) \sup_{c,x^c} \left(\int_{\mathbb{R}^\nu} |\tilde{\lambda}^c_{x^c}(k)|^2 dk \right)^{\frac{1}{2}},
$$

and therefore clearly (4.41).

We fix n such that $2d(E+\epsilon)(\#\mathcal{B})C^n (n!)^{-\frac{1}{2}} \leq \frac{\epsilon}{4}$ bounding the contribution from all R_n (one for each b) to the second term on the right hand side of (4.36) by $\frac{\epsilon}{4}$.

Next we look at the term T_n. Clearly

$$
ad^m_{c_1,\cdots,c_m}(I \otimes P_\Omega) = \int_{X^c} \oplus K_f(x^c) dx^c =: D^c
$$

with $X^c = X^{c_1} + \cdots + X^{c_m}$ and $K_f(\cdot)$ being a bounded weakly measurable compact operator–valued function with $||K_f(x^c)|| \to 0$ for $|x^c| \to \infty$.

Up to a small bounded operator on $\mathcal{H}^c = L^2(X^c) \otimes \mathcal{F}$

$$
D^c \Big((p^c)^2 + 1 \Big)^{-1} \otimes I \approx D^c \, K^c \otimes I,
$$

where K^c is on the form $K^c = g^c(x^c) \Big((p^c)^2 + 1 \Big)^{-1}$, $g^c \in L^\infty(X^c)$ being compactly supported.

Again up to a small bounded operator we can write

$$D^c\, K^c \otimes I \approx \int\limits_{X^c} \oplus Q K_f(x^c) Q dx^c\ K^c \otimes I,$$

where Q is a finite dimensional projection on $\mathcal{F}$ (by the Lebesgue dominated convergence theorem).

Finally using that $(p^c)^2$ is H–bounded we conclude that

$$\begin{aligned} &-2d(E+\epsilon)(m!)^{-1} P_b(R) \otimes I i^m\, ad^m_{c_1,\cdots,c_m}(I \otimes P_\Omega)\eta_1(H-E) \approx \\ &\sum_{h^c, K_f} (P_b(R) h^c(x^c)) \otimes K_f\, \eta_1(H-E), \end{aligned} \tag{4.43}$$

where the summation is finite and ranges over compactly supported $h^c \in L^\infty(X^c)$ and compact operators K_f on $\mathcal{F}$.

Of course the approximation in (4.43) is uniform with respect to R. We fix for each term an approximation such that the total contribution from the errors to the second term on the right hand side of (4.36) is bounded by $\frac{\epsilon}{4}$ (uniformly in R).

Since we can write $q_b\big(\frac{x}{R}\big) h^c(x^c) = q_b\big(\frac{x}{R}\big) h^d_R\big(x^d\big)$ with $X^d = X^b + X^c$ and $h^d_R \in L^\infty\big(X^d\big)$ being compactly supported (cf. Lemma 4.3 4)), and for this d

$$P_b(R) = P_b(R) F\Big((p_d)^2 < d(E+\epsilon) - 2\epsilon\Big) + o(1)$$

(cf. (2.10), (4.16) and Lemma 4.3 5)), we conclude

$$\begin{aligned} &(P_b(R) h^c(x^c)) \otimes K_f \\ &= \Big(P_b(R) F\Big((p_d)^2 < d(E+\epsilon) - 2\epsilon\Big) h^d_R\Big(x^d\Big)\Big) \otimes K_f + o(1). \end{aligned} \tag{4.44}$$

Now we fix R_0 such that the contributions from the terms $o(1)$ on the right hand side of (4.44) and (4.40) plus the one from $\eta_1(H-E) o_{HH}(1) \eta_1(H-E)$ (with $o_{HH}(1)$ given in (4.36)) altogether are bounded by $\frac{\epsilon}{4}$ for $R \geq R_0$.

Finally by Corollary 4.2 we can find a compact operator $K = K(R)$ on $\mathcal{H}$ such that for a small $\delta > 0$

$$||K - \sum_{b,d,K_f} P_{b,d,K_f} \otimes K_f \eta_\delta(H-E)|| \leq \frac{\epsilon}{4};$$

$$P_{b,d,K_f} = P_b(R) F\Big((p_d)^2 < d(E+\epsilon) - 2\epsilon\Big) h^d_R\Big(x^d\Big).$$

Upon symmetrizing all previously encountered terms we readily obtain the result using that the four $\frac{\epsilon}{4}$–bounds appeared up to this point adds to ϵ. □

The proof of Proposition 4.4 was involved due to the appearance of the operator T. Partly as a warm up for the next section we close Section 4 by specializing to the case

$T = I$ and $\lambda^b = 0$ (for all b). We shall derive spectral properties for the generalized Schrödinger operator $H_{el} := -\triangle + V$ acting (as a factor) on $L^2(X)$. Notice that in this case $H = H_{el} \otimes I + I \otimes H_f$ and that the set of thresholds of H as defined by (4.6) coincides with the set $\mathcal{F}_{el}$ of thresholds of H_{el} (defined similarly in terms of sub-Hamiltonians on the Hilbert space $L^2(X)$).

Corollary 4.5 *(A Mourre estimate) Under the condition (4.3) there exists for any given $E \in \mathbb{R}$ and $\epsilon > 0$ a positive number R_0 so that for any $R \geq R_0$ we can find an open neighbourhood $\mathcal{U} = \mathcal{U}(R)$ of E and a compact operator $K = K(R)$ on $L^2(X)$ such that*

$$\begin{aligned} &f(H_{el})i[H_{el}, B_R]f(H_{el}) \\ &\geq f(H_{el})\{(2d(E) - \epsilon)I - K\}f(H_{el}) \end{aligned} \tag{4.45}$$

for all real–valued $f \in C_0^\infty(\mathcal{U})$.

Moreover the eigenvalues of H_{el} (counted with multiplicity) can only accumulate at $\mathcal{F}_{el}$, the latter set being closed and countable.

