Griffiths' Singularities in Diluted Ising Models on the Cayley Tree

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Abstract

The Griffiths' singularities are exhibited for a class of diluted ferromagnetic Ising models defined on the Cayley tree (Bethe lattice). For the deterministic model the Lee-Yang circle theorem is explicitly proven for the magnetization at the origin and it is shown that, in the thermodynamic limit, the Lee-Yang singularities become dense in the entire unit circle for the whole ferromagnetic phase. Smoothness (infinite differentiability) of the quenched magnetization m at the origin with respect to the external magnetic field is also proven for convenient choices of temperature and disorder. From our analysis one also concludes that the existence of metastable states is impossible for the random models in consideration.

Key words: Lee-Yang singularities; Griffiths' singularities; infinite differentiability; metastable states.

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1 Introduction

For systems with quenched impurities the equilibrium properties depend not only on the usual parameters (temperature, magnetic field, etc.) but also on the impurity distribution. The presence of randomness may affect their critical behavior by rounding first order transitions [AW] and may also produce singularities of unconventional type: a whole region of the phase diagram corresponding to a pure phase may be populated by "weak" singularities (see e.g. [F]).

Griffiths [G] was the first to realize and actually prove such phenomenon. He considered a random Ising ferromagnet in $\mathbb{Z}^d$ described by the Hamiltonian

$$H(\sigma;\ \xi) := -\sum_{\langle xy \rangle} J_{xy}\ \sigma_x \sigma_y - h \sum_x \sigma_x \tag{1.1}$$

where $\sigma : \mathbb{Z}^d \ni x \longmapsto \sigma_x \in \{1, -1\}$ is a configuration of spins which are coupled by a nearest neighbor interaction of the form

$$J_{xy} = \ \xi_x \xi_y \tag{1.2}$$

with $\xi = \left\{\xi_x,\ x \in \mathbb{Z}^d\right\}$ being independent and equally distributed Bernoulli random variables

$$\xi_x = \begin{cases} 1, & \text{with probability p} \\ 0, & \text{with probability } 1 - \text{p} \end{cases} . \tag{1.3}$$

Griffiths has shown [G] that the free energy for this model cannot be analytically continued from $h > 0$ to $h < 0$ for all temperatures below the critical temperature $T_c(p = 1)$ for the deterministic model and all $p < p_c$, the threshold of site percolation. This fact is also true beyond p_c, as it has been shown, independently, by Sütő [S] and Fröhlich [F]. As a consequence, Sütő claimed that a metastable phase cannot take place in diluted Ising models.

The presence of Griffiths' singularities is now recognized to be a common feature of disordered systems (see [F] and references therein). Their appearance may be explained by the occurrence, with probability one, of arbitrarily large regions inside which the system is strongly correlated.

The aim of this article is to exhibit the Griffiths' singularities in a caricature model for the system (1.1)-(1.3). We consider a class of bond diluted Ising models given by the Hamiltonian (1.1) defined on the Bethe lattice whose disorder is introduced as follows.

Let $\mathcal{C}_k$ be an homogeneous rooted Cayley tree (Bethe lattice) of order k, i.e., a tree with $k+1$ bonds attached to each site which is not the root (called origin).

A bond b is classified according to its distance from the origin. Thus, b belongs to the generation M (or to the equivalent class indexed by M) if a directed path starting from the origin pass exactly by $M-1$ bonds before it reaches b.

To each generation M a Bernoulli random variable ξ_M

$$\xi_M = \begin{cases} 1, & \text{with probability } \mathrm{p_M} \\ 0, & \text{with probability } \mathrm{q_M} = 1 - \mathrm{p_M} \end{cases} \tag{1.4}$$

is assigned and we then set $J_{xy} = \xi_M$ if $\langle xy \rangle \equiv b$ is a bond at the generation M and 0 otherwise. See Figure 1.

For any event A, which is an element of the σ-algebra generated by the space of spin configurations, we define the quenched expected value of A as

$$\overline{A} \equiv \mathbb{E}[A] = \mathbb{E}_\xi \langle A \rangle (\xi) \tag{1.5}$$

where $\mathbb{E}_\xi$ is the expectation with respect to the disorder variables and

$$\langle A \rangle (\xi) = \frac{1}{Z(\xi)} \sum_\sigma A(\sigma)\, e^{-\beta H(\sigma;\, \xi)} \tag{1.6}$$

is the thermal average with

$$Z(\xi) = \sum_\sigma\ e^{-\beta H(\sigma;\, \xi)}$$

being the partition function.

The expectation (1.5) is a function of the inverse temperature β, the external magnetic field h and the sequence of Bernoulli parameters $\mathbf{p} = \{p_j\}_{j=1}^{\infty}$. Let $\pi = \{\mathbf{p} :\ 0 < p_j \leq 1,\ j = 1, 2, \ldots\}$ and

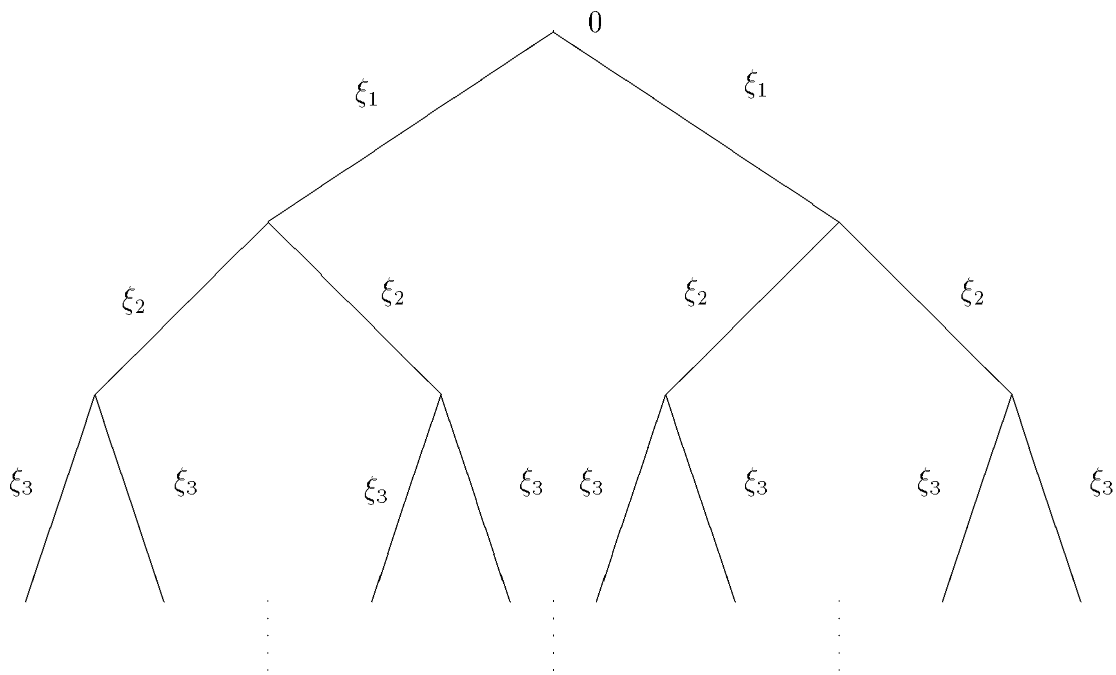

Figure 1: The homogeneous rooted Cayley tree for $k = 2$ and the couplings ξ_M.

notice that $\prod_j p_j \leq 1$. For $0 \leq a \leq 1$, let $\pi_a = \left\{ \mathbf{p} \in \pi : \lim_{n\to\infty} p_1 \ldots p_n = a \right\}$. Also, for convenience, we write $\zeta = e^{-2\beta}$ and $z = e^{-2\beta h}$.

We shall first take the expectation in a finite tree $\mathcal{C}_{k,M}$, $M = 1, 2, \ldots$, with free boundary conditions and study the function $\overline{A} : [0,1] \times \mathbb{C} \times \pi \longmapsto \overline{A}(\zeta, z, \mathbf{p}) \in \mathbb{C}$ in the limit as M goes to infinite. We recall that $\overline{A}$ is not translation invariant, even at the thermodynamic limit.

As in [G], instead of analyzing the free energy, we shall be concerned with the magnetization at origin $m := \mathbb{E}[\sigma_0] = \mathbb{E}_\xi \langle \sigma_0 \rangle (\xi)$ which can be written as

$$m \;=\; F \,+\, I, \tag{1.7}$$

with $F := \lim_{N\to\infty} F_N$, where

$$F_N \;:=\; \sum_{M=0}^{N-1} a_M \, \langle \sigma_0 \rangle_M (1) \tag{1.8}$$

gives the contribution to m due to finite clusters and $I := \lim_{N\to\infty} I_N$, where

$$I_N \;:= p_1 \ldots p_N \, \langle \sigma_0 \rangle_N (1) \tag{1.9}$$

is the contribution of the (unique) infinite cluster. Here, for any observable A, $\langle A \rangle_M$ is the thermal average (1.6) with $\mathcal{C}_k$ replaced by the finite tree $\mathcal{C}_{k,M}$ and a_M, $M \in \mathbb{N}$, is given by

$$\begin{aligned} a_0 \;&:=\; 1 - p_1 \\ a_N \;&:=\; p_1 \ldots p_N \,(1 - p_{N+1}), \qquad N \geq 1. \end{aligned}$$

Equation (1.7) is a consequence of the fact that $\langle\sigma_0\rangle(\xi) = \langle\sigma_0\rangle_M(1)$ for all ξ such that $\xi_1 = \cdots = \xi_M = 1$ and $\xi_{M+1} = 0$. In addition, since

$$\sum_{n=0}^{N-1} a_n + p_1 \dots p_N = 1$$

for any $N \in \mathbb{N}$, we note that

$$\sum_{n=0}^{\infty} a_n = 1 - a$$

for all $p \in \pi_a$. Clearly, $\{a_n\}_{n=0}^{\infty}$ is a summable sequence.

There are two different manners of m fail to be analytic in z depending on the behavior of F and I. Only F is responsible for the Griffiths' singularities. We shall see that, since $\langle\sigma_0\rangle_M(1)$ is bounded, F is always a continuous function of z, $z \in \mathbb{R}_+$, for all $\mathbf{p} \in \pi$. However, for $\mathbf{p} \in \pi_a$ with $a > 0$, I is discontinuous at $z = 1$ for $\zeta < \zeta_c$ with ζ_c being the critical point of the deterministic model (see Theorem 1.6 for precise statements). Note that, since a single random variable has been assigned to bonds at the same generation, I is equal to the expected value $\langle\sigma_0\rangle(1)$ times a, the probability that an infinite cluster occurs in a one dimensional percolation problem.

To compute (1.7) one has to calculate thermal expectations only for "perfect" finite trees. This simplification is analogous to that introduced by McCoy and Wu [MW] for the random Ising model in $\mathbb{Z}^2$. For comparison, the equation (1.7) for the model defined by (1.1)-(1.3) reads

$$m = \sum_{\psi} P_\psi \; m_\psi,$$

with m_ψ being the magnetization density in the cluster ψ (a finite connected piece of $\mathbb{Z}^d$) and P_ψ the probability that an isolated cluster ψ contains a given fixed site. It is quite difficult to evaluate m_ψ for a general graph in $\mathbb{Z}^d$ and Lee-Yang circle Theorem [LY] has been used in Griffiths' proof to estimate these quantities.

In this paper we will prove the following two theorems, which illustrate explicitly Lee-Yang and Griffiths theorems for the particular model introduced above. In order to simplify the presentation of these and all other results of this work, our analysis will be restricted to the Cayley tree $\mathcal{C}_k$with $k = 2$.

Theorem 1.1 (Analyticity) *The magnetization at origin, $m = m(\zeta,\ z,\ \mathbf{p})$, given by (1.7) is an analytic function of z in the domain $\{z \in \mathbb{C},\ |z| < 1\} \cup \{z \in \mathbb{C},\ |z| > 1\}$ for all $\zeta \in [0,1]$ and $\mathbf{p} \in \pi$.*

$\square$

Griffiths [G] has argued that the Lee-Yang singularities, in the thermodynamic limit, accumulate at $z = 1$ for all $T < T_c$ (see also [KG]). The next theorem shows that the singularities in our model are dense over the arc $A_K := \left\{z = e^{i\theta} : K \le |\theta| \le \pi\right\}$ in the unit circle for some $K = K(\zeta)$ and do accumulate at $z = 1$ for all $\zeta \le \zeta_c \equiv 1/3$. More precisely

Theorem 1.2 (Distribution of Singularities) *Let $\zeta \in [0,1]$ and $\mathbf{p} \in \pi$ such that $a_n \neq 0$ for infinitely many values of n. Then, the magnetization at origin, $m = m(\zeta, z, \mathbf{p})$, cannot be analytically continued from $\{|z| < 1\}$ to $\{|z| > 1\}$ (or vice-versa) along the arc of circle A_K where $K = K(\zeta)$ is given by*

$$K(\zeta) := \begin{cases} 3 \arccos\left(\dfrac{3 - 8\zeta + 3\zeta^2}{2\zeta}\right) - 4 \arctan\left(\dfrac{\sqrt{3(3-\zeta)(3\zeta-1)}}{5 - 3\zeta}\right) & \text{if} \quad \zeta_c < \zeta < \zeta_l, \\ \\ 0 & \text{if} \quad \zeta \leq \zeta_c, \end{cases}$$

with $\zeta_l \in (1/3,\ 1)$, $\zeta_l \simeq 0.46409$.

In addition, for all $\zeta > \zeta_c$ and $\mathbf{p} \in \pi$, m is an analytic function of z for all $z \in \mathbb{C} \setminus A_\kappa$, where $\kappa = \kappa(\zeta)$ is given by

$$\kappa := 4 \arctan\left(\frac{\zeta \sin \overline{\phi}}{1 + \zeta \cos \overline{\phi}}\right) - \overline{\phi} \tag{1.10}$$

with

$$\cos \overline{\phi} := \frac{1}{2\zeta} - \frac{3}{2}\zeta.$$

□

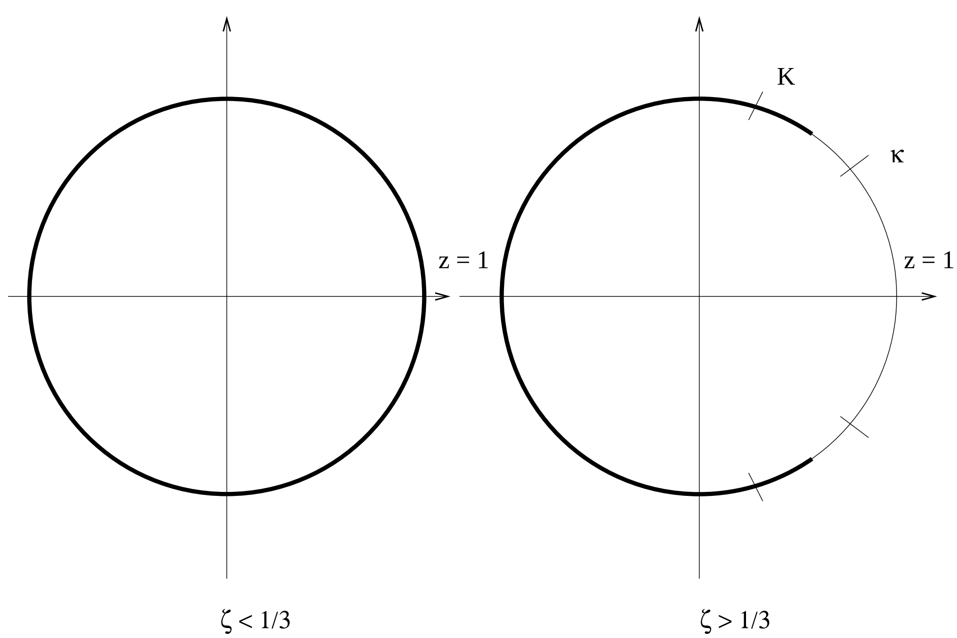

Figure 2: The limit set of singularities (bold arcs) in two regimes.

Remark 1.3 *The proof of Theorem 1.2 is based on a detailed description of the singularities of $\langle \sigma_0 \rangle_M (1)$ which are located, by Theorem 1.1, in the unit circle. These are isolated poles which* *accumulate in the arc A_K. The measure properties of this set will not be described in this work. Our estimates can be easily adapted to provide better values for K and ζ_l. We believe that ζ_l can be pushed up to 1. Note that $A_\kappa \backslash A_K \neq \emptyset$ for all $\zeta_c < \zeta < \zeta_l$ with $\lim_{\zeta \searrow \zeta_c} K(\zeta) = \lim_{\zeta \searrow \zeta_c} \kappa(\zeta) = 0$ (see Figure 2).*

Remark 1.4 *Due to the dilution, free boundary conditions were imposed. It is interesting to note that the circle theorem is no longer valid for boundary conditions corresponding to a different magnetic field (eventually infinite) acting on the spins at the boundary.*

Remark 1.5 *As discussed by Sütő, Theorem 1.2 immediately implies absence of metastability for our diluted model. We recall that our models distinguish the contribution of the finite clusters from the contribution of the infinite cluster. When $\mathbf{p} = 1$ the system reduces to the deterministic Ising ferromagnet on the Cayley tree which exhibits metastability as in the mean field theory.*

Recently, Dreifus, Klein and Perez [DKP] have established infinite differentiability of the quenched magnetization for general disordered spin systems using a modified high temperature expansion which converges in the presence of Griffiths' singularities. We here are able to study the right and left derivatives of m at $z = 1$ when spontaneous magnetization occurs and the high temperature expansion does not converge. In view of Theorems 1.1 and 1.2 it suffices to examine only the neighborhood of $z = 1$ in $z \in \mathbb{R}_+$ when $\zeta < \zeta_c$.

Theorem 1.6 (Continuity and Differentiability) *For any $0 \leq \zeta < \zeta_c$, $z \in \mathbb{R}_+$ and $\mathbf{p} \in \pi_0$, the quenched magnetization m given by (1.7) is always a continuous function of z with $m(1) = 0$. Moreover, F is at least n -times differentiable in $z \in \mathbb{R}_+$ for $0 < \zeta < \zeta_c$ and $\mathbf{p} \in \pi$, provided the sequence $\{a_j\, \mu^{nj}\}_{j=0}^{\infty}$ with $\mu = \mu(\zeta, z)$ given by*

$$\mu \;:=\; 4\zeta H z \;+\; 2\left[\left(\frac{4\zeta H z + 1}{4}\right)^2 + \zeta H\right]^{1/2}$$

is summable, i.e.,

$$\frac{1}{n!}\left|\frac{d^n F}{dz^n}\right| \leq C D^n \sum_{j=0}^{\infty} \mu^{nj}\, a_j.$$

for some finite constants $C = C(\zeta)$ and $D = D(\zeta)$. Here $H = H(\zeta) \leq 0.53/\zeta^2$.

