METASTABILITY OF THE THREE DIMENSIONAL ISING MODEL ON A TORUS AT VERY LOW TEMPERATURES

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ABSTRACT. We study the metastability of the stochastic three dimensional Ising model on a finite torus under a small positive magnetic field at very low temperatures.

1. INTRODUCTION

The theoretical study of metastability has become a field of active research during the last years [2,10,11,12,13,14,15]. The crucial problem is to understand the exit path of a system close to a phase transition relaxing from a metastable state towards a stable state. For the physical motivation of this problem, we refer the reader to [14,15]. The basic situation to investigate the metastability is the stochastic Ising model evolving according to a Glauber or Metropolis dynamics. The first step was accomplished by Neves and Schonmann [12,13,15] who studied the case of the two dimensional Ising model in a finite volume at very low temperatures. They put forward the essential role played by the droplets: small droplets are likely to shrink and disappear whereas big ones tend to grow. The threshold between these opposite behaviours corresponds to the critical droplets whose energy is exactly the energy barrier the system has to overcome to escape from the metastable state. The nucleation occurs through the formation of such a critical droplet. Several more complicated models have also been investigated in the regime of low temperatures, but still in two dimensions: an anisotropic case [10] and a case with second nearest

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neighbours interactions [11]. It became progressively clear that all the relevant information to describe the exit path was to be found in the energy landscape associated to the model.

In fact, these Metropolis dynamics at very low temperatures have been thoroughly studied in a more abstract setting in the context of stochastic optimization. Large deviations techniques and the Wentzell–Freidlin theory provide the most powerful tools to analyze these dynamics depending upon a small parameter. The central idea is to decompose the space into a hierarchy of particular subsets called cycles. Roughly speaking, the cycles are the most stable sets we can build for the perturbed process. Their fundamental property is that the process can't escape from a cycle without having forgotten its entrance point. The limiting dynamics is completely determined by the cycle decomposition. For simulated annealing, the best way to cool down the temperature schedule heavily depends upon some quantities associated to the cycle decomposition [3,4,5,6]. For parallel simulated annealing and generalizations, see also [17,18,19]. For genetic algorithms, the cycle decomposition is the key for understanding the behaviour of the process and for controlling the algorithm in order to ensure the convergence towards the global maxima [8]. For the study of metastability, a general model independent method to search efficiently for the exit path was developed in [7]. Our technique focus on the crucial elements of the cycle decomposition necessary to find the exit path. More precisely, we seek only the sequence of cycles the system might visit during the exit. Such a precise and economical method is indispensable to deal with the three dimensional Ising model: the energy landscape is so huge and complex that the least inaccuracy, the least deviation from the metastable path is disastrous and leads to inextricable problems. To apply our general method to a particular model, we need to solve some global and local variational problems. This combinatorial part for the three dimensional Ising model was handled in [1].

This paper is organized as follows. We first describe the model and we recall the main definitions concerning the cycle decomposition. We describe briefly the general method for finding the exit path. We prove some essential facts for the cycles of the Ising model which are valid in any dimension. We then handle the two dimensional case; most results of this section have already been proved by Neves and Schonmann [12,13,15]. We sharpen their results and reprove them using a technique which will also work in three dimensions. We finally handle the three dimensional situation. Partial results for the three dimensional case had been announced or proved by Neves and Schonmann. Yet a systematic and efficient approach was necessary in order to make one's way in this very complex energy landscape.

2. Description of the model and main result

We consider a finite box $\Lambda = \{1 \cdots N\}^d$ of side N in the d–dimensional integer lattice $\mathbb{Z}^d$. We will work either in two dimensions ($d = 2$) or three dimensions ($d = 3$). A point of this box is called a site. Sites will be denoted by the letters x, y, z. We wrap this box into a torus and we define a neighbourhood relation on Λ by: $x \sim y$ if all the coordinates of x and y are equal except one which differs by 1 or $N - 1$. At each site x of Λ there is a spin

taking the values -1 or $+1$. The set of all possible spins configurations is $X = \{-1, +1\}^\Lambda$. Configurations of spins will be denoted by the letters η, σ, ξ. The value of the spin at site x for a configuration σ is denoted by $\sigma(x)$.

We might see a configuration as a subset of points of the integer lattice $\mathbb{Z}^d$, these points being the sites of the configurations having a positive spin. The inclusion relation is then defined on the configurations by

$$\eta \subset \sigma \quad \Longleftrightarrow \quad \forall x \in \Lambda \quad \eta(x) = +1 \Rightarrow \sigma(x) = +1 \,.$$

The elementary sets operations $\cap$, $\cup$, $\setminus$ are defined by

$$\forall x \in \Lambda \qquad (\sigma \cup \eta)(x) \;=\; \begin{cases} +1 & \text{if } \sigma(x) = +1 \text{ or } \eta(x) = +1 \\ -1 & \text{if } \sigma(x) = \eta(x) = -1 \end{cases}$$

$$\forall x \in \Lambda \qquad (\sigma \cap \eta)(x) \;=\; \begin{cases} +1 & \text{if } \sigma(x) = \eta(x) = +1 \\ -1 & \text{if } \sigma(x) = -1 \text{ or } \eta(x) = -1 \end{cases}$$

$$\forall x \in \Lambda \qquad (\sigma \setminus \eta)(x) \;=\; \begin{cases} +1 & \text{if } \sigma(x) = +1 \text{ and } \eta(x) = -1 \\ -1 & \text{if } \sigma(x) = -1 \text{ or } \eta(x) = +1 \end{cases}$$

There is a natural correspondence between configurations and polyominoes. To a configuration we associate the polyomino which is the union of the unit cubes centered at the sites having a positive spin. The main difference between configurations and polyominoes is that the polyominoes are defined up to translations. We will constantly use the notations introduced in [1] for polyominoes. When we say that a polyomino is included in a configuration, we mean that one of its representative is included in it. For instance, the square $l \times l$ is included in σ if there exists (α, β) such that $l \times l(\alpha, \beta) \subset \sigma$.

The energy of a configuration σ is

$$E(\sigma) \;=\; -\frac{1}{2} \sum_{\{x,y\}:x\sim y} \sigma(x)\sigma(y) \;-\; \frac{h}{2} \sum_{x\in\Lambda} \sigma(x)$$

where $h > 0$ is the external magnetic field.
The Gibbs probability measure μ at inverse temperature β is

$$\forall \sigma \in X \qquad \mu(\sigma) \;=\; \frac{1}{Z} \exp -\beta E(\sigma)$$

where Z is the partition function.
We build the stochastic Ising model by letting the spins evolve randomly in time. The spin at site x flips with the rate $c(x, \sigma)$ when the system is in the configuration σ. The

standard construction yields a continuous time Markov process $(\sigma_t)_{t\geq 0}$ with state space X and infinitesimal generator L defined by

$$\forall f : \Lambda \to \mathbb{R} \qquad (Lf)(\sigma) \;=\; \sum_{x\in\Lambda} c(x,\sigma)\big(f(\sigma^x) - f(\sigma)\big)$$

where σ^x is the configuration σ flipped at x i.e.

$$\sigma^x(y) \;=\; \begin{cases} +\sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x \end{cases}$$

We will consider the Metropolis dynamics associated to the rates

$$c(x,\sigma) = \exp -\beta(\Delta_x E(\sigma))^+$$

where $(\quad)^+$ denotes the positive part and

$$\Delta_x E(\sigma) \;=\; E(\sigma^x) - E(\sigma) \;=\; \sigma(x)\big(\sum_{y:y\sim x} \sigma(y) + h\big).$$

Since these rates satisfy the detailed balance condition $\mu(\sigma)c(x,\sigma) \;=\; \mu(\sigma^x)c(x,\sigma^x)$, we know that the Gibbs measure μ is the only invariant measure for the process (σ_t).
Here we will deal with the equivalent discrete time version of the Metropolis dynamics. We first define a kernel q on X by $q(\eta,\sigma) = N^{-d}$ if there exists x in Λ such that $\eta^x = \sigma$ (i.e η and σ differ by one spin flip) and $q(\eta,\sigma) = 0$ otherwise. The kernel q is irreducible and symmetric. The transition kernel p_β at inverse temperature β is the Markov kernel on X defined by

$$\begin{aligned} \forall \eta \neq \sigma \qquad p_\beta(\eta,\sigma) \;&=\; q(\eta,\sigma)\, \exp -\beta\big(E(\sigma) - E(\eta)\big)^+ \\ \forall \eta \qquad p_\beta(\eta,\eta) \;&=\; 1 - \sum_{\sigma\neq\eta} p_\beta(\eta,\sigma). \end{aligned}$$

We will study the time homogeneous Markov chain $(\sigma_n)_{n\in\mathbb{N}}$ having for transition probabilities the Markov kernel p_β.
We denote by $\underline{-1}$ (respectively $\underline{+1}$) the configuration with all spins down (resp. up).
For D an arbitrary subset of X we define the time $\tau(D,m)$ of exit from D after time m

$$\tau(D,m) \;=\; \min\{\, n \geq m : \sigma_n \notin D \,\}$$

(we make the convention that $\tau(D) = \tau(D,0)$).
For instance, $\tau(\underline{+1}^c)$ is the hitting time of the ground state $\underline{+1}$.
We define also the time $\theta(D,m)$ of the last visit to the set D before time m

$$\theta(D,m) \;=\; \max\{\, n \leq m : \sigma_n \in D \,\}$$

(if the chain has not visited D before m, we take $\theta(D,m)=0$).
For instance, $\theta(\underline{-1},\tau(\underline{+1}^c))$ is the last visit to the metastable state $\underline{-1}$ before reaching $\underline{+1}$. We are interested in the laws of $\tau(\underline{+1}^c)$ starting from $\underline{-1}$ (time of the relaxation to equilibrium) and of the way of escaping from the metastable state $\underline{-1}$ to reach $\underline{+1}$, that is the law of the exit path $((\sigma_n),\ \theta(\underline{-1},\tau(\underline{+1}^c)) \le n \le \tau(\underline{+1}^c))$, in the limit of vanishing temperature $\beta\to\infty$.

Our main result (theorem 7.33) is the following.

Theorem 2.1. *(the exit path)*
For any positive ϵ, the following events take place with probability converging to one exponentially fast as β goes to infinity:
- $\exp\beta(E(n_c)-\epsilon) \le \tau(\underline{+1}^c) \le \exp\beta(E(n_c)+\epsilon)$, *where* $n_c = j_c(j_c+\delta_c)(j_c+1)+l_c(l_c+1)+1$ *is the critical volume,* $l_c=\lfloor 2/h\rfloor$ *is the two dimensional critical length,* $j_c=\lfloor 4/h\rfloor$ *is the three dimensional critical length,* $\delta_c=0$ *if* $4+\sqrt{16+h^2} < h(2\lfloor 4/h\rfloor+1)$ *and* $\delta_c=1$ *otherwise;*
- $\exp\beta(E_{2c}-\epsilon) \le \tau(\underline{+1}^c)-\theta(\underline{-1},\tau(\underline{+1}^c)) \le \exp\beta(E_{2c}+\epsilon)$, *where* E_{2c} *is the two dimensional global energy barrier given in proposition* 6.15*;*
- *during the exit path* $\sigma_n, \theta\le n\le\tau$, *the process crosses the set of critical configurations* $\mathcal{S}_{n_c}$ *at exactly one point* σ_c*; it does not contain another configuration of volume* n_c*;*
- *if we let* $n_*=\min\{n\ge\theta : v(\sigma_n)=n_c\}$, $n^*=\max\{n\le\tau : v(\sigma_n)=n_c\}$, *we have that* $\sigma_n=\sigma_{n_*}=\sigma_{n^*}$ *for all* n *in* $\{n_*\cdots n^*\}$ *and* $n^*-n_*\le\exp(\beta\epsilon)$*;*
- *all the configurations of the exit path before time* n_* *are of volume less than* n_c*, all configurations after* n^* *are of volume greater than* n_c*;*
- *during the ascending part, the process goes through an increasing sequence of quasicubes* $(j\times(j+\delta)\times(j+\theta),\ j\le j_c)$*. It visits a quasicube* $j\times(j+\delta)\times(j+\theta)$ *approximately* $\exp\beta(j-1)h$ *times if* $j\le l_c$ *and* $\exp\beta(E_2(l_c(l_c+1)+1)-E_2(j(j+\delta)))$ *times if* $j>l_c$*. When it definitely leaves a quasicube, it fills the right side of the quasicube in order to obtain the next larger quasicube, this filling occurs by following the two dimensional nucleation mechanism; in particular, it does not make a parallelepiped which is not a quasicube (there is no growth mistake);*
- *during the descending part, the process goes through an increasing sequence of parallelepipeds* $(j_1\times j_2\times j_3,\ \min(j_1,j_2,j_3)>j_c)$*. It visits a parallelepiped* $j_1\times j_2\times j_3$ *approximately* $\exp\beta E_{2c}$ *times. When it definitely leaves a quasicube, it fills a side of the parallelepiped chosen at random in order to obtain a larger parallelepiped, this filling occurs by following the two dimensional nucleation mechanism;*
- *the process reaches the thermal equilibrium within each cycle of* $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$ *it crosses: the distribution of the process before the exit of the cycle is very close to the Gibbs distribution at inverse temperature* β *restricted to the cycle.*

Let us notice a completely new phenomenon occurring in dimension three. Before growing a quasicube or a parallelepiped whose sides are larger than l_c, the system will

build every configuration reachable below the energy barrier E_{2c}. This energy barrier goes to infinity like $4/h$ when h goes to zero and can thus be terribly high. In particular, the system will form small clusters very far from the main cluster. There will be small sparks appearing everywhere on the torus. It will also try to build very strange shapes around the main cluster, like small antennae and wires. For h small enough, it will write the complete works of Shakespeare together with the ninth Symphony of Ludwig Van Beethoven.