Proof We proceed by induction with respect to the ordering of $\mathcal{B}$ (cf. the proof of Lemma 4.1). So suppose we know the statements of the corollary for sub–Hamiltonians (leaving the start of induction to the reader), then we need to verify them for H_{el}. Since (by assumption) the eigenvalues of any sub–Hamiltonian can only accumulate at the set of thresholds of this operator we obtain that $\mathcal{F}_{el}$ is closed and countable.

Now we can verify the other statements without further reference to the induction hypothesis. We apply (4.28) to the invariant subspace $L^2(X) = (I \otimes P_\Omega)\mathcal{H}$ yielding

$$\begin{aligned} &f(H_{el})i[H_{el}, B_R]f(H_{el}) \\ &\geq f(H_{el})\{(2d(E + \epsilon) - 7\epsilon)I - K\}f(H_{el}). \end{aligned} \tag{4.46}$$

If $E \in \mathcal{F}_{el}$ then clearly (4.45) follows from (4.46). Using that $\mathcal{F}_{el}$ is closed then also for $E \notin \mathcal{F}_{el}$ (4.46) implies (4.45).

The remaining statement that the eigenvalues of H_{el} can only accumulate at $\mathcal{F}_{el}$ follows readily from (4.45) and a virial theorem (cf. Corollary 2.3). □

5 The electron–boson model, continued

We proceed somewhat similar to Section 3 skipping some arguments given there. The basic set–up is the same as the one in Section 4, but we shall need the following stronger conditions compared to (4.2) and (4.3).

We require

$$
\begin{aligned}
&\lambda^{b(i,j)}_{(\cdot)}(\cdot) \in L^\infty\Big(X^b, L^2_\omega\Big);\\
&\lambda^{b(i,j)}_{x^b}(k) = |k|^{-j}\left(\frac{\partial}{\partial|k|}\right)^i \lambda^b_{x^b}(k),\ i,j \in \{0,1,2\}, i+j \le 2,\\
&||\partial^\alpha_y\Big(|y|^\alpha \lambda^{b(i,j)}_y(\cdot)\Big)||_\omega \to 0 \text{ for } |y| \to \infty;\\
&i+j+|\alpha| \le 2,
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
&v^b\Big(-\triangle^b + 1\Big)^{-1} \text{ compact},\ v^b \text{ real-valued},\\
&|\partial^\alpha_y\Big(|y|^{|\alpha|} v^b(y)\Big)| \to 0 \text{ for } |y| \to \infty;\ |\alpha| \le 2.
\end{aligned}
\tag{5.2}
$$

As in Section 3 the function $\lambda^b_{x^b}(\cdot)$ in (5.1) (for fixed x^b) is considered as a distribution on $C^\infty_0(\mathbb{R}^\nu \setminus \{0\})$. Moreover all the above derivatives with respect to x^b are only required to exist outside a compact subset $\mathcal{K}^b \subset X^b$ in the sense of distribution. More precisely we consider in the case of (5.1) $\lambda^{b(i,j)}$ as a distribution on $C^\infty_0\big(X^b \setminus \mathcal{K}^b\big) \otimes C^\infty_0(\mathbb{R}^\nu \setminus \{0\})$ and in the case of (5.2) v^b as a distribution on $C^\infty_0\big(X^b \setminus \mathcal{K}^b\big)$.

We shall need the following analogue of (3.12). Assume that for all $b \in \mathcal{B}$

$$
\begin{aligned}
&\beta^b := \sup_{x^b \in X^b} \int_{\mathbb{R}^\nu} |\sum_{c \subset b} \lambda^{c(1)}_{x^c}(k)|^2 dk < 1;\\
&\lambda^{c(1)}_{x^c} = \lambda^{c(1,0)}_{x^c} + \frac{\nu-1}{2}\lambda^{c(0,1)}_{x^c}.
\end{aligned}
\tag{5.3}
$$

For later convenience we abbreviate

$$
\lambda_x = \sum_{b \in \mathcal{B}} \lambda^b_{x^b},\ \lambda^{(1)}_x = \sum_{b \in \mathcal{B}} \lambda^{b(1)}_{x^b},\ \beta = \beta^{b_{\max}}.
\tag{5.4}
$$

We shall verify Assumptions 2.1 for some fixed energy $E \notin \mathcal{F}(H)$ in Assumptions 2.1 4) assuming that $\mathcal{F}(H)$ is closed and countable. The latter condition can be verified inductively, cf. the proof of Corollary 4.5.

For that let B_R be defined as the closure in $L^2(X)$ of the expression (4.24) on $C^\infty_0(X)$ and $B_f = d\Gamma(B)$ as in Section 3. We shall consider

$$
A = CB_R \otimes I + I \otimes B_f
\tag{5.5}
$$

defined as the closure on $\mathcal{C}$, the latter given by (4.27). Here C is an arbitrary constant chosen such that (with $d(E)$ given by (4.7))

$$
C2d(E) \ge 1.
\tag{5.6}
$$

It is readily proved (cf. [RS, Theorem X.49]) that $-iA$ generates a contraction semigroup.

With B_n given by (3.7) we define

$$A_n = CB_R \otimes I + I \otimes d\Gamma(B_n)$$

as the closure on $\mathcal{C}$, which constitutes a family of self–adjoint operators approaching A as $n \to \infty$.

Let

$$M = I \otimes (N + P_\Omega) \, (\geq I). \tag{5.7}$$

We can now verify Assumptions 2.1 for R large enough. We shall mainly focus on Assumption 2.1 4) referring to Section 3 for a more detailed account. (Many arguments there may readily be modified). We notice that the invariance of $\mathcal{D}(H)$ and $\mathcal{D}(M)$ under e^{itA_n} follows from the formula

$$e^{itA_n} = e^{itCB_R} \otimes I + I \otimes e^{itd\Gamma(B_n)} \tag{5.8}$$

since by explicit computations these properties hold for both terms on the right hand side. Moreover (5.8) implies invariance of $\mathcal{C}$, and therefore it suffices to calculate commutators as forms on $\mathcal{C}$ (cf. Section 3).