In addition, for all $\mathbf{p} \in \pi_a$, with $a > 0$, m is a discontinuous function of z at $z = 1$ with

$$\lim_{\varepsilon \to 0}[m(1-\varepsilon) - m(1+\varepsilon)] \;=\; 2\,a\,\frac{(1-3\zeta)^{1/2}(1+\zeta)^{1/2}}{1-\zeta}.$$

□

Remark 1.7 *The summability condition on $\{a_j\, \mu^{jn}\}_{j=0}^{\infty}$ can be fulfilled in two different regimes:*

1. *When* $\lim_{z\to 1} m(z) = 0$*, the Bernoulli parameters* $\mathbf{p} \in \pi_0$*, can be chosen such that*

$$\frac{-1}{N}\sum_{j=1}^{N} \ln p_j > n \ln \mu + \rho$$

holds for all $N > N_0$*, for some* $N_0 = N_0(n, \mu) < \infty$ *and* $\rho > 0$*. Notice that, if* $p_j \sim e^{-j^{\varepsilon}}$ *with* $\varepsilon > 0$*, the magnetization* m *is infinitely differentiable.*

2. *For* $\mathbf{p} \in \pi_a$*, with* $a > 0$*, if*

$$p_j > 1 - \mu^{-nj}, \qquad j = 1,\ 2,\ \ldots$$

the right and left derivatives of m *at* $z = 1$ *exist up to order* n*. If* $p_j \sim 1 - e^{-j^{1+\varepsilon}}$*, with* $\varepsilon > 0$*, the right and left derivatives of all orders exist.*

Lee-Yang circle theorem (Theorem 1.1) is proven in Section 3, Griffiths theorem (Theorem 1.2) in Sections 4 and 5 and Theorem 1.6 will be proven in Section 6. It plays an important role in our proofs the recurrence relation

$$\Delta_j = \tau_z(\Delta_{j+1}), \qquad j \ = \ 0, 1, \ldots, M-1, \tag{1.11}$$

with $\Delta_M = 1$ and $z \in \mathbb{C}$, where, for any $\zeta \in [0,1]$ and $z\ ,\ y \in \mathbb{C}$,

$$\tau_z(y) \ \equiv h(zy) := \ \left(\frac{\zeta + zy}{1 + \zeta zy}\right)^2 .$$

Writing $\Delta_j \ = \ e^{-2\beta H_j}$, H_j is the effective magnetic field at generation j. Since the couplings are random, so are the quantities Δ_j's. Notice that Δ_j depends on the generation level M of the Cayley tree through its initial condition. The distribution of Δ_0, when $M \to \infty$, is worth studying by its own right. In the next theorem it will be described the analytic properties of the moments $\mathcal{M}_s = \mathbb{E}_{\xi}\left[\Delta_0(\xi)^s\right]$, $s \in \mathbb{N}$, of Δ_0 which can be written as

$$\mathcal{M}_s \ = \ \mathcal{F}_s \ + \ \mathcal{I}_s, \tag{1.12}$$

where $\mathcal{F}_s \ := \ \lim_{M\to\infty} \mathcal{F}_{s,\,M}$ with

$$\mathcal{F}_{s,M} \ := \ \sum_{n=0}^{M-1} a_n \ \left(\tau_z^{(n)}(1)\right)^s \tag{1.13}$$

and $\mathcal{I}_s \ := \ \lim_{M\to\infty} \mathcal{I}_{s,\,M}$ with

$$\mathcal{I}_{s,M} \ := \ p_1 \ldots p_M \ \left(\tau_z^{(M)}(1)\right)^s . \tag{1.14}$$

Theorem 1.8 (Moments) *For fixed* $s \in \mathbb{N}$*, let* $\mathcal{M}_s :\ \mathbb{C}\times[0,1]\times\pi \ni (z,\zeta,\mathbf{p}) \longmapsto \mathcal{M}_s(z,\zeta,\mathbf{p}) \in \mathbb{C}$ *be given by (1.12). Then* $\mathcal{M}_s$ *is an analytic function in* z *for* $|z| < 1$*. In addition, for* $\zeta \in [0,\ 1]$*,* $\mathcal{M}_s$ *is a holomorphic function of* z *in the open set* $\{z \in \mathbb{C};\ \ |z| > 1\} \setminus \overline{\mathcal{Z}}$*,* $\overline{\mathcal{Z}}$ *being the closure of*

$$\mathcal{Z} := \bigcup_{a\in\mathbb{N}} \mathcal{Z}_a$$

with $\mathcal{Z}_a$ *being the set of the* $2^{a+1}-1$ *solutions of the equation*

$$\underbrace{\tau_z \circ \cdots \circ \tau_z}_{a-\text{times}}(1) + \frac{1}{\zeta z} = 0.$$

Furthermore, $\mathcal{Z}_a$ *are self-conjugated sets (i.e.,* $\overline{z} \in \mathcal{Z}_a$ *if* $z \in \mathcal{Z}_a$*) satisfying* $\mathcal{Z}_a \cap [0,\infty) = \emptyset$ *and* $\mathcal{Z}^0 \equiv \overline{\mathcal{Z}} \setminus \mathcal{Z} \subset S^1$*. For* $\zeta \in [0,\ \zeta_0)$ *with* $\zeta_0 \in [0,1/3]$ *(*$\zeta_0 \simeq 0.29559$*)*[1] *we estimate*

$$\mathcal{Z}_a \subset \left\{ z \in \mathbb{C} : 1 < |z| < (1/\zeta)^{1/(a+1)} \right\}.$$

We conjecture that, for $\zeta \in [0,1/3]$*, the accumulation points* $\mathcal{Z}^0 = S^1$*.*

$\square$

This theorem will be proven in Section 7.

As a consequence of the recurrence relation (1.11), we are also able to compute the two-point function explicitly. The next result shows that the fluctuation on the Cayley tree is bounded by that of the one-dimensional lattice model.

Theorem 1.9 (Two-Point Function) *The truncated two-point function*

$$\langle \sigma_0;\ \sigma_x \rangle_N (\xi) := \langle \sigma_0\ \sigma_x \rangle_N (\xi) - \langle \sigma_0 \rangle_N (\xi)\ \ \langle \sigma_x \rangle_N (\xi)$$

can be written as

$$\langle \sigma_0;\ \sigma_x \rangle_N (\xi) = s(z\Delta_0)\ \prod_{j=1}^{n} t(\zeta_j, z\Delta_j) \tag{1.15}$$

where $n = n(x) =$*dist*$(0,\ x)$ *is the generation of the site* x*,*

$$s(x) := 1 - \left(\frac{1-x}{1+x} \right)^2$$

and

$$t(x) = t(\zeta, x) := \frac{\zeta^{-1} - \zeta}{x^{-1} + x + \zeta^{-1} + \zeta}.$$

For all $\zeta \in [0,1]$*,* $z \in \mathbb{R}_+$ *and* $\mathbf{p} \in \pi_a$ *,* $0 \le a \le 1$*, the following identity holds for the expected value* $\mathbb{E}_\xi \langle \sigma_0;\ \sigma_x \rangle (\xi) = \lim_{N\to\infty} \mathbb{E}_\xi \langle \sigma_0;\ \sigma_x \rangle_N (\xi)$*:*

$$\mathbb{E}_\xi \langle \sigma_0;\ \sigma_x \rangle (\xi) = \sum_{k=n}^{\infty} a_k\ s(w_k)\ \left[\prod_{j=1}^{n} t(w_{k-j}) \right] + a\ s(w)\ \ [t(w)]^n \tag{1.16}$$

where $w = w(\zeta, z)$ *is the limit point of the sequence*

$$w_n = z\, \underbrace{\tau_z \circ \cdots \circ \tau_z}_{n-\text{times}}(1), \qquad s \in \mathbb{N}.$$

[1] Concerning this restriction see the Remark 7.11.

□

Remark 1.10 *A simple upper bound on (1.16) for the paramagnetic phase can be obtained by using Perez's self-avoiding random walk estimate [P]. Only one term, corresponding to the single self-avoiding path connecting* 0 *to* x*, contributes to the correlation functions in* $\mathcal{C}_k$*. This fact shows that the Ising model on the Cayley tree behaves, on what concerns the asymptotic behavior of the correlations, as a one dimensional system. As a consequence, the correlation length is always finite, even at the transition point. (Note that* $|t(\zeta, x)| \leq 1/2$ *if* $\zeta \in [1/3, 1]$ *and* $x \in \mathbb{R}_+$*).*

Our last results shows that the quenched susceptibility at origin $\chi := \lim_{N\to\infty} \chi_N$ given by

$$\chi_N := \sum_{x \in \mathcal{C}_{2,N}} \mathbb{E}_\xi \langle \sigma_0;\ \sigma_x \rangle (\xi) \tag{1.17}$$

diverges as $|\zeta - \zeta_c|^\gamma$ at the critical point $\zeta_c = 1/3$ with an exponent given by the classical theory $\gamma = 1$. More precisely

Theorem 1.11 (Susceptibility) *The quenched susceptibility at origin* χ *is finite for all* $\zeta \in [0, 1]$*,* $z \in \mathbb{R}_+ \backslash \{1\}$ *and* $\mathbf{p} \in \pi$*. Moreover,* $\lim_{z\to 1} \chi$ *is also finite provided* $\zeta \neq \zeta_c$*.*

In addition, for $\mathbf{p} \in \pi_a$*,* $0 < a \leq 1$*, the following asymptotic behavior*

$$\lim_{z\to 1} \chi \sim C_\eta \, |\eta|^\gamma \qquad \text{as} \qquad \eta := \zeta - \zeta_c \to 0 \tag{1.18}$$

holds with $\gamma = 1$*,* $\lim_{\eta \searrow 0} C_\eta = 4a/3$ *and* $\lim_{\eta \nearrow 0} C_\eta = 2a/3$ *provided the condition*

$$\sum_{n=0}^{\infty} n \, a_n < \infty \tag{1.19}$$

is satisfied. Non classical exponent $\gamma = \gamma(\mathbf{p})$ *appears if (1.19) is violated.*

□

The proof of Theorems 1.9 and 1.11 will be given in Section 8.

2 Some Basic Results and Definitions

In this section we present some basic mathematical facts which we will use in the proofs of our main results. Some useful definitions are also introduced.

For $z \in \mathbb{C}$ consider the map $\tau_z : \mathbb{C} \mapsto \mathbb{C}$ given by $\mathbb{C} \ni u \to \tau_z(u)$ with

$$\tau_z(u) := h(zu),$$

where, for $w \in \mathbb{C}$,

$$h(w) := \left(\frac{\zeta + w}{1 + \zeta w} \right)^2 , \tag{2.1}$$

with $0 \leq \zeta \leq 1$.

Let us denote by $\tau_z^{(n)}$ the n-th composition of τ_z with itself, for $n \in \mathbb{N}$:

$$\tau_z^{(n)} := \underbrace{\tau_z \circ \cdots \circ \tau_z}_{n}, \tag{2.2}$$

with $\tau_z^{(0)}(u) = u$, $u \in \mathbb{C}$.

Let us start with some elementary remarks concerning the function h.

First, note that h has a unique double pole at $w = -1/\zeta$. Moreover, $h(w)$ is bounded on $\mathbb{C} \setminus \mathcal{O}$, where $\mathcal{O}$ is any open set containing the pole. For, note that

$$h(w) = \frac{1}{\zeta^2} \left(\frac{v}{w + 1/\zeta} + 1 \right)^2, \tag{2.3}$$

where $v := \zeta - 1/\zeta$. Hence, for $|w + 1/\zeta| > \epsilon > 0$, one has

$$|h(w)| < \frac{1}{\zeta^2} \left(\frac{|v|}{\epsilon} + 1 \right)^2. \tag{2.4}$$

This fact means also that h has a removable singularity at the complex infinity and could actually be considered as a map of the Riemann sphere S^2 into itself. As such a map h is holomorphic in $S^2 \setminus \{-1/\zeta\}$. Moreover, one has in S^2

$$h(w^{-1}) = h(w)^{-1}, \tag{2.5}$$

and

$$h(\overline{w}) = \overline{h(w)}. \tag{2.6}$$

Both relations together imply that h maps the unit circle $S^1 := \{w \in S^2 : \ |w| = 1\}$ into itself, since for $w \in S^1$,

$$\overline{h(w)} = h(\overline{w}) = h(w^{-1}) = h(w)^{-1}, \tag{2.7}$$

and so $\overline{h(w)}h(w) = 1$.

Let us denote by $D_{<1}$ the open unit disk in S^2 and by $D_{>1}$ the complement of its closure. More generally, for $a > 0$ call

$$D_{<a} := \{w \in S^2 : \ |w| < a\}, \tag{2.8}$$

$$D_{>a} := \{w \in S^2 : \ |w| > a\}. \tag{2.9}$$

For further purposes define also for $a, b \in \mathbb{R}_+$, $a < b$,

$$D_{a,\, b} := \{w \in S^2 : \ a < |w| < b\}. \tag{2.10}$$

The following theorem is very important for this work.

Theorem 2.1 *For the whole interval* $0 \leq \zeta < 1$, *the function* h *maps* $D_{<1}$ *into itself and* $D_{>1}$ *into itself.*

$\square$

Proof. By (2.5) it is enough to prove the claim for $D_{<1}$. For $0 \leq \zeta < 1$ the function h is analytic in $D_{<1}$. If $w \in S^1$, the boundary of $D_{<1}$, we have seen that $|h(w)| = 1$. Therefore, since h is not constant in $D_{<1}$, we conclude by the Maximum Modulus Theorem (see e.g. [T]) that $|h(w)| < 1$ for all $w \in D_{<1}$. $\blacksquare$

One can also trivially see that h maps $\mathbb{R}_+$ into itself. So, we have also established the following picture, which will play a central role in our discussions:

Corollary 2.2 *As a function of z, $\tau_z^{(n)}(1)$ maps each of the sets $D_{<1}$, S^1 and $D_{>1}$ into itself for all $n \in \mathbb{N}$ and all $\zeta \in [0,\ 1)$. Moreover, each of the sets $[0,\ 1)$ and $(1,\ \infty) \subset \mathbb{R}_+$ is also mapped into itself. Finally, for $z = 1$ one has $\tau_1^{(n)}(1) = 1$, for all $n \in \mathbb{N}$.*

$\square$

We will also make use of the following observation. The function h is the ratio of two polynomials in z of degree 2 and, hence, $\tau_z^{(n)}(1)$ is the ratio of two polynomials of degree $2^{n+1} - 2$. We will write

$$\tau_z^{(n)}(1) \quad = \quad \frac{P_n(z)}{Q_n(z)}, \tag{2.11}$$

where $P_n(z)$ and $Q_n(z)$ are polynomials in z of degree $2^{n+1} - 2$. To be more explicit, P_n and Q_n are defined recursively through

$$\begin{aligned} P_0(z) \quad &:= \quad 1, \\ Q_0(z) \quad &:= \quad 1, \\ P_{n+1}(z) \quad &:= \quad \left(\zeta Q_n(z) + z P_n(z)\right)^2, \qquad n \in \mathbb{N}, \\ Q_{n+1}(z) \quad &:= \quad \left(Q_n(z) + \zeta z P_n(z)\right)^2, \qquad n \in \mathbb{N}. \end{aligned} \tag{2.12}$$

3 Analyticity of m

This section is devoted to the proof of the analyticity of the quenched magnetization at the origin, as described in Theorem 1.1. We start by computing the partition function $Z_M(\xi)$ in a finite tree with M generations and $\xi \equiv 1$ (deterministic case).

Let $\widehat{Z}_j = (\widehat{Z}_j^+, \widehat{Z}_j^-)$, $j = 0,\ 1, \ldots, M$, be a sequence of two-component vectors defined recursively by

$$\begin{aligned} \widehat{Z}_{j-1}^{\sigma} \quad &:= \quad \left(\sum_{\sigma'=\pm 1} e^{\beta\sigma\sigma'} \ e^{\beta h \sigma'} \ \widehat{Z}_j^{\sigma'} \right)^2 \\ &= \quad (\zeta z)^{-1} \left(\zeta^{(1-\sigma)/2} \ \widehat{Z}_j^+ + \zeta^{(1+\sigma)/2} \ z \ \widehat{Z}_j^- \right)^2 \end{aligned} \tag{3.1}$$

with $\widehat{Z}_M^+ = \widehat{Z}_M^- = 1$.

If the spin configurations are summed starting from the branches towards the root, the partition function $Z_M(\xi)$ at $\xi \equiv 1$ can be written as

$$Z_M(1) = z^{-1/2}\ \widehat{Z}_0^+ + z^{1/2}\ \widehat{Z}_0^- \tag{3.2}$$

To compute the one point-function

$$\langle \sigma_0 \rangle_M (\xi) = \frac{1}{Z_M(\xi)} \sum_\sigma \sigma_0\ e^{-\beta H(\sigma;\,\xi)}, \tag{3.3}$$

we repeat the calculation leading to (3.2) for the numerator in (3.3). Except by the last summation on the spin at the root, all remaining ones give exactly the same expressions. We thus have

$$\langle \sigma_0 \rangle_M (1) = \frac{z^{-1/2}\ \widehat{Z}_0^+ - z^{1/2}\ \widehat{Z}_0^-}{z^{-1/2}\ \widehat{Z}_0^+ + z^{1/2}\ \widehat{Z}_0^-} \equiv \frac{1 - z\widehat{\Delta}_0}{1 + z\widehat{\Delta}_0} \tag{3.4}$$

where $\widehat{\Delta}_j = \widehat{Z}_j^- \big/ \widehat{Z}_j^+$ (the hat is to distinguish it from the random variable Δ_j defined below).

From (3.1), the sequence $\left\{\widehat{\Delta}_j\right\}_{j=0}^M$ satisfies the following recurrence relation:

$$\widehat{\Delta}_{j-1} = \tau_z(\widehat{\Delta}_j), \qquad j = 1, \ldots, M, \tag{3.5}$$

with $\widehat{\Delta}_M = 1$. We have thus proven the following proposition.

Proposition 3.1 *The one point function $\langle \sigma_0 \rangle_M (1)$, defined on a finite tree $\mathcal{C}_{2,M}$ with no disorder $(\xi \equiv 1)$, can be written as*

$$\langle \sigma_0 \rangle_M (1) = \frac{1 - z\ \tau_z^{(M)}(1)}{1 + z\ \tau_z^{(M)}(1)}. \tag{3.6}$$

$\square$

Remark 3.2 *The one point function (3.3) for a given $\xi \not\equiv 1$ can still be given by expression (3.6) provided $\widehat{\Delta}_j$, $j = 0, \ldots, M$, are replaced by random variables $\Delta_j = \Delta_j(\xi)$ defined recursively by*

$$\Delta_{j-1} = \tau_{j,z}(\Delta_j), \qquad j = 1, \ldots, M, \tag{3.7}$$

with $\Delta_M = 1$, where

$$\tau_{j,z}(u) = h_j(zu) \tag{3.8}$$

and h_j is given by (2.1) with ζ replaced by $\zeta_j = e^{-2\beta\xi_j}$. In the thermodynamic limit, $\langle \sigma_0 \rangle (\xi) = \lim\limits_{M\to\infty} \langle \sigma_0 \rangle_M (\xi)$ is thus governed by the distribution of $\lim\limits_{M\to\infty} \Delta_0$.

We will present the proof of Theorem 1.1 in the next two subsections. The first is dedicated to the proof of analyticity of F and the second to the proof of analyticity of I.

3.1 Analyticity of F in $S^2 \setminus S^1$

Using (3.6) we can write (1.8) as

$$F_N(z) := \sum_{n=0}^{N-1} a_n \ \frac{1 - z\tau_z^{(n)}(1)}{1 + z\tau_z^{(n)}(1)}. \tag{3.9}$$

Consider now any open set $\mathcal{B}$ such that $\overline{\mathcal{B}} \subset D_{<1}$. One has

$$\inf_{z\in\mathcal{B}} \left| z\tau_z^{(n)}(1) + 1 \right| > 1 - \sup\{|w|, \ \ w \in \mathcal{B}\} =: C^{-1} > 0, \tag{3.10}$$

since, by Corollary 2.2, $\tau_z^{(n)}(1) \in D_{<1}$ for all $z \in \mathcal{B}$ and all $n \in \mathbb{N}$. Moreover, under the same hypotheses $|1 - z\tau_z^{(n)}(1)| \leq 2$. Therefore, using the fact that $\{a_n\}_{n=0}^{\infty}$ is a summable sequence, we have

$$|F_M(z) - F_N(z)| \leq 2C \sum_{n=N}^{M-1} |a_n| < \epsilon, \tag{3.11}$$

for any prescribed $\epsilon > 0$, by choosing $M > N$, and N large enough. Hence, $F_N(s)$ is a uniform Cauchy sequence of analytic functions in $\mathcal{B}$ and, consequently, its limit exists and is analytic in $\mathcal{B}$.

Since $\mathcal{B}$ is generic, this shows that $F(z)$ is analytic in the whole set $D_{<1}$. To prove that $F(z)$ is also analytic in $D_{>1}$ we observe that, by (2.5), one has

$$\tau_z^{(n)}(1) = \left(\tau_{z^{-1}}^{(n)}(1)\right)^{-1} \tag{3.12}$$

and, hence,

$$F_N(z) = -F_N(1/z), \qquad N \in \mathbb{N}. \tag{3.13}$$

3.2 Analyticity of I in $S^2 \setminus S^1$

Let us consider the closed set $D_1 := D_{<1} \cup S^1 \subset \mathbb{C}$ and the map $\tau_z : D_1 \mapsto D_1$, for $z \in D_{<1}$ fixed.

Proposition 3.3 *The map τ_z is contractive on D_1 for $0 \leq \zeta \leq 1$ and $|z| < \rho_-(\zeta)$, where $\rho_-(\zeta) : [0,\ 1] \to [0,\ 1]$ is given by*

$$\rho_-(\zeta) := \frac{1 + \zeta - \zeta^2 - \sqrt{(1 + 2\zeta - \zeta^2)(1 - \zeta^2)}}{\zeta^2}. \tag{3.14}$$

Therefore, by the Banach contraction theorem, the sequences $\left\{\tau_z^{(n)}(w)\right\}_{n\in\mathbb{N}}$, with $w \in D_1$, converge to $\tau \in D_1$ which is the unique solution in D_1 of the fixed-point equation

$$\tau \ = \ \tau_z(\tau) \ = \ h(z\tau). \tag{3.15}$$

Moreover, for $|z| \leq \rho_0 < \rho_-(\zeta)$ and $|w|,\ |v| \leq 1$ one has

$$|h(zw) - h(zv)| \leq q_Q \ |w - v|, \tag{3.16}$$

with

$$q_Q := 2(1 - \zeta^2)\frac{\rho_0}{(1 - \zeta\rho_0)^2} < 1 \tag{3.17}$$

$\square$

Proof. We want to analyze $|\tau_z(w) - \tau_z(v)| = |h(zw) - h(zv)|$ for $w,\, v \in D_1$. A computation shows that

$$h(zw) - h(zv) = Q(z,\, w,\, v)\ (w - v), \tag{3.18}$$

where

$$Q(z,\, w,\, v) := \left[\frac{\zeta + zw}{1 + \zeta zw} + \frac{\zeta + zv}{1 + \zeta zv}\right]\ \frac{z(1-\zeta^2)}{(1+\zeta zw)(1+\zeta zv)}. \tag{3.19}$$

We prove that τ_z is contractive provided we find conditions which guarantee that $|Q(z,\, w,\, v)| < q$ for some constant q with $q < 1$.