3. Decomposition into cycles

The most efficient tool to analyze the dynamical behaviour of a Markov chain with exponentially vanishing transition probabilities is the hierarchical decomposition of the state space into cycles. This fundamental notion is due to Freidlin and Wentzell [9] and was sharpened in the context of the simulated annealing by Catoni. We recall here briefly the main definitions introduced by Catoni [3] related to the hierarchical decomposition into cycles of the energy landscape (X, q, E) (that is a finite set X equipped with a communication kernel q and the energy E). We are here in the reversible case and the cycle decomposition can be built directly with the help of the energy E.

Definition 3.1. (communication at level λ)
Let λ be a real number. The communication relation at level λ on the landscape (X, q, E) is the equivalence relation $\mathcal{R}_\lambda$ defined by $\sigma \mathcal{R}_\lambda \eta$ if and only if there exists a sequence $\xi_0, \cdots, \xi_r$ of points in X such that

$$\xi_0 = \sigma,\ \xi_r = \eta, \quad \forall i \in \{0 \cdots r-1\} \quad q(\xi_i, \xi_{i+1}) > 0, \quad \forall i \in \{0 \cdots r\} \quad E(\xi_i) \leq \lambda.$$

Definition 3.2. (cycles)
A cycle π is a subset of X which is an equivalence class of a relation $\mathcal{R}_\lambda$ for some λ in $\mathbb{R}$. The smallest such λ is denoted by $\lambda(\pi)$ and is called the level of the cycle π. For σ in X and λ in $\mathbb{R}$ we denote by $\pi(\sigma, \lambda)$ the equivalence class of σ for the relation $\mathcal{R}_\lambda$.

Proposition 3.3. *The sets reduced to one single point are cycles. Two cycles are either disjoint or comparable for the inclusion relation.*

Notation 3.4. Since X is finite, the set $\{ E(\sigma) : \sigma \in X \}$ is also finite. For λ in $\mathbb{R}$ we set

$$\begin{aligned} \operatorname{pred} \lambda &= \max\{ E(\sigma) : E(\sigma) < \lambda,\ \sigma \in X \}, \\ \operatorname{succ} \lambda &= \min\{ E(\sigma) : E(\sigma) > \lambda,\ \sigma \in X \}. \end{aligned}$$

Definition 3.5. (maximal partition)
Let D be a subset of X. The partition of D into its maximal subcycles is denoted by $\mathcal{M}(D)$. For σ in D we denote by $\pi(\sigma, D)$ the unique cycle of $\mathcal{M}(D)$ containing σ (for σ in D^c we make the convention that $\pi(\sigma, D) = \{\sigma\}$).

Remark. In case D is a cycle, we have $\mathcal{M}(D) = \{D\}$.

Remark. There is no ambiguity between the two notations $\pi(\sigma, \lambda)$ and $\pi(\sigma, D)$, since λ and D are different objects.

Definition 3.6. (communication altitude)
The communication altitude between two points σ and η of X is the smallest λ such that $\sigma \mathcal{R}_\lambda \eta$. It is denoted by $E(\sigma, \eta)$. Equivalently, it is the level of the smallest cycle containing the points σ and η. The communication altitude between two sets A and B is defined by $E(A, B) = \min\{ E(a, b) : a \in A,\, b \in B \}$.

Definition 3.7. Let π be a cycle. We define

- its energy $E(\pi) = \min\{ E(\sigma) : \sigma \in \pi \}$,
- its bottom $F(\pi) = \{ \sigma \in \pi : E(\sigma) = E(\pi) \}$,
- its boundary $B(\pi) = \{ \sigma \notin \pi : \exists\, \eta \in \pi \quad q(\eta, \sigma) > 0 \}$,
- its height $H(\pi) = \min\{ (E(\sigma) - E(\pi))^+ : \sigma \in B(\pi) \}$,
- its principal boundary $\widetilde{B}(\pi) = \{ \sigma \in B(\pi) : E(\sigma) \leq E(\pi) + H(\pi) \}$.

Remark. If the height $H(\pi)$ is positive, then the principal boundary $\widetilde{B}(\pi)$ is exactly the set $\{ \sigma \in B(\pi) : E(\sigma) = E(\pi) + H(\pi) \}$.

The main part of the work here consists in analyzing the specific energy landscape of the Ising model and some crucial facts concerning its cycle decomposition.

4. The exit saddle path

In this section, we make a brief summary of the general strategy devised in [7] in order to describe the exit path of the process. We take into account the simplifications induced by the reversibility of the process and just state the essential points. A good way to describe the exit path is to look at the points of entrance and exit of the cycles of $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ the system visits during the exit path. The knowledge of this sequence of points includes the knowledge of the sequence of the cycles visited by the process. Our work will consists in finding the most probable of these sequences i.e. those whose probability does not vanish as β goes to infinity.

Definition 4.1. (saddle path)
We define recursively a sequence of random times and points:

$$
\begin{array}{lll}
\tau_0 = \theta(\underline{-1}, \tau(\underline{+1}^c)) + 1 & s_0 = X_{\tau_0 - 1} & s_1 = X_{\tau_0} \\
\tau_1 = \tau(\pi(s_1, \{\underline{-1}, \underline{+1}\}^c), \tau_0) & s_2 = X_{\tau_1 - 1} & s_3 = X_{\tau_1} \\
\quad \vdots & \quad \vdots & \quad \vdots \\
\tau_k = \tau(\pi(s_{2k-1}, \{\underline{-1}, \underline{+1}\}^c), \tau_{k-1}) & s_{2k} = X_{\tau_k - 1} & s_{2k+1} = X_{\tau_k} \\
\quad \vdots & \quad \vdots & \quad \vdots \\
\tau_r = \tau(\underline{+1}^c) & s_{2r} = X_{\tau_r - 1} & s_{2r+1} = X_{\tau_r}.
\end{array}
$$

Clearly we have $s_0 = \underline{-1}$ and $s_{2r+1} = \underline{+1}$. The sequence $(\underline{-1}, s_1, \cdots, s_{2r}, \underline{+1})$ is called the saddle path of (X_n) relative to $\underline{+1}^c, \underline{-1}$ and denoted by $\mathcal{S}(X, \underline{+1}^c, \underline{-1})$. We define a cost function V on the set of the saddle paths:

$$V(s_0, \cdots, s_{2r+1}) = -H(\pi(\underline{-1}, \underline{+1}^c)) + \big(E(s_1) - E(s_0)\big)^+ + \sum_{k=1}^{r} \Big(E(s_{2k+1}) - H(\pi(s_{2k}, \{\underline{-1}, \underline{+1}\}^c)) - E(\pi(s_{2k}, \{\underline{-1}, \underline{+1}\}^c))\Big).$$

We proved in [7] that the law of the saddle path satisfies a large deviation principle with rate function V. As a consequence, with a probability converging to one exponentially fast as β goes to infinity, the exit saddle path follows a saddle path of null cost. What we have to do is to search for all these saddle paths. The task of determining the set of all the saddle paths of null V–cost can be performed in the following way:

i) find the points σ' of the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$;

ii) for each such point σ' in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$, determine all the sequences of cycles $\pi_0 \cdots \pi_r$ in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ such that $\sigma' \in \pi_0$, $\widetilde{B}(\pi_k) \cap \pi_{k+1} \neq \emptyset$ for k in $\{0 \cdots r-1\}$, $\underline{+1} \in \widetilde{B}(\pi_r)$.

iii) for each such cycle path $\pi_0, \cdots, \pi_r$ determine all the saddle paths $(\sigma', s_1, \cdots, s_{2r}, \underline{+1})$ such that $(s_{2k-1}, s_{2k}) \in \pi_k$ for k in $\{1 \cdots r-1\}$, and (s_{2k}, s_{2k+1}) is an optimal saddle path exiting from π_k for all k in $\{0 \cdots r\}$ i.e. $s_{2k} \in \pi_k$, $s_{2k+1} \in \widetilde{B}(\pi_k)$, $q(s_{2k}, s_{2k+1}) > 0$.

iv) for each point σ' in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ such that step *ii)* has succeeded, determine all the sequences of cycles $\pi'_0 \cdots \pi'_{r'}$ in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ such that $\sigma' \in \pi'_0$, $\widetilde{B}(\pi'_k) \cap \pi'_{k+1} \neq \emptyset$ for k in $\{0 \cdots r'-1\}$, $\underline{-1} \in \widetilde{B}(\pi_{r'})$.

v) for each such cycle path $\pi'_0, \cdots, \pi'_{r'}$ determine all the saddle paths $(\sigma', s'_1, \cdots, s'_{2r'}, \underline{-1})$ such that $(s'_{2k-1}, s'_{2k}) \in \pi'_k$ for k in $\{1 \cdots r'-1\}$, and (s'_{2k}, s'_{2k+1}) is an optimal saddle path exiting from π'_k for all k in $\{0 \cdots r'\}$ i.e. $s'_{2k} \in \pi'_k$, $s'_{2k+1} \in \widetilde{B}(\pi'_k)$, $q(s'_{2k}, s'_{2k+1}) > 0$.

vi) the set of all the saddle paths of null V–cost passing trough σ' is the set of the saddle paths $(s'_{2r'+1}, \cdots, s'_1, \sigma', \sigma', s_1, \cdots, s_{2r+1})$ obtained by glueing together a reversed saddle path obtained at step *v)* with a saddle path obtained at step *iii)*.

The global variational problem consists in finding the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$. The local variational problems consist in finding the principal boundaries of the cycles appearing in the cycle paths starting at the points of this principal boundary. Notice that we took advantage of the reversibility of the process to complete steps *iv)* and *vi)*. We used the fact that the portion of the saddle path between $\underline{-1}$ and σ' in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$ can be obtained by reversing a saddle path starting at σ' and reaching $\underline{-1}$ (see last section of [7]). In the general case, the variational condition imposed on this portion of the saddle path is not easy to manipulate.

Remark that step *ii)* may fail: it might happen that there is no cycle path of null cost in $\mathcal{M}(\underline{-1}, \underline{+1})$ starting at σ' reaching $\underline{+1}$. In that case, there is no saddle path of null cost

passing through σ' and it is not necessary to perform the step v): such a point σ' leads to a dead–end. What we do at step ii) is to look at all the cycles we can reach from σ' in the graph over $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$ defined by the relation

$$\pi_1 \to \pi_2 \qquad \Longleftrightarrow \qquad \widetilde{B}(\pi_1) \cap \pi_2 \neq \emptyset$$

i.e. we determine the orbit of the cycle $\{\sigma'\} = \pi(\sigma', \{\underline{-1},\underline{+1}\}^c)$ in this graph (i.e. the minimal stable subgraph containing $\{\sigma'\}$). The good cycle paths are the cycle paths included in this graph which end at a cycle whose principal boundary contains $\underline{+1}$.

5. The bottom of the cycles

The results of this section are valid in any dimension d. They describe some essential facts concerning the geometry of the cycles of the energy landscape of the Ising model. Theorem 5.3 concerns the points of the cycle we might reach starting from a configuration belonging to the bottom of the cycle. Theorem 5.5 gives a condition implying that the bottom of the cycle is reduced to one point. These theorems will be crucial to handle the two and three dimensional cases. Their proofs rely heavily on the following inequality.

Theorem 5.1. *(an energy inequality)*
For any configurations η, σ, ξ of X such that $\eta \subset \sigma \subset \xi$, we have (for positive h)

$$E(\sigma \setminus \eta) \, - \, E(\sigma) \; \leq \; E(\xi \setminus \eta) \, - \, E(\xi).$$

Remark. This inequality might be interpreted as follows. The variation of energy when we turn down a fixed set of spins of a configuration σ increases when we enlarge σ.