We obtain

$$G = \int_X \oplus \left\{ a\left(\lambda_x^{(1)}\right) + a^\star\left(\lambda_x^{(1)}\right) \right\} dx - I \otimes P_\Omega + Ci[H, B_R \otimes I]. \tag{5.9}$$

Clearly by (5.1) and (4.25) G is bounded relative to H.

To verify Assumption 2.1 4) we let again T be given by (3.13) (with $\lambda_x^{(1)}$ given in (5.4)). Then using (5.7) and (5.9) we split for any real–valued $f \in C_0^\infty(\mathbb{R})$ (cf. (3.15))

$$\begin{aligned}
&M + f(H)Gf(H) = P - L + f(H)(L + Q)f(H);\\
&P = I \otimes N + \int_X \oplus \left\{ a\left(\lambda_x^{(1)}\right) + a^\star\left(\lambda_x^{(1)}\right) \right\} dx + TI \otimes P_\Omega T^{-1},\\
&L = \int_X \oplus \left\{ a\left(\lambda_x^{(1)}\right) + a^\star\left(\lambda_x^{(1)}\right) \right\} dx - I \otimes P_\Omega + TI \otimes P_\Omega T^{-1},\\
&Q = Ci[H, B_R \otimes I] - TI \otimes P_\Omega T^{-1}.
\end{aligned} \tag{5.10}$$

Due to (5.3) and (3.14) $P \geq (1 - \beta) =: 2\alpha > 0$. By (4.28) and (5.6)

$$f(H)Qf(H) \geq -Cf(H)\{K + \epsilon I\}f(H), \tag{5.11}$$

for $\epsilon > 0$ small, $K = K(\epsilon, R)$, R large, and finally for all f supported sufficiently close to E. We fix $\epsilon = \frac{\alpha}{2}C^{-1}$ and a corresponding large R. Then we may proceed exactly as in Section 3 after the formula (3.15). Using (3.16) with the ϵ there chosen such that $\epsilon^2||B||^2 \leq \frac{\alpha}{2}$ we obtain now from (5.10) and (5.11) the inequality of Assumption 2.1 4) (with the above α and K replaced by CK).

We have now outlined a verification of Assumptions 2.1 for our electron–boson Hamiltonian. Leaving it to the reader to use Corollary 2.3 (see the first part of the proof of Theorem 3.1) and to perform an induction argument (cf. the proof of Corollary 4.5, the start of induction using Theorem 3.1 with $m = 1$) we summarize our results as follows.

Theorem 5.1 *Suppose (5.1), (5.2) and (5.3). Then any eigenstate of H belongs to $\mathcal{D}\Big(M^{\frac{1}{2}}\Big)$, with M given by (5.7), and the eigenvalues of H (counted with multiplicity) can only accumulate at the set of thresholds $\mathcal{F}(H)$, the latter being closed and countable.*

Moreover the bounds of Theorem 2.4 hold with A given by (5.5) and E not being an eigenvalue nor threshold energy of H. In particular $\mathcal{H}_{sc}$, the singular continuous subspace of H, is empty.

6 Extended models

We shall now consider an extension of the model of the previous section. Up to inclusion of spin, polarization and various cut–offs the standard (Dirac) model of quantum electrodynamics (see [BFS]) can be put on this form as to be discussed at the end of this section. We consider again

$$H = p^2 \otimes I + I \otimes H_f + \tilde{V} \text{ on } \mathcal{H} = L^2(X) \otimes \mathcal{F}, \tag{6.1}$$

where the "potential" $\tilde{V} = \sum\limits_{b \in \mathcal{B}} \tilde{V}_b$ now is more general. We demand

$$\begin{aligned}
\tilde{V}_b &= V_b + \bar{U}_b \cdot p^b + p^b \cdot \bar{U}_b + \big(\bar{W}_b\big)^2; \\
V_b &= \int\limits_{X^b} \oplus \Big\{ a\Big(\lambda_{x^b}^{V_b}\Big) + a^\star\Big(\lambda_{x^b}^{V_b}\Big) + v^{V_b}\Big(x^b\Big) \Big\} dx^b, \\
\big(\bar{U}_b\big)_l &= \int\limits_{X^b} \oplus \Big\{ a\Big(\lambda_{x^b}^{U_{bl}}\Big) + a^\star\Big(\lambda_{x^b}^{U_{bl}}\Big) + v^{U_{bl}}\Big(x^b\Big) \Big\} dx^b, \\
\big(\bar{W}_b\big)_l &= \int\limits_{X^b} \oplus \Big\{ a\Big(\lambda_{x^b}^{W_{bl}}\Big) + a^\star\Big(\lambda_{x^b}^{W_{bl}}\Big) + v^{W_{bl}}\Big(x^b\Big) \Big\} dx^b,
\end{aligned} \tag{6.2}$$

where we write (with respect to an orthonormal basis in X^b)

$$\bar{U}_b = (U_{b1}, \cdots, U_{b\,\dim X^b}),\ \bar{W}_b = (W_{b1}, \cdots, W_{b\,\dim X^b}), p^b = \Big(p_1^b, \cdots, p_{\dim X^b}^b\Big).$$

We aim at an analogue of Theorem 5.1 for H on this form by verifying Assumptions 2.1 with the same inputs for A, A_n and M.