Since $\left|\dfrac{\zeta + zw}{1+\zeta zw}\right| \leq 1$ for $|z|,\, |w| \leq 1$ one has

$$|Q(z,\, w,\, v)| \leq 2(1-\zeta^2)\frac{|z|}{(1-\zeta|z|)^2}. \tag{3.20}$$

Now we note that the condition

$$2(1-\zeta^2)\frac{|z|}{(1-\zeta|z|)^2} < 1 \tag{3.21}$$

means that

$$-\zeta^2|z|^2 + 2|z|(1+\zeta-\zeta^2) - 1 < 0. \tag{3.22}$$

The polynomial in the left hand side can be written as

$$-\zeta^2\left(|z| - \rho_-(\zeta)\right)\left(|z| - \rho_+(\zeta)\right), \tag{3.23}$$

where

$$\rho_\pm(\zeta) := \frac{1+\zeta-\zeta^2 \pm \sqrt{(1+2\zeta-\zeta^2)(1-\zeta^2)}}{\zeta^2}. \tag{3.24}$$

Note that $1 + 2\zeta - \zeta^2 = (\sqrt{2} + 1 - \zeta)(\zeta + \sqrt{2} - 1) \geq 0$ for $\zeta \in [0,\, 1]$.

We show now that, for $\zeta \in (0,\, 1)$, one has $\rho_+(\zeta) > 1$ and $\rho_-(\zeta) < 1$. If this is the case, then expression (3.23) is negative, under the hypotheses. These conditions mean

$$1 + \zeta - 2\zeta^2 \;>\; -\sqrt{(1+2\zeta-\zeta^2)(1-\zeta^2)}, \quad \text{and} \tag{3.25}$$

$$1 + \zeta - 2\zeta^2 \;<\; \sqrt{(1+2\zeta-\zeta^2)(1-\zeta^2)}, \tag{3.26}$$

respectively. Now, these last two conditions together are equivalent to the condition

$$(1+\zeta-2\zeta^2)^2 - (1+2\zeta-\zeta^2)(1-\zeta^2) < 0. \tag{3.27}$$

The left hand side equals $-\zeta^2(1-\zeta)(1+3\zeta)$ which is manifestly negative for $0 < \zeta < 1$.

We have established that, under the hypotheses, $|\tau_z(w) - \tau_z(v)| < q\,|w - v|$ for some fixed q with $0 < q < 1$ what means that τ_z is contractive.

Since the right hand side of (3.20) is an increasing function of $|z|$ for $0 \le |z| < 1/\zeta$, we have

$$|Q(z,\, w,\, v)| \le q_Q := 2(1-\zeta^2)\frac{\rho_0}{(1-\zeta\rho_0)^2} < 1 \tag{3.28}$$

for $0 \le |z| < \rho_0 < \rho_-(\zeta)$. This completes the proof of the proposition.

■

Remark 3.4 *One has to show that $\rho_-(\zeta) > 0$, what means that the region of validity for z of the last proposition is in fact non-empty. Since*

$$1 + \zeta - \zeta^2 = \left(\frac{1+\sqrt{5}}{2} - \zeta\right)\left(\frac{\sqrt{5}-1}{2} + \zeta\right) \ge 0 \tag{3.29}$$

for $\zeta \in [0,\, 1]$, it is enough to show that

$$(1+\zeta-\zeta^2)^2 - (1+2\zeta-\zeta^2)(1-\zeta^2) \ge 0. \tag{3.30}$$

But the left hand size equals ζ^2 and so, positivity of $\rho_-(\zeta)$ is proven. Actually, one finds numerically that $\rho_-(\zeta) > \rho_0 \simeq 0.41$ for all $\zeta \in [0,\, 1]$.

Finally we notice that $\lim_{\zeta\to 0} \rho_-(\zeta) = 1/2$ and that $\rho_-(1) = 1$. This last fact means that the region $\{z \in \mathbb{C} :\ |z| < \rho_-(\zeta)\}$ converges to $D_{<1}$ when $\zeta \to 1$.

We are now in position to establish the following important result:

Theorem 3.5 *The sequence $\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, of analytic functions on $D_{<1}$ converges uniformly to an analytic function $\tau = \tau(z)$ on the whole set $D_{<1}$.*

□

Proof. The sequence $\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, is a sequence of analytic functions of $z \in D_{<1}$ which is uniformly bounded in this domain, since $|\tau_z^{(n)}(1)| < 1$. Proposition 3.3 says that this sequence converges on the open subset $D_{\rho_-} \subset D_{<1}$. Now, according to Vitali's Convergence Theorem (see, e.g. [T]), these facts guarantee that the sequence converges uniformly on the whole $D_{<1}$ to a function $\tau(z)$, analytic on $D_{<1}$.

■

Theorem 3.6 *For all $\zeta \in (0,\, 1]$ the function $\tau = \tau(z)$ fulfills the fixed point equation $\tau = h(z\tau)$ on the whole set $D_{<1}$. As a consequence, $\tau(z)$ has no zeros on $D_{<1}$.*

□

Proof. The function $G(z) := \tau(z) - h(z\tau(z))$ is analytic on $D_{<1}$. According with Proposition 3.3, the function $\tau = \tau(z)$ fulfills the fixed point equation $\tau = h(z\tau)$ on D_{ρ_-}. Hence $G(z) = 0$ on D_{ρ_-} and, consequently, $G(z)$ is identically zero on the whole set $D_{<1}$. Since $h(0) \ne 0$, $\tau(z)$ cannot have a zero in $D_{<1}$.

■

Remark 3.7 *The fixed point equation $\tau = h(z\tau)$ is equivalent to the third degree equation*

$$\tau\,(1+\zeta\,z\,\tau)^2 \;-\; (\zeta+z\,\tau)^2 \;=\; 0. \tag{3.31}$$

Solving this equation and identifying the adequate roots would furnish the explicit expression of τ as a function of $z \in D_{<1}$. Unfortunately this explicit form is too complicated to be useful.

The bound $|\tau(z)| \leq 1$ holds on $D_{<1}$. Therefore,

$$I^{<}(z) \;:=\; \lim_{N\to\infty} I_N(z) \quad = \quad a\,\frac{1-z\,\tau(z)}{1+z\,\tau(z)} \tag{3.32}$$

is analytic on $D_{<1}$. Since $I_N(z) = -I_N(z^{-1})$ for any $N \in \mathbb{N}$ we conclude that, for $z \in D_{>1}$, we have

$$I^{>}(z) \;:=\; \lim_{N\to\infty} I_N(z) \;=\; -I^{<}(z^{-1}) \tag{3.33}$$

which is analytic on $D_{>1}$.

This completes the proof of Theorem 1.1.

■

According to Theorem 1.1, the singularities of $m(z)$, if they exist, must be localized in the unit circle S^1. In particular, the partial sums $F_N(z)$ may have singularities at points z for which $z\tau_z^{(n)}(1)+1=0$ for some $0 \leq n \leq N$. As we will next see, these singularities exist indeed and are poles. We will call these singularities the Lee-Yang singularities of the magnetization at the origin $m(z)$.

4 The Lee-Yang Singularities of m

For $n \in \mathbb{N}$ we are interested in solutions of the following equation

$$z\;\tau_z^{(n)}(1) \quad = \quad -1. \tag{4.1}$$

Since $z\tau_z^{(n)}(1)$, as a function of z, maps both sets $D_{<1}$ and $D_{>1}$ into itself, the solutions of (4.1), if they exist, must lie on $S^1 := \{z \in \mathbb{C};\ \ |z| = 1\}$. To show the existence of solutions we observe that equation (4.1) is equivalent to the polynomial equation

$$z\,P_n(z) + Q_n(z) = 0, \tag{4.2}$$

which has at least 1 and at most $2^{n+1}-1$ distinct solutions in $\mathbb{C}$. By the previous argument these solutions must all lie in $S^1 \subset \mathbb{C}$. We will establish below that the number of distinct solutions is precisely $2^{n+1}-1$.

Let us denote by $\mathfrak{S}_n \subset S^1$ the set of all solutions of (4.1) for a given $n \in \mathbb{N}$. Since $\overline{\tau_z^{(n)}(1)} = \tau_{\overline{z}}^{(n)}(1)$ we conclude that if $z_0 \in \mathfrak{S}_n$ then $\overline{z_0} \in \mathfrak{S}_n$ as well. Since $\tau_{-1}^{(n)}(1) = 1$, it is also clear that $-1 \in \mathfrak{S}_n$.

Writing $z = e^{i\phi}$, $0 \leq \phi \leq 2\pi$ for the elements of S^1, let us enumerate the elements of $\mathfrak{S}_n$ in increasing order according to the angle ϕ. We write

$$\mathfrak{S}_n \; = \; \left\{ z(n,\, l) = e^{i\phi(n,\, l)} \in S^1 \text{ with } 0 \leq \phi(n,\, 1) \leq \cdots \leq \phi(n,\, 2^{n+1}-1) \leq 2\pi \right\},$$

where

$$z(n,\, l)\; \tau^{(n)}_{z(n,\, l)}(1) \; = \; -1.$$

Our previous remarks say that

$$\phi(n,\, 2^n) = \pi$$

and

$$\phi(n,\, a) = 2\pi - \phi(n,\, 2^{n+1}-1-a),$$

for all $a \, = \, 1, \, \ldots, \, 2^n$.

Let us denote by $\mathfrak{R}_n \subset [0,\, \pi]$ the set

$$\mathfrak{R}_n := \{\phi(n,\, l) \in [0,\, \pi], \quad 1 \leq l \leq 2^n\}.$$

In the next two subsections we will establish the following theorem:

Theorem 4.1 *The set $\mathfrak{R}_n$ becomes a dense set on $[0,\, \pi]$ when $n \to \infty$ for $0 \leq \zeta \leq 1/3$.*

$\square$

To start with the analysis, let us look in more detail at the sequence $z\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, for $z \in S^1$. Firstly, consider the function $h(w)$ for $w \in S^1$. Since $1/w = \overline{w}$ we can write

$$h(w) = w^2 \left(\frac{1+\zeta\overline{w}}{1+\zeta w} \right)^2 .$$

Defining $w =: e^{i\theta}$ and $h(w) =: e^{i\theta'}$, it is easy to verify the relation

$$\theta' \; = \; 2\theta - 4 \arctan \left(\frac{\zeta \sin\theta}{1+\zeta\cos\theta} \right).$$

Hence, defining ϕ through $z =: e^{i\phi}$ and ϕ_j through $z\,\tau_z^{(j)}(1) =: e^{i\phi_j}$, $j \in \mathbb{N}$, and taking into account that $z\,\tau_z^{(j+1)}(1) = z\,h(z\,\tau_z^{(j)}(1))$, we have

$$\begin{aligned} \phi_0 \quad &:= \quad \phi, && \phi \in [0, \pi], \\ \phi_{i+1} \quad &:= \quad \phi_0 + L(\phi_i) && i \in \mathbb{N}, \end{aligned} \tag{4.3}$$

where

$$L(x) := 2x - 4 \arctan \left(\frac{\zeta \sin\, x}{1+\zeta\cos\, x} \right), \tag{4.4}$$

$x \in \mathbb{R}$ and $\zeta \in [0,\, 1)$. These equations will be treated as a discrete dynamical system on $\mathbb{R}$.

4.1 Properties of the Discrete Dynamical System

Proposition 4.2 *For all $x \in \mathbb{R}$ and $\zeta \in [0,\ 1)$, L is a continuous monotone increasing function of x such that*

$$L'(x) \geq L'(0) \equiv \lambda(\zeta), \tag{4.5}$$

where

$$\lambda(\zeta) \ := \ 2\left(\frac{1-\zeta}{1+\zeta}\right). \tag{4.6}$$

Moreover, the function L has the following properties:

(i) $L(x+2\pi) = 4\pi + L(x)$, $x \in \mathbb{R}$,

(ii) $L''(x) \geq 0$, *for* $x \in [0,\ \pi]$,

(iii) $L(x) = 4\pi - L(2\pi - x)$, *for* $x \in [0,\ 2\pi]$,

(iv) $\lim\limits_{\zeta \uparrow 1} L(x) = \begin{cases} 0 & , \quad 0 \leq x < \pi \\ 2\pi & , \quad x = \pi \\ 4\pi & , \quad \pi < x \leq 2\pi \end{cases}$.

□

Proof. By definition

$$L'(x) = 2\left(\frac{1-\zeta^2}{1+2\zeta\cos x + \zeta^2}\right) \geq \lambda(\zeta). \tag{4.7}$$

This relation says that L is a strictly positive, strictly increasing function in $\mathbb{R}_+ \setminus \{0\}$.

We also have

$$L''(x) = \frac{4\zeta(1-\zeta^2)\sin x}{(1+2\zeta\cos x+\zeta^2)^2} \geq 0$$

for $0 \leq x < \pi$. Note that, for $\zeta = 1$ and $0 \leq x < \pi$, L satisfies the first order differential equation $L'(x) = 0$ with initial data $L(0) = 0$. This shows item (**iv**) since $L(\pi) = 2\pi$ for any ζ. For (**iii**), we use

$$\arctan\left(\frac{\zeta\sin(2\pi - x)}{1+\zeta\cos(2\pi - x)}\right) = -\arctan\left(\frac{\zeta\sin x}{1+\zeta\cos x}\right) .$$

Item (**i**) is clear.

■

Corollary 4.3 *For $x \in \mathbb{R}_+$ and $\zeta \in [0,\ 1]$ one has the simple bound*

$$L(x) \ \geq \ \lambda(\zeta)\ x. \tag{4.8}$$

□

In view of Proposition 4.2 and since $\lambda(\zeta) > 1$ for $0 \le \zeta < 1/3$, the Liapunov exponent of the discrete dynamical system (4.3) is positive in this range of ζ. We have also the following

Lemma 4.4 *If $\phi_0 \in \mathbb{R}_+$ then $\phi_i \in \mathbb{R}_+$ for all $i \in \mathbb{N}$, $i \ge 0$.*

□

Proof. Let us assume, by an induction argument, that $\phi_i \in \mathbb{R}_+$. By Corollary 4.3 one has

$$\phi_{i+1} \ge \phi_0 + \lambda(\zeta)\ \phi_i \ge 0, \tag{4.9}$$

by the hypothesis. Since this holds for all $i \in \mathbb{N}$ the lemma is proven.

■

From now on we will consider only starting points in $\mathbb{R}_+$, what reduces the dynamical system to a dynamical system on $\mathbb{R}_+$.

Lemma 4.5 *Let ϕ^2 and ϕ^1 two starting points with $\phi^2 > \phi^1 \ge 0$. Denote by ϕ_i^a, with $i \in \mathbb{N}$ and $a = 1, 2$ the sequences generated by the dynamical system having ϕ^a as starting point. Then for all $i \in \mathbb{N}$ one has $\phi_i^2 > \phi_i^1 \ge 0$.*

□

Proof. Assume, by an induction argument, that $\phi_i^2 > \phi_i^1$. Then

$$\phi_{i+1}^2 - \phi_{i+1}^1 = (\phi^2 - \phi^1) + L(\phi_i^2) - L(\phi_i^1). \tag{4.10}$$

For $y \ge x \ge 0$ we can write

$$L(y) - L(x) = \int_x^y\ L'(s)\ ds \ge \lambda(\zeta)\ (y - x), \tag{4.11}$$

by Proposition 4.2. Therefore

$$\phi_{i+1}^2 - \phi_{i+1}^1 \ge (\phi^2 - \phi^1) + \lambda(\zeta)(\phi_i^2 - \phi_i^1) > 0, \tag{4.12}$$

by the hypothesis.

■

Let us denote by $\phi_i(\phi)$ the i-th iterated of the discrete dynamical system having ϕ as starting point. Note that $\phi_i(\phi)$ is a continuous function of the starting point ϕ, since it is a finite composition of continuous functions. This fact and the previous lemma immediately imply:

Lemma 4.6 *For all $i \in \mathbb{N}$, $\phi_i(\phi)$ is a strictly increasing continuous function on $\mathbb{R}_+$.*

$\square$

This lemma has an important consequence:

Corollary 4.7 *The sets $\mathfrak{R}_n$, $n \in \mathbb{N}$, are composed by 2^n distinct elements, what in particular says that equation (4.1) has, for each $n \in \mathbb{N}$, precisely $2^{n+1}-1$ distinct solutions in S^1. Each set $\mathfrak{R}_n$ can be ordered such that $0 \leq \phi(n,\ l) < \phi(n,\ l') \leq \pi$ for $l < l'$ and the elements $\phi(n,\ l)$ are such that*

$$\phi_n(\phi(n,\ l)) = \pi + 2\pi(l-1) \tag{4.13}$$

for $1 \leq l \leq 2^n$. All these facts hold for all $\zeta \in [0,\ 1)$.

$\square$

Proof. The sets $\mathfrak{R}_n$, $n \in \mathbb{N}$, defined above can also be characterized as the sets of all $\phi(n,\ l) \in [0,\ \pi]$, $1 \leq l \leq 2^n$, for which $\phi_n(\phi(n,\ l)) = \pi + 2\pi n_l$ for some $n_l \in \mathbb{N}$, depending on l. To see that $n_l = l-1$ we note that, due to Lemma 4.6, $\phi_n(\phi)$ is a strictly increasing continuous function of ϕ with $\phi_n(0) = 0$ and $\phi_n(\pi) = (2^{n+1}-1)\pi$. Therefore, $\phi_n :\ [0,\ \pi] \to [0,\ (2^{n+1}-1)\pi]$ is an invertible map and we can define

$$\phi(n,\ l) := \phi_n^{-1}(\pi + 2\pi(l-1)) \tag{4.14}$$

for $1 \leq l \leq 2^n$. Clearly $\phi(n,\ l) < \phi(n,\ l')$ if $l < l'$.

$\blacksquare$

4.2 The Sets $\mathfrak{R}_n$ in the Ferromagnetic Region $(0 \leq \zeta \leq 1/3)$

In this subsection we will show that, in the ferromagnetic region $(0 \leq \zeta \leq 1/3)$, the Lee-Yang singularities become dense in the whole unit circle S^1 when the thermodynamic limit is taken.

The next two theorems are crucial for understanding the behavior of the Lee-Yang singularities when n goes to infinity.

Theorem 4.8 *With the above definitions we have*

$$\lim_{n\to\infty} (\phi(n,\ l+1) - \phi(n,\ l)) = 0 \tag{4.15}$$

uniformly in l, for all $0 \leq \zeta \leq 1/3$.

$\square$

Proof. Call $\phi_i(n,\ l) := \phi_i(\phi(n,\ l))$, the i -th iteration started from the point $\phi(n,\ l)$. We have for $0 \leq i \leq n-1$

$$\phi_{i+1}(n,\ l+1) - \phi_{i+1}(n,\ l) \quad = \quad (\phi(n,\ l+1) - \phi(n,\ l)) + L(\phi_i(n,\ l+1)) - L(\phi_i(n,\ l)). \tag{4.16}$$

Since $\phi_i(n,\ l+1) \geq \phi_i(n,\ l)$, we can write

$$L(\phi_i(n,\ l+1)) - L(\phi_i(n,\ l)) \geq \lambda(\zeta)(\phi_i(n,\ l+1) - \phi_i(n,\ l)), \tag{4.17}$$

by Lemma 4.5 and (4.11). Hence

$$\phi_{i+1}(n,\ l+1) - \phi_{i+1}(n,\ l) \geq (\phi(n,\ l+1) - \phi(n,\ l)) + \lambda(\zeta)\,(\phi_i(n,\ l+1) - \phi_i(n,\ l))\,. \tag{4.18}$$

Iterating this inequality we obtain

$$\phi_{i+1}(n,\ l+1) - \phi_{i+1}(n,\ l) \geq \left(\sum_{a=0}^{i} \lambda(\zeta)^a\right)(\phi(n,\ l+1) - \phi(n,\ l)). \tag{4.19}$$

Since $\phi_n(n,\ l+1) - \phi_n(n,\ l) = 2\pi$, we get

$$\phi(n,\ l+1) - \phi(n,\ l) \leq \frac{2\pi}{\sum\limits_{a=0}^{n-1} \lambda(\zeta)^a}. \tag{4.20}$$

If $\lambda(\zeta) \geq 1$, what happens for $0 \leq \zeta \leq 1/3$, we have

$$\lim_{n\to\infty} (\phi(n,\ l+1) - \phi(n,\ l)) = 0, \tag{4.21}$$

uniformly in l.

■

We can also prove the following theorem:

Theorem 4.9 *With the above definitions we have*

$$\lim_{n\to\infty} \phi(n,\ 1) = 0 \tag{4.22}$$

for all $0 \leq \zeta \leq 1/3$.

□

Proof. Since

$$\phi_{i+1} = \phi(n,\ 1) + L(\phi_i) \tag{4.23}$$

with $0 \leq i \leq n-1$, we have, using Corollary 4.3, that

$$\phi_{i+1} \geq \phi(n,\ 1) + \lambda(\zeta)\phi_i, \tag{4.24}$$

from which we conclude that

$$\phi_{i+1} \geq \left(\sum_{a=0}^{i} \lambda(\zeta)^a\right) \phi(n,\ 1). \tag{4.25}$$

Hence, taking $i = n - 1$

$$\phi(n,\ 1) \le \frac{\pi}{\sum_{a=0}^{n-1} \lambda(\zeta)^a} \tag{4.26}$$

from which the theorem follows.

■

Both Theorems 4.8 and 4.9 together with the fact that $\pi \in \mathfrak{R}_n$ for all $n \in \mathbb{N}$ imply Theorem 4.1.

■

Note that, by (4.26) we conclude that

$$\limsup_{n\to\infty}\ \phi(n,\ 1)\ \le\ \pi\,\frac{3\zeta - 1}{1+\zeta}. \tag{4.27}$$

for $1/3 < \zeta \le 1$.