Proof. We compute $E(\sigma \setminus \eta)$:

$$\begin{aligned}
E(\sigma \setminus \eta) \; &= \; -\frac{1}{2} \sum_{\{x,y\}:x\sim y} (\sigma \setminus \eta)(x)(\sigma \setminus \eta)(y) \, - \, \frac{h}{2} \sum_{x\in\Lambda} (\sigma \setminus \eta)(x) \\
&= \; -\frac{1}{2}\Bigg(\sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=\sigma(y)=-1}} 1 \, + \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=1,\sigma(y)=-1}} \eta(x) \, + \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=\sigma(y)=1}} \eta(x)\eta(y) \Bigg) \\
&\qquad\qquad - \, \frac{h}{2}\Bigg(\sum_{x:\sigma(x)=-1} -1 \, + \sum_{x:\sigma(x)=1} -\eta(x) \Bigg) \\
&= \; -\frac{1}{2}\Bigg(\sum_{\{x,y\}:x\sim y} \sigma(x)\sigma(y) \, + \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=1,\sigma(y)=-1}} (1+\eta(x)) \, + \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=\sigma(y)=1}} (\eta(x)\eta(y)-1) \Bigg) \\
&\qquad\qquad - \, \frac{h}{2}\Bigg(\sum_{x} \sigma(x) \, + \sum_{x:\sigma(x)=1} -(1+\eta(x)) \Bigg)
\end{aligned}$$

whence

$$E(\sigma \setminus \eta) - E(\sigma) \;=\; -\frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=1,\sigma(y)=-1}} (1+\eta(x)) + \frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=\sigma(y)=1}} (1-\eta(x)\eta(y)) + \frac{h}{2} \sum_{x:\sigma(x)=1} (1+\eta(x)).$$

Taking into account that $\eta \subset \sigma$ we can simplify the summation sets to get

$$E(\sigma \setminus \eta) - E(\sigma) \;=\; -\frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(y)=-1}} (1+\eta(x)) + \frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=\sigma(y)=1}} (1-\eta(x)\eta(y)) + \frac{h}{2} \sum_{x} (1+\eta(x)).$$

Only the first two terms of the righthand side depend on σ. Yet for any configuration ξ such that $\sigma \subset \xi$ we have

$$-\frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(y)=-1}} (1+\eta(x)) \;\le\; -\frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \xi(y)=-1}} (1+\eta(x)),$$

$$\frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \sigma(x)=\sigma(y)=1}} (1-\eta(x)\eta(y)) \;\le\; \frac{1}{2} \sum_{\substack{\{x,y\}:x\sim y \\ \xi(x)=\xi(y)=1}} (1-\eta(x)\eta(y))$$

so that finally $E(\sigma \setminus \eta) - E(\sigma) \le E(\xi \setminus \eta) - E(\xi)$. $\square$

Notations 5.2. Let x be a site in Λ. We define the spin flip operator $F(x) : X \to X$ by $F(x)(\sigma) = \sigma^x$ where we recall that

$$\sigma^x(y) \;=\; \begin{cases} +\sigma(y) & \text{if } y \neq x \\ -\sigma(y) & \text{if } y = x \end{cases}$$

Let $x_1, \cdots, x_r$ be a sequence of sites in Λ. We define the operator $F(x_1, \cdots, x_r) : X \to X$ by $F(x_1, \cdots, x_r) = F(x_r) \circ \cdots \circ F(x_1)$.
Let σ belong to X. To σ we associate two operators $S_+(\sigma)$ and $S_-(\sigma)$ on the sequences of sites. The sequence $S_-(\sigma)(x_1, \cdots, x_r)$ (respectively $S_+ \cdots)$) is the subsequence of $(x_1, \cdots, x_r)$ consisting of the sites x_i such that $\sigma(x_i) = -1$ (resp. $\sigma(x_i) = +1)$).

Remark. For any sequence of sites $(x_1, \cdots, x_r)$ and any configuration σ we have

$$F(S_+(\sigma)(x_1, \cdots, x_r))(\sigma) \;\subset\; \sigma \;\subset\; F(S_-(\sigma)(x_1, \cdots, x_r))(\sigma).$$

Theorem 5.3. *Let π be a cycle and let σ be a configuration of the bottom $F(\pi)$ of π. Let $(x_1, \cdots, x_r)$ be a sequence of sites such that $F(x_1, \cdots, x_j)(\sigma)$ belongs to π for all j in $\{1 \cdots r\}$. Then $F(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$ and $F(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)$ also belong to π. Equivalently, if we let $\eta = F(x_1, \cdots, x_r)(\sigma)$, then $\eta \cup \sigma$ and $\eta \cap \sigma$ are in π.*

Proof. We prove the result by induction on the length r of the sequence of sites $(x_1, \cdots, x_r)$. The result is obviously true for $r = 0$ and $r = 1$. Assume it is true at rank r and let $(x_1, \cdots, x_{r+1})$ be a sequence of sites such that $F(x_1, \cdots, x_j)(\sigma)$ belongs to π for all j in $\{1 \cdots r+1\}$. Let $\xi = F(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$ and let η be the unique configuration included in σ such that $\sigma \setminus \eta = F(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)$. We have $\eta \subset \sigma \subset \xi$ and theorem 5.1 yields $E(\xi) - E(\xi \setminus \eta) \leq E(\sigma) - E(\sigma \setminus \eta)$ or equivalently

$$E(F(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) + E(F(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \leq E(F(x_1, \cdots, x_{r+1})(\sigma)) + E(\sigma). \quad (1)$$

By hypothesis, $\xi \setminus \eta = F(x_1, \cdots, x_{r+1})(\sigma) \in \pi$ whence $E(\xi \setminus \eta) \leq \lambda(\pi)$.

- First case: $\sigma(x_{r+1}) = -1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1, \cdots, x_{r+1}) &= (S_-(\sigma)(x_1, \cdots, x_r), x_{r+1}), \\ S_+(\sigma)(x_1, \cdots, x_{r+1}) &= S_+(\sigma)(x_1, \cdots, x_r). \end{aligned}$$

By the induction hypothesis, we know that

$$F(S_-(\sigma)(x_1, \cdots, x_r))(\sigma) \in \pi, \qquad F(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma) \in \pi.$$

Now $\xi = F(x_{r+1}) \circ F(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$ i.e. ξ communicates by one spin flip with a configuration of π. Moreover $\sigma \setminus \eta$ belongs to π whence $E(\sigma) \leq E(\sigma \setminus \eta)$ (because $\sigma \in F(\pi)$). Inequality (1) shows that $E(\xi)$ is less than the level of π so that ξ is in π.

- Second case: $\sigma(x_{r+1}) = +1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1, \cdots, x_{r+1}) &= S_-(\sigma)(x_1, \cdots, x_r), \\ S_+(\sigma)(x_1, \cdots, x_{r+1}) &= (S_+(\sigma)(x_1, \cdots, x_r), x_{r+1}). \end{aligned}$$

By the induction hypothesis, we know that

$$F(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma) \in \pi, \qquad F(S_+(\sigma)(x_1, \cdots, x_r))(\sigma) \in \pi.$$

Inequality (1) shows that $E(\sigma \setminus \eta)$ is less than the level of π. Since $\sigma \setminus \eta$ differs by one spin flip (at site x_{r+1}) from $F(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)$ which is in π, $\sigma \setminus \eta$ is also in π.
Thus the induction is completed. □

Corollary 5.4. *Let π be a cycle an let σ be an element of $F(\pi)$. If η is a minimal (respectively maximal) element of π with respect to the inclusion, then there exists a sequence of sites $(x_1, \cdots, x_r)$ such that $\eta = F(x_1, \cdots, x_r)(\sigma)$, $F(x_1, \cdots, x_j)(\sigma) \in \pi$ and $\sigma(x_j) = +1$ (resp. $\sigma(x_j) = -1$) for all j in $\{1 \cdots r\}$. In particular, $\eta \subset \sigma$ (resp. $\sigma \subset \eta$).*

Theorem 5.5. *Let π be a cycle and let σ be a configuration of π satisfying:*
for each ϵ in $\{-1, +1\}$ and each sequence of sites $(x_1, \cdots, x_l)$ in Λ^l such that

$$F(x_1, \cdots, x_l)(\sigma) \neq \sigma, \qquad \forall j \in \{1 \cdots l\} \quad \sigma(x_j) = \epsilon, \quad F(x_1, \cdots, x_j)(\sigma) \in \pi,$$

we have $E(F(x_1, \cdots, x_l)) > E(\sigma)$.
Then the configuration σ also satisfies:
for each sequence of sites $(x_1, \cdots, x_r)$ in Λ^r such that $F(x_1, \cdots, x_j)(\sigma)$ belongs to π for all j in $\{1 \cdots r\}$, the configurations $F(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)$ and $F(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)$ are in π and

$$\begin{aligned}
E(F(x_1, \cdots, x_r)(\sigma)) &\geq E(F(S_-(\sigma)(x_1, \cdots, x_r))(\sigma)) \geq E(\sigma), \\
E(F(x_1, \cdots, x_r)(\sigma)) &\geq E(F(S_+(\sigma)(x_1, \cdots, x_r))(\sigma)) \geq E(\sigma), \\
E(F(x_1, \cdots, x_r)(\sigma)) &= E(\sigma) \implies F(x_1, \cdots, x_r)(\sigma) = \sigma.
\end{aligned}$$

The bottom $F(\pi)$ of the cycle π is then reduced to this single configuration σ.

Proof. We prove the result by induction on the length r of the sequence of sites $(x_1, \cdots, x_r)$ appearing in the conclusion of the theorem. For $r = 0$ and $r = 1$ it is true. Assume it is true until rank r and let $(x_1, \cdots, x_{r+1})$ be a sequence of sites such that $F(x_1, \cdots, x_j)(\sigma)$ belongs to π for all j in $\{1 \cdots r+1\}$. The induction hypothesis yields that for all j in $\{1 \cdots r\}$ the configurations $F(S_-(\sigma)(x_1, \cdots, x_j))(\sigma))$, $F(S_+(\sigma)(x_1, \cdots, x_j))(\sigma))$ are in π and moreover

$$\begin{aligned}
E(F(x_1, \cdots, x_j)(\sigma) &\geq E(F(S_-(\sigma)(x_1, \cdots, x_j))(\sigma)) \geq E(\sigma), \\
E(F(x_1, \cdots, x_j)(\sigma) &\geq E(F(S_+(\sigma)(x_1, \cdots, x_j))(\sigma)) \geq E(\sigma), \\
E(F(x_1, \cdots, x_j)(\sigma) &= E(\sigma) \implies F(x_1, \cdots, x_j)(\sigma) = \sigma.
\end{aligned}$$

Now theorem 5.1 yields

$$\begin{aligned}
E(F(x_1, \cdots, x_{r+1})(\sigma)) - E(F(S_-(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) \geq \\
(2) \qquad\qquad E(F(S_+(\sigma)(x_1, \cdots, x_{r+1}))(\sigma)) - E(\sigma)
\end{aligned}$$

By hypothesis, $F(x_1, \cdots, x_{r+1})(\sigma)$ is in π so that $E(F(x_1, \cdots, x_{r+1})(\sigma) \leq \lambda(\pi)$.

- First case: $\sigma(x_{r+1}) = -1$. In this situation, we have

$$\begin{aligned}
S_-(\sigma)(x_1, \cdots, x_{r+1}) &= (S_-(\sigma)(x_1, \cdots, x_r), x_{r+1}), \\
S_+(\sigma)(x_1, \cdots, x_{r+1}) &= S_+(\sigma)(x_1, \cdots, x_r).
\end{aligned}$$

By the induction hypothesis, we know that $F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)$ is in π. Since for all j in $\{1\cdots r+1\}$, $F(S_+(\sigma)(x_1,\cdots,x_j))(\sigma)$ is in π, the hypothesis on σ implies that $E(F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) \geq E(\sigma)$. Inequality (2) implies

$$E(F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) \leq E(F(x_1,\cdots,x_{r+1})(\sigma)) \leq \lambda(\pi)$$

and $F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma))$ is in π, since its energy is less than the level of π and it differs by one spin flip (at site x_{r+1}) from a configuration of π. The hypothesis on σ finally implies that $E(F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) \geq E(\sigma)$.

- Second case: $\sigma(x_{r+1}) = +1$. In this situation, we have

$$\begin{aligned} S_-(\sigma)(x_1,\cdots,x_{r+1}) &= S_-(\sigma)(x_1,\cdots,x_r), \\ S_+(\sigma)(x_1,\cdots,x_{r+1}) &= (S_+(\sigma)(x_1,\cdots,x_r),x_{r+1}). \end{aligned}$$

By the induction hypothesis, we know that $F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)$ is in π. Since for all j in $\{1\cdots r+1\}$, $F(S_-(\sigma)(x_1,\cdots,x_j))(\sigma)$ is in π, the hypothesis on σ implies that $E(F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) \geq E(\sigma)$. Inequality (2) implies

$$E(F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) \leq E(F(x_1,\cdots,x_{r+1})(\sigma)) \leq \lambda(\pi)$$

and $F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma))$ is in π, since its energy is less than the level of π and it differs by one spin flip (at site x_{r+1}) from a configuration of π. The hypothesis on σ finally implies that $E(F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) \geq E(\sigma)$.
In addition the case of equality in inequality (2) is the case where

$$E(F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) = E(F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma)) = E(\sigma).$$

The hypothesis on σ then implies that

$$F(S_-(\sigma)(x_1,\cdots,x_{r+1}))(\sigma) = F(S_+(\sigma)(x_1,\cdots,x_{r+1}))(\sigma) = \sigma$$

or equivalently $F(x_1,\cdots,x_{r+1})(\sigma) = \sigma$. The induction is completed. □

6. Dimension two

A polyomino associated to a configuration is a finite union of squares. For σ a configuration we will denote by $p(\sigma)$ the perimeter of the associated polyomino and by $a(\sigma)$ its area. We will rely heavily on the notations and results of [1]. Let us recall briefly some essential points.