We impose the conditions (5.1) and (5.2) for the first term V_b on the right hand side of (6.2) and need to specify conditions on the "vectors" $\bar{U}_b$ and $\bar{W}_b$, which again will allow us to apply the Kato–Rellich theorem. A related problem is to find conditions such that our new G is bounded relatively to H. For the latter purpose we (formally) compute the following analogue of (5.9):

$$
\begin{aligned}
&G = T_1 + T_2 + T_3 - I \otimes P_\Omega + Ci[H, B_R \otimes I];\\
&T_1 = \int_X \oplus \Big\{ a\Big(\lambda_x^{V(1)}\Big) + a^\star\Big(\lambda_x^{V(1)}\Big)\Big\} dx,\\
&T_2 = \sum_{b\in\mathcal{B}} \sum_{1\le l\le \dim X^b} i\Big\{ \big[U_{bl}, B_f\big] p_l^b + p_l^b \big[U_{bl}, B_f\big]\Big\},\\
&T_3 = \sum_{b\in\mathcal{B}} \sum_{1\le l\le \dim X^b} i\big\{ W_{bl}\big[W_{bl}, B_f\big] + \big[W_{bl}, B_f\big] W_{bl}\big\}.
\end{aligned}
\tag{6.3}
$$

Here $\lambda_x^{V(1)} = \sum\limits_{b\in\mathcal{B}} \lambda_{x^b}^{V_b(1)}$ is given as in (5.3) and (5.4). With a similar notation the commutators are given by

$$
\begin{aligned}
i\big[U_{bl}, B_f\big] &= \int_{X^b} \oplus \Big\{ a\Big(\lambda_{x^b}^{U_{bl}(1)}\Big) + a^\star\Big(\lambda_{x^b}^{U_{bl}(1)}\Big)\Big\} dx^b,\\
i\big[W_{bl}, B_f\big] &= \int_{X^b} \oplus \Big\{ a\Big(\lambda_{x^b}^{W_{bl}(1)}\Big) + a^\star\Big(\lambda_{x^b}^{W_{bl}(1)}\Big)\Big\} dx^b.
\end{aligned}
\tag{6.4}
$$

Moreover we may write

$$
\begin{aligned}
&ip_l^b\big[U_{bl}, B_f\big] = i\big[U_{bl}, B_f\big] p_l^b +\\
&\int_{X^b} \oplus \Big\{ a\Big(\frac{\partial}{\partial x_l}\lambda_{x^b}^{U_{bl}(1)}\Big) + a^\star\Big(\frac{\partial}{\partial x_l}\lambda_{x^b}^{U_{bl}(1)}\Big)\Big\} dx^b.
\end{aligned}
\tag{6.5}
$$

Motivated by the above computations we can now specify conditions on the new terms. Clearly (6.4), (6.5) and (3.2) suggest the condition

$$
\partial_{x^b}^\beta \lambda_{x^b}^{U_{bl}(1)}(\cdot) \in L^\infty\big(X^b, L^2_\omega\big);\ |\beta| \le 1,
\tag{6.6}
$$

since it assures that the term T_2 on the right hand side of (6.3) is H–bounded. (Notice that $\big(-\triangle^b + 1\big)^{\frac{1}{2}} \otimes \big(H_f + 1\big)^{\frac{1}{2}}$ is H–bounded.)

Similarly the condition

$$
\partial_{x^b}^\beta \lambda_{x^b}^{U_{bl}}(\cdot) \in L^\infty\Big(X^b, L^2_\omega\Big);\ |\beta| \le 1,
$$

assures relative boundedness of the contribution to H from the terms involving $\lambda^{U_{bl}}$ in the definition of $\tilde{V}_b$.

To obtain relative boundedness of the term T_3 we use the bounds

$$
\begin{aligned}
&||\big(H_f+1\big)\int\limits_X \oplus a(\lambda_x)dx\big(H_f+1\big)^{-\frac{3}{2}}|| \leq 2\sup_{x\in X}||(1+\omega)\lambda_x||_\omega,\\
&||\big(H_f+1\big)\int\limits_X \oplus a^\star(\lambda_x)dx\big(H_f+1\big)^{-\frac{3}{2}}|| \leq 2\sup_{x\in X}||(1+\omega)\lambda_x||_\omega,
\end{aligned}
\tag{6.7}
$$

obtained from (3.2), (3.5) and commutations. By interpolating (3.2) and (6.7) we obtain

$$
\begin{aligned}
&||\big(H_f+1\big)^{\frac{1}{2}}\int\limits_X \oplus a(\lambda_x)dx\big(H_f+1\big)^{-1}|| \leq 2^{\frac{1}{2}}\sup_{x\in X}||(1+\omega)\lambda_x||_\omega,\\
&||\big(H_f+1\big)^{\frac{1}{2}}\int\limits_X \oplus a^\star(\lambda_x)dx\big(H_f+1\big)^{-1}|| \leq 2^{\frac{1}{2}}\sup_{x\in X}||(1+\omega)\lambda_x||_\omega.
\end{aligned}
\tag{6.8}
$$

We expand each term of T_3 into a sum of products using (6.4). Those terms that are products of two factors each one either being of the form $a(\lambda_x)$ or $a^\star(\lambda_x)$ are handled by inserting

$$I=\big(H_f+1\big)^{-\frac{1}{2}}\big(H_f+1\big)^{\frac{1}{2}}$$

between the factors and then invoke (3.2) and (6.8). This procedure leads to the conditions

$$(1+\omega)\lambda^{W_{bl}},\ (1+\omega)\lambda^{W_{bl}(1)}\ \in\ L^\infty\big(X^b,L^2_\omega\big). \tag{6.9}$$

To handle the contribution from the remaining terms $a\left(\lambda_{x^b}^{W_{bl}(1)}\right)v_{x^b}^{W_{bl}}$ and $a^\star\left(\lambda_{x^b}^{W_{bl}(1)}\right)v_{x^b}^{W_{bl}}$ we impose

$$\big(v^{W_{bl}}\big)^2\big(-\triangle^b+1\big)^{-1}\ \text{bounded}, \tag{6.10}$$

which by interpolation implies boundedness of $v^{W_{bl}}\big(-\triangle^b+1\big)^{-\frac{1}{2}}$. Combining this property with (6.9) we readily treat the above terms.

Moreover we remark that (6.9) and (6.10) assure the relative boundedness of the contribution from the term involving $\bar{W}_b$ in the definition of $\tilde{V}_b$.