Remark 4.10 *Notice that both Theorems 4.8 and 4.9 hold also at the critical point $\zeta = 1/3$.*

4.3 The Sets $\mathfrak{R}_n$ in the Paramagnetic Region $(1/3 < \zeta \le 1)$

In Section 5 we will see that, for each value of ζ in the paramagnetic region $(1/3 < \zeta \le 1)$, the sets $\mathfrak{S}_n$ are excluded from a neighborhood of $z = 1$. However, one should expect that, in analogy to the situation in the ferromagnetic region, the sets $\mathfrak{S}_n$ become dense in some proper subset of S^1 when the limit $n \to \infty$ is taken.

In this subsection we will prove a weaker statement, namely that there exists a proper subset A_κ of S^1 such that $\mathfrak{S}_n \bigcap A_\kappa$ becomes dense on A_κ when $n \to \infty$. Let us start capturing the main ingredients for the proof, as we learned from the last subsection.

The proofs presented in the previous subsection show clearly that the phenomenon of the Lee-Yang singularities becoming dense is closely related to the positivity of the Liapunov exponents of our discrete dynamical system. In order to obtain analogous results for the region $1/3 < \zeta \le 1$ one has to take into account that, for these values of ζ, the derivative $L'(x)$ is not larger than 1 for all points $x \in [0,\ 2\pi]$. However, if the trajectory of the dynamical system visits frequently enough regions where L' is large, a positive lower bound for the Liapunov exponent can be obtained with few iterations. This can be the case if, for instance, the points of the trajectory successively jump from a region where $L' < 1$ to other where $L' > 1$, large enough. Below we will follow this sort of idea in order to advance into the paramagnetic region $1/3 < \zeta \le 1$.

Consider the sets $\mathfrak{R}_n$ for $1/3 < \zeta \le 1$. For our discrete dynamical system we have

$$\phi_{i+2}\ =\ \phi_0 + L(\phi_0 + L(\phi_i)), \qquad i \ge 1. \tag{4.28}$$

Define, for $\phi \in [0,\ \pi]$ and $x \in \mathbb{R}_+$,

$$M(\phi,\ x) := L(\phi + L(x)). \tag{4.29}$$

Since

$$M_x(\phi,\ x) \ = \ L'(\phi + L(x))\, L'(x), \tag{4.30}$$

one has $M_x(\phi,\ x + 2\pi) = M_x(\phi,\ x)$, $x \in \mathbb{R}$.

Assume that, for some convenient set $[\kappa^0,\ \pi] \subset [0,\ \pi]$, $0 \le \kappa^0 < \pi$, one has

$$M^0 \ := \ \inf_{\phi\in[\kappa^0,\ \pi]}\ \inf_{x\in\mathbb{R}}\ M_x(\phi,\ x) \ \ge \ 1. \tag{4.31}$$

This basic assumption will be proven at the end of this subsection. Let us first draw some of its general consequences.

For $\kappa^0 \le a < b \le \pi$ and $0 \le x < y$ one has

$$\begin{aligned} M(b,\ y) - M(a,\ x) \ &= \ (M(b,\ y) - M(b,\ x)) + (M(b,\ x) - M(a,\ x)) \\ &= \ \int_x^y \ M_x(b,\ s)\, ds + \int_a^b \ M_\phi(\mu,\ x)\, d\mu. \end{aligned} \tag{4.32}$$

But $M_\phi(\mu,\ x) = L'(\mu + L(x)) \ge 0$ and so

$$M(b,\ y) - M(a,\ x) \ge \int_x^y \ M_x(b,\ s)\, ds \ \ge M^0\ (y - x) \tag{4.33}$$

with M^0 given by (4.31).

Under the assumption (4.31) and using (4.33) we have, as in (4.16),

$$\begin{aligned} \phi_{i+2}(n,\ l+1) - \phi_{i+2}(n,\ l) \ &= \ (\phi(n,\ l+1) - \phi(n,\ l)) + \\ &\quad\ M(\phi(n,\ l+1),\ \phi_i(n,\ l+1)) - M(\phi(n,\ l),\ \phi_i(n,\ l)) \\ &\ge \ (\phi(n,\ l+1) - \phi(n,\ l)) + M^0(\phi_i(n,\ l+1) - \phi_i(n,\ l)) \end{aligned} \tag{4.34}$$

for $1 \le i \le n-2$, provided $\phi(n,\ l)$, $\phi(n,\ l+1) \in \mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$.

Iterating the inequality (4.34) with i even, we get

$$\phi_{i+2}(n,\ l+1) - \phi_{i+2}(n,\ l) \ge \left[\sum_{a=0}^{i/2} (M^0)^a\right] (\phi(n,\ l+1) - \phi(n,\ l)), \tag{4.35}$$

which gives

$$\phi(n,\ l+1) - \phi(n,\ l) \ \le \ \frac{2\pi}{\displaystyle\sum_{a=0}^{n/2} (M^0)^a} \tag{4.36}$$

for n even and

$$\phi(n,\ l+1) - \phi(n,\ l) \ \le \ \left(\frac{3-\zeta}{1+\zeta}\right) \frac{2\pi}{\displaystyle\sum_{a=0}^{(n-1)/2} \ (M^0)^a} \tag{4.37}$$

for n odd.

Since, by the hypotheses, the right hand side goes to zero when $n \to \infty$ uniformly in l, we have proven that, under the circumstances that provide (4.31), the set $\mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$ becomes dense on itself when $n \to \infty$. To establish that $\mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$ becomes indeed dense over the whole interval $[\kappa^0,\ \pi]$ we will also need the following result:

Theorem 4.11 *Let $l^0 = l^0(n)$ be the label of the first singularity $\phi(n,\, l^0)$ in $\mathfrak{R}_n$ which exceeds κ^0, i.e.,*

$$\phi(n,\, l^0) = \min_{\phi}\left\{\phi \in \mathfrak{R}_n : \phi > \kappa^0\right\} \ .$$

Then,

$$\lim_{n\to\infty} \phi(n,\, l^0) = \kappa^0 \ .$$

□

Proof. Let us consider the dynamical system (4.28) with ϕ_0 replaced by κ^0:

$$\kappa_{i+2} = \kappa^0 + M(\kappa^0, \kappa_i) \ .$$

As in (4.34), we write

$$\begin{aligned} \phi_{i+2}(n,\, l^0) - \kappa_{i+2} \ &= \ \phi(n,\, l^0) - \kappa^0 + M\left(\phi(n,\, l^0),\ \phi_i(n,\, l^0)\right) - M(\kappa^0, \kappa_i) \\ &\geq \ \phi(n,\, l^0) - \kappa^0 + M^0\ \left(\phi_i(n,\, l^0) - \kappa_i\right). \end{aligned} \tag{4.38}$$

The inequality in (4.38) follows from (4.33).

Repeating the steps (4.35)-(4.37) we conclude the proof of Theorem 4.11. Recall only that $\phi(n,\, l^0) > \kappa^0 \geq \phi(n,\, l^0 - 1)$ implies $\phi_j(n,\, l^0) > \kappa_j \geq \phi_j(n,\, l^0-1)$ which gives $\phi_n(n,\, l^0) - \kappa_n \leq 2\pi$.

■

We shall next establish that $\mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$ becomes indeed dense over the whole interval $[\kappa^0,\ \pi]$ under the above assumptions when $n \to \infty$. After the above results, we only need to show that the sets $\mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$ contain other elements than π for the considered values of ζ and for n large enough.

Since we already know that, for $\zeta = 1/3$, $\mathfrak{R}_n$ becomes dense on S^1 when n goes to infinite, we can argue that, for $\zeta \equiv \zeta_1$ larger but close enough to $1/3$, there is a fixed n_0 large enough such that $\mathfrak{R}_{n_0} \bigcap [\kappa^0,\ \pi] \setminus \{\pi\} \neq \emptyset$. This is because the elements of $\mathfrak{R}_{n_0}$ depend continuously on ζ. We have seen that $\mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$ becomes dense on itself for $n \to \infty$ and by Theorem 4.11, we conclude that for $\zeta \equiv \zeta_1$ the set $\mathfrak{R}_n \bigcap [\kappa^0,\ \pi]$ becomes dense on $[\kappa^0,\ \pi]$ when $n \to \infty$. This sort of argument can, of course, be repeated for larger values of ζ covering the whole interval $1/3 < \zeta < \zeta_l$.

It remains now only to prove that the basic assumption (4.31) indeed holds for suitable values of ζ. For this a careful analysis of our discrete dynamical system is necessary.

Let us introduce the following sets

$$\begin{aligned} \mathcal{B} \ &:= \ \{x \in \mathbb{R} : \lambda(\zeta)\ L'(x) > 1\} \\ \mathcal{R} \ &:= \ \mathbb{R} \setminus \mathcal{B} \end{aligned}$$

where $\lambda(\zeta)$ was defined in (4.6). By (4.5) these definitions say that $L'(x)L'(y) > 1$ for any $y \in \mathbb{R}$ provided $x \in \mathcal{B}$.

Let us first show that the set $\mathcal{B}$ is non-empty if $\zeta \in (1/3,\ 1]$. A simple computation shows that

$$\mathcal{B} = \left\{ x \in \mathbb{R} : \quad \cos x < \frac{3\zeta^2 - 8\zeta + 3}{2\zeta} \right\} =: (\gamma,\ 2\pi - \gamma) + 2\pi\mathbb{Z}, \tag{4.39}$$

where $\gamma = \gamma(\zeta)$ is the solution in $[0,\ \pi]$ of the equation

$$\cos\gamma := \begin{cases} 1, & \text{if } 0 \le \zeta \le 1/3, \\ \dfrac{3\zeta^2 - 8\zeta + 3}{2\zeta}, & \text{if } 1/3 < \zeta \le 1 \end{cases} . \tag{4.40}$$

Now, notice that

$$-1 \le \frac{3\zeta^2 - 8\zeta + 3}{2\zeta} \le 1, \tag{4.41}$$

provided $1/3 \le \zeta \le 1$, as one easily checks. This says that $\mathcal{B}$ is a non-empty set for $1/3 \le \zeta \le 1$. One checks easily too that, as a function of ζ, $\dfrac{3\zeta^2 - 8\zeta + 3}{2\zeta}$ is a continuous and strictly decreasing function for $0 \le \zeta \le 1$.

From (4.30) and (4.5) we conclude that, for $x \in \mathcal{B}$, one has $M_x(\phi,\ x) > 1$ for any $\phi \in [0,\ \pi]$.

What happens if $x \in \mathcal{R}$? We will next show that for $x \in \mathcal{R}$ one has $\phi + L(x) \in \mathcal{B}$, provided $1/3 < \zeta < \zeta_l$, where $\zeta_l \simeq 0.46409$, and ϕ is large enough. Therefore, under these circumstances we also have the bound $M_x(\phi,\ x) > 1$. We express these results in the following theorem:

Theorem 4.12 *There exist a constant ζ_l, $1/3 < \zeta_l < 1$, $\zeta_l \simeq 0.46409$, and a continuous strictly increasing function $K(\zeta)$ on the interval $[1/3,\ \zeta_l]$ with $K(1/3) = 0$, $K(\zeta_l) = \pi$, such that, for $\zeta \in [1/3,\ \zeta_l)$ and $\phi \in (K(\zeta),\ \pi]$, one has*

$$M_x(\phi,\ x) > 1 \tag{4.42}$$

for any $x \in [0,\ 2\pi]$.

□

This theorem says that we can adopt $\kappa^0 = K(\zeta)$ in our previous analysis.

Proof of Theorem 4.12. As we have already discussed, the case $x \in \mathcal{B}$ is trivial, by the definitions. For $x \in \mathcal{R}$ we want to show that, under the hypotheses, one has $\phi + L(x) \in \mathcal{B}$, i.e.,

$$\gamma < (\phi + L(x)) \bmod 2\pi < 2\pi - \gamma. \tag{4.43}$$

We will split the proof in two parts. The first for $x \in [0,\ \gamma]$ and the second for $x \in [2\pi - \gamma,\ 2\pi]$.

Denote

$$K(\zeta) := \gamma(\zeta) + L(\gamma(\zeta)). \tag{4.44}$$

A simple computation shows

$$K(\zeta) = 3 \arccos\left(\frac{3 - 8\zeta + 3\zeta^2}{2\zeta} \right) - 4 \arctan\left(\frac{\sqrt{3(3-\zeta)(3\zeta - 1)}}{5 - 3\zeta} \right). \tag{4.45}$$

Moreover,

$$\begin{aligned} K'(\zeta) &= (1 + L'(\gamma(\zeta)))\,\gamma'(\zeta) \\ &= (1 + L'(\gamma(\zeta)))\,\frac{3\,(1-\zeta^2)}{\zeta \sin\gamma(\zeta)} \;>\; 0 \end{aligned}$$

for $1/3 < \zeta < 1$. Therefore, K is a continuous strictly increasing function of ζ in the interval $[1/3, 1]$ with $K(1) = 5\pi/3 > \pi$ for $1/3 < \zeta < \zeta_l$. A numerical computation indicates that $\zeta_l \simeq 0.46409$.

1. The case $x \in [0,\ \gamma]$.

Here we want to show that

$$\gamma < \phi + L(x) < 2\pi - \gamma. \tag{4.46}$$

Since $L \geq 0$, the first inequality is clear since, by the hypotheses,

$$\phi > K(\zeta) = \gamma + L(\gamma) \geq \gamma - L(x). \tag{4.47}$$

For the second inequality in (4.46) we notice that for $x \in [0,\ \gamma]$ one has $L(x) \leq L(\gamma)$ since L is an increasing function. Hence, the inequality $\gamma + \phi + L(x) < 2\pi$ holds if

$$\gamma + L(\gamma) < \pi, \tag{4.48}$$

since $\phi \leq \pi$. But inequality (4.48) is always true for $1/3 \leq \zeta < \zeta_l$. This completes the proof of the Theorem 4.12 for $x \in [0,\ \gamma]$.

2. The case $x \in [2\pi - \gamma,\ 2\pi]$.

For $x \in [2\pi - \gamma,\ 2\pi]$ we write $x = 2\pi - y$, where $y \in [0,\ \gamma]$ and notice that $L(x) = 4\pi - L(y)$. We want to show that

$$\gamma < \phi - L(y) < 2\pi - \gamma. \tag{4.49}$$

The second inequality is trivial, because

$$\phi + \gamma - L(y) < 2\pi, \tag{4.50}$$

since ϕ and γ lie in $[0,\ \pi]$ and $L \geq 0$. The inequality $\gamma + L(y) < \phi$ will hold if

$$\phi > \gamma(\zeta) + L(\gamma(\zeta)) = K(\zeta), \tag{4.51}$$

which is one of the conditions of Theorem 4.12 and can always be satisfied provided $1/3 \leq \zeta < \zeta_l$. This completes the proof of the Theorem 4.12.

■

5 Analyticity of m around $z = 1$ for $1/3 < \zeta \leq 1$

We shall now determine the domain of analyticity of the magnetization m in the paramagnetic phase. We aim to show that, for $1/3 < \zeta \leq 1$ and $\mathbf{p} \in \pi$, m can be analytically continued through the arc $S^1 \setminus A_\kappa$ which, in view of Theorem 1.1, proves the analytic statement of Theorem 1.2.

We begin by showing some preliminary results.

Theorem 5.1 *Given $1/3 < \zeta \leq 1$, let $\kappa = \kappa(\zeta)$ be given by*

$$\kappa = 4 \arctan \left(\frac{\zeta \sin \overline{\phi}}{1 + \zeta \cos \overline{\phi}} \right) - \overline{\phi} \tag{5.1}$$

where

$$\cos \overline{\phi} = (1/2\zeta) \left(1 - 3\zeta^2\right) . \tag{5.2}$$

Then, for all $\phi_0 < \kappa$, the fixed point equation

$$\phi = \phi_0 + L(\phi) \tag{5.3}$$

admits a unique stable solution $\phi^* = \phi^*(\zeta, \phi_0)$ to which the dynamical system (4.3) converges. Moreover, the solution ϕ^* is a monotonically increasing function of the initial condition ϕ_0 with $0 < \phi^* < \overline{\phi} \leq \pi$.

□

Remark 5.2 *Theorem 5.1 implies that the arc $S^1 \setminus A_\kappa$ is free of Lee-Yang singularities since (4.1) can never be satisfied for any $n \in \mathbb{N}$.*

Proof. By Proposition 4.2, L is convex on the domain $0 \leq x < \pi$ with $L'(0) < 1$ if $1/3 < \zeta \leq 1$. Therefore, there always exists $\kappa = \kappa(\zeta) > 0$ such that, for all $\phi_0 \leq \kappa$ the graph of $\phi_0 + L(x)$ intercepts the graph of x at least once in this domain. The constant κ is obtained by imposing

$$\overline{\phi} = \kappa + L(\overline{\phi})$$

and the tangency condition

$$L'(\overline{\phi}) = 2 \left(\frac{1 - \zeta^2}{1 + 2\zeta \cos \overline{\phi} + \zeta^2} \right) = 1 \ .$$

These equations determine uniquely κ and the fixed point solution $\overline{\phi}$ to (5.3) at $\phi_0 = \kappa$, in accordance to Theorem 5.1.

Next we will show that, the iterates of the discrete map (4.3) converges to the fixed point ϕ^*.

Let $\phi^* = \phi^*(\zeta, \phi_0)$ be smallest solution to (5.3). It follows from Proposition 4.2 that

$$\phi < \phi_0 + L(\phi) < \phi^* < \overline{\phi}$$

holds for all $\phi \leq \phi^*$ and $\phi_0 < \kappa$. In particular, using (5.3), $\phi_0 = \phi^* - L(\phi^*) < \phi^*$. So, (4.3) maps the interval $[0, \phi^*]$ into itself and it is contractive in this domain:

$$|L(\phi) - L(\sigma)| \leq L'(\phi^*) \, |\phi - \sigma| \equiv q^* \, |\phi - \sigma|$$

for all ϕ, $\sigma \in [0, \phi^*]$. This implies convergence of the sequence $\{\phi_j\}_{j \geq 0}$ to the solution ϕ^* since $q^* < L'(\overline{\phi}) \equiv 1$ and concludes the proof of Theorem 5.1.

■

We are now in position to determine the domain of analyticity of the magnetization m according to Theorem 1.2. We begin by showing the following result.

Lemma 5.3 *For $1/3 < \zeta \leq 1$ and $\mathbf{p} \in \pi$, the sequence $\{m_N = F_N + I_N, N \in \mathbb{N}\}$ converges to a continuous function $\widetilde{m} = \widetilde{F} + \widetilde{I}$ on S^1/A_κ where $\kappa = \kappa(\zeta)$ is given by (5.1).*

$\square$

Proof. Let $z = e^{i\phi_0}$ with $\phi_0 \in [0, \pi)$ and recall that $w_j = e^{i\phi_j} \equiv z\tau_z^{(j)}(1)$, $j \in \mathbb{N}$, are such that ϕ_j satisfy the discrete map (4.3). By Theorem 5.1, we have

$$\left|1 + z\tau_z^{(j)}(1)\right| \geq \delta \tag{5.4}$$

for some $\delta = \delta(\zeta, \phi_0) > 0$ provided $1/3 < \zeta \leq 1$ and $0 \leq \phi_0 < \kappa$. As a consequence, for any $\varepsilon > 0$,

$$|F_M(z) - F_N(z)| \leq \sum_{n=N}^{M-1} a_n \left|\frac{1 - z\tau_z^{(j)}(1)}{1 + z\tau_z^{(j)}(1)}\right| \leq \frac{2}{\delta} \sum_{n=N}^{M-1} a_n \leq \varepsilon,$$

for $M > N$, N large enough, since $\{a_n\}_{n\in\mathbb{N}}$ is summable. This says that the sequence of continuous functions F_N converges uniformly to a continuous function $\widetilde{F}$ on $S^1 \setminus A_\kappa$ with $\kappa = \kappa(\zeta)$ for all $1/3 < \zeta \leq 1$.

Analogously, under the same conditions, we have

$$|I_M(z) - I_N(z)| \leq 2\left|\frac{w_N - w_M}{(1 + w_M)(1 + w_N)}\right| \leq \frac{2}{\delta^2} |w_N - w_M| \leq \varepsilon$$

for any $\varepsilon > 0$ provided M and N are large enough, as a consequence of Theorem 5.1. Thus, the sequence of continuous functions I_N, $N \in \mathbb{N}$, converges uniformly to a continuous function $\widetilde{I}$ on $S^1 \setminus A_\kappa$.

This concludes the proof of Theorem 5.3.

$\blacksquare$

By using the "Edge-of-the-Wedge" theorem, the analytic function m studied in Section 1.1 can be analytically continued through the arc $S^1 \setminus A_\kappa$ provided $\widetilde{m}$ is the limit of m when z approximates the arc $S^1 \setminus A_\kappa$. We shall establish this in the following theorem.

Theorem 5.4 *Let $1/3 < \zeta \leq 1$ and $\mathbf{p} \in \pi$. Let $m = m(\zeta, z, \mathbf{p})$ be the analytic functions on z stated in Theorem 1.1 for the domain $\{|z| < 1\} \cup \{|z| > 1\}$. If $z_0 \in S^1 \setminus A_\kappa$, we have*

$$\lim_{z\to z_0} |m(z) - \widetilde{m}(z_0)| = 0$$

$\square$

Proof. Given $z \in \mathbb{C}$ define a sequence w_j, $j \in \mathbb{N}$, recursively by

$$w_j = zh(w_{j-1}) \tag{5.5}$$

with $w_0 = z$. Let us denote by $w_{0,j}$, $j \in \mathbb{N}$, the sequence given by (5.5) with $z = z_0 \in S^1 \setminus A_\kappa$. Note that $|w_{0,j}| = 1$ and if one writes $w_{0,j} = e^{i\phi_j}$, the sequence ϕ_j, $j \in \mathbb{N}$, satisfies (4.3).