Summary of the combinatorial results.

The rectangle of sides l_1 and l_2 is denoted by $l_1 \times l_2$. By $l_1 \times l_2 +_1 k$ (resp. $l_1 \times l_2 +_2 k$) we denote the rectangle $l_1 \times l_2$ plus a vertical (resp. horizontal) bar of length k sticked against the side l_2 (resp. l_1) starting from the bottom (resp. from the left). When we speak of $l_1 \times l_2 +_1 k$, we assume implicitly that the length of the bar is less than l_2. When we don't specify the direction, i.e. we write $l_1 \times l_2 + k$, it means that the direction is not relevant, and the statement holds for both $l_1 \times l_2 +_1 k$ and $l_1 \times l_2 +_2 k$. We use the operator $+^i$ to specify the position of the bar: $l_1 \times l_2 +_1^i k$ means that the bar starts at the $(i+1)$–th square along the side of $l_1 \times l_2$. Finally, $l_1 \times l_2 \oplus_1 k$ is the set of all the polyominoes obtained by sliding the bar of length k along the side l_2 of the rectangle (in such a way that the polyomino is always included in $(l_1 + 1) \times l_2$). For A a set of polyominoes, we denote by $\overline{A}$ its orbit under the action of the planar isometries which leave the integer lattice $\mathbb{Z}^2$ invariant. By $\overline{A}^{12}$ we denote its orbit under the action of the two symmetries with respect to the axis.

Proposition 6.1. *For each integer n there exists a unique 3–uple (l, k, ϵ) such that*

$$\epsilon \in \{0, 1\}, \quad 0 \leq k < l + \epsilon \quad \textit{and} \quad n = l(l + \epsilon) + k.$$

The set of the polyominoes of area n is C_n; the set M_n of the minimal polyominoes of area n is the set of the polyominoes of C_n having minimal perimeter. Let $n = l(l + \epsilon) + k$ be the decomposition of n. The canonical polyomino of area n is

$$m_n = \begin{cases} l \times l +_1 1 \times k & \text{if } \epsilon = 0 \\ (l + 1) \times l +_2 k \times 1 & \text{if } \epsilon = 1 \end{cases}$$

Theorem 6.2. *The polyomino m_n is minimal.*

We define several important subsets of M_n. The set S_n of the standard polyominoes is

$$S_n \;=\; \begin{cases} \overline{l \times l \oplus_1 1 \times k} & \text{if } \epsilon = 0 \\ \overline{(l + 1) \times l \oplus_2 k \times 1} & \text{if } \epsilon = 1 \end{cases}$$

The set $\widetilde{M}_n$ of the principal polyominoes is

$$\widetilde{M}_n \;=\; \overline{l \times (l + \epsilon) \oplus_1 1 \times k} \bigcup \overline{l \times (l + \epsilon) \oplus_2 k \times 1}.$$

Theorem 6.3. *The set M_n is reduced to $\{ m_n \}$ if and only if n is of the form l^2.*
The set M_n coincides with S_n if and only if n is of the form $l^2, l(l + 1) - 1, l(l + 1)$, in which case $S_n = \overline{m}_n$. The set M_n coincide with $\widetilde{M}_n$ if and only if the integer n is of the form $l^2, l(l + 1) - 1, l(l + 1), (l + 1)^2 - 1$.

Lemma 6.4. *For n of the form $l^2, l(l+1)$, we have*

$$\{\, c \in M_{n-1} : q(M_n, c) > 0 \,\} \;=\; S_{n-1},$$
$$\{\, c \in M_{n+1} : q(M_n, c) > 0 \,\} \;=\; \widetilde{M}_{n+1}$$

(where $q(M_n, c) = \min\{q(d, c) : q \in M_n\}$).

Theorem 6.5. *The principal polyominoes can be completely shrinked through the minimal polyominoes: for any integer n and for any principal polyomino c in $\widetilde{M}_n$, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$ and $c_n = c$ and $q(c_{i-1}, c_i) > 0$ for i in $\{1 \cdots n\}$.*

Theorem 6.6. *The standard polyominoes can be grown or shrinked arbitrarily far through minimal polyominoes: for any integers $m \leq n$ and for any standard polyomino c in S_m, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$, $c_m = c$ and $q(c_{i-1}, c_i) > 0$ for i in $\{1 \cdots n\}$.*

Notation 6.7. We have also some results concerning the best way to shrink or to grow a rectangle, that is on the sets

$$M(l_1 \times l_2, -k) \;=\; \{\, c \in C_{l_1 l_2 - k} : c \subset l_1 \times l_2,\ p(c) \text{ minimal} \,\},$$
$$M(l_1 \times l_2, +k) \;=\; \{\, c \in C_{l_1 l_2 + k} : l_1 \times l_2 \subset c,\ p(c) \text{ minimal} \,\}.$$

We denote by $E(l_1 \times l_2, k)$ the energy of an element of $M(l_1 \times l_2, k)$ for k in $\mathbb{Z}$.
A natural way to remove (add) k squares (for $k < l_1, k < l_2$) is to remove (add) a line on a side of the rectangle; thus we define

$$S(l_1 \times l_2, -k) \;=\; \overline{(l_1 - 1) \times l_2 \oplus_1 1 \times (l_2 - k)}^{12} \bigcup \overline{l_1 \times (l_2 - 1) \oplus_2 (l_1 - k) \times 1}^{12},$$
$$S(l_1 \times l_2, k) \;=\; \overline{\{\, l_1 \times l_2 \oplus_2 k \times 1,\, l_1 \times l_2 \oplus_1 1 \times k \,\}}^{12}.$$

Theorem 6.8. *Let l_1, l_2, k be positive integers such that $k < l_1$, $k < l_2$.*
The set $M(l_1 \times l_2, -k)$ is the set of the polyominoes obtained by removing successively k corner squares from $l_1 \times l_2$. In particular, $S(l_1 \times l_2, -k)$ is included in $M(l_1 \times l_2, -k)$.

Theorem 6.9. *Let l_1, l_2, k be positive integers such that $k < l_1$, $k < l_2$.*
The set $M(l_1 \times l_2, +k)$ is equal to the set $S(l_1 \times l_2, k)$.

Application to the two dimensional Ising model.

We first express the energy of the Ising model with the help of the perimeter and the area of the polyomino associated to the configuration.

Lemma 6.10. *For any configuration σ in X, we have*

$$E(\sigma) \;=\; -\frac{1}{2}\sum_{\{x,y\}:x\sim y}\sigma(x)\sigma(y) \;-\; \frac{h}{2}\sum_{x\in\Lambda}\sigma(x) \;=\; p(\sigma) \;-\; ha(\sigma) \;+\; N^2(h/2-1)\,.$$

We do not change the dynamics nor the cycle decomposition by adding a constant to the energy E. In dimension two, we will work with the energy $E(\sigma) = p(\sigma) - ha(\sigma)$. We denote by $E(n)$ the minimal energy of a configuration of C_n (so that $E(n) = E(\sigma) = p(n) - hn$ for any σ in M_n).

Hypothesis on the magnetic field h and the size N. We suppose that h is small compared to the unity and that for any configurations η, σ, the equality $E(\eta) = E(\sigma)$ implies $a(\eta) = a(\sigma)$ and $p(\eta) = p(\sigma)$. Whenever we take the integral part of a quantity involving h, we assume that this quantity is not an integer. For instance $2/h$ is not an integer. Finally, N is large enough to ensure that the combinatorial results proved on the infinite lattice $\mathbb{Z}^2$ remain valid on the torus until the critical area $2/h(2/h+1)$. This is obviously true if $h^2N > 6$.

Remark. A careful study would yield the weaker condition $h^2N^2 > K$ for some constant K: this would require to push further the techniques used in [1].

We now follow the general strategy outlined in section 4. Our first aim is to determine the communication altitude between $\underline{-1}$ and $\underline{+1}$ (Corollary 6.14) and to compute it explicitly (Proposition 6.15).

Proposition 6.11. *Let $m \le n$ be two integers. The communication altitude between the sets C_m and C_n is greater or equal than* $\max\{\, E(r) : m \le r \le n \,\}$.

Proof. Let $\sigma_0, \cdots, \sigma_s$ be a sequence of configurations such that $\sigma_0 \in C_m$, $\sigma_s \in C_n$ and $q(\sigma_i, \sigma_{i+1}) > 0$ for i in $\{0 \cdots s-1\}$. Necessarily we have $|a(\sigma_i) - a(\sigma_{i+1})| \le 1$ so that the sequence of integers $a(\sigma_0), \cdots, a(\sigma_s)$ takes all the values between $m = a(\sigma_0)$ and $n = a(\sigma_s)$. Henceforth

$$\max_{0\le i\le s} E(\sigma_i) \;\ge\; \max_{0\le i\le s} E(a(\sigma_i)) \;\ge\; \max_{m\le r\le n} E(r)\,. \quad \square$$

Proposition 6.12. *Let σ belong to S_m and let m_1, m_2 in $\mathbb{N}$ be such that $m_1 \le m \le m_2$. There exist σ_1, σ_2 in S_{m_1}, S_{m_2} satisfying*

$$E(\sigma_1, \sigma) \;=\; \max\{\, E(n) : m_1 \le n \le m \,\}\,,$$
$$E(\sigma, \sigma_2) \;=\; \max\{\, E(n) : m \le n \le m_2 \,\}\,.$$

Proof. We only deal with σ_1 and S_{m_1}, the other case being similar. Theorem 6.6 yields the existence of a sequence $\sigma_{m_1}, \cdots, \sigma_{m-1}, \sigma_m$ such that

$$\sigma_m = \sigma\,, \qquad \forall j \in \{m_1 \cdots m-1\} \quad \sigma_j \in M_j\,, \quad q(\sigma_j, \sigma_{j+1}) > 0\,.$$

Taking $\sigma_1 = \sigma_{m_1}$ we have

$$E(\sigma_1, \sigma) \leq \max_{m_1 \leq n \leq m} E(\sigma_n) \leq \max_{m_1 \leq n \leq m} E(n).$$

The reverse inequality is a consequence of proposition 6.11. □

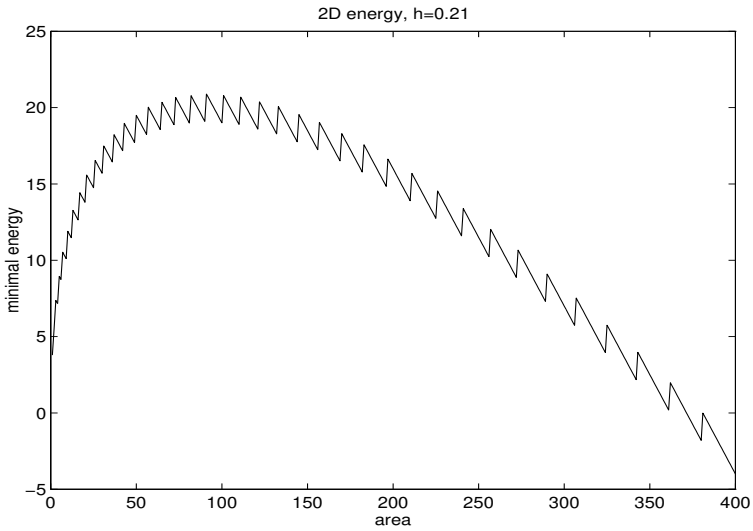
Corollary 6.13. *Let $m \leq n$ be two integers. The communication altitude between the sets C_m and C_n (or S_m and S_n) is*

$$E(C_m, C_n) = E(S_m, S_n) = \max\{ E(r) : m \leq r \leq n \} = E(m_m, m_n).$$

Proof. This is a straightforward consequence of propositions 6.11 and 6.12. □

Corollary 6.14. $E(\underline{-1}, \underline{+1}) = \max\{ E(n) : 0 \leq n \leq N^2 \}$.

The figure below shows a graph of $E(n)$ for $h = 0.21$.