It remains to elaborate on the last term on the right hand side of (6.3). Formally (cf. (4.25))

$$
\begin{aligned}
&i[H, B_R \otimes I] := \Big(2p\omega_\star\Big(\frac{x}{R}\Big)p - \frac{1}{2R^2}(\triangle(\nabla\cdot\omega))\Big(\frac{x}{R}\Big)\Big) \otimes I + \sum_{b\in\mathcal{B}} \tilde{V}_b';\\
&\tilde{V}_b' = T_{1b}' + T_{2b}' + T_{3b}',\\
&T_{1b}' = i[V_b, B_R\otimes I],\\
&T_{2b}' = i\Big[\bar{U}_b\cdot p^b + p^b\cdot\bar{U}_b, B_R\otimes I\Big],\\
&T_{b3}' = i\Big[\big(\bar{W}_b\big)^2, B_R\otimes I\Big],
\end{aligned}
\tag{6.11}
$$

where we may compute

$$
\begin{aligned}
&T_{1b}' = V_b' = -R\omega\Big(\frac{x}{R}\Big)^b\cdot\int_{X^b}\oplus\Big\{a\Big(\nabla^b\lambda_{x^b}^{V_b}\Big) + a^\star\Big(\nabla^b\lambda_{x^b}^{V_b}\Big) + \Big(\nabla^b v^{V_b}\Big)\Big(x^b\Big)\Big\}dx^b,\\
&T_{2b}' = \sum_{1\le l\le \dim X^b}\Big\{U_{bl}'p_l^b + p_l^bU_{bl}' + U_{bl}B_{Rl}' + B_{Rl}'U_{bl}\Big\};\\
&B_{Rl}' = \frac{1}{2}\Big(\omega_l'\Big(\frac{x}{R}\Big)\cdot p + p\cdot\omega_l'\Big(\frac{x}{R}\Big)\Big),\ \omega_l' = \frac{\partial}{\partial x_l}\omega,\\
&T_{3b}' = \sum_{1\le l\le\dim X^b}\big\{W_{bl}W_{bl}' + W_{bl}'W_{bl}\big\}.
\end{aligned}
$$

Here U_{bl}' and W_{bl}' are given as V_b' by replacing V_b by U_{bl} and W_{bl}, respectively.

The conditions

$$
\begin{aligned}
&\partial_{x^b}^\alpha\Big(|x^b|^{|\alpha|}\partial_{x^b}^\beta\lambda_{x^b}^{U_{bl}}(\cdot)\Big)\in L^\infty\Big(X^b, L^2_\omega\Big),\\
&\partial_{x^b}^\alpha\Big(|x^b|^{|\alpha|}\partial_{x^b}^\beta v^{U_{bl}}(\cdot)\Big)\in L^\infty\Big(X^b\Big),\\
&\partial_{x^b}^\alpha\Big(|x^b|^{|\alpha|}(1+w)\lambda_{x^b}^{W_{bl}}(\cdot)\Big)\in L^\infty\Big(X^b, L^2_\omega\Big)\\
&\partial_{x^b}^\alpha\Big(|x^b|^{|\alpha|}v^{W_{bl}}(\cdot)\Big)\in L^\infty\Big(X^b\Big);\\
&|\beta|,\ |\alpha|\le 1,
\end{aligned}
\tag{6.12}
$$

and Lemma 4.3 assure H–boundedness of the contribution from the terms T_{2b}' and T_{3b}'. Actually due to Lemma 4.3 and (4.23) we here only need boundedness outside a compact set in X^b.

Since we shall verify more than just relative boundedness (6.6), (6.9), (6.10) and (6.12) would not suffice. For example we shall need decay at infinity in (6.9) and (6.12). Also we shall need assumptions for $|\alpha| = 2$.

Explicitly (and as a conclusion of the previous discussion) we impose in addition to (5.1) and (5.2) for V_b the following conditions.

Let for $i,j \in \{0,1,2\}, i+j \leq 2$,

$$\lambda_{x^b}^{U_{bl}(i,j)}(k) = |k|^{-j}\left(\frac{\partial}{\partial|k|}\right)^i \lambda_{x^b}^{U_{bl}}(k).$$

Let $\lambda_{x^b}^{W_{bl}(i,j)}(k)$ be defined similarly.

As for $\bar{U}_b$ we demand

$$\begin{aligned}
&\partial_{x^b}^{\beta}\lambda_{x^b}^{U_{bl}(i,j)}(\cdot) \in L^{\infty}\Big(X^b, L^2_{\omega}\Big);\ i+j \leq 2, |\beta| \leq 1,\\
&||\partial_y^{\alpha}\Big(|y|^{\alpha}\partial_y^{\beta}\lambda_y^{U_{bl}(i,j)}(\cdot)\Big)||_{\omega} \to 0 \text{ for } |y| \to \infty;\\
&i+j+|\alpha| \leq 2, |\beta| \leq 1,
\end{aligned} \tag{6.13}$$

and

$$\begin{aligned}
&\Big(\partial_{x^b}^{\beta} v^{U_{bl}}\Big(x^b\Big)\Big)\Big(-\triangle^b + 1\Big)^{-\frac{1+\beta}{2}} \text{ compact; } |\beta| \leq 1,\ \ v^{U_{bl}} \text{ real-valued,}\\
&|\partial_y^{\alpha}\Big(|y|^{|\alpha|}\partial_y^{\beta} v^{U_{bl}}(y)\Big)| \to 0 \text{ for } |y| \to \infty;\ |\alpha| \leq 2, |\beta| \leq 1.
\end{aligned} \tag{6.14}$$

As for $\bar{W}_b$ we demand

$$\begin{aligned}
&(1+\omega)\lambda_{x^b}^{W_{bl}(i,j)} \in L^{\infty}\Big(X^b, L^2_{\omega}\Big);\ i+j \leq 2,\\
&||\partial_y^{\alpha}\Big(|y|^{\alpha}(1+\omega)\lambda_y^{W_{bl}(i,j)}\Big)||_{\omega} \to 0 \text{ for } |y| \to \infty;\\
&i+j+|\alpha| \leq 2,
\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}
&\Big(v^{W_{bl}}\Big)^2\Big(-\triangle^b + 1\Big)^{-1} \text{ compact, } \ v^{W_{bl}} \text{ real-valued,}\\
&|\partial_y^{\alpha}\Big(|y|^{|\alpha|} v^{W_{bl}}(y)\Big)| \to 0 \text{ for } |y| \to \infty;\ |\alpha| \leq 2.
\end{aligned} \tag{6.16}$$