We have

$$|w_0 - w_{0,0}| = |z - z_0| \tag{5.6}$$

and for $j = 1, 2, \ldots$,

$$\begin{aligned} |w_{j+1} - w_{0,j+1}| &= |zh(w_j) - z_0 h(w_{0,j})| \\ &\leq |z - z_0| + |z|\,|h(w_j) - h(w_{0,j})| \\ &= |z - z_0| + |zQ(w_j, w_{0,j})|\,|w_j - w_{0,j}| \end{aligned} \tag{5.7}$$

where $Q(w, w') \equiv Q(1, w, w')$ is defined in (3.19).

For fixed $\varepsilon > 0$ and $0 < q < 1$ we let $z \in \mathbb{C}$ be such that

$$|z - z_0| < \varepsilon(1 - q). \tag{5.8}$$

Note that, by Theorem 5.1,

$$\begin{aligned} |Q(w_{0,j}, w_{0,j})| &= \frac{2(1-\zeta^2)}{1 + 2\zeta\cos\phi_j + \zeta^2} \\ &\leq \frac{2(1-\zeta^2)}{1 + 2\zeta\cos\phi^* + \zeta^2} \\ &= q^* \end{aligned}$$

holds for all $j \in \mathbb{N}$. Moreover, since $\phi^* < \overline{\phi}$, and recalling equation (5.2),

$$q^* < \frac{2(1-\zeta^2)}{1 + 2\zeta\cos\overline{\phi} + \zeta^2} = 1.$$

Now, using the fact that $Q(w, w')$ is a continuous function of w and w', there always exist numbers $0 < q < 1$ and $\varepsilon > 0$ such that, for z satisfying (5.8), we have

$$\sup_{w \in S^1 \setminus A_\kappa} \; \sup_{w' : |w' - w| \leq \varepsilon} |zQ(w', w)| < q. \tag{5.9}$$

We now show, by using induction, that

$$\lim_{z \to z_0} |w_j - w_{0,j}| = 0 \tag{5.10}$$

holds for any $j \in \mathbb{N}$.

Clearly, $|w_0 - w_{0,0}| < \varepsilon$. Now, let us assume that $|w_j - w_{0,j}| < \varepsilon$ for some $j \in \mathbb{N}$. Then $|zQ(w_j, w_{0,j})| < q$ and, from (5.7) and (5.8), we get

$$\begin{aligned} |w_{j+1} - w_{0,j+1}| &\leq |z - z_0| + q\,|w_j - w_{0,j}| \\ &< (1-q)\varepsilon + q\varepsilon \\ &= \varepsilon \end{aligned}$$

uniformly in $j \in \mathbb{N}$. Since ε is arbitrary, this establishes (5.10).

Let us now define

$$D^j(z,z_0) := \left| \frac{1 - z\tau_z^{(j)}(1)}{1 + z\tau_z^{(j)}(1)} - \frac{1 - z_0\tau_{z_0}^{(j)}(1)}{1 + z_0\tau_{z_0}^{(j)}(1)} \right|.$$

If $|z - z_0| < (1-q)\varepsilon\delta/2$ with δ as in (5.4), we have (recall $w_j := z\tau_z^{(j)}(1)$)

$$D^j(z,z_0) = \frac{2\,|w_j - w_{0,j}|}{|(1+w_j)(1+w_{0,j})|} < \varepsilon$$

and consequently

$$\begin{aligned} |m_N(z) - \widetilde{m}_N(z_0)| &\leq \left|F_N - \widetilde{F}_N\right| + \left|I_N - \widetilde{I}_N\right| \\ &= \sum_{j=0}^{N-1} a_j\, D^j(z,z_0) + p_1 \dots p_N\, D^N(z,z_0) \\ &< \varepsilon \left\{ \sum_{j=0}^{N-1} a_j \, + p_1 \dots p_N \right\} \\ &= \varepsilon \end{aligned}$$

holds uniformly in N.

This concludes the proof of Theorem 5.4 and the proof of the analyticity statement of Theorem 1.2.

■

6 Continuity and Differentiability of m

The aim of this section is to establish Theorem 1.6. The quenched magnetization at origin m will be here considered as a real valued function of z: $m : \mathbb{R}_+ \longmapsto \mathbb{R}$.

Let us set

$$r(x) = \frac{1-x}{1+x} \tag{6.1}$$

and write $m_N = (F_N + I_N)$ where, in terms of r,

$$F_N = \sum_{n=0}^{N-1} a_n\; r\left(z\tau_z^{(n)}(1)\right) \tag{6.2}$$

and

$$I_N = p_1 \dots p_N\; r\left(z\tau_z^{(N)}(1)\right). \tag{6.3}$$

Our study on the smoothness of the quenched magnetization m will be divided in three subsections. In the first it will be examined under which conditions m is discontinuous at $z = 1$ and its value will be computed. The second subsection will establish smoothness of F under the assumption of Theorem 6.7 which will be proven in the last subsection.

6.1 Jump of the Magnetization

This subsection is devoted to show that, for $z \in \mathbb{R}_+$ and $0 \leq \zeta < 1/3$, $m = \lim_{N \to \infty} m_N$ is a continuous function of z if $\mathbf{p} \in \pi_0$ and discontinuous at $z = 1$ if $\mathbf{p} \in \pi_a$ with $a > 0$.

We start by noting the following facts about r (the proof will be omitted):

Proposition 6.1 *$r : x \in \mathbb{R}_+ \longmapsto r(x) \in \mathbb{R}$ given by (6.1) is a continuous monotone decreasing function of x with*

$$r(x) = -r(1/x) \leq 1.$$

□

Now, we observe that, in view of Corollary 2.2 and (2.5),

$$z\tau_z^{(n)}(1) = \frac{1}{(1/z)\tau_{1/z}^{(n)}(1)}. \tag{6.4}$$

Proposition 6.1 and (6.4) implies

$$m(z) = -m(1/z) \tag{6.5}$$

and since $\sum_{n=0}^{N-1} a_n + p_1 \dots p_N = 1$,

$$-1 \leq m(z) \leq 1.$$

In addition, $m(1) = 0$ as a consequence of $\tau_1^{(n)}(1) = 1$ for all $n \in \mathbb{N}$.

Let $\zeta \in [0, 1/3)$, $z \in \mathbb{R}_+$ and $\mathbf{p} \in \pi_0$. By Proposition 6.1 and recalling that $\lim_{n \to \infty} p_1 \dots p_n = 0$, we have

$$\begin{aligned} |m_N - m_M| &\leq \sum_{n=M+1}^{N-1} a_n \, |r(w_n)| + p_1 \dots p_M \, |r(w_M) - p_{M+1} \dots p_N r(w_N)| \\ &\leq \sum_{n=M}^{N-1} a_n + p_1 \dots p_M (1 + p_{M+1} \dots p_N) < \varepsilon \end{aligned} \tag{6.6}$$

for any $\varepsilon > 0$ provided N and M, $M < N$, are large enough. Here $w_n := z\tau_z^{(n)}(1)$ is a continuous function of z. The sequence $\{m_N\}_{N \geq 0}$ is thus a uniform Cauchy sequence of continuous functions which establishes the continuity of m in $z \in \mathbb{R}_+$.

Note that this statement is also true for the sequence $\{F_N\}_{N \geq 0}$ if π_0 is replaced by π in the assumptions. So, in order m to be discontinuous at $z = 1$, I_N must converge, as $N \to \infty$, to different limits, $I^>$ and $I^<$, depending on whether z is larger or smaller than 1. This requires $\mathbf{p} \in \pi_a$ with $a > 0$ and $\zeta \in [0, 1/3)$.

By definition $w_n := z\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, satisfy $w_n = zh\,(w_{n-1})$ with $w_0 = z$. The following proposition gives some properties of h which will be useful for this and the next subsection. Its proof follows by an easy computation and will be omitted.

Proposition 6.2 *For $\zeta \in [0,1/3)$, $h : w \in \mathbb{R}_+ \longmapsto h(w) \in \mathbb{R}_+$ given by (2.1) is a continuous monotone increasing function of w, bounded from below and above by ζ^2 and $1/\zeta^2$, respectively, with $h(1) = 1$. Its first derivative*

$$h'(w) = \frac{2(\zeta + w)(1-\zeta^2)}{(1+\zeta w)^3}$$

has a maximum value at $\hat{w} = (1-3\zeta^2)/\,2\zeta > 1$ given by

$$h'(\hat{w}) = \frac{8}{27\zeta(1-\zeta^2)}$$

with $h''(w) > 0$ for $0 \le w < \hat{w}$ and $h''(w) < 0$ for $w > \hat{w}$. In addition, $h'(1) = 2\,(1-\zeta)\,/(1+\zeta)$ is strictly larger than 1 for all $0 \le \zeta < 1/3$.

$\square$

We shall now describe in the next proposition the limit points of w_n. By (6.4), we can restrict ourselves to $z \in [0,1)$. This implies, in view of Corollary 2.2, that $w_n \in [0,1)$ for all $n \in \mathbb{N}$.

Proposition 6.3 *Let $\zeta \in [0,1/3)$ and $z \in [0,1)$. The sequence w_n, $n \in \mathbb{N}$, converges to the unique solution $w = w(\zeta, z)$ of the fixed point equation $w = zh(w)$ which behaves as*

$$\underline{w} := \lim_{z \nearrow 1} w = \frac{1-2\zeta-\zeta^2}{2\zeta^2} - \frac{1-\zeta^2}{2\zeta^2}\left(\frac{1-3\zeta}{1+\zeta}\right)^{1/2}$$

$\square$

Proof. Let $z \in [0,1]$ and $0 \le \zeta < 1/3$. From Proposition 6.2, the graph of $zh(x)$ intercepts the graph of the identity function x at a single point $w = w(\zeta, z)$ with $w < z = w_0$ and $zh(x) < x$ for all $x \in (w, z]$. Therefore, the sequence $\{w_n\}_{n \ge 0}$ is strictly decreasing:

$$w < w_{n+1} < w_n \qquad , \qquad n \in \mathbb{N}$$

Since $h'(1) > 1$ for $0 \le \zeta < 1/3$, the graph of $h(y)$ intercepts the graph of y at $\underline{w} = \lim_{z \nearrow 1} w(\zeta, z) < 1$ with the property that $h(y) < y$ for all $y \in (\underline{w}, 1)$ (recall Proposition 6.2). We thus have

$$w_n = zh(w_{n-1}) < zw_{n-1} < z^{n+1}$$

where the last inequality follows by iteration . Therefore, there exist $n_0 = n_0(\zeta, z) < \infty$ such that $w < w_n < \underline{w}$ holds for all $n > n_0$.

For any $x, y \in [0, \underline{w}]$, we have

$$\begin{aligned} |zh(x) - zh(y)| &\le z \sup_{0 \le t < \underline{w}} |h'(t)|\,|x-y| \\ &\le q\,|x-y| \ . \end{aligned} \tag{6.7}$$

for some $q < 1$ since $\sup_{0 \le t < \underline{w}} |h'(t)| < 1$. This implies convergence of w_n, $n \in \mathbb{N}$, to the unique solution w of the fixed point equation by the Banach contraction theorem.

By an equation analogous to (6.7) it follows that w_n converges to w for all $1/3 \le \zeta \le 1$.

To conclude Proposition 6.3, we note that $\lim_{z \nearrow 1} w = \underline{w}$ is equal to 1 for $1/3 \le \zeta \le 1$, since $h'(1) \le 1$, and for $0 \le \zeta < 1/3$ is a nontrivial solution of $h(w) = w$. This equation is equivalent to

$$(w-1)\left[\zeta^2 w^2 - \left(1 - 2\zeta - \zeta^2\right) w + \zeta^2\right] = 0$$

which, when $0 \le \zeta < 1/3$ has three real solutions. w is the only one strictly smaller than 1.

■

Remark 6.4 *Proposition 6.3 can be extended to any $z \in \mathbb{R}_+$ by using the relation*

$$w(\zeta, z) = 1 \,/ w(\zeta, 1/z)$$

so, $\overline{w} := \lim_{z \searrow 1} w(\zeta, z) = 1/\underline{w}$.

We now turn to the magnetization jump. As a Corollary of the result established in (6.6) we have

$$\lim_{z \nearrow 1} \lim_{N \to \infty} F_N = 0$$

for all $\mathbf{p} \in \pi$. Therefore, in view of Proposition 6.3 , for $\mathbf{p} \in \pi_a$ with $a > 0$, we have

$$\lim_{z \nearrow 1} \lim_{N \to \infty} m_N = \lim_{z \nearrow 1} \lim_{N \to \infty} I_N = a\ r(\underline{w})$$

Analogously, by Proposition 6.1 and (6.4),

$$\lim_{z \searrow 1} \lim_{N \to \infty} m_N = \lim_{z \searrow 1} \lim_{N \to \infty} I_N = a\ r(\overline{w}) = -a\ r(\underline{w}).$$

Hence,

$$\lim_{\varepsilon \searrow 0} (m(z - \varepsilon) - m(z + \varepsilon)) = 2a\, r(\underline{w}) = 2a \frac{(1 - 3\zeta)^{1/2}(1 + \zeta)^{1/2}}{1 - \zeta}.$$

This concludes the proof of the continuity and the magnetization jump statement of Theorem 1.6.

■

6.2 Differentiability of F

We have shown that F is a continuous function of z for all $z \in \mathbb{R}_+$ and $\mathbf{p} \in \pi$. Here we will establish the differentiability statement of Theorem 1.6 for this function. We recall that, in view of Theorem 1.2, we need only consider $\zeta \in [0, 1/3)$.

We shall use very often the recursion relation

$$w_n = zh \circ w_{n-1} \qquad , \qquad n = 1, 2, \ldots \tag{6.8}$$

so, with the initial condition $w_0 = z$, we have $w_n = z\tau_z^{(n)}(1)$.

The proof that F is a k–differentiable function requires an induction procedure indexed by the number of iteration of (6.8) which involves derivatives of (6.8) up to the order k. The following notation will be used frequently

$$W_n^k := \frac{1}{k!} \left| D^k w_n \right| \qquad , \quad k \in \mathbb{N} \tag{6.9}$$

($W_n^0 \equiv w_n$) where D denotes the differential operator d/dz.

By the chain rule, the derivative of (6.8) gives

$$Dw_n = h(w_{n-1}) + z\, h'(w_{n-1})\, Dw_{n-1} \quad . \tag{6.10}$$

To illustrate our procedure the flow of the first derivative, Dw_n, $n \in \mathbb{N}$, will be controlled in absolute value in the following Theorem.

Theorem 6.5 *Given $z \in \mathbb{R}_+$ and $\zeta \in [0, 1/3)$,*

$$W_n^1 \le C_1 \, \frac{\hat{\mu}}{1-\hat{\mu}} \hat{\mu}^n \tag{6.11}$$

holds with $C_1 = 1$ and $\hat{\mu} = h'(1)$ for $z \in [0,1]$ and with $C_1 = w$ (solution of $w = zh(w)$) and $\hat{\mu} = zh'(\hat{z})$ for $z > 1$.

□

Proof. From Corollary 2.2, $w_n \le 1$ for all $n \in \mathbb{N}$ provided $z \le 1$. This fact and Proposition 6.2 implies that (6.10) can be bounded by

$$\begin{aligned} W_n^1 &\le 1 + h'(1)\, W_{n-1}^1 \\ &\le \frac{(h'(1))^{n+1}}{1 - h'(1)} \end{aligned}$$

for all $z \in [0, 1]$.

Now, let $z > 1$. By Propositions 6.2 and 6.3

$$W_n^1 \le w + z\, h'(\hat{w})\, W_{n-1}^1 \tag{6.12}$$

which, after iterating, can be written as (6.11) with $C_1 = w$ and $\hat{\mu} = z\, h'(\hat{w})$.

Theorem 6.5 is then proven.

■

Taking the k-th derivative of $w_1(z) = zh(z)$, gives us

$$D^k w_1 = z\, h^{(k)} + k\, h^{(k-1)} \tag{6.13}$$

The next lemma controls derivative of arbitrary order of h. This will be an ingredient for our inductive procedure and provide us with an estimate on (6.13) which is sharper than the one stated in Theorem 6.9.

Lemma 6.6 *Let $z \in \mathbb{R}_+$, $\zeta \in [0, 1/3)$ and $k \in \mathbb{N}$, $k \geq 2$. The following holds*

$$\left|h^{(k)}\right| \leq C_k\, k!\, \zeta^k$$

with $C_k = C_k(\zeta) = |k(1-\zeta^2)^2 - 1 + \zeta^4| / \zeta^2$. Therefore, in view of Proposition 6.2

$$\left|h^{(k)}\right| \leq H\, k!\, (2\zeta)^k \tag{6.14}$$

holds for any $k \geq 1$ with $H = \max\left(8/(27\zeta^2), H_{\geq 2}\right)$ where $H_{\geq 2} = \sup_{k\geq 2} C_k 2^{-k}$.

$\square$

Proof. For $k \in \mathbb{N}$, we write $h = a \cdot b$ where $a =: (\zeta + z)^2$ and $b =: (1+\zeta z)^{-2}$ and use the Leibnitz formula:

$$h^{(k)} = \sum_{p=0}^{k} \binom{k}{p} a^{(k)} b^{(k-p)}$$

where derivatives of b can be explicitly evaluated:

$$b^{(j)}(z) = (-1)^j (j+1)! \frac{\zeta^j}{(1+\zeta z)^{2+j}} \qquad , \qquad j = 0, 1, \ldots$$

We thus have

$$h^{(k)} = (-1)^k k! \left(\frac{\zeta}{1+\zeta z}\right)^k c_k \tag{6.15}$$

where

$$c_k = \frac{1-\zeta^2}{\zeta^2 (1+\zeta z)^2} \left[k\left(1-\zeta^2\right) - 1 - \zeta^2 - 2z\zeta\right].$$

Now, the local minima or maxima of $h^{(k)}$ are located at the zeros of $h^{(k+1)}$ which are given by the solutions of the equation

$$c_{k+1} = 0 \quad . \tag{6.16}$$

It turns out that there is only one solution to (6.16):

$$\overline{z} = \overline{z}(k, \zeta) = \frac{k(1-\zeta^2) - 2\zeta^2}{2\zeta}$$

with

$$h^{(k)}(\overline{z}) = \frac{-4}{\zeta^2 (k+2)^2} k! \left(\frac{-2\zeta}{(k+2)(1-\zeta^2)}\right)^k \quad . \tag{6.17}$$

So, $h^{(k)}$ is a strictly monotone function of z in both interval $[0,\overline{z})$ and $[\overline{z},\infty)$ for all $k \geq 1$.

Note that $\overline{z}$ is a monotone increasing function of k with $\overline{z}(k,\zeta) \geq \overline{z}(1,\zeta) > 1$, for all $0 \leq \zeta < 1/3$. In addition, since $\lim_{z\to\infty} h^{(k)}(z) = 0$ for all $k \geq 1$, we conclude

$$\left|h^{(k)}(z)\right| \leq \max\left(\left|h^{(k)}(0)\right|, \left|h^{(k)}(\overline{z})\right|\right)$$

where, by an explicit calculation,

$$\left|h^{(k)}(0)\right| = k!\, \left(k(1-\zeta^2)^2 - (1-\zeta^4)\right)\, (-\zeta)^{k-2} \quad . \tag{6.18}$$

Now, taking the ratio between (6.18) and (6.17), we have

$$\frac{\left|h^{(k)}(0)\right|}{\left|h^{(k)}(\overline{z})\right|} = \left[(1-\zeta^2)\left(1+\frac{k}{2}\right)\right]^k \left(1+\frac{k}{2}\right)^2 \left(k(1-\zeta^2)^2 - (1-\zeta^4)\right) > 1$$

for all $k \geq 2$ since $(1-\zeta^2)(1+k/2) \geq 2(1-\zeta^2) > 16/9$ and

$$\left(1+\frac{k}{2}\right)^2 \left(k(1-\zeta^2)^2 - (1-\zeta^4)\right) \geq 4\left(1-4\zeta^2+3\zeta^4\right) > \frac{20}{9}$$

provided $0 \leq \zeta < 1/3$.

This concludes the proof of Lemma 6.6.

$\blacksquare$

Substituting (6.15) into (6.13), yields

$$\frac{1}{k!} D^k w_1 = d_k \left(\frac{-\zeta}{1+\zeta z}\right)^k$$

where

$$d_k = \frac{1-\zeta^2}{\zeta^3(1+\zeta z)}\left[2 + \frac{1-\zeta^2}{1+\zeta z}(\zeta z - k)\right] \quad .$$

We thus have

$$W_1^k \leq W_1\ (2\zeta)^k \tag{6.19}$$

with $W_1 = \sup_{k\geq 1} d_k \left[2(1-\zeta^2)\right]^{-k}$ which can be bounded by

$$W_1 \leq \frac{(1-\zeta^2)^2}{e\zeta^2(1+z\zeta)^3 \ln\left[2(1+\zeta z)\right]} \quad .$$

To show smoothness of F, we need to estimate the k-th derivative, $D^k(f\circ g)$, of the composition of two functions f and g. If $\mathcal{P}_k$ denotes the set of all partitions $\mathbf{P} = (P_1,\ldots,P_s)$ of $\{1,2,\ldots,k\}$ and $|P|$ the cardinality of the set P, we have

$$D^k(f\circ g) = \sum_{\mathbf{P}\in\mathcal{P}_k}\left(\prod_{j=1}^s D^{|P_j|}g\right) f^{(s)}\circ g. \tag{6.20}$$

Now, let us assume that f and g are smooth functions such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \left|f^{(n)}(x)\right| &\leq C_f \, n! \, K_f^n \\ \left|g^{(n)}(x)\right| &\leq C_g \, n! \, K_g^n \end{aligned} \tag{6.21}$$

hold for some finite constants C_f, K_f, C_g and K_g.