To compute $E(n)$ we apply theorem 6.2. The minimal perimeter $p(n)$ of a polyomino of area n is

$$p(n) = \begin{cases} 2(2l + \epsilon) & \text{if } k = 0 \\ 2(2l + \epsilon) + 2 & \text{if } k > 0 \end{cases} \tag{3}$$

where (l, k, ϵ) is the unique 3–uple in $\mathbb{N}^3$ satisfying $n = l(l + \epsilon) + k$, $\epsilon \in \{0, 1\}$, $k < l + \epsilon$. Let E_1 be the corresponding one dimensional energy i.e. $E_1(k) = 2 \cdot 1(k > 0) - hk$ (where $1(A)$ denotes the characteristic function of A). We have $E(n) = E(l(l+\epsilon)) + E_1(k)$. Starting with E_1 we find

$$\max_{k_1 \leq k \leq k_2} E_1(k) = \begin{cases} 2 - hk_1 & \text{if } k_1 \geq 2 \\ 0 & \text{if } k_2 < 1 \\ 2 - h & \text{if } k_1 \leq 1 \leq k_2 \end{cases}$$

We next compute the energy barrier between two consecutive quasisquares:

$$\max_{l^2\le n\le l(l+1)} E(n) \;=\; E(l^2) + \max_{0\le k\le l} E_1(k) \;=\; E(l^2) + 2 - h\,,$$
$$\max_{l(l+1)\le n\le (l+1)^2} E(n) \;=\; E(l(l+1)) + 2 - h\,.$$

Thus

$$\max_{l(l+1)\le n\le (l+1)^2} E(n) \;-\; \max_{l^2\le n\le l(l+1)} E(n) \;=\; 2 - hl\,,$$
$$\max_{l^2\le n\le l(l+1)} E(n) \;-\; \max_{l(l-1)\le n\le l^2} E(n) \;=\; 2 - hl\,.$$

Putting $l_c = \lfloor 2/h \rfloor$, we see that these energy variations are negative for $l > l_c$ and positive for $l < l_c$. From this, we deduce the energy barrier between two remote quasisquares:

$$\max_{l_1(l_1+\epsilon_1)\le n\le l_2(l_2+\epsilon_2)} E(n) \;=\; \begin{cases} E(l_2(l_2+\epsilon_2-1)+1) & \text{if } (l_2,\epsilon_2)\le (l_c,1) \\ E(l_c(l_c+1)+1) & \text{if } (l_1,\epsilon_1)\le (l_c,1) < (l_2,\epsilon_2) \\ E(l_1(l_1+\epsilon_1)+1) & \text{if } (l_c,1) < (l_1,\epsilon_1) \end{cases}$$

(we use the lexicographical order on the pairs (l,ϵ)).
We finally obtain the global energy barrier.

Proposition 6.15. *The altitude of communication between $\underline{-1}$ and $\underline{+1}$ is equal to*

$$E(\underline{-1},\underline{+1}) \;=\; E(n_c) \;= E(l_c(l_c+1)+1) \;=\; (4 - h\lfloor 2/h\rfloor)\lfloor 2/h\rfloor - h\lfloor 2/h\rfloor + 4 - h$$

where the critical length l_c is $\lfloor 2/h\rfloor$ and the critical area n_c is $l_c(l_c+1)+1$.

Remark. In particular, $E(\underline{-1},\underline{+1}) \sim 4/h$ when h goes to zero: the energy barrier the system has to overcome goes to infinity like h^{-1}.

From this we deduce the level of the greatest cycle containing $\underline{-1}$ and not $\underline{+1}$.

Corollary 6.16. *The level of the cycle $\pi(\underline{-1},\underline{+1}^c)$ is $\mathrm{pred}\, E(n_c)$.*

Remark. We recall that $\mathrm{pred}\,\lambda = \max\{\, E(\sigma) : E(\sigma) < \lambda,\ \sigma \in X \,\}$, see notation 3.4.

Notation 6.17. If Y is a subset of X, its minimal and maximal areas $\underline{a}(Y)$ and $\overline{a}(Y)$ are

$$\underline{a}(Y) \;=\; \min\{\, a(\sigma) : \sigma \in Y \,\}\,, \quad \overline{a}(Y) \;=\; \max\{\, a(\sigma) : \sigma \in Y \,\}\,.$$

Since $E(\underline{-1}, C_n) \ge E(n_c)$ for any $n \ge n_c$ (by proposition 6.11), then all the configurations of the cycle $\pi(\underline{-1},\underline{+1}^c)$ have an area less then $n_c - 1$ i.e. $\overline{a}(\pi(\underline{-1},\underline{+1}^c)) \le n_c - 1$. To complete step i) of the general strategy, we determine the configurations of the principal boundary of the cycle $\pi(\underline{-1},\underline{+1}^c)$.

Theorem 6.18. *The principal boundary $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$ of the cycle $\pi(\underline{-1},\underline{+1}^c)$ is the set $\widetilde{M}_{n_c}$ of the principal configurations of area n_c. For σ in $\widetilde{M}_{n_c}$ there exists a unique configuration in $\pi(\underline{-1},\underline{+1}^c)$ communicating with σ, which is the quasisquare $l_c \times (l_c+1)$ or $(l_c+1)\times l_c$ included in σ.*

Proof. Let σ belong to $\widetilde{M}_{n_c}$. Then $E(\sigma) = E(n_c)$ and theorem 6.5 yields the existence of a sequence $\sigma_0, \cdots, \sigma_{n_c}$ such that

$$\sigma_0 = \underline{-1}, \quad \sigma_{n_c} = \sigma, \quad \forall j \in \{0 \cdots n_c - 1\} \quad \sigma_j \in M_j, \quad q(\sigma_j, \sigma_{j+1}) > 0.$$

In particular, we have $\max\{\, E(\sigma_n) \,:\, 0 \le n < n_c \,\} < E(n_c)$ so that σ_{n_c-1} belongs to $\pi(\underline{-1},\underline{+1}^c)$. Since $q(\sigma_{n_c-1}, \sigma_{n_c}) > 0$, the configuration σ_{n_c} is in the principal boundary of $\pi(\underline{-1},\underline{+1}^c)$. Thus $\widetilde{M}_{n_c} \subset \widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$.
Conversely, let σ belong to $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$. Necessarily $E(\sigma) = E(n_c)$ so that σ is of area n_c and it is a minimal configuration. In addition there must exist η in $\pi(\underline{-1},\underline{+1}^c)$ communicating with σ. This η is in C_{n_c-1} and satisfies $E(\eta) < E(\sigma)$. Thus $p(\eta) \le p(\sigma) - h = p(n_c) - h = 4l_c + 4 - h$ and $p(\eta) \ge p(n_c - 1) = 4l_c + 2$. However the perimeter is an even integer. The only possibility is $p(\eta) = 4l_c + 2$ whence η is minimal. Yet $n_c - 1 = l_c(l_c+1)$ and by theorem 6.3, $M_{n_c-1} = S_{n_c-1}$ so that η belongs to S_{n_c-1} and it is a quasisquare $l_c \times (l_c+1)$ or $(l_c+1)\times l_c$. Lemma 6.4 shows that the only points of M_{n_c} which communicate with M_{n_c-1} are the configurations of $\widetilde{M}_{n_c}$. Thus σ is a principal configuration of area n_c and $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c)) \subset \widetilde{M}_{n_c}$. □

We now proceed to steps *ii)* and *iii)* of the general strategy. On the set of the cycles included in $\{\underline{-1},\underline{+1}\}^c$, we consider the graph G defined by

$$(\pi_1 \to \pi_2) \in G \qquad \Longleftrightarrow \qquad \widetilde{B}(\pi_1) \cap \pi_2 \neq \emptyset.$$

We denote by G^+ the restriction of G to $\pi(\underline{-1},\underline{+1}^c)^c$ (obtained by deleting all the arrows whose starting cycle is included in $\pi(\underline{-1},\underline{+1}^c)$). According to the general method for determining the saddle paths of minimal cost exiting from $\underline{-1}$ and reaching $\underline{+1}$, the next step is to find, for each point σ' in the principal boundary of $\pi(\underline{-1},\underline{+1}^c)$, the orbit of $\{\sigma'\}$ in G^+ (i.e. the minimal stable subgraph of G^+ containing $\{\sigma'\}$). If there does not exist a path in this graph leading from $\{\sigma'\}$ to $\underline{+1}$, then σ' is a dead–end and the last exit from $\pi(\underline{-1},\underline{+1}^c)$ before reaching $\underline{+1}$ won't take place through σ'. Otherwise, σ' is a possible global saddle between $\underline{-1}$ and $\underline{+1}$ and we have to determine all the paths in G^+ leading from $\{\sigma'\}$ to $\underline{+1}$; then we must search for all the cycle paths in $\{\underline{-1},\underline{+1}\}^c$ of null cost realizing the exit of $\pi(\underline{-1},\underline{+1}^c)$ at σ'. More precisely, if we denote by G^- the restriction of G to $\pi(\underline{+1},\underline{-1}^c)^c$ (obtained by deleting all the arrows whose starting cycle is in $\pi(\underline{+1},\underline{-1}^c)$), we have to determine the minimal stable subgraph of G^- containing σ' (i.e. the orbit of σ' in the graph G^-) and all the paths in this graph starting at $\{\sigma'\}$ and

ending at $\underline{-1}$. To achieve these goals, we will next describe a list of relevant cycles, which are included in $\{\underline{-1}, \underline{+1}\}^c$ and we will precise their principal boundaries. Notice that we don't know *a priori* that these cycles are in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. We will then use the following simple result.

Lemma 6.19. *Let $\pi_0, \cdots, \pi_r$ be a sequence of cycles included in $\{\underline{-1}, \underline{+1}\}^c$ such that $\widetilde{B}(\pi_i) \cap \pi_{i+1} \neq \emptyset$ for i in $\{0, \cdots r-1\}$ and $\widetilde{B}(\pi_r) \cap \{\underline{-1}, \underline{+1}\} \neq \emptyset$. Then all the cycles $\pi_0, \cdots, \pi_r$ are maximal cycles in $\{\underline{-1}, \underline{+1}\}^c$.*

This result will yield "instantaneously" the list of the relevant cycles and will designate those cycles of the list which are in $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. Our list of cycles will be large enough to ensure that for each configuration σ' in $\widetilde{B}(\pi(\underline{-1}, \underline{+1}^c))$, it includes all the vertices of the minimal stable subgraph of G containing $\{\sigma'\}$. Moreover, we will handle separately the case of the configurations in S_{n_c} and in $\widetilde{M}_{n_c} \setminus S_{n_c}$: it turns out that step ii) succeeds for the standard configurations and fails for the principal non standard configurations. The latter are dead–ends.

The standard configurations. We start by describing the relevant list of cycles for determining the minimal stable subgraph of G containing the standard configurations of area n_c. These are the cycles around the rectangles $l_1 \times l_2$ and the cycles $\{l_1 \times l_2 +^i k\}$ with $l_1 > l_c, l_2 > l_c$.

Theorem 6.20. *Let l_1, l_2 be two integers strictly greater than l_c.*
The cycle $\pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\begin{aligned} \underline{a}(\pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1))) &= l_1 l_2 - l_c + 1\,, \\ \forall k \in \{1 \cdots l_c - 1\} \qquad M(l_1 \times l_2, -k) &\subset \pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1))\,, \\ \overline{a}(\pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1))) &= l_1 l_2\,. \end{aligned}$$

The bottom of this cycle is $\{l_1 \times l_2\}$ and its principal boundary is $\overline{l_1 \times l_2 \oplus 1}^{12}$.

Proof. We check that the rectangle $l_1 \times l_2$ and the cycle $\pi(l_1 \times l_2, \mathrm{pred}\, E(l_1 \times l_2 + 1))$ satisfy the hypothesis of theorem 5.5. Let $x_1, \cdots, x_r$ be a sequence of sites such that $F(x_1, \cdots, x_j)(l_1 \times l_2)$ is in $\pi(l_1 \times l_2, \mathrm{pred}\, E(l_1 \times l_2 + 1))$ for j in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\mathrm{pred}\, E(l_1 \times l_2 + 1)$). We put $\eta_j = F(x_1, \cdots, x_j)(l_1 \times l_2)$ for j in $\{0 \cdots r\}$.
• First case: all the sites are outside $l_1 \times l_2$. We have that $l_1 \times l_2 \subset \eta_j$ whence $E(\eta_j) \geq E(l_1 \times l_2, a(\eta_j) - l_1 l_2)$ and (see notation 6.7)

$$\mathrm{pred}\, E(l_1 \times l_2 + 1) \geq \max_{0 \leq j \leq r} E(\eta_j) \geq \max_{0 \leq j \leq r} E(l_1 \times l_2, a(\eta_j) - l_1 l_2).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|a(\eta_{j+1}) - a(\eta_j)| \leq 1$ and $(a(\eta_j),\ 0 \leq j \leq r)$ takes all the values between $l_1 l_2$ and $a(\eta_r)$. Henceforth

$$\max_{0 \leq j \leq r} E(l_1 \times l_2, a(\eta_j) - l_1 l_2) \ \geq \ \max\{\, E(l_1 \times l_2, k) : 0 \leq k \leq a(\eta_r) - l_1 l_2 \,\}$$

and the area of η_r must satisfy

$$\max\{\, E(l_1 \times l_2, k) : 0 \leq k \leq a(\eta_r) - l_1 l_2 \,\} \ \leq \ \operatorname{pred} E(l_1 \times l_2 + 1).$$

By theorem 6.9, we have $E(l_1 \times l_2, 1) = E(l_1 \times l_2 + 1) > \operatorname{pred} E(l_1 \times l_2 + 1)$ so that $a(\eta_r) = l_1 l_2$ and $\eta_r = l_1 \times l_2$.