To verify the Kato–Rellich criterion

$$||\tilde{V}^b\phi|| \leq \big(1-\epsilon^b\big)||\Big(\big(p^b\big)^2 \otimes I + I \otimes H_f\Big)\phi|| + C_{\epsilon^b}||\phi|| \tag{6.17}$$

(for some $\epsilon^b > 0$; $b \in \mathcal{B}$) one would need an additional smallness assumption. It may readily be shown that the following bound implies (6.17):

$$2^{\frac{5}{2}}\sum_{b\in\mathcal{B}}\sum_{1\leq l\leq \dim X^b}\Big(\sup_{x^b\in X^b}||\lambda_{x^b}^{U_{bl}}||_{\omega} + \sup_{x^b\in X^b}||(1+\omega)\lambda_{x^b}^{W_{bl}}||^2_{\omega}\Big) < 1.$$

With the above assumptions we may justify the previous computations. In particular G is given by the H–bounded expression (6.3). To verify Assumption 2.1 4) we notice that the decay conditions of (5.1), (5.2) and (6.13)–(6.16), and the property (4.23), imply

that the terms $\tilde{V}_b'$ in (6.11) in the form sense vanish for $R \to \infty$. Explicitly using the notation of the proof of Proposition 4.4 $\tilde{V}_b' = o_{HH}(1)$. The other terms on the right hand side of (6.11) may be treated as in the proof of Proposition 4.4 (with $T = I$), and since we may generalize the previous results of Section 4 to the present more general class of Hamiltonians we conclude (4.28) with $T = I$ under the present assumptions (5.1), (5.2) and (6.13)–(6.17).

We shall need a lower bound for the form $T_1+T_2+T_3$ on the right hand side of (6.3) (substituting (5.3)). Due to the appearance of momentum operators in the definition of T_2 we need at this point to restrict our attention to a bounded energy regime. So let $I_0 := (-\infty, E_0]$ for some fixed E_0. Similarly to Section 5 we assume that $\mathcal{F}(H) \cap I_0$ is closed and countable. We claim that the latter property may be proved inductively under the following additional smallness condition replacing (5.3).

We demand that for all $b \in \mathcal{B}$

$$
\begin{aligned}
&F_0^b \Big(T_1^b + T_2^b + T_3^b \Big) F_0^b \geq -\beta^b F_0^b; \\
&F_0^b = F\Big(H^b < E_0 + 1 \Big),\ \beta^b \in [0,1),
\end{aligned}
\tag{6.18}
$$

where for $b = b_{\max}$ $\left(T_1^b + T_2^b + T_3^b\right)$ is given by $T_1 + T_2 + T_3$ on the right hand side of (6.3) and for $b \neq b_{\max}$ by similarly expressions in terms of the potentials $\tilde{V}_c$ with $c \subset b$ (cf. (5.3)).

We notice that (6.17) and (6.18) are smallness conditions in the sense that they are satisfied upon replacing the potential $\tilde{V}$ subjected to the previous conditions by $e\tilde{V}$ for any sufficiently small constant $e > 0$.

Assuming (6.18) and that $E \in I_0 \backslash \mathcal{F}(H)$ we verify Assumption 2.1 4) by choosing the constant C (defining A and A_n) such that (5.6) holds (with $d(E)$ defined similarly). Then we decompose (with $F_0 = F_0^{b_{\max}}$)

$$
\begin{aligned}
&M + f(H)Gf(H) = P - L + f(H)(L+Q)f(H); \\
&P = I \otimes N + F_0(T_1 + T_2 + T_3)F_0 + I \otimes P_\Omega, \\
&L = F_0(T_1 + T_2 + T_3)F_0, \\
&Q = Ci[H, B_R \otimes I] - I \otimes P_\Omega.
\end{aligned}
\tag{6.19}
$$

Again $P \geq (1-\beta) =: 2\alpha$ (with $\beta = \beta^{b_{\max}}$). Since (due to arguments given above) we can use (5.11) again we can from this point proceed as in Section 5.

We conclude

Theorem 6.1 *Let $E_0 \in \mathbb{R}$ and $I_0 := (-\infty, E_0]$. Suppose (5.1) and (5.2) for the terms V_b, (6.13)–(6.16) for the "vectors" $\bar{U}_b$ and $\bar{W}_b$, respectively, and the smallness conditions (6.17) and (6.18). Then any eigenstate of H belongs to $\mathcal{D}\Big(M^{\frac{1}{2}}\Big)$, with M given by (5.7), and the eigenvalues of H in I_0 (counted with multiplicity) can only accumulate at the set of thresholds in I_0, $\mathcal{F}(H) \cap I_0$, the latter being closed and countable.*

Moreover the bounds of Theorem 2.4 hold with A given by (5.5) and $E \in I_0$ not being an eigenvalue nor threshold energy of H. In particular the restriction of H to the spectral subspace $F(H < E_0)\mathcal{H}$ does not have singular continuous spectrum.