In addition, let us define a polynomial of degree k

$$R_k(x) := \frac{1}{k!} \sum_{\mathbf{P} \in \mathcal{P}_k} \left(\prod_{j=1}^{s} |P_j|! \right) s! x^s \quad . \tag{6.22}$$

From (6.21), (6.22) and since $\sum_j |P_j| = k$ for any partition $\mathbf{P} \in \mathcal{P}_k$, the absolute value of (6.20) can be bounded by

$$\begin{aligned} \left|D^k (f \circ g)\right| &\leq C_f \sum_{\mathbf{P} \in \mathcal{P}_k} \left(\prod_{j=1}^{s} C_g |P_j|! K_g^{|P_j|} \right) s! K_f^s \\ &= C_f \, K_g^k \sum_{\mathbf{P} \in \mathcal{P}_k} \left(\prod_{j=1}^{s} |P_j|! \right) s! \left(C_g \, K_f\right)^s \\ &= k! \, C_f \, K_g^k \; R_k \left(C_g \, K_f\right) \ . \end{aligned} \tag{6.23}$$

To conclude the estimate on the derivatives of $f \circ g$ we need an estimate on the polynomial R_k. It is convenient to write (6.22) in a more suitable form.

If n_j denotes the number of times a set P with $|P| = j$ occurs in the partition $\mathbf{P}$, R_k can be rewritten as

$$R_k(x) = \sum_{\mathbf{n} \in \mathcal{N}_k} \frac{\left(\sum_j n_j\right)!}{n_1! \ldots n_k!} x^{\sum_j n_j} \tag{6.24}$$

where the summation is over all k-component vectors $\mathbf{n} = (n_1, \ldots n_k)$ in the set

$$\mathcal{N}_k = \left\{ \mathbf{n} \in \mathbb{N}^k : n_1 + 2n_2 + \cdots + k \, n_k = k \right\}.$$

The following theorem is our main technical result of this section. Essentially, it compares R_k with the polynomial $(1 + x)^k$.

Theorem 6.7 *Given $k \in \mathbb{N}_+$, let $R_k : x \in \mathbb{R}_+ \longmapsto \mathbb{R}_+$ be the polynomial defined by (6.24). Then, for any $\delta > 1$, the following holds*

$$R_k(x) \leq C_\delta \; (\delta + x)^k \tag{6.25}$$

with

$$C_\delta = 1 + \frac{x}{\delta - 1} \quad .$$

$\square$

Theorem 6.7 will be proven in the next subsection.

Applying Theorem 6.7 to the inequality (6.23), gives an upper bound of the form (6.21). For any $\delta > 1$, we thus have

$$\left|D^k (f \circ g)\right| \leq C_{f \circ g}\, k!\, K^k_{f \circ g} \tag{6.26}$$

where

$$C_{f \circ g} = C_f \left(1 + C_g \frac{K_f}{\delta - 1}\right) \tag{6.27}$$

and

$$K_{f \circ g} = (\delta + C_g\, K_f)\, K_g \quad . \tag{6.28}$$

The estimate (6.26) can be used to control inductively derivatives of w_n in (6.8) provided f is replaced by h and g replaced by w_{n-1}. This yields the following theorem.

Theorem 6.8 *For fixed n, $k \in \mathbb{N}$, $k \geq 1$, let W^j_n be given by (6.9) and suppose*

$$W^j_n \leq W_n\, \mu^j_n \qquad , \qquad j = 1, \ldots, k \tag{6.29}$$

holds with W_n and μ_n positive constants. Then, given $\delta > 1$ and H as in Lemma 6.14,

$$W^j_{n+1} \leq W_{n+1}\, \mu^j_{n+1}$$

holds for all $j = 1, \ldots, k$, with

$$W_{n+1} = H \left(z + \frac{1}{\delta}\right)\left(1 + \frac{2\zeta}{\delta - 1} W_n\right) \tag{6.30}$$

and

$$\mu_{n+1} = (\delta + 2\zeta W_n)\, \mu_n \quad . \tag{6.31}$$

$\square$

Proof: Differentiating (6.8) j-times and taking absolute value, gives

$$\frac{1}{j!}\left|D^j w_{n+1}\right| \leq z\, \frac{1}{j!}\left|D^j (h \circ w_n)\right| + \frac{1}{(j-1)!}\left|D^{j-1} (h \circ w_n)\right| \quad .$$

Applying equations (6.26) - (6.28) and Lemma 6.14, we have

$$\begin{aligned} W^j_{n+1} &\leq H \left(z + \frac{1}{\delta + 2\zeta W_n}\right)\left(1 + \frac{2\zeta}{\delta - 1} W_n\right)(\delta + 2\zeta W_n)^j \\ &\leq H \left(z + \frac{1}{\delta}\right)\left(1 + \frac{2\zeta}{\delta - 1} W_n\right)(\delta + 2\zeta W_n)^j \end{aligned}$$

by the positivity of W_n, $n \in \mathbb{N}$. This concludes the proof of Theorem 6.8.

$\blacksquare$

Theorem 6.5 and equation (6.19) are particular cases of the following result obtained by the induction described in Theorem 6.8.

Theorem 6.9 *Given $0 < \zeta < 1/3$, $z \in \mathbb{R}_+$ and $n,\ k \in \mathbb{N}$, $k \geq 1$. There exist a constant $C = C(\zeta, z) < \infty$ such that*

$$W_n^j \leq C\ \mu^{nj} \tag{6.32}$$

holds for all $j = 1, \ldots, k$ with

$$\mu(z) \ := \ 4\zeta H z \ + \ 2\left[\left(\frac{4\zeta H z + 1}{4}\right)^2 + \zeta H\right]^{1/2}$$

and H as in Lemma 6.14. Moreover, μ and C satisfies the following relation $\mu = 1 + 4\zeta C$.

□

Proof: Since $w_0 = z$, we have

$$W_0^j \leq \begin{cases} 1 & \text{if} \qquad j = 1 \\ 0 & \text{otherwise} \end{cases}$$

which certainly satisfies (6.29) with $W_0 = 1$ and $\mu_0 = 1$. This will be the initial conditions of Theorem 6.8.

We now observe that the sequence $\{W_n\}_{n\in\mathbb{N}}$, defined by the recurrence (6.30) is a convergent sequence provided δ is such that $2\zeta H\left(z + 1/\delta\right)/(\delta - 1) < 1$. For definiteness, let δ be given by

$$\frac{2\zeta H}{\delta - 1}\left(z + \frac{1}{\delta}\right) = \frac{1}{2}. \tag{6.33}$$

Iterating equation (6.30), yields

$$\begin{aligned} W_{n+1} &= \widetilde{H}\left(1 + \frac{1}{2} + \cdots + \left(\frac{1}{2}\right)^{n+1} W_0\right) \\ &\leq 2\widetilde{H} \end{aligned} \tag{6.34}$$

where $\widetilde{H} = H\left(z + 1/\delta\right)$ which can be written as

$$\widetilde{H} = \frac{\delta - 1}{4\zeta} = \frac{4\zeta H z - 1}{8\zeta} + \frac{1}{4\zeta}\left[\left(\frac{4\zeta H z + 1}{4}\right)^2 + \zeta H\right]^{1/2}$$

by solving equation (6.33).

In view of (6.31) and (6.34), we have

$$\mu_{n+1} = \mu_0 \prod_{j=0}^{n}\left(\delta + 2\zeta W_j\right) \leq \left(\delta + 4\zeta\widetilde{H}\right)^{n+1} = \left(1 + 8\zeta\widetilde{H}\right)^{n+1} \tag{6.35}$$

which concludes the proof of Theorem 6.9.

■

We are now ready to finish the proof of Theorem 1.6.

Proof of Theorem 1.6 (Conclusion): Differentiating (6.2) k-times, gives

$$\frac{1}{k!}D^k F_N = \sum_{n=1}^{N-1} a_n \; \frac{1}{k!}D^k\left[r\circ w_n\right] \tag{6.36}$$

which can be estimated by using (6.26) provided we compute $r^{(j)}$, $j\in\mathbb{N}$. By an explicitly computation, we have

$$r^{(j)}(x) = (-1)^j j!\,\frac{2}{(1+x)^j}$$

and, in view of Proposition 6.3

$$\frac{1}{j!}\left|r^{(j)}(w_n)\right| \leq 2\eta^j$$

with $\eta = (1+w)^{-1}$ if $z\leq 1$ and $\eta = 2^{-1}$ if $z>1$.

Now, (6.26) with f replaced by r and g replaced by w_n, implies

$$\frac{1}{k!}\left|D^k\left[r\circ(w_n)\right]\right| \leq 2\left(1+\frac{C\eta}{\delta-1}\right)(\delta+C\eta)^k\,\mu^{nk} \tag{6.37}$$

with $\delta>1$.

In view of (6.36) and (6.37), there exist finite constants, C_1 and C_2, such that

$$\frac{1}{k!}\left|D^kF_N - D^kF_M\right| \leq C_1C_2^k\sum_{n=M+1}^{N} a_n\mu^{kn} < \varepsilon$$

for any $\varepsilon>0$, provided $\left\{a_n\mu^{kn}\right\}_{n\geq 0}$ is a summable sequence and $M,\ N>N_0$ for some $N_0 = N_0(\zeta\,,z\,,\mathbf{p},k)<\infty$. The sequence $\left\{D^kF_N\right\}_{N\geq 0}$, for any $k\in\mathbb{N}$, is a uniform Cauchy sequence of continuous function of z in $\mathbb{R}_+$. This concludes the proof of Theorem 1.6.

■

6.3 Proof of Theorem 6.7

Theorem 6.7 will be proven firstly assuming that the following upper bound

$$R_k(x)\leq x\sum_{j=0}^{k-1}R_j(x) \tag{6.38}$$

holds for all $x\in\mathbb{R}_+$, with $R_0(x)\equiv 1$. The inequality (6.38) will be established afterward.

We prove (6.25) by induction. For $k=0$,

$$R_0(x)\equiv 1\leq 1+\frac{x}{\delta-1}.$$

Now, let us assume (6.25) valid for $k = 0, \ldots, n$, $n \in \mathbb{N}$. From (6.38), we have

$$\begin{aligned} R_{n+1}(x) &\le x\left(1+\frac{x}{\delta-1}\right)\sum_{j=0}^{n}(\delta+x)^j \\ &= x\left(1+\frac{x}{\delta-1}\right)\frac{(\delta+x)^{n+1}-1}{x+\delta-1} \\ &\le \left(1+\frac{x}{\delta-1}\right)(\delta+x)^{n+1} \end{aligned} \tag{6.39}$$

since $x/(x+\delta-1) < 1$. This proves Theorem 6.7 under the assumption (6.38).

■

Proof of (6.38): Define

$$y_\ell = \sum_{j=1}^{\ell} n_j \qquad , \qquad \ell = 1, \ldots, \mathrm{k}$$

and notice that the coefficients of R_k can be written as

$$\frac{\left(\sum_j n_j\right)!}{n_1! \ldots n_k!} = \frac{y_1!}{n_1!}\frac{y_2!}{n_2! y_1!} \cdots \frac{y_k!}{n_k! y_{k-1}!} = \binom{y_2}{n_2} \cdots \binom{y_k}{n_k}$$

With this notation, (6.24) can thus be written as

$$R_k(x) = \sum_{n_1} x^{n_1} \sum_{n_2} \binom{y_2}{n_2} x^{n_2} \ldots \sum_{n_k} \binom{y_k}{n_k} x^{n_k}\ \delta_k\left(n_1 + 2n_2 + \cdots + kn_k\right) \tag{6.40}$$

where $\delta_k(m) = 1$ if $m = k$ and 0 otherwise.

For $1 \le i < j \le k$, let $R_k^{[i,j]}(x)$ be the polynomial of order k in x obtained from (6.40) by setting $n_1, \ldots, n_{i-1}, n_{j+1}, \ldots, n_k$ all equal to 0, i.e.,

$$R_k^{[i,j]}(x) = \sum_{n_i} x^{n_i} \sum_{n_{i+1}} \binom{y_{i+1}}{n_{i+1}} x^{n_{i+1}} \ldots \sum_{n_j} \binom{y_j}{n_j} x^{n_j}\ \delta_k\left(in_i + (i+1)n_{i+1} + \cdots + jn_j\right)$$

Now, note that $n_k = 0, 1$ are the only values which satisfy the equation $n_1 + 2n_2 + \cdots + kn_k = k$. In the case $n_k = 1$, we have $n_1 = \cdots = n_{k-1} = 0$ and $y_k = 1$. Equation (6.40) can thus be rewritten as

$$\begin{aligned} R_k(x) &= R_k^{[1,k-1]}(x) + x \\ &= R_k^{[2,k-2]}(x) + xR_{k-1}(x) + x \end{aligned} \tag{6.41}$$

In the second line of (6.41), $R_k^{[1,k-1]}(x)$ has been written as a sum of two terms:

$$R_k^{[1,k-1]}(x) = R_k^{[2,k-1]}(x) + \sum_{n_1 \ne 0} x^{n_1} \ldots \sum_{n_{k-1}} \binom{y_{k-1}}{n_{k-1}} x^{n_{k-1}}\ \delta_k\left(n_1 + \cdots + (k-1)\, n_{k-1}\right) \quad .$$

By making the change of variables $n_1' = n_1 - 1$ the second term can be written as xR_{k-1}. For the first term, we note that $y_2 = n_2$ and $2n_2 + \cdots + (k-1)\, n_{k-1} = k$ holds only with $n_{k-1} = 0$. This implies (6.41).

This procedure can be repeated for the sum over n_2 in $R_k^{[2,k-2]}(x)$. For the term with $n_2 \geq 1$, we change variable to $n_2' = n_2 - 1$

$$x \sum_{n_2 \geq 1} x^{n_2 - 1} \ldots \sum_{n_{k-1}} \binom{y_{k-2}}{n_{k-2}} x^{n_{k-2}}\ \delta_{k-2}\left(2n_2 - 2 + \cdots + (k-2)\, n_{k-2}\right) = xR_{k-2}^{[2,k-2]}$$

and notice that $R_{k-2}^{[2,k-2]} \leq R_{k-2}$ by the positivity of x.

(6.41) can thus be written as

$$R_k(x) \leq R_k^{[3,k-3]}(x) + x\left(R_{k-2} + R_{k-1}(x) + 1\right)$$

where we have used that $3n_3 + \cdots + (k-2)\, n_{k-2} = k$ holds only with $n_{k-2} = 0$.

Continuing this process, we get

$$R_k(x) \leq x\left(R_{[k/2]}(x) + \cdots + R_{k-1}(x) + 1\right)$$

where $[v]$ is the integer part of $v \in \mathbb{R}$. Since $R_j(x) \geq 0$ for $x \in \mathbb{R}_+$, this concludes the proof of (6.38).

■

7 The Analyticity Domain of $\mathcal{M}_r$

In this section we will prove Theorem 1.8 concerning the analyticity domain of $\mathcal{M}_r$. We split the proof in two subsections, the first dedicated to the function $\mathcal{I}_r$ and the second to the function $\mathcal{F}_r$.

7.1 The Analyticity Domain of $\mathcal{I}_r$

In this subsection we will establish the analyticity results on the functions $\mathcal{I}_r(z)$, $r \in \mathbb{N}$.

Analyticity of $\mathcal{I}_r(z)$ on $D_{<1}$ has already been proven in Theorem 3.5. Let us now consider $\mathcal{I}_r(z)$ on $D_{>1}$. For $r \in \mathbb{N}$, $\mathbf{p} \in \pi_a$ and $z \in D_{>1}$, we have

$$\mathcal{I}_r(z) \ = \ a \lim_{n \to \infty} \left[\tau_z^{(n)}(1)\right]^r \ = \ a \lim_{n \to \infty} \left[\tau_{z^{-1}}^{(n)}(1)\right]^{-r} \ = \ a \left[\tau(z^{-1})\right]^{-r} . \tag{7.1}$$

Therefore, in view of Theorem 3.6 which says that $\tau(w)$ has no zeros for $w \in D_{<1}$, $\mathcal{I}_r(z)$ is an analytic function on $D_{>1}$.

7.2 The Analyticity Domain of $\mathcal{F}_r$

In this subsection we will establish the analyticity results on the functions $\mathcal{F}_r(z)$, $r \in \mathbb{N}$. The proof will be divided in two subsections according to whether $z \in D_{<1}$ or $z \in D_{>1}$.

7.2.1 Analyticity in $D_{<1}$

We now prove our first result on the analyticity of $\mathcal{F}_r(z)$.

Theorem 7.1 (The First Analyticity Theorem) *The function $\mathcal{F}_r(z)$ defined in (1.13) is an analytic function of z in the open set $D_{<1}$ for all $\mathbf{p} \in \pi$ and $\zeta \in [0, 1)$.*

□

Proof. The partial sums

$$\mathcal{F}_{r,\,N}(z) := \sum_{n=0}^{N} a_n \, \left(\tau_z^{(n)}(1)\right)^r , \qquad r,\ N \in \mathbb{N}, \tag{7.2}$$

are analytic in $D_{<1}$ since the pole of h lies in $D_{>1}$. Moreover, by Corollary 2.2, one has $|\tau_z(1)| < 1$ for $z \in D_{<1}$ and so, for any $\varepsilon > 0$ one has

$$|\mathcal{F}_{r,\,N}(z) - \mathcal{F}_{r,\,M}(z)| < \sum_{n\,=\,N+1}^{M} a_n \le \varepsilon,$$

for all N and M, $N < M$, large enough, since the sequence $\{a_n,\ n \in \mathbb{N}\}$ is summable. So, the partial sums form a uniform Cauchy sequence of analytic functions and, therefore, their limit exists and is analytic in $D_{<1}$.

■

7.2.2 Analyticity in $D_{>1}\backslash\overline{\mathcal{Z}}$

We now prove the analyticity results on the function $\mathcal{F}_r(z)$ in $D_{>1}$. We start by studying the location of the singularities of $\mathcal{F}_r(z)$. First we prove the following lemma.

Lemma 7.2 *Let $\mathcal{B} \subset \mathbb{C}$ be an open set such that $\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, are analytic on $\mathcal{B}$ and*

$$\inf_{z\in\mathcal{B}}\ \inf_{n\in\mathbb{N}} \left| \tau_z^{(n)}(1) - \left(\frac{-1}{z\zeta}\right) \right| > 0. \tag{7.3}$$

Then $\mathcal{F}_r(z)$ is an analytic function in $\mathcal{B}$.

□

Remark 7.3 *Note that $\inf_{n\in\mathbb{N}} \left|\tau_z^{(n)}(1) - (-z^{-1}/\zeta)\right| = 0$ is not a necessary condition for $\mathcal{F}_r$ to be singular at z. For instance, for $z = -1$, one has $\left|\tau_z^{(n)}(1) - (-z^{-1}/\zeta)\right| = 1/\zeta - 1$ for all $n \in \mathbb{N}$. However, as we will see below, $\mathcal{F}_r(z)$ is not analytic at $z = -1$ because -1 is an accumulation point of poles of $\mathcal{F}_r$.*

Proof of Lemma 7.2. If condition (7.3) holds for a given $\mathcal{B}$ it implies that there exists a neighborhood $\mathcal{O}$ of $-1/\zeta$ such that, for all $z \in \mathcal{B}$ and for all $n \in \mathbb{N}$,

$$z\tau_z^{(n)}(1) \in \mathbb{C} \setminus \mathcal{O}.$$

Since $\tau_z^{(n+1)}(1) = h(z\tau_z^{(n)}(1))$, this means that there exists a positive constant M, such that for all $z \in \mathcal{B}$ and all $n \in \mathbb{N}$,

$$|\tau_z^{(n)}(1)| \leq M.$$

Hence, by the same argument used in the proof of Theorem 7.1, the partial sums $\mathcal{F}_{r,\,N}(z)$ form a uniform Cauchy sequence of analytic functions in $\mathcal{B}$ and consequently, their limit exists and is analytic in $\mathcal{B}$.

■

Our task now is to localize the points $z \in \mathbb{C}$ where the condition

$$\inf_{n\in\mathbb{N}} \left| \tau_z^{(n)}(1) - \left(\frac{-1}{z\zeta} \right) \right| = 0 \tag{7.4}$$

holds. The next proposition shows that, for each $n \in \mathbb{N}$, there exist at least one point $z \in \mathbb{C}$ such that $\left|\tau_z^{(n)}(1) - (-z^{-1}/\zeta)\right| = 0$ holds.