- Second case: all the sites are inside $l_1 \times l_2$. Now $\eta_j \subset l_1 \times l_2$ so that

$$\begin{aligned} \operatorname{pred} E(l_1 \times l_2 + 1) \ &\geq \ \max_{0 \leq j \leq r} E(\eta_j) \ \geq \ \max_{0 \leq j \leq r} E(l_1 \times l_2, a(\eta_j) - l_1 l_2) \\ &\geq \ \max\{\, E(l_1 \times l_2, k) : a(\eta_r) - l_1 l_2 \leq k \leq 0 \,\}. \end{aligned}$$

Theorem 6.8 shows that for $0 \leq k \leq l_1$, $E(l_1 \times l_2, -k) = E(l_1 \times (l_2 - 1) \oplus_2 (l_1 - k) \times 1)$ so that $E(l_1 \times l_2 + 1) - E(l_1 \times l_2, -k) = 2 - h(k+1)$. For this quantity to be positive, we must have $k \leq l_c - 1$ whence $a(\eta_r) \geq l_1 l_2 - (l_c - 1)$. In addition for any k, $0 \leq k \leq l_c - 1$, $E(l_1 \times l_2, -k) - E(l_1 \times l_2) = hk > 0$ so that $E(\eta_r) > E(l_1 \times l_2)$ whenever $\eta_r \neq l_1 \times l_2$. We have proved that $l_1 \times l_2$ and the cycle $\pi(l_1 \times l_2, \operatorname{pred} E(l_1 \times l_2 + 1))$ satisfy the hypothesis of theorem 5.5. Thus the bottom of the cycle is $\{l_1 \times l_2\}$. We have also proved the inequalities on $\overline{a}$ and $\underline{a}$. That $M(l_1 \times l_2, -k)$ is included in the cycle for $0 \leq k \leq l_c - 1$ is obvious: each configuration of this set communicates with $l_1 \times l_2$ under the level $\operatorname{pred} E(l_1 \times l_2 + 1)$ (cf theorem 6.8: these configurations are obtained by deleting successively k corner squares from $l_1 \times l_2$). Finally a configuration ξ of the principal boundary of this cycle is of energy $E(\xi) = E(l_1 \times l_2 + 1)$ so that its area is equal to $l_1 l_2 + 1$, and its perimeter to $2(l_1 + l_2) + 2$. Let η be a configuration of the cycle such that $q(\eta, \xi) > 0$. Necessarily, the area of η is $l_1 l_2$. Thus η is a configuration of maximal area of the cycle and as such it is a maximal configuration of the cycle for the inclusion relation. By theorem 5.3 the rectangle $l_1 \times l_2$ is included in η whence in fact $\eta = l_1 \times l_2$. Thus ξ belongs to $M(l_1 \times l_2, +1)$. Theorem 6.9 shows that this set is equal to $\overline{l_1 \times l_2 \oplus 1}^{12}$. Conversely each configuration of this set communicates with $l_1 \times l_2$ and belongs to the principal boundary of the cycle. $\square$

Lemma 6.21. *Let l_1, l_2, k, i be integers with $l_1 > 0$, $l_2 > 0$, $k > 1$, $i \geq 0$, $l_1 > k + i$. The cycle $\{l_1 \times l_2 +_2^i k\}$ has one or two configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{l_1 \times l_2 +_2^i (k+1)\}$,*
- *if $i = l_1 - 1 - k$ then $\widetilde{B} = \{l_1 \times l_2 +_2^{i-1} (k+1)\}$,*
- *if $0 < i < l_1 - 1 - k$ then $\widetilde{B} = \{l_1 \times l_2 +_2^{i-1} (k+1),\ l_1 \times l_2 +_2^i (k+1)\}$.*

Lemma 6.22. *Let l_1, l_2, i be integers with $l_1 > 0$, $l_2 > 0$, $i \geq 0$, $l_1 > i+1$.*
The cycle $\{l_1 \times l_2 +_2^i 1\}$ has two or three configurations in its principal boundary $\widetilde{B}$:

- *if $i = 0$ then $\widetilde{B} = \{l_1 \times l_2\,,\ l_1 \times l_2 +_2^i 2\}$,*
- *if $i = l_1 - 1 - k$ then $\widetilde{B} = \{l_1 \times l_2\,,\ l_1 \times l_2 +_2^{i-1} 2\}$,*
- *if $0 < i < l_1 - 1 - k$ then $\widetilde{B} = \{l_1 \times l_2\,,\ l_1 \times l_2 +_2^{i-1} 2\,,\ l_1 \times l_2 +_2^i 2\}$.*

Remark. Results similar to those stated in lemmas 6.21 and 6.22 hold for the configurations $l_1 \times l_2 +_1^i 1$ and also for any configuration in $\overline{l_1 \times l_2 \oplus 1}$.

Corollary 6.23. *The following cycles are maximal cycles of $\{\underline{-1}, \underline{+1}\}^c$: (with $l_1 > l_c$ and $l_2 > l_c$)*

$$\begin{aligned}
&\{\eta\}\,, \qquad \eta \in \overline{l_1 \times l_2 \oplus_2 k}, \qquad 0 < k < l_1 - l_c + 1\,,\\
&\{\eta\}\,, \qquad \eta \in \overline{l_1 \times l_2 \oplus_1 k}, \qquad 0 < k < l_2 - l_c + 1\,,\\
&\pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1))\,.
\end{aligned}$$

Proof. This corollary is a consequence of lemma 6.19 together with theorem 6.20 and lemmas 6.21 and 6.22. Notice that we have to put together the descriptions of the cycles of theorem 6.20, lemmas 6.21, 6.22 in order to check that for each cycle π in the above list, there is a sequence of cycles $\pi_0, \cdots, \pi_r$ such that $\pi_0 = \pi$, $\widetilde{B}(\pi_i) \cap \pi_{i+1} \neq \emptyset$, $0 \leq i < r$ and $\underline{+1} \in \widetilde{B}(\pi_r)$. $\square$

Corollary 6.24. *Let σ belong to S_{n_c+1}. The minimal stable subgraph $G^+(\sigma)$ of G^+ containing σ is the restriction of G to the vertices listed in corollary 6.23. The arrows of $G^+(\sigma)$ are (where the rectangles $l_1 \times l_2$ are configurations containing σ)*

$$\begin{aligned}
&\{l_1 \times l_2 +_2^i k\} \to \{l_1 \times l_2 +_2^i \quad k+1\}, \qquad 0 \leq i < l_1 - k, \quad 0 < k < l_1 - l_c\,,\\
&\{l_1 \times l_2 +_2^i k\} \to \{l_1 \times l_2 +_2^{i-1} k+1\}, \qquad 0 < i \leq l_1 - k, \quad 0 < k < l_1 - l_c\,,\\
&\{l_1 \times l_2 +_1^i k\} \to \{l_1 \times l_2 +_1^i \quad k+1\}, \qquad 0 \leq i < l_2 - k, \quad 0 < k < l_2 - l_c\,,\\
&\{l_1 \times l_2 +_1^i k\} \to \{l_1 \times l_2 +_1^{i-1} k+1\}, \qquad 0 < i \leq l_2 - k, \quad 0 < k < l_2 - l_c\,,\\
&\{l_1 \times l_2 +_2^i l_1 - l_c\} \to \pi(l_1 \times (l_2+1), pred\, E(l_1 \times (l_2+1) + 1)), \quad 0 \leq i \leq l_c\,,\\
&\{l_1 \times l_2 +_1^i l_2 - l_c\} \to \pi((l_1+1) \times l_2, pred\, E((l_1+1) \times l_2 + 1)), \quad 0 \leq i \leq l_c\,,\\
&\pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1)) \leftrightarrow \{\eta\}\,, \qquad \eta \in l_1 \times l_2 \oplus_1 1\,,\\
&\pi(l_1 \times l_2, pred\, E(l_1 \times l_2 + 1)) \leftrightarrow \{\eta\}\,, \qquad \eta \in l_1 \times l_2 \oplus_2 1\,.
\end{aligned}$$

The symbol $\leftrightarrow$ means that both arrows $\to$ and $\leftarrow$ are present. The above list should be completed with all the isometric arrows (obtained by applying the same isometry to both ends of an arrow).

The only loops in the graph $G^+(\sigma)$ are

$$\pi(l_1 \times l_2, \operatorname{pred} E(l_1 \times l_2 + 1)) \leftrightarrow \{\eta\}\,, \qquad \eta \in l_1 \times l_2 \oplus_1 1 \cup l_1 \times l_2 \oplus_2 1\,.$$

Any other arrow $\pi_1 \to \pi_2$ of $G^+(\sigma)$ satisfies $\overline{a}(\pi_1) < \underline{a}(\pi_2)$. As a consequence a path in $G^+(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{+1}\}$.

We are done with the standard configurations. We have to examine the remaining configurations of the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$ i.e. the principal non standard configurations.

The principal non standard configurations. We now do the same work for the configurations in $\widetilde{M}_{n_c} \setminus S_{n_c}$. The relevant cycles are the cycles around the rectangles $l_c \times (l_c + 2)$ and around the configurations of $\widetilde{M}_{n_c+1} \setminus S_{n_c+1}$.

Theorem 6.25. *The cycle $\pi(l_c \times (l_c + 2), pred\, E(n_c))$ is the greatest cycle containing $l_c \times (l_c + 2)$ included in $\{\underline{-1}, \underline{+1}\}^c$. Moreover,*

$$\begin{aligned} \underline{a}(\pi(l_c \times (l_c + 2), pred\, E(n_c))) &= l_c(l_c + 1) + 2\,, \\ \forall k \in \{1 \cdots l_c - 2\} \qquad M(l_c \times (l_c + 2), -k) &\subset \pi(l_c \times (l_c + 2), pred\, E(n_c))\,, \\ \overline{a}(\pi(l_c \times (l_c + 2), pred\, E(n_c))) &= l_c(l_c + 2)\,. \end{aligned}$$

The bottom of this cycle is $\{l_c \times (l_c+2)\}$; its principal boundary is $M(l_c \times (l_c+2), -(l_c-1))$ and thus contains $\widetilde{M}_{n_c} = \overbrace{l_c \times (l_c + 1) \oplus_2 1}^{12}$.

Remark. Obviously, similar statements are true for the rectangle $(l_c + 2) \times l_c$.

Proof. We check that the rectangle $l_c \times (l_c + 2)$ and the cycle $\pi(l_c \times (l_c + 2), \operatorname{pred} E(n_c))$ satisfy the hypothesis of theorem 5.5. Let $x_1, \cdots, x_r$ be a sequence of sites such that $F(x_1, \cdots, x_j)(l_c \times (l_c + 2))$ is in $\pi(l_c \times (l_c + 2), \operatorname{pred} E(n_c))$ for j in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\operatorname{pred} E(n_c)$). We put $\eta_j = F(x_1, \cdots, x_j)(l_c \times (l_c + 2))$ for j in $\{0 \cdots r\}$.

- First case: all the sites are outside $l_c \times (l_c + 2)$. We have that $l_c \times (l_c + 2) \subset \eta_j$ whence $E(\eta_j) \geq E(l_c \times (l_c + 2), a(\eta_j) - l_c(l_c + 2))$ and

$$\operatorname{pred} E(n_c) \;\geq\; \max_{0 \leq j \leq r} E(\eta_j) \;\geq\; \max_{0 \leq j \leq r} E(l_c \times (l_c + 2), a(\eta_j) - l_c(l_c + 2)).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|a(\eta_{j+1}) - a(\eta_j)| \leq 1$ and $(a(\eta_j),\ 0 \leq j \leq r)$ takes all the values between $l_c(l_c + 2)$ and $a(\eta_r)$. Henceforth

$$\max_{0 \leq j \leq r} E(l_c \times (l_c+2), a(\eta_j) - l_c(l_c+2)) \;\geq\; \max\{\, E(l_c \times (l_c+2), k) : 0 \leq k \leq a(\eta_r) - l_c(l_c+2)\,\}$$

and the area of η_r must satisfy

$$\max\{\, E(l_c \times (l_c+2), k) : 0 \le k \le a(\eta_r) - l_c(l_c+2) \,\} \;\le\; \operatorname{pred} E(n_c).$$

By theorem 6.9, we have $E(l_c \times (l_c+2), 1) = E(l_c \times (l_c+2)+1)$ whence $E(l_c \times (l_c+2), 1) - E(n_c) = 2 - hl_c$ which is strictly positive since $l_c = \lfloor 2/h \rfloor$. Thus $a(\eta_r) = l_c(l_c+2)$ and $\eta_r = l_c \times (l_c+2)$.

- Second case: all the sites are inside $l_c \times (l_c+2)$. Now $\eta_j \subset l_c \times (l_c+2)$ so that

$$\begin{aligned} \operatorname{pred} E(n_c) &\ge \max_{0\le j\le r} E(\eta_j) \ge \max_{0\le j\le r} E(l_c \times (l_c+2), a(\eta_j) - l_c(l_c+2)) \\ &\ge \max\{\, E(l_c \times (l_c+2), k) : a(\eta_r) - l_c(l_c+2) \le k \le 0 \,\}. \end{aligned}$$

Theorem 6.8 shows that for $0 \le k \le l_c$, $E(l_c \times (l_c+2), -k) = E(l_c \times (l_c+1) \oplus_2 (l_c - k) \times 1)$ so that $E(n_c) - E(l_c \times (l_c+2), -k) = h(l_c - k - 1)$. For this quantity to be positive, we must have $k < l_c - 1$ whence $a(\eta_r) \ge l_c(l_c+2) - (l_c - 2)$. In addition for any k, $0 \le k \le l_c - 2$, $E(l_c \times (l_c+2), -k) - E(l_c \times (l_c+2)) = hk > 0$ so that $E(\eta_r) > E(l_c \times (l_c+2))$ whenever $\eta_r \ne l_c \times (l_c+2)$.