For high energies we can obtain a similar result under some other assumptions on the potential. The idea is to verify Assumptions 2.1 with $M = I$ and

$$A = A_n = \tfrac{1}{2}(x \cdot p + p \cdot x) \otimes I + I \otimes d\Gamma(-(k \cdot p_k + p_k \cdot k)) \tag{6.20}$$

(cf. [BFS]). Formally

$$i[H, A] = 2\Big(H - \tilde{V}\Big) + i\Big[\tilde{V}, A\Big]. \tag{6.21}$$

If $\tilde{V}$ and $i\Big[\tilde{V}, A\Big]$ are "small" then the right hand side of (6.21) is large for high energies and hence in particular positive. To implement this idea we need the following conditions on $\tilde{V}$:

$$\begin{aligned} &\Big(x^b \cdot \nabla_{x^b}\Big)^{m_1} (k \cdot \nabla_k)^{m_2} \lambda_{x^b}^{V_b}(k) \in L^\infty\Big(X^b, L^2_\omega\Big); \\ &m_1, m_2 \in \{0, 1, 2\}, m_1 + m_2 \leq 2. \end{aligned} \tag{6.22}$$

$$\Big(\big(x^b \cdot \nabla_{x^b}\big)^m v^{V_b}\big(x^b\big)\Big)\big(-\triangle^b + 1\big)^{-1} \text{ bounded; } m \in \{0, 1, 2\}. \tag{6.23}$$

$$\begin{aligned} &\Big(x^b \cdot \nabla_{x^b}\Big)^{m_1} \partial_{x^b}^\beta (k \cdot \nabla_k)^{m_2} \lambda_{x^b}^{U_{bl}}(k) \in L^\infty\Big(X^b, L^2_\omega\Big); \\ &m_1, m_2 \in \{0, 1, 2\}, m_1 + m_2 \leq 2, |\beta| \leq 1. \end{aligned} \tag{6.24}$$

$$\begin{aligned} &\Big(\Big(x^b \cdot \nabla_{x^b}\Big)^m \partial_{x^b}^\beta v^{U_{bl}}\Big(x^b\Big)\Big)\Big(-\triangle^b + 1\Big)^{-\frac{1+\beta}{2}} \text{ bounded;} \\ &m \in \{0, 1, 2\}, |\beta| \leq 1. \end{aligned} \tag{6.25}$$

$$\begin{aligned} &\Big(x^b \cdot \nabla_{x^b}\Big)^{m_1} (1 + \omega)(k \cdot \nabla_k)^{m_2} \lambda_{x^b}^{W_{bl}} \in L^\infty\Big(X^b, L^2_\omega\Big); \\ &m_1, m_2 \in \{0, 1, 2\}, m_1 + m_2 \leq 2. \end{aligned} \tag{6.26}$$

$$\begin{aligned} &\Big(\Big(x^b \cdot \nabla_{x^b}\Big)^m \Big(v^{W_{bl}}\Big(x^b\Big)\Big)^j\Big)\Big(-\triangle^b + 1\Big)^{-\frac{j}{2}} \text{ bounded;} \\ &m \in \{0, 1, 2\}, j \in \{1, 2\}. \end{aligned} \tag{6.27}$$

Also we need (6.17) and the form inequality to be valid for some (large) $E_0 \in \mathbb{R}$, any $b \in \mathcal{B}$ and some $\alpha^b \in (0, 2)$

$$\begin{aligned} &\bar{F}_0^b\Big(i\Big[\tilde{V}^b, A^b\Big] - 2\tilde{V}^b\Big)\bar{F}_0^b \geq -\Big(2 - \alpha^b\Big)H^b\bar{F}_0^b; \\ &\bar{F}_0^b = F\Big(H^b \geq E_0 - 2\Big). \end{aligned} \tag{6.28}$$

Here A^b is given by (6.20) for $b = b_{\max}$ and similarly for $b \neq b_{\max}$.

Similarly to a previous discussion of the conditions (6.17) and (6.18) the conditions (6.22)–(6.27) imply (6.17) and (6.28) upon introducing a small enough coupling constant.

With the above conditions it is now straightforward to justify the form identity (6.21) (on the domain $\mathcal{D}(H)$) and the inequality (with $\alpha = \alpha^{b_{\max}}$)

$$\eta_1(H-E)i[H,A]\eta_1(H-E) \geq \alpha(E_0-2)\eta_1(H-E)^2;\ E \geq E_0. \tag{6.29}$$

Hence we have verified Assumption 2.1 4). Moreover the strong form of (6.29) implies absence of eigenvalues $\geq E_0$ (by a virial theorem). We may argue similarly for sub–Hamiltonians. Therefore we have outlined a proof of

Theorem 6.2 *Suppose (6.22)–(6.27) and the smallness conditions (6.17) and (6.28), the latter for some $E_0 > 2$. Let $\bar{I}_0 := [E_0, \infty)$. Then $(\sigma_{pp}(H) \cap \mathcal{F}(H)) \cap \bar{I}_0 = \emptyset$. Moreover the bounds of Theorem 2.4 hold for any $E \in \bar{I}_0$ and with A given by (6.20). In particular the restriction of H to the spectral subspace $F(H \geq E_0)\mathcal{H}$ is purely absolutely continuous.*

Under all of the previous conditions (5.1),(5.2), (6.13)–(6.16), (6.22)–(6.27) we may combine Theorems 6.1 and 6.2 upon introducing a small coupling constant. This is done by using the E_0 of Theorem 6.2 as input in Theorem 6.1 and would imply absence of singular continuous spectrum and also the same conclusion on the set of eigenvalues and the set of thresholds as in Theorem 5.1. We end this section essentially by proceeding in this way for an example in a slightly more general framework, namely the standard model of quantum electrodynamics (see [BFS] for a more detailed account).

We consider the following model for an atom coupled to a photon field: Including spin and polarization the Hilbert space is given by $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{H}_f$ where $\mathcal{H}_{el}$ is the N times anti–symmetric tensor product of the 1–electron space $L^2(\mathbb{R}^3_x) \otimes \mathbb{C}^2$ while $\mathcal{H}_f$ denotes the symmetric Fock space built from the 1–photon space $L^2(\mathbb{R}^3_k) \otimes \mathbb{C}^2$. (The first factor $\mathbb{C}^2$ accounts for spin while the latter factor accounts for polarization.) Let m and e denote the electron mass and charge, respectively, and σ_j the triple consisting of the standard Pauli matrices (each one considered as acting on the spin space $\mathbb{C}^2$ of the j'th electron). We shall use units in which $h(\text{bar}), c = 1$, and therefore $e = \sqrt{\alpha}$ where the feinstructure constant $\alpha \simeq (137)^{-1}$. We shall consider e as a small parameter.