Proposition 7.4 *For each $n \in \mathbb{N}$ there is at least one and at most $2^{n+1} - 1$ distinct solutions in $\mathbb{C}$ of the equation*

$$\tau_z^{(n)}(1) - \left(\frac{-1}{z\zeta} \right) = 0. \tag{7.5}$$

□

Proof. As observed in Section 2, $\tau_z^{(n)}(1)$ is the ratio of two polynomials of degree $2^{n+1} - 2$:

$$\tau_z^{(n)}(1) = \frac{P_n(z)}{Q_n(z)}.$$

Therefore, (7.5) means that

$$zP_n(z) + \frac{1}{\zeta} Q_n(z) = 0, \tag{7.6}$$

which has at least one and at most $2^{n+1} - 1$ possible distinct solutions in $\mathbb{C}$, since the left hand side is a polynomial of degree $2^{n+1} - 1$. Clearly, for $n = 0$, the unique solution is the pole of h, $z = -1/\zeta$.

■

Let us denote by $\mathcal{Z}_n$ the finite set of all z's satisfying (7.5) for $n \in \mathbb{N}$. Define

$$\mathcal{Z} \;:=\; \bigcup_{a\in\mathbb{N}} \mathcal{Z}_a$$

and let $\overline{\mathcal{Z}}$ be the closure of $\mathcal{Z}$. Note that an infinite countable union of closed sets as $\mathcal{Z}$, is not in general a closed set. The set difference $\mathcal{Z}^0 := \overline{\mathcal{Z}} \setminus \mathcal{Z}$ is the set of accumulation points of $\mathcal{Z}$.

To establish that $\mathcal{F}_r$ is holomorphic in $\mathbb{C} \setminus \overline{\mathcal{Z}}$ we need to describe where the elements of $\mathcal{Z}$ and $\mathcal{Z}^0$ are located.

Theorem 7.5 *For all $\zeta \in (0,\ 1)$ and $k \in \mathbb{N}$ one has $\mathcal{Z}_k \subset D_{>1}$ with the accumulation points of $\mathcal{Z}$ lying in the unit circle, i. e., $\mathcal{Z}^0 \subset S^1$.*

In addition,

$$\mathcal{Z}_k \cap [1,\ \infty) \ = \ \emptyset$$

holds for all $k \in \mathbb{N}$. The sets $\mathcal{Z}_k$ are also self-conjugated in the sense that $\overline{z} \in \mathcal{Z}_k$ if $z \in \mathcal{Z}_k$ and, for all $k \in \mathbb{N}$, one has

$$\mathcal{Z}_k \cap (-\infty,\ -1) \ \neq \ \emptyset.$$

This last fact together with $\mathcal{Z}^0 \subset S^1$ implies that $-1 \in \mathcal{Z}^0$.

□

Proof. Firstly, observe that the equation

$$\tau_z^{(k)}(1) = -\frac{1}{z\zeta} \tag{7.7}$$

has no solutions for $z \in D_1 := D_{<1} \cup S^1$, since $\tau_z^{(k)}(1)$ maps D_1 into itself and $\zeta < 1$. This equation has no solutions for $|z| \geq 1/\zeta$ either, since in this region, by Corollary 2.2, the left hand side of (7.7) lies in $D_{>1}$, but the right hand side in D_1. We conclude that

$$\mathcal{Z} \ \subset \ D_{1,\,1/\zeta}.$$

Analogously, (7.7) has no solutions in $(1,\ \infty)$, since there its left hand side is positive while the right hand side is negative.

To prove that $\mathcal{Z}^0 \subset S^1$ we proceed as follows. The elements of $\mathcal{Z}_n$ are the poles of $\tau_z^{(n+1)}(1)$ in $D_{>1}$. Therefore, if $w \in \mathcal{Z}_n$, w^{-1} is a zero of $\tau_z^{(n+1)}(1)$ in $D_{<1}$. The sequence of functions $\tau_z^{(n)}$ converges uniformly in $D_{<1}$ to an analytic function $\tau = \tau(z)$ (Theorem 3.5). According to a theorem of Hurwitz (see, e.g., [T]) the fact that τ has no zeros on $D_{<1}$ (Theorem 3.6) implies that the set of zeros of the functions $\tau_z^{(n)}$ has no limit points on $D_{<1}$ and therefore, if they exist, they must lie on S^1.

The sets $\mathcal{Z}_k$ are all self-conjugated by the definition of h, since the left hand side of (7.6) is a polynomial with real coefficients. This polynomial is also of odd degree, namely $2^{k+1} - 1$. Both facts together imply that this polynomial has at least one real root. However, according to the previous remarks, the real roots cannot lie in $[-1,\ \infty)$. So, they must be negative and lower than -1. From the fact that $\mathcal{Z}^0 \subset S^1$ it follows that these negative roots must converge to -1 when $k \to \infty$ and so $-1 \in \mathcal{Z}^0$. This shows, in particular, that $\mathcal{Z}^0 \neq \emptyset$.

■

Remark 7.6 *Numerical computations seem to indicate that $\mathcal{Z}^0 \ = \ S^1$ for the whole ferromagnetic region $0 \leq \zeta < 1/3$. Unfortunately we were not able to prove this conjecture.*

After these considerations on the location of the singularities of $\tau_z^{(n)}$we turn to the proof of the analyticity statement on $\mathcal{F}_r$.

As above, let $\tau(w)$, $w \in D_{<1}$, be the uniform limit function of the sequence of functions $\tau_w^{(n)}(1)$, $n \in \mathbb{N}$.

Lemma 7.7 *Let $\mathcal{B}$ be an open set such that $\overline{\mathcal{B}} \subset D_{>1} \setminus \overline{\mathcal{Z}}$, where $\overline{\mathcal{B}}$ is the closure of $\mathcal{B}$. Then the sequence $\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, converges to the function $1/\tau(z^{-1})$ uniformly on $\mathcal{B}$.*

Proof . For $z \in \mathcal{B}$ call $w \equiv z^{-1}$ the "image" of z, and let the open set $\mathcal{B}_i$ given by

$$\mathcal{B}_i \;:=\; \Big\{ w \in D_{<1} \text{ such that w}^{-1} \in \mathcal{B} \Big\}$$

be the "image set" of $\mathcal{B}$.

From Theorem 3.6 we know that $\tau(w)$ has no zeros on $D_{<1}$. Hence, for $w \in \mathcal{B}_i$, $|\tau\,(w)\,| > \rho_1$, for some positive ρ_1. Next, we argue that there is also a positive constant ρ_2 such that $|\tau_w^{(n)}(1)| > \rho_2$ for all $n \in \mathbb{N}$ and all $w \in \mathcal{B}_i$. For, notice that no element of the sequence of functions $\tau_w^{(n)}(1)$ has a zero in $\mathcal{B}_i$. Since the sequence converges uniformly and the limit function has no zeros in $\mathcal{B}_i$, there must be a positive constant such that $|\tau_w^{(n)}(1)| > \rho_2$ for all $w \in \mathcal{B}_i$ and $n \in \mathbb{N}$. The desired uniformity follows now from

$$\left| \tau_z^{(n)} - \frac{1}{\tau(z^{-1})} \right| = \left| \frac{\tau(w) - \tau_w^{(n)}(1)}{\tau(w)\;\tau_w^{(n)}(1)} \right| \leq \frac{1}{\rho_1\,\rho_2}\,|\tau(w) - \tau_w^{(n)}(1)| < \varepsilon \tag{7.8}$$

for any prescribed $\varepsilon > 0$, uniformly on $\mathcal{B}_i$ and for n large enough since, on $\mathcal{B}_i$, the sequence $\tau_w^{(n)}(1)$ converges uniformly to $\tau(w)$.

■

Lemma 7.8 *Let $\mathcal{B}$ be as in Lemma 7.7. Then the condition*

$$\inf_{z\in\overline{\mathcal{B}}}\; \inf_{n\in\mathbb{N}} \left| \tau_z^{(n)}(1) - \left(\frac{-1}{z\zeta} \right) \right| = 0 \tag{7.9}$$

is impossible for any $\zeta \in (0,\ 1]$.

□

Proof . By the definition of $\mathcal{B}$, the condition $\tau_z^{(n)}(1) + 1/(z\zeta) = 0$ does not hold for any finite n. Therefore, condition (7.9) says that

$$\inf_{z\in\overline{\mathcal{B}}} \left| \frac{1}{\tau(z^{-1})} \;-\; \left(\frac{-1}{z\zeta} \right) \right| \;=\; 0.$$

Since $\overline{\mathcal{B}}$ is closed and $1\,/\tau(z^{-1}) \;+\; 1/(z\zeta)$ is a continuous function of z on $\overline{\mathcal{B}}$, this implies the existence of a point $u \in \overline{\mathcal{B}}$ such that $\tau(u^{-1}) = -u\,\zeta$. Since $\tau(w)$ fulfills the fixed point equation $\tilde{\tau} = h(w\,\tilde{\tau})$ for $w \in D_{<1}$, we conclude that $-u\,\zeta = h(-\zeta) \equiv 0$. But this is impossible for $\zeta > 0$ and $u \in D_{>1}$.

■

Theorem 7.9 (The Second Analyticity Theorem) *The function $\mathcal{F}_r(z)$ defined in (1.13) is a holomorphic function of z in the open set $D_{>1} \setminus \overline{\mathcal{Z}}$ for all $\mathbf{p} \in \pi$ and $\zeta \in (0, 1]$.*

If $a_n \neq 0$ the singularities of $\mathcal{F}_r$ at the points of $\mathcal{Z}_n$ are poles of order not greater than $(2^{n+1} - 1)\, r$.

□

Proof of Theorem 7.9. Pick an open set $\mathcal{B}$ such that $\overline{\mathcal{B}} \subset D_{>1} \setminus \overline{\mathcal{Z}}$, where $\overline{\mathcal{B}}$ is the closure of $\mathcal{B}$. Then, analyticity of $\mathcal{F}_r$ on $\mathcal{B}$ follows from Lemma 7.2 and Lemma 7.8. Since $D_{>1} \setminus \overline{\mathcal{Z}}$ can be covered by such open sets the theorem is proven.

■

7.2.3 Estimates on the Location of the Poles of $\tau_z^{(n)}(1)$

To give an idea of how fast the sets $\mathcal{Z}_k$ accumulate on S^1 we will make use of the following Contraction Theorem.

First, some definitions. For $w \in \mathbb{C}$, $w \neq 1/\zeta$, define

$$g(w) := \frac{\zeta - w}{\zeta w - 1}. \tag{7.10}$$

and let $u^{1/2}$ be some branch of the square root function in $\mathbb{C}$. Define

$$h_+^{-1}(u) \ := \ g(u^{1/2}), \qquad h_-^{-1}(u) \ := \ g(-u^{1/2}). \tag{7.11}$$

Then one has $h \circ h_\pm^{-1}(z) = z$, $\forall z \in \mathbb{C}$.

Theorem 7.10 *There exists a number $\zeta_0 \in [0,\ 1/3)$, whose approximate value is $\zeta_0 \simeq 0.29559$, such that for $\zeta \in [0,\ \zeta_0)$ there exists a strictly positive function $e(\zeta)$ such that for all $u \in D_{1,\, a(\zeta)}$ with*

$$a(\zeta) := \frac{1}{\zeta} + e(\zeta)$$

one has

$$|h_\pm^{-1}(u)| \ < \ |u|. \tag{7.12}$$

□

Remark 7.11 *Numerical computations indicate to be impossible to improve the region of validity of equation (7.12) to $\zeta \geq \zeta_0$.*

Remark 7.12 *As already observed, the inequality (7.12) becomes an equality in S^1, which is the internal boundary of $D_{1,\, a(\zeta)}$. It is important to note also that the set $D_{1,\, a(\zeta)}$ contains the the pole $z = -1/\zeta$, of h.*

We will present the proof of the Contraction Theorem in Subsection 7.2.4. Let us now explore some of its consequences.

Theorem 7.13 *Let ζ_0 as in Theorem 7.10. For all $\zeta \in (0,\ \zeta_0)$ and $k \in \mathbb{N}$*

$$\mathcal{Z}_k \subset D_{1,\, r_k}, \tag{7.13}$$

holds with

$$r_k = \zeta^{\frac{-1}{k+1}}.$$

Since the $\mathcal{Z}_k$ are finite sets, this clearly shows explicitly that the accumulation points $\mathcal{Z}^0$ of $\mathcal{Z}$ lie in the unit circle.

□

Proof. The proof of Theorem 7.13 makes use of the Contraction Theorem which requires the following technical lemma. We note that, from equation (7.7), there exists a finite sequence of signals $\{s(l) \in \{-,+\},\ 1 \le l \le k\}$ such that

$$\frac{1}{z} h^{-1}_{s(k)} \left(\frac{1}{z} h^{-1}_{s(k-1)} \left(\frac{1}{z} \cdots \frac{1}{z} h^{-1}_{s(1)} \left(-\frac{1}{z\zeta} \right) \cdots \right) \right) = 1. \tag{7.14}$$

Lemma 7.14 *Given $z \in \mathcal{Z}_k$, $k \in \mathbb{N}$, $k \ge 1$, consider a sequence of signals $\{s(l) \in \{-,+\},\ 1 \le l \le k\}$ satisfying (7.14) and define*

$$w_l := \frac{1}{z} h^{-1}_{s(l)} \left(\frac{1}{z} h^{-1}_{s(l-1)} \left(\frac{1}{z} \cdots \frac{1}{z} h^{-1}_{s(1)} \left(-\frac{1}{z\zeta} \right) \cdots \right) \right) \quad , l \in \{1, \ldots, k\}.$$

Then all w_l's belong to $D_{1,\, a(\zeta)}$, except, of course, w_k which is equal to 1.

□

Proof. To prove Lemma 7.14 we take, without loss of generality, $k > 1$ and note that, since $z \in D_{1, 1/\zeta}$, one has $h^{-1}_{s(1)}(-z^{-1}/\zeta) \in D_{1,1/\zeta}$, by the Contraction Theorem. Hence, $w_1 \in D_{\zeta,\, 1/\zeta}$.

On the other hand w_1 cannot belong to $D_{\zeta,\, 1} \cup S^1$ for the following reason: $h^{-1}_{\pm}$ maps $D_{<1} \cup S^1$ into itself and, since $1/|z| < 1$, this implies $w_2 \in D_{<1}$, what, in turn, would imply that all subsequent w_l's would also be in $D_{<1}$ and so they could not reach the point 1 after a finite number of steps.

Therefore, $w_1 \in D_{1,\, 1/\zeta}$ and we can apply again the Contraction Theorem and claim that $h^{-1}_{s(2)}(w_1) \in D_{1,\, 1/\zeta}$. So, we conclude again that $w_2 \in D_{\zeta,\, 1/\zeta}$. If $k = 2$ then, by the hypotheses, $w_2 = 1$. Otherwise, if $k > 2$, repeating the previous arguments, we conclude again that $w_2 \in D_{1,\, 1/\zeta}$. This argumentation can be repeated a finite number of times and therefore the lemma is proven.

■

Lemma 7.14 implies that all w_l, $1 \le l < k$, lie in the region of validity of the Contraction Theorem. Therefore, applying the inequality (7.12) repeatedly in (7.14) we get

$$1 \ < \ \frac{1}{|z|^{k+1}} \, \frac{1}{\zeta},$$

which proves (7.13). The fact that $\mathcal{Z}^0 \subset S^1$ follows easily from (7.13). This completes the proof of Theorem 7.13.

■

The Contraction Theorem has another consequence.

Theorem 7.15 *For the sets $\mathcal{Z}_k$ defined above we have, in the region of validity of the Contraction Theorem, that*

$$\mathcal{Z}_l \,\cap\, \mathcal{Z}_m \;=\; \emptyset,$$

for all $l,\, m \in \mathbb{N}$, $l \neq m$.

□

Proof. Let $m > l$ and pick a $z \in \mathcal{Z}_l \,\cap\, \mathcal{Z}_m$. Then one has

$$\tau_z^{(m-l)}\left(-\frac{1}{z\zeta}\right) \;=\; -\frac{1}{z\zeta}.$$

This means also that there exists a finite sequence $\{s(a) \in \{-,+\},\ 1 \leq a \leq m-l\}$ such that

$$-\frac{1}{z\zeta} = \frac{1}{z}h^{-1}_{s(m-l)}\left(\frac{1}{z}h^{-1}_{s(m-l-1)}\left(\frac{1}{z}\cdots\frac{1}{z}h^{-1}_{s(1)}\left(-\frac{1}{z\zeta}\right)\cdots\right)\right). \tag{7.15}$$

Applying the Contraction Theorem to the last equality we get

$$\frac{1}{|z|\zeta} < \frac{1}{|z|^{m-l+1}\zeta},$$

what means $|z|^{m-l} < 1$, a contradiction since $\mathcal{Z}_l \cap \mathcal{Z}_m \subset D_{1,\,1/\zeta}$.

■

7.2.4 The Contraction Theorem

This subsection is devoted to the proof of Theorem 7.10 which is implied by the following theorem.

Theorem 7.16 *There exists a $\zeta_0 \in [0,\ 1/3)$, whose approximate value is $\zeta_0 \simeq 0.29559$, such that there is a strictly positive function $f(\zeta)$ in the interval $\zeta \in [0,\ \zeta_0)$ such that, for $w \in D_{1,\,\frac{1}{\sqrt{\zeta}}+f(\zeta)}$, i.e., for*

$$1 < |w| < \frac{1}{\sqrt{\zeta}} + f(\zeta),$$

one has

$$|g(w)| \;<\; |w|^2.$$

□

In view of (7.10), (7.11) and Theorem 7.16, for $u \in \mathbb{C}$ such that

$$1 < |u| < \left(\frac{1}{\sqrt{\zeta}} + f(\zeta) \right)^2 =: \frac{1}{\zeta} + \epsilon(\zeta),$$

one has

$$|h_{\pm}^{-1}(u)| = |g(\pm u^{1/2})| < |u|.$$

This concludes the proof of Theorem 7.10.

■

Proof of Theorem 7.16. Writing $w = x + iy$, with $x, y \in \mathbb{R}$, a simple computation shows that we have

$$|g(w)|^2 = \frac{(\zeta^2 + |w|^2) - 2\zeta x}{(1 + \zeta^2 |w|^2) - 2\zeta x}. \tag{7.16}$$

To analyze the right hand side, consider the following lemma.

Lemma 7.17 *Let $t : \mathbb{R} \longmapsto \mathbb{R}$, be given by*

$$t(a) := \frac{\alpha - \beta a}{\gamma - \beta a},$$

for α, β and $\gamma \in \mathbb{R}$ with $\alpha > \gamma > 0$ and $\beta \geq 0$. Consider the open interval $J := (-\infty, \gamma/\beta)$. Then, t is a continuous, strictly positive, nowhere constant for $\beta > 0$ and increasing function in J.

□

Proof. Continuity on the interval J is obvious. A computation shows that

$$t'(a) = \frac{\beta(\alpha - \gamma)}{(\gamma - \beta a)^2}.$$

All claims follow immediately from continuity and from this relation.

■

Looking at (7.16) we can make the following identifications:

$$\alpha \equiv \zeta^2 + |w|^2; \quad \beta \equiv 2\zeta; \quad \gamma \equiv 1 + \zeta^2 |w|^2; \quad a \equiv x. \tag{7.17}$$

Note that, in view of this

$$\alpha - \gamma = (1 - \zeta^2)(|w|^2 - 1) > 0, \tag{7.18}$$

for all $w \in D_{>1}$. Moreover, the condition $a < \gamma/\beta$ means

$$x < \frac{1 + \zeta^2 |w|^2}{2\zeta}.$$

This condition is always satisfied if $|w| < 1/\zeta$, since $x \leq |w|$ and since the stronger condition

$$|w| < \frac{1+\zeta^2|w|^2}{2\zeta}$$

is equivalent to the condition $(1-\zeta|w|)^2 > 0$.

In view of Lemma 7.17 we conclude that, for $w \in D_{1,\,1/\zeta}$,

$$|g(w)|^2 \leq \frac{\zeta^2+|w|^2-2\zeta|w|}{1+\zeta^2|w|^2-2\zeta|w|} = \frac{(\zeta-|w|)^2}{(1-\zeta|w|)^2}.$$

Finally, note that $D_{1,\,1/\sqrt{\zeta}+f(\zeta)} \subset D_{1,1/\zeta}$ for a convenient f.

Now we come to the technically most important result of this section.

Theorem 7.18 *Under the assumptions of Theorem 7.16 one has*

$$\frac{(\zeta-|w|)^2}{(1-\zeta|w|)^2} < |w|^4. \tag{7.19}$$

□

Proof. Condition (7.19) is equivalent to the condition that the polynomial

$$P(t) := (\zeta-t)^2 - t^4(1-\zeta t)^2 \tag{7.20}$$

is strictly negative for $1 < t < \frac{1}{\sqrt{\zeta}} + f(\zeta)$. Writing it explicitly one has

$$\begin{aligned} P(t) &= -\zeta^2t^6 + 2\zeta t^5 - t^4 + t^2 - 2\zeta t + \zeta^2 \\ &= \left((\zeta-t)+t^2(1-\zeta t)\right)\left((\zeta-t)-t^2(1-\zeta t)\right). \end{aligned}$$

Using the fact that ± 1 is a root of the polynomial $(\zeta-t) \pm t^2(1-\zeta t)$, we can write

$$P(t) = (t-1)(t+1)\left(-\zeta t^2 + (1-\zeta)t - \zeta\right)\left(\zeta t^2 - (1+\zeta)t + \zeta\right).$$

This last relation allows us to find explicitly the roots of $P(t)$: they are the elements of the set $\{-1,\ +1,\ a_-,\ a_+,\ b_-,\ b_+\}$, where

$$a_\pm := \left(\frac{1-\zeta}{2\zeta}\right) \pm \frac{1}{2\zeta}\sqrt{(1+\zeta)(1-3\zeta)}$$

and

$$b_\pm := \left(\frac{1+\zeta}{2\zeta}\right) \pm \frac{1}{2\zeta}\sqrt{(1-\zeta)(1+3\zeta)}.$$

The following lemma gives the possible signs of the polynomial $P(t)$ on the real axis.