We have proved that $l_c \times (l_c+2)$ and the cycle $\pi(l_c \times (l_c+2), \operatorname{pred} E(n_c))$ satisfy the hypothesis of theorem 5.5. It follows that the bottom of the cycle is $\{l_c \times (l_c+2)\}$. The inequalities on $\overline{a}$ and $\underline{a}$ have also been proved. That $M(l_c \times (l_c+2), -k)$ is included in the cycle for $0 \le k \le l_c - 2$ is obvious: each configuration of this set communicates with $l_c \times (l_c+2)$ under the level $\operatorname{pred} E(n_c)$ (theorem 6.8 shows that all these configurations are obtained by deleting successively k corner squares from $l_c \times (l_c+2)$). Finally a configuration ξ of the principal boundary of this cycle is of energy $E(\xi) = E(n_c)$ so that its area is equal to n_c, and its perimeter to $p(n_c)$. Let η be a configuration of the cycle such that $q(\eta, \xi) > 0$. Necessarily, the area of η is $n_c + 1$. Thus η is a configuration of minimal area of the cycle and as such it is a minimal configuration of the cycle for the inclusion relation. By theorem 5.3 it is included in the rectangle $l_c \times (l_c+2)$. Thus ξ belongs to $M(l_c \times (l_c+2), -(l_c-1))$. Conversely each configuration of this set communicates with $M(l_c \times (l_c+2), -(l_c-1))$ and belongs to the principal boundary of the cycle. Finally, theorem 6.8 shows that this set contains $\widetilde{M}_{n_c}$. $\square$

Corollary 6.26. *Let σ belong to $\widetilde{M}_{n_c+1} \setminus S_{n_c+1}$. Suppose $l_c \times (l_c+1) \subset \sigma \subset l_c \times (l_c+2)$. Let $G^+(\sigma)$ be the minimal stable subgraph of G^+ containing $\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c)$. The only arrows of $G^+(\sigma)$ entering $\pi(\underline{-1}, \underline{+1}^c)$ are*

$$\{\xi\} \to \pi(l_c \times (l_c+1), \{\underline{-1}, \underline{+1}\}^c), \quad \xi \in \widetilde{M}_{n_c} \setminus S_{n_c}, \quad l_c \times (l_c+1) \subset \xi.$$

The remaining arrows of $G^+(\sigma)$ are

$$\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c) \leftrightarrow \{\eta\}, \quad \eta \in M(l_c \times (l_c+2), -(l_c-1)).$$

There is no arrow in $G^+(\sigma)$ ending at $\{\underline{+1}\}$.

Remark. In this statement, the rectangle $l_c \times (l_c+1)$ should be understood as a configuration and not as a polyomino. That is, $l_c \times (l_c+1) \subset \xi$ means that the same (fixed) rectangle is included in σ and ξ.

Proof. That these arrows belong to $G^+(\sigma)$ is a straightforward consequence of theorem 6.25 which implies in particular that $\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c) = \pi(l_c \times (l_c+2), \{\underline{-1}, \underline{+1}\}^c)$. We have to check that there are no other arrow. Let η belong to the principal boundary of $\pi(l_c \times (l_c+2), \operatorname{pred} E(n_c))$. If η is in $\widetilde{M}_{n_c}$, then all arrows of G starting at the cycle $\{\eta\}$ are present in the above list (lemma 6.22). If η is not in $\widetilde{M}_{n_c}$, we claim that

$$\{\eta\} \to \pi(l_c \times (l_c+2), \operatorname{pred} E(n_c))$$

is the unique arrow of G starting at $\{\eta\}$. Let ξ be a point such that $q(\eta,\xi) > 0$ and $E(\xi) \leq E(\eta)$. Since $a(\eta) = n_c$, then $a(\xi)$ is equal to $n_c - 1$ or $n_c + 1$. Moreover η is minimal, and the inequality $E(\xi) \leq E(\eta)$ implies in both cases that ξ is also minimal. Thus ξ cannot be of area $n_c - 1$ (by lemma 6.4, the only configurations of M_{n_c} communicating with M_{n_c-1} are the principal configurations $\widetilde{M}_{n_c}$). Thus $a(\xi) = n_c + 1$. We next show that ξ belongs to $\pi(l_c \times (l_c+2), \operatorname{pred} E(n_c))$. Let $x_1, \cdots, x_r$ be a sequence of sites inside σ such that $F(x_1, \cdots, x_j)(l_c \times (l_c+2))$ is in $\pi(l_c \times (l_c+2), \operatorname{pred} E(n_c))$ for j in $\{1 \cdots r-1\}$ and $\eta = F(x_1, \cdots, x_r)(l_c \times (l_c+2))$. We put $\eta_j = F(x_1, \cdots, x_j)(l_c \times (l_c+2))$, $0 \leq j \leq r$. Since η is in the boundary of the cycle, we have $E(\eta_{r-1}) < E(\eta_r)$. Moreover η_{r-1} is a minimal configuration of the cycle, of area $n_c + 1$, so that the last spin flip at site x_r has decreased the area. Let x_{r+1} be the unique site such that $\xi = F(x_{r+1})(\eta)$. We have $\xi = F(x_{r+1})(\eta_r) = F(x_{r+1}, x_r)(\eta_{r-1}) = F(x_r, x_{r+1})(\eta_{r-1}) = F(x_r)(\eta'_r)$ where $\eta'_r = F(x_{r+1})(\eta_{r-1})$. The energy inequality 5.1 yields $E(\eta'_r) - E(\eta_{r-1}) \leq E(\xi) - E(\eta)$ whence $E(\eta'_r) \leq E(\eta_{r-1})$. It follows that η'_r is in the cycle, as well as ξ (their energies are less or equal than $E(n_c)$ and they both communicate with a configuration of the cycle). □

Corollary 6.27. *The principal non standard configurations of area n_c (i.e. the set $\widetilde{M}_{n_c} \setminus S_{n_c}$) are dead–ends: there is no saddle path of null cost between $\underline{-1}$ and $\underline{+1}$ passing through them.*

Corollary 6.28. *The set of the global saddle points between $\underline{-1}$ and $\underline{+1}$ is exactly S_{n_c}. These configurations are the critical two dimensional configurations.*

Steps *ii)* and *iii)* are now completed and we proceed to steps *iv)* and *v)*.

The ascending part. For each configuration σ' in S_{n_c}, we must determine the minimal stable subgraph of G^- containing σ' and all the paths in this graph starting at $\{\sigma'\}$ and ending at $\underline{-1}$. Our exposition is similar as before: we first list the set of the relevant cycles and we use lemma 6.19 to find those belonging to $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. We finally check that we have in hand all the vertices of G^-.

Theorem 6.29. *Let l be an integer less or equal than l_c.*
The cycle $\pi(l \times l, pred\, E(l(l-1)+1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\begin{aligned} \underline{a}(\pi(l \times l, pred\, E(l(l-1)+1))) &= l(l-1)+2\,, \\ \forall k \in \{1 \cdots l-2\} \qquad M(l \times l, -k) &\subset \pi(l \times l, pred\, E(l(l-1)+1))\,, \\ \overline{a}(\pi(l \times l, pred\, E(l(l-1)+1))) &= l^2\,. \end{aligned}$$

The bottom of this cycle is $\{l \times l\}$; its principal boundary is $M(l \times l, -(l-1))$ and thus contains $S_{l(l-1)+1}$.

Proof. We apply corollary 6.13 and formula (3). For any n greater than l^2, we have

$$E(l \times l, C_n) \geq E(l^2+1) \geq E(l(l-1)+1)$$

whence $\overline{a}(\pi(l \times l, \mathrm{pred}\, E(l(l-1)+1))) = l^2$. Analogously, we have for any n smaller than $l(l-1)+2$, $E(l \times l, C_n) \geq E(l(l-1)+1)$; moreover,

$$E(l \times l, C_{l(l-1)+2}) = E(l(l-1)+2) < E(l(l-1)+1$$

so that $\underline{a}(\pi(l \times l, \mathrm{pred}\, E(l(l-1)+1))) = l(l-1)+2$. To prove that the bottom of the cycle is $\{l \times l\}$, we could proceed as before and use theorem 5.5. However, a direct application of our geometrical results (theorem 6.2) yields that the minimum $\min\{\, E(n) : l(l-1)+2 \leq n \leq l^2 \,\}$ is equal to $E(l^2)$; the unique configuration of energy $E(l^2)$ is the square $l \times l$. Theorem 5.3 implies also that the altitude of communication between two different squares $l \times l$ is greater than $E(l^2+1)$ (one has to make a spin–flip outside the initial square to create another square). □

Theorem 6.30. *Let l be an integer less or equal than l_c.*
The cycle $\pi(l \times (l+1), pred\, E(l^2+1))$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\begin{aligned} \underline{a}(\pi(l \times (l+1), pred\, E(l^2+1))) &= l^2+2\,, \\ \forall k \in \{1 \cdots l-2\} \qquad M(l \times (l+1), -k) &\subset \pi(l \times (l+1), pred\, E(l^2+1))\,, \\ \overline{a}(\pi(l \times (l+1), pred\, E(l^2+1))) &= l(l+1)\,. \end{aligned}$$

The bottom of this cycle is $\{l \times (l+1)\}$; its principal boundary is $M(l \times (l+1), -(l-1))$ and thus contains S_{l^2+1}.

Proof. The proof is similar as the proof of theorem 6.29.

Remark. Similar statements hold for the quasisquares $(l+1) \times l$.

Corollary 6.31. *The following cycles are maximal cycles of* $\{\underline{-1},\underline{+1}\}^c$ *(where* $l \le l_c$*):*

$$\begin{aligned}
&\{\eta\}, \qquad \eta \in M(l\times(l+1),-(l-1))\,,\\
&\{\eta\}, \qquad \eta \in M((l+1)\times l,-(l-1))\,,\\
&\{\eta\}, \qquad \eta \in M(l\times l,-(l-1))\,,\\
&\pi(l\times l, pred E(l(l-1)+1))\,,\\
&\pi(l\times(l+1), pred E(l^2+1))\,,\\
&\pi((l+1)\times l, pred E(l^2+1))\,.
\end{aligned}$$

Proof. This corollary is a consequence of lemma 6.19 together with theorems 6.29 and 6.30. Notice that we have to put together the descriptions of the cycles of theorems 6.29, 6.30 in order to check that for each cycle π in the above list, there is a sequence of cycles $\pi_0,\cdots,\pi_r$ such that $\pi_0=\pi$, $\widetilde{B}(\pi_i)\cap\pi_{i+1}\neq\emptyset$, $0\le i<r$ and $\underline{-1}\in\widetilde{B}(\pi_r)$. □

Corollary 6.32. *Let* σ *belong to* S_{n_c-1}*. The minimal stable subgraph* $G^-(\sigma)$ *of* G^- *containing* σ *is the restriction of* G *to the vertices listed in corollary* 6.31*. The arrows of* $G^-(\sigma)$ *are (where each square* $l\times l$ *or quasisquare* $l\times(l+1)$ *appearing on a line must be included in* σ*);*

$$\begin{aligned}
&\{\eta\}\leftrightarrow\pi(l\times(l+1), pred E(l^2+1))\,, && \eta\in M(l\times(l+1),-(l-1))\,,\\
&\{\eta\}\leftrightarrow\pi((l+1)\times l, pred E(l^2+1))\,, && \eta\in M((l+1)\times l,-(l-1))\,,\\
&\{\eta\}\leftrightarrow\pi(l\times l, pred E(l(l-1)+1))\,, && \eta\in M(l\times l,-(l-1))\,,\\
&\{\eta\}\to\pi(l\times l, pred E(l(l-1)+1))\,, && \eta\in \widetilde{M}_{l^2+1}\,,\\
&\{\eta\}\to\pi(l\times(l+1), pred E(l^2+1))\,, && \eta\in \overline{l\times(l+1)\oplus_2 1}\,,\\
&\{\eta\}\to\pi((l+1)\times l, pred E(l^2+1))\,, && \eta\in \overline{(l+1)\times l\oplus_1 1}\,,
\end{aligned}$$

The only loops in the graph $G^-(\sigma)$ are loops around two cycles (corresponding to the arrows described in the first three lines of the list). Any other arrow $\pi_1\to\pi_2$ of $G^-(\sigma)$ satisfies $\overline{a}(\pi_1)>\underline{a}(\pi_2)$. As a consequence a path in $G^-(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{-1}\}$.