The Hamiltonian reads

$$H = \sum_{j=1}^{N} \tfrac{1}{2m}[\sigma_j \cdot (-i\nabla_j - eA(x_j))]^2 + I \otimes H_f + e^2 V(x) \otimes I, \tag{6.30}$$

where $A(y)$ is a quantized vector potential to be elaborated on below and with Z being the charge of the nucleus

(6.31) $V(x) = \sum_{j=1}^{N} -\frac{Z}{|x_j|} + \sum_{1\leq i<j\leq N} \frac{1}{|x_i - x_j|}.$

The theory of Dirac amounts to putting

(6.32) $A(y) = \sum_{\mu=1,2} \int \frac{1}{2\pi\sqrt{\omega(k)}} \left[\epsilon_\mu(k) e^{iky} a_\mu(k) + \epsilon_\mu(k) e^{-iky} a_\mu^\star(k)\right] dk,$

where the vectors $\epsilon_1(k)$, $\epsilon_2(k)$ and $k|k|^{-1}$ constitute an orthonormal basis in $\mathbb{R}^3$, and the operators $a_1(k), a_1^\star(k), a_2(k)$ and $a_2^\star(k)$ obey

$$\left[a_1(k), a_2\left(k'\right)\right] = \left[a_1(k), a_2^\star\left(k'\right)\right] = 0,$$
$$\left[a_1(k), a_1^\star\left(k'\right)\right] = \left[a_2(k), a_2^\star\left(k'\right)\right] = \delta\left(k - k'\right).$$

Since the latter operators are only formally defined (cf. [RS, Section X.7]) the expression (6.32) is a formal object. In order to have a well–defined operator one makes the so–called ultraviolet cut–off. Then the form is

(6.33) $A(y) = \sum_{\mu=1,2} \left[a_\mu\left(\bar{\lambda}_{y\mu}\right) + a_\mu^\star\left(\bar{\lambda}_{y\mu}\right)\right]; \bar{\lambda}_{y\mu}(k) = (\lambda_{y\mu 1}(k), \lambda_{y\mu 2}(k), \lambda_{y\mu 3}(k)),$

where

$$\lambda_{y\mu l} = e^{-iky} \tilde{\lambda}_{\mu l}(k)$$

for a suitable real–valued function $\tilde{\lambda}_{\mu l} \in L^2\left(\mathbb{R}^3_k\right)$.

The methods discussed previously in this section fail to handle $\lambda_{y\mu l}$ of this form. The problem is the lack of decay as $y \to \infty$ of the y–dependence which may be overcome by cutting off suitably at $y = \infty$. There is another problem at $k = 0$ (the infrared region) for the concrete value of $\tilde{\lambda}_{\mu l}$ indicated above which may be overcome by cutting off suitably in this region.

In any case imposing similar conditions as before on each component $\lambda_{y\mu l}(k)$ the methods may be generalized to the present extended model. Notice that we may rewrite the first term on the right hand side of (6.30) as

(6.34)

$$\sum_{j=1}^{N} \frac{1}{2m} [\sigma_j \cdot (-i\nabla_j - eA(x_j))]^2 = -\triangle \otimes I + \sum_{j=1}^{N} \tilde{V}_j;$$

$$\triangle = \sum_{j=1}^{N} \frac{1}{2m} \triangle_j, \tilde{V}_j = V_j + \bar{U}_j \cdot p_j + p_j \cdot \bar{U}_j + \left(\bar{W}_j\right)^2, p_j = -i\nabla_j,$$

$$V_j = -\frac{e}{2m} \sigma_j \cdot \text{curl} A(x_j), \bar{U}_j = -\frac{e}{2m} A(x_j), \bar{W}_j = \frac{e}{\sqrt{2m}} A(x_j).$$

Except for the additional spin and polarization structure the form of (6.30) and (6.34) fits into the framework given by (6.1) and (6.2). The conditions we need on $\lambda_{y\mu l}(k)$ may be derived from (6.34) by comparing with (5.1),(5.2), (6.13)–(6.16), (6.22)–(6.27). Notice that the last term on the right side of (6.30) is in agreement with (5.2) and (6.23). We consider the following simplified condition given in terms of some $\epsilon > 0$:

$$\begin{aligned} &\left(1+y^2\right)^{\frac{|\alpha|+\epsilon}{2}} \partial_y^{\alpha+\beta} (1+\omega(k))^{1-|\beta|} |k|^{m-j} \left(\frac{\partial}{\partial |k|}\right)^{m+i} \lambda_{y\mu l}(k) \in L^\infty\left(\mathbb{R}^3_y, L^2_\omega\right); \\ &|\alpha| + m + i + j \leq 2,\ |\beta| \leq 1. \end{aligned} \tag{6.35}$$

For the above model our methods imply

Theorem 6.3 *Consider the Hamiltonian H (6.30) with $V(x)$ given by (6.31) and $A(y)$ by (6.33), each component $\lambda_{y\mu l}$ obeying (6.35). Then for small enough $e > 0$ the following statements hold:*

The eigenvalues of H (counted with multiplicity) can only accumulate at the set of thresholds $\mathcal{F}(H)$, the latter being closed and countable. Moreover there is a limiting absorption principle away from the eigenvalues and thresholds. In particular the singular continuous subspace of H is empty.

Remarks 6.4

1) *We have not attempted to prove bounds on $e > 0$ comparably with the physically relevant value $e \simeq (137)^{-\frac{1}{2}}$. It is an open problem whether the smallness of e is strictly needed for the conclusion of Theorem 6.3.*

2) *For molecules consisting of static nuclei there is a similar model (see [BFS]). In this case the Coulomb potentials describing the interaction between the electrons and the nuclei do not fulfil (6.23). On the other hand the weaker analogue condition (5.2) of Theorem 6.1 is fulfilled. Consequently the conclusion of that theorem holds for the molecule–photon model too.*

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