Lemma 7.19 *For $a_\pm$ and $b_\pm$ defined above as functions of ζ, one has the following results:*

1. *For $0 \leq \zeta < 1/3$ one has $0 \leq a_- \leq 1$.*
2. *For $0 \leq \zeta < 1$ one has $0 \leq b_- \leq 1$.*
3. *For $0 \leq \zeta < 1/3$ one has $b_+ - a_+ \geq 1$.*
4. *For $0 \leq \zeta < \zeta_0$ one has $a_+ - 1/\sqrt{\zeta} > 0$, where $\zeta_0 \in [0,\ 1/3)$, $\zeta_0 \simeq 0.29559$. This in particular says that $a_+ > 1$ and, by 3., $b_+ > 1$.*

□

Before we prove this lemma, let us finish the proof of Theorem 7.18 and, hence, of the Theorem 7.16. According to Lemma 7.19, $P(t) < 0$ if $1 < t < a_+$. This follows from the localization of the roots and from the fact that P is a polynomial of even degree with a negative leading term. See Figure 3.

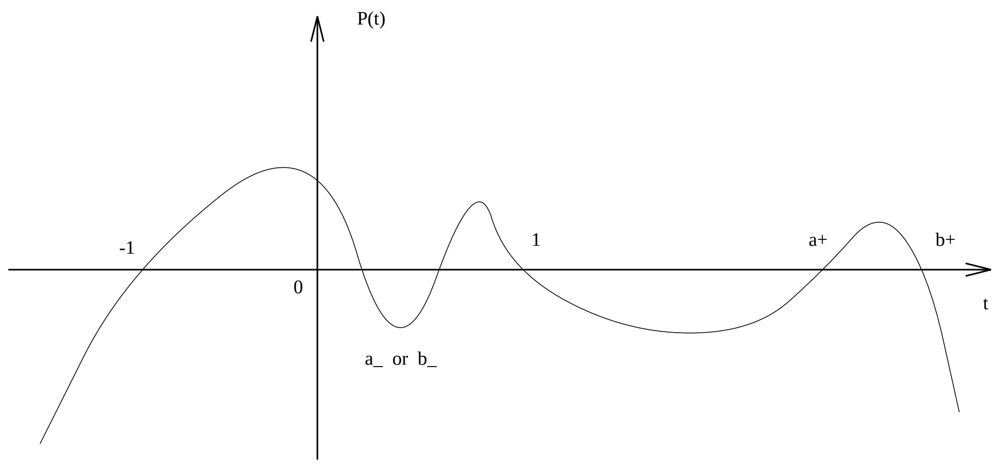

Figure 3: The graphic of $P(t)$.

The proof if completed by defining $f(\zeta) := a_+(\zeta) - 1/\sqrt{\zeta}$, which, according with item 4 of Lemma 7.19, is strictly positive for $0 \leq \zeta < \zeta_0$. This completes also the proof of the Contraction Theorem.

■

Proof of Lemma 7.19. We will prove each of the items of Lemma 7.19 separately.

Proof of item 1.

The hypothesis that $a_- \geq 0$ is equivalent to the hypothesis that

$$(1-\zeta)^2 \geq (1+\zeta)(1-3\zeta)$$

for $0 \leq \zeta < 1/3$. This last relation means $4\zeta^2 \geq 0$, which is obviously verified.

The hypothesis that $a_- \leq 1$ is equivalent to the hypothesis that

$$1 - \sqrt{(1+\zeta)(1-3\zeta)} \leq 3\zeta$$

which is equivalent to

$$(1-3\zeta)^2 \leq 1 - 2\zeta - 3\zeta^2.$$

This means $4\zeta(3\zeta - 1) \leq 0$, what is true for $0 \leq \zeta < 1/3$, the equality holding only if $\zeta = 0$.

Proof of item 2.

The hypothesis that $b_- \geq 0$ is equivalent to the hypothesis

$$(1+\zeta)^2 \geq (1-\zeta)(1+3\zeta),$$

which is equivalent to $4\zeta^2 \geq 0$, which, of course, is always true.

The hypothesis that $b_- \leq 1$ is equivalent to the hypothesis

$$(1-\zeta)^2 \leq (1-\zeta)(1+3\zeta).$$

This means that $4\zeta(\zeta - 1) \leq 0$, which is always true for $0 \leq \zeta \leq 1$.

Proof of item 3.

Since

$$b_+ - a_- = 1 + \frac{1}{2\zeta}\left(\sqrt{1+2\zeta-3\zeta^2} - \sqrt{1-2\zeta-3\zeta^2}\right),$$

item *3*. is proven, provided the term between parenthesis above is positive. This is implied by

$$1 + 2\zeta - 3\zeta^2 \geq 1 - 2\zeta - 3\zeta^2,$$

which is always true for $\zeta \geq 0$.

Proof of item 4.

The condition $a_+ > 1\big/\sqrt{\zeta}$ means that

$$\sqrt{(1+\zeta)(1-3\zeta)} > \zeta + 2\sqrt{\zeta} - 1. \tag{7.21}$$

The right hand side is strictly negative for $0 \leq \zeta < (\sqrt{2}-1)^2 \simeq 0.171$. So, in this region the condition above is automatically satisfied. On the other hand, if the right hand side of (7.21) is positive, we can square both sides and arrive to the equivalent condition

$$4s(s^3 + s^2 + s - 1) < 0 \tag{7.22}$$

where $s = \sqrt{\zeta}$. The polynomial $s^3 + s^2 + s - 1$ has one real root at $s_0 \simeq 0.543689$ and two complex roots at $s_\pm \simeq -0.77 \pm 1.115i$. We call $\zeta_0 := s_0^2$, which gives $\zeta_0 \simeq 0.295597$. Thus, condition (7.22) is satisfied for $0 < \zeta < \zeta_0$. With this the proof of Lemma 7.19 is complete.

■

8 The Two-Point Function

This section is dedicated to the proof of Theorems 1.9 and 1.11. We shall consider the truncated two-point function

$$\langle \sigma_0; \sigma_x \rangle_M (\xi) = \langle \sigma_0 \sigma_x \rangle_M (\xi) - \langle \sigma_0 \rangle_M (\xi) \ \langle \sigma_x \rangle_M (\xi) \tag{8.1}$$

where

$$\langle \sigma_0 \sigma_x \rangle_M (\xi) = \frac{1}{Z_M(\xi)} \sum_\sigma \sigma_0 \ \sigma_x \ e^{-\beta H(\sigma; \xi)} \tag{8.2}$$

for all $x \in \mathcal{C}_{2,M}$ with M large enough.

Let us first restrict to the case $\xi \equiv 1$. To compute the numerator of (8.2), we repeat the steps in the calculation of the partition function $Z_M(\xi)$ that led to (3.2).

Let $\widetilde{Z}_j = \left(\widetilde{Z}_j^+, \widetilde{Z}_j^-\right)$, $j = 0, \ldots, M$, be a sequence of two-component vectors defined recursively as in the following.

For $j = n_0 + 1, \ldots, M$, with $n_0 =$dist$(0, \ x)$, $\widetilde{Z}_j^\sigma = \widehat{Z}_j^\sigma$ satisfy the equation (3.1), i.e.,

$$\widehat{Z}_{j-1}^\sigma = (\zeta z)^{-1} \left(\zeta^{(1-\sigma)/2} \ \widehat{Z}_j^+ + \zeta^{(1+\sigma)/2} \ z \ \widehat{Z}_j^- \right)^2 \tag{8.3}$$

with initial conditions $\widehat{Z}_M^+ = \widehat{Z}_M^- = 1$.

For $j \le n_0$, we consider a linear transformation of the form

$$\widetilde{Z}_{j-1}^\sigma = \widehat{Z}_{j-1}^\sigma \frac{\zeta^{(1-\sigma)/2} \ \widetilde{Z}_j^+ + \zeta^{(1+\sigma)/2} \ z \ \widetilde{Z}_j^-}{\zeta^{(1-\sigma)/2} \ \widehat{Z}_j^+ + \zeta^{(1+\sigma)/2} \ z \ \widehat{Z}_j^-} \tag{8.4}$$

with $\widetilde{Z}_{n_0}^\sigma = \sigma \widehat{Z}_{n_0}^\sigma$.

We now observe that the sum over all spin configurations in the numerator of (8.2), except by spin at the origin, is determined by (8.4) and the sum over all spin configurations in the denominator is determined by (8.3). The two point function (8.2) can thus be written in the following form

$$\langle \sigma_0 \sigma_x \rangle_M (1) = \frac{\widetilde{Z}_0^+ - z \ \widetilde{Z}_0^-}{\widehat{Z}_0^+ + z \ \widehat{Z}_0^-}. \tag{8.5}$$

In the following lemma the equation (8.5) will be reorganized and reexpressed in terms of $\widehat{Z}_j^\sigma$. We recall the definition $\widehat{\Delta}_j := \widehat{Z}_j^- \Big/ \widehat{Z}_j^+$.

Lemma 8.1 *The sequence of vectors $\widetilde{Z}_n$, $n = 1, \ldots, n_0$, defined by (8.4), can be written as*

$$\widetilde{Z}_{n-1}^\sigma = (\mathcal{A}_n + \sigma \mathcal{B}_n) \, \widehat{Z}_{n-1}^\sigma, \tag{8.6}$$

where

$$\mathcal{A}_n = A_n \mathcal{B}_{n+1} + \ldots + A_{n_0 - 1} \mathcal{B}_{n_0} + A_{n_0} \tag{8.7}$$

and

$$\mathcal{B}_j = B_j B_{j+1} \ldots B_{n_0} \qquad , \qquad j = n, \ldots, n_0 \tag{8.8}$$

Here $A_j = A_j(\zeta, z)$ and $B_j = B_j(\zeta, z)$, $j = 1, \dots, n_0$, are given by

$$A_j = \frac{\left(z\widehat{\Delta}_j\right)^{-1} - z\widehat{\Delta}_j}{\left(z\widehat{\Delta}_j\right)^{-1} + z\widehat{\Delta}_j + \zeta^{-1} + \zeta} \tag{8.9}$$

and

$$B_j = \frac{\zeta^{-1} - \zeta}{\left(z\widehat{\Delta}_j\right)^{-1} + z\widehat{\Delta}_j + \zeta^{-1} + \zeta}. \tag{8.10}$$

□

Proof. We shall prove Lemma 8.1 by induction. We let $j = n_0$ and observe that, by multiplying the numerator and the denominator of (8.4) by $\left(\zeta^{-(1-\sigma)/2}\left(z\widehat{\Delta}_{n_0}\right)^{-1} + \zeta^{-(1+\sigma)/2}\right)/\widehat{Z}^{+}_{n_0}$, it can be written as

$$\frac{\widetilde{Z}^{\sigma}_{n_0-1}}{\widehat{Z}^{\sigma}_{n_0-1}} = \frac{\left(z\widehat{\Delta}_{n_0}\right)^{-1} - z\widehat{\Delta}_{n_0} + \zeta^{-\sigma} - \zeta^{\sigma}}{\left(z\widehat{\Delta}_{n_0}\right)^{-1} + z\widehat{\Delta}_{n_0} + \zeta^{-\sigma} + \zeta^{\sigma}} = A_{n_0} + \sigma B_{n_0}. \tag{8.11}$$

Now, let $j = n+1$. Assuming (8.6) valid, (8.4) can be written as

$$\frac{\widetilde{Z}^{\sigma}_{n}}{\widehat{Z}^{\sigma}_{n}} = \mathcal{A}_{n+1} + \frac{\zeta^{(1-\sigma)/2} - \zeta^{(1+\sigma)/2}\, z\, \widehat{\Delta}_{n+1}}{\zeta^{(1-\sigma)/2} + \zeta^{(1+\sigma)/2}\, z\, \widehat{\Delta}_{n+1}}\mathcal{B}_{n+1}. \tag{8.12}$$

Multiplying the numerator and the denominator by $\zeta^{-(1-\sigma)/2}\left(z\widehat{\Delta}_{n+1}\right)^{-1} + \zeta^{-(1+\sigma)/2}$, gives

$$\frac{\widetilde{Z}^{\sigma}_{n}}{\widehat{Z}^{\sigma}_{n}} = \mathcal{A}_{n+1} + \left(A_n + \sigma B_n\right)\mathcal{B}_{n+1}$$

which, in view of (8.7) and (8.8), concludes the proof of Lemma 8.1.

■

Now we proceed with the proof of Theorem 1.9. Using (8.6) to compute (8.5) gives

$$\begin{aligned} \langle \sigma_0 \sigma_x \rangle_M (1) &= \frac{\left(\mathcal{A}_1 + \mathcal{B}_1\right)\widehat{Z}^{+}_0 - z\left(\mathcal{A}_1 - \mathcal{B}_1\right)\widehat{Z}^{-}_0}{\widehat{Z}^{+}_0 + z\widehat{Z}^{-}_0} \\ &= \mathcal{A}_1 \frac{1 - z\widehat{\Delta}_0}{1 + z\widehat{\Delta}_0} + \mathcal{B}_1. \end{aligned} \tag{8.13}$$

The one-point function at x can be analogously represented by equations of the form (8.5) and (8.13). It can thus be written as

$$\langle \sigma_x \rangle_M (1) = \frac{\widetilde{Z}^{+}_0 + z\,\widetilde{Z}^{-}_0}{\widehat{Z}^{+}_0 + z\,\widehat{Z}^{-}_0} = \mathcal{A}_1 + \mathcal{B}_1 \frac{1 - z\widehat{\Delta}_0}{1 + z\widehat{\Delta}_0}. \tag{8.14}$$

Inserting (8.13) and (8.14) into (8.1) and taking into account (3.4), yields

$$\langle \sigma_0; \sigma_x \rangle_M (1) = \left[1 - \left(\frac{1 - z\widehat{\Delta}_0}{1 + z\widehat{\Delta}_0} \right)^2 \right] \mathcal{B}_1. \tag{8.15}$$

For any given value of ξ the truncated two-point function is still given by (8.15) provided that, for each B_j in $\mathcal{B}_1$, $\widehat{\Delta}_j$ is replaced by the random variables Δ_j and the variable ζ is replaced by $\zeta_j = e^{-2\beta\xi_j}$ (see remark after Proposition 3.1). In the thermodynamic limit, $M \to \infty$, the random variables Δ_j's in (8.15) can be replaced by their limits $\lim_{M\to\infty} \Delta_j$ (recall that $\mathcal{B}_1$ depends on $\{\Delta_j\}_{j=0}^{n_0}$, each of which defined by a recursion relation with initial condition $\Delta_M = 1$ dependent on the generation M). Since Δ_j is bounded from above and below, the convergence in distribution is guaranteed by the convergence of their moments [Fe]. It follows from Theorem 1.8 that $\lim_{M\to\infty} \mathbb{E}_\xi \Delta_j$ exists and is a real analytic function of z in $z \in \mathbb{R}_+$.

In order to take the expectation value of (8.15) we shall use the following property: for any bounded function $f(\xi) = f(\xi_j, \xi')$ of the random variables $\xi = (\xi_j, \xi')$, we have

$$\mathbb{E}_\xi f(\xi) B_j(\xi) = p_j \mathbb{E}_{\xi'} f(1, \xi') \frac{\zeta^{-1} - \zeta}{(z\Delta_j)^{-1} + z\Delta_j + \zeta^{-1} + \zeta} \tag{8.16}$$

since $\zeta_j^{-1} - \zeta_j = e^{2\beta\xi_j} - e^{-2\beta\xi_j} = 0$ if $\xi_j = 0$.

Define

$$s(x) \;:=\; 1 - \left(\frac{1 - x}{1 + x} \right)^2 \tag{8.17}$$

and

$$t(x) \;=\; t(\zeta, x) \;:=\; \frac{\zeta^{-1} - \zeta}{x^{-1} + x + \zeta^{-1} + \zeta}. \tag{8.18}$$

Using (8.16) in the expected value of (8.15) gives

$$\begin{aligned} \mathbb{E}_\xi \langle \sigma_0;\ \sigma_x \rangle_M (\xi) \;&=\; p_1 \dots p_{n_0}\ \mathbb{E}_{\xi'} s(z\Delta_0) \left[\prod_{j=1}^{n_0} t(z\Delta_j) \right] \\ &=\; \sum_{k=n_0}^{M-1} a_k\ s(w_k) \left[\prod_{j=1}^{n_0} t(w_{k-j}) \right] + p_1 \dots p_M\ s(w_M) \left[\prod_{j=1}^{n_0} t(w_{M-j}) \right] \end{aligned} \tag{8.19}$$

where $w_n = z\tau_z^{(n)}(1), n \in \mathbb{N}$. Here, we have first taken partial expectations with respect to the variables $\xi_1, \dots \xi_{n_0}$ and use the fact that the sequence Δ_j, $j \in \mathbb{N}$, satisfies the recurrence relation $z\Delta_{j-1} = zh_j(z\Delta_j)$ with $h_j(1) = 1$ in the remaining expectations. Equation (1.16) then follows since $s(x)$ and $t(x)$ are continuous in $\mathbb{R}_+$ and w_n converges to the solution $w = w(\zeta, z)$ of the fixed point equation $w = zh(w)$ in this domain provided $z \in \mathbb{R}_+$.

This concludes the proof of Theorem 1.9.

■

We now turn to the proof of Theorem 1.11 on the quenched susceptibility at origin χ. We note the following facts on the function s and t (the proof will be omitted):

Proposition 8.2 *The function $s : w \in \mathbb{R}_+ \longmapsto s(w) \in \mathbb{R}_+$ and $t : (\zeta, w) \in [0,1] \times \mathbb{R}_+ \longmapsto t(\zeta, w) \in \mathbb{R}_+$ given by (8.17) and (8.18), respectively, have a maximum value at $w = 1$ with $s(1) = 1$ and $t(\zeta, 1) = (1-\zeta)/(1+\zeta)$, are monotonically increasing function of w in $[0,1]$ and satisfy $s(w) = s(w^{-1})$ and $t(\zeta, w) = t(\zeta, w^{-1})$.*

□

For $z \in [0,1)$ we recall that $w_n, n \in \mathbb{N}$, is a monotonically decreasing sequence with $w_n < 1$ and for each n, $w_n = w_n(z)$ is monotonically decreasing function of z in this domain. So, in view of Proposition 8.2,

$$2t(\zeta, w_n) \leq 2\frac{1-\zeta}{1+\zeta} < 1 \tag{8.20}$$

holds for all $\zeta \in (1/3, 1]$ and $z \in \mathbb{R}_+ \backslash \{1\}$.

In addition, for $\zeta \in [0, 1/3)$ (ferromagnetic phase) and $z \in [0,1)$, w_n converges to a nontrivial solution $w(\zeta, z)$ of $w = zh(w)$ (see Proposition 6.3) with $w \nearrow \underline{w}$ as $z \nearrow 1$ with

$$\underline{w} = \frac{1 - 2\zeta - \zeta^2}{2\zeta^2} - \frac{1-\zeta^2}{2\zeta^2}\left(\frac{1-3\zeta}{1+\zeta}\right)^{1/2} .$$

This and Proposition 8.2 implies that there exist finite number $n_0 = n_0(\zeta, z) \in \mathbb{N}$ such that for $n > n_0$

$$2t(\zeta, w_n) \leq 2t(\zeta, \underline{w}) = \frac{2\zeta}{1-\zeta} < 1 \tag{8.21}$$

holds for all $\zeta \in [0, 1/3)$ and $z \in \mathbb{R}_+ \backslash \{1\}$

Substituting equation (1.16) into (1.17) and taking into account the inequalities (8.20) and (8.21), gives

$$\begin{aligned}
\chi &= \sum_{n=0}^{\infty} 2^n \sum_{k=n}^{\infty} a_k \; s(w_k) \; \left[\prod_{j=1}^{n} t(w_{k-j})\right] + a \; s(w) \sum_{n=0}^{\infty} \; [2t(w)]^n \\
&= \sum_{k=0}^{\infty} a_k \; s(w_k) \sum_{n=0}^{k} \; 2^n \left[\prod_{j=1}^{n} t(w_{k-j})\right] + \frac{a \; s(w)}{1-2t(w)} .
\end{aligned} \tag{8.22}$$

Note that

$$\lim_{z \to 1} \frac{a \; s(w)}{1 - 2t(w)} \sim C_\eta \, |\eta| \qquad \text{as} \qquad \eta := \zeta - \zeta_c \to 0$$

holds with $\lim_{\eta \searrow 0} C_\eta = 4a/3$ and $\lim_{\eta \nearrow 0} C_\eta = 2a/3$. Note also that, for $\mathbf{p} \in \pi_a$, $0 < a \leq 1$, such that $\sum_n n \, a_n < \infty$, the first term on the second line of (8.22) does not diverge at $z = 1$ for any $\zeta \in [0,1]$ since, by the dominated convergence theorem,

$$\lim_{x \nearrow 1} \sum_{k=0}^{\infty} \frac{1-x^k}{1-x} \, a_k < \infty .$$

This concludes the proof of Theorem 1.11.

■

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