The exit path. We have finally reached the last step $vi)$ which consists in doing the synthesis of all the results gathered in the previous steps $i)-v)$. The set of all the cycle paths of null cost is obtained by glueing together an ascending part and a descending part. We define a graph H over the cycles of $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$ by

$$\pi_1\to\pi_2\in H \qquad\Longleftrightarrow\qquad \exists\,\sigma\in S_{n_c}\quad \pi_1\to\pi_2\in G^+(\sigma)\quad\text{or}\quad \pi_2\to\pi_1\in G^-(\sigma)$$

i.e. the graph H is the union for σ in S_{n_c} of the graphs $G^+(\sigma)$ (described in corollary 6.24) and the reversed graphs $G^-(\sigma)$ (described in corollary 6.32). The set of the cycle paths of null cost is the set of all the paths in the graph H joining $\{\underline{-1}\}$ to $\{\underline{+1}\}$. In addition, there exists only one optimal saddle path between two cycles associated to each arrow of the graph H. Thus the graph H contains all the information necessary to obtain the set of the saddle paths of null cost between $\underline{-1}$ and $\underline{+1}$. We describe for instance the canonical saddle path, which follows the sequence of the canonical configurations:

$$\underline{-1} \to m_1,\, m_1 \to m_2,\, m_2 \to m_3,\, m_3 \to m_4,\, m_4 \to m_5,\, m_5 \to m_6,\, m_6 \to m_7,$$
$$m_7 \to m_8,\, m_9 \to m_{10},\, m_{10} \to m_{11},\, m_{12} \to m_{13},\, m_{13} \to m_{14},\, m_{16} \to m_{17},\, \cdots,$$
$$m_{k^2} \to m_{k^2+1},\, m_{k^2+1} \to m_{k^2+2},\, m_{k(k+1)} \to m_{k(k+1)+1},\, \cdots,$$
$$m_{l_c^2} \to m_{l_c^2+1},\, m_{l_c^2+1} \to m_{l_c^2+2},\, m_{l_c(l_c+1)} \to m_{l_c(l_c+1)+1},$$
$$m_{l_c(l_c+1)+1} \to m_{l_c(l_c+1)+2},\, m_{(l_c+1)^2} \to m_{(l_c+1)^2+1},\, \cdots,\, m_{N^2}.$$

We state here some consequences of the information provided by the graph H which describes the set of all the saddle paths of null cost. Most of them had already been proved by Neves and Schonmann with completely different methods (mainly coupling techniques). We stress that the graph H provides the most complete information available on the limiting dynamics. Once we know this graph, the results obtained in a general framework [7] may be applied in a systematic fashion to obtain various of estimates. We let the process $(\sigma_n)_{n\in\mathbb{N}}$ start from $\underline{-1}$. We recall that $\tau(\underline{+1}^c)$ is the hitting time of the ground state $\underline{+1}$ and $\theta(\underline{-1}, \tau(\underline{+1}^c))$ is the last visit to the metastable state $\underline{-1}$ before reaching $\underline{+1}$.

Theorem 6.33. *(the exit path)*
For any positive ϵ, the following events take place with probability converging to one exponentially fast as β goes to infinity:

- $\exp\beta(E(n_c)-\epsilon) \le \tau(\underline{+1}^c) \le \exp\beta(E(n_c)+\epsilon)$;
- $\exp\beta(2-h-\epsilon) \le \tau(\underline{+1}^c) - \theta(\underline{-1}, \tau(\underline{+1}^c)) \le \exp\beta(2-h+\epsilon)$;
- *during the exit path $\sigma_n, \theta \le n \le \tau$, the process crosses the set of critical configurations S_{n_c} at exactly one point σ_c; it does not cross $C_{n_c} \setminus S_{n_c}$;*
- *if we let $n_* = \min\{n \ge \theta : a(\sigma_n) = n_c\}$, $n^* = \max\{n \le \tau : a(\sigma_n) = n_c\}$, we have that $\sigma_n = \sigma_{n_*} = \sigma_{n^*}$ for all n in $\{n_* \cdots n^*\}$ and $n^* - n_* \le \exp(\beta\epsilon)$;*
- *all the configurations of the exit path before time n_* are of area less than n_c, all configurations after n^* are of area greater than n_c;*
- *during the whole exit path, the configuration stays connected i.e. there is only one cluster of spins in the system; this cluster is a quasisquare $l \times (l+\epsilon)$ minus at most $l-1$ corner squares before n_* and a rectangle $l_1 \times l_2$ minus at most $l_c - 1$ corner squares or plus a line after n^*;*
- *during the ascending part, the process goes through an increasing sequence of quasisquares $(l \times (l+\epsilon),\, l \le l_c)$, and visits a quasisquare $l \times (l+\epsilon)$ approximately* $\exp\beta(l-1)h$

times; when it leaves definitely a quasisquare $l \times (l+1)$, it makes a spin flip along the largest side of the quasisquare in order to create a square $(l+1) \times (l+1)$ and it does not create a rectangle $l \times (l+2)$;

- *during the descending part, the process goes through an increasing sequence of rectangles ($l_1 \times l_2$, $\min(l_1, l_2) > l_c$); he visits each of them $\exp \beta(2-h)$ times; when it leaves definitely a rectangle, he choses randomly the next rectangle it creates, either $l_1 \times (l_2+1)$ or $l_1 \times (l_2+1)$;*
- *the process reaches the thermal equilibrium within each cycle of $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ it crosses: the distribution of the process before the exit of the cycle is very close to the Gibbs distribution at inverse temperature β restricted to the cycle.*

7. Dimension three

A polyomino associated to a configuration is a finite union of cubes. For σ a configuration we will denote by $a(\sigma)$ the area of the associated polyomino and by $v(\sigma)$ its volume. We will rely heavily on the notations and results of [1]. Let us recall briefly some essential points.

Summary of the combinatorial results.

The parallelepiped of sides j_1, j_2, j_3 is denoted by $j_1 \times j_2 \times j_3$. By $j_1 \times j_2 \times j_3 +_i c$ we denote the parallelepiped $j_1 \times j_2 \times j_3$ with the polyomino c sticked against the side of $j_1 \times j_2 \times j_3$ whose outward normal is the vector e_i (where e_1, e_2, e_3 is the standard basis of $\mathbb{Z}^3$). The polyomino c is sticked in the position which minimizes its coordinates with respect to the other vectors (for the precise definition, see [1]). When we don't specify the direction, i.e. we write $j_1 \times j_2 \times j_3 + c$, it means that the direction is not relevant, and the statement holds for the three possible polyominoes. Most often, c will be a two dimensional polyomino, for instance $l_1 \times l_2 + k$. In this case, we first transform it into a three dimensional polyomino by replacing its unit squares by unit cubes. In this way, two dimensional polyominoes might be seen as three dimensional "planar" polyominoes. We make also the following convention: when we speak of $j_1 \times j_2 \times j_3 +_3 c$, we assume implicitly that the two dimensional polyomino c is included in the rectangle $j_1 \times j_2$ so that the resulting polyomino is included in $j_1 \times j_2 \times (j_3+1)$. By $j_1 \times j_2 \times j_3 \oplus_3 l_1 \times l_2$ we denote the set of all the polyominoes obtained by translating the rectangle $l_1 \times l_2$ along the side $j_1 \times j_2$ (in such a way that the polyomino is always included in $j_1 \times j_2 \times (j_3+1)$). For A a set of polyomino, we denote by $\overline{A}$ its orbit under the action of the spatial isometries which leave the integer lattice $\mathbb{Z}^3$ invariant. By $\overline{A}^{123}$ we denote its orbit under the action of the three symmetries with respect to the planes $(e_1, e_2), (e_2, e_3), (e_1, e_3)$.

Proposition 7.1. *For each integer n there exists a unique 6–uple $(j, l, k, \delta, \theta, \epsilon)$ such that $\delta, \theta, \epsilon \in \{0, 1\}$, $\delta \leq \theta$, $k < l+\epsilon, l(l+\epsilon)+k < (j+\delta)(j+\theta)$ and $n = j(j+\delta)(j+\theta)+l(l+\epsilon)+k$.*

The set of the polyominoes of volume n is $\mathcal{C}_n$; the set $\mathcal{M}_n$ of the minimal polyominoes of

volume n is the set of the polyominoes of $\mathcal{C}_n$ having minimal area. Let $n = j(j+\delta)(j+\theta)+l(l+\epsilon)+k$ be the decomposition of n. We put $r = l(l+\epsilon)+k$. The canonical polyomino $\mathfrak{m}_n$ is obtained by adding the two dimensional canonical polyomino m_r to the right side of a quasicube of volume $j(j+\delta)(j+\theta)$. The general formula is

$$\mathfrak{m}_n = (j+\theta) \times (j+\delta) \times j +_{1+\delta+\theta} \big((l+\epsilon) \times l +_{1+\epsilon} k\big).$$

The polyomino $\mathfrak{m}_n$ is called the canonical polyomino of volume n.

Theorem 7.2. *The polyomino $\mathfrak{m}_n$ is minimal.*

We define several important subsets of $\mathcal{M}_n$. The set $\mathcal{S}_n$ of the standard polyominoes is

$$\mathcal{S}_n = \overline{(j+\theta) \times (j+\delta) \times j \oplus_{1+\delta+\theta} \overline{(l+\epsilon) \times l \oplus_{1+\epsilon} k}}$$

and the set $\widetilde{\mathcal{M}}_n$ of the principal polyominoes is

$$\widetilde{\mathcal{M}}_n = \bigcup_{\substack{t=1,2,3 \\ u=1,2}} \overline{(j+\theta) \times (j+\delta) \times j \oplus_t \overline{(l+\epsilon) \times l \oplus_u k}}$$

Theorem 7.3. *The set $\mathcal{M}_n$ is reduced to $\{\mathfrak{m}_n\}$ if and only if n is of the form j^3.*
The set $\mathcal{M}_n$ is equal to $\mathcal{S}_n$ if and only if n is of the form $j^3, j^2(j+1), j(j+1)^2$ or $j^3-1, j^2(j+1)-1, j(j+1)^2-1$ (in which case $S_n = \overline{\mathfrak{m}}_n$), or $j^3+l^2, j^3+l(l+1)$.
The set $\mathcal{M}_n$ coincides with $\widetilde{\mathcal{M}}_n$ if and only if n is of the form quasicube+quasisquare

$$\begin{array}{ccc} j^3 & j^3+l^2 & j^3+l(l+1) \\ j^2(j+1) & j^2(j+1)+l^2 & j^2(j+1)+l(l+1) \\ j(j+1)^2 & j(j+1)^2+l^2 & j(j+1)^2+l(l+1) \end{array}$$

(where the quasisquare l^2 or $l(l+1)$ is strictly smaller than the greatest face j^2, $j(j+1)$ or $(j+1)^2$ of the quasicube) or quasicube minus one $j^3-1, j^2(j+1)-1, j(j+1)^2-1$.

Theorem 7.4. *The principal polyominoes can be completely shrinked through the minimal polyominoes: for any integer n and for any principal polyomino c in $\widetilde{\mathcal{M}}_n$, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$ and $c_n = c$ and $q(c_{i-1}, c_i) > 0$ for i in $\{1 \cdots n\}$.*

Theorem 7.5. *The standard polyominoes can be grown or shrinked arbitrarily far through minimal polyominoes: for any integers $m \leq n$ and for any standard polyomino c in $\mathcal{S}_m$, there exists an increasing sequence $c_0, \cdots, c_n$ of minimal polyominoes such that $c_0 = \emptyset$, $c_m = c$ and $q(c_{i-1}, c_i) > 0$ for i in $\{1 \cdots n\}$.*

Lemma 7.6. *For n a quasicube plus a quasisquare i.e. of the form $j(j+\delta)(j+\theta)+l(l+\epsilon)$ (where $0 < l(l+\epsilon) < (j+\delta)(j+\theta)$) we have*

$$\begin{aligned} \{\, c \in \mathcal{M}_{n-1} : q(\widetilde{\mathcal{M}}_n \setminus S_n, c) = 1 \,\} &\supset \widetilde{\mathcal{M}}_{n-1} \setminus S_{n-1}, \\ \{\, c \in \mathcal{M}_{n-1} : q(\mathcal{S}_n, c) > 0 \,\} &\supset \mathcal{S}_{n-1}, \\ \{\, c \in \mathcal{M}_{n+1} : q(\mathcal{M}_n, c) > 0 \,\} &= \widetilde{\mathcal{M}}_{n+1}, \end{aligned}$$

where $q(A,c) = \min\{q(d,c) : q \in A\}$.

Notation 7.7. We have also some results concerning the best way to shrink or to grow a parallelepiped plus a rectangle, that is on the sets (where σ is either a parallelepiped or a parallelepiped plus a rectangle):

$$\begin{aligned} \mathcal{M}(\sigma, -k) &= \{\, c \in C_{v(\sigma)-k} : c \subset \sigma,\ a(c) \text{ minimal} \,\}, \\ \mathcal{M}(\sigma, +k) &= \{\, c \in C_{v(\sigma)+k} : \sigma \subset c,\ a(c) \text{ minimal} \,\}. \end{aligned}$$

We denote by $E(\sigma, k)$ the energy of an element of $\mathcal{M}(\sigma, k)$ for k in $\mathbb{Z}$.

The next two results are restatements of Theorem 3.26, Corollary 3.27, 3.28 and Theorem 3.30 of [1].

Theorem 7.8. *Let j_1, j_2, j_3, r be positive integers such that $r < \min(j_1 j_2, j_2 j_3, j_3 j_1)$. The set $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$ is the set of the polyominoes obtained by removing from $j_1 \times j_2 \times j_3$ as many bars as possible, and then removing a succession of corner cubes until reaching the volume $j_1 j_2 j_3 - r$. In particular, a polyomino obtained from $j_1 \times j_2 \times j_3$ by the successive removal of r cubes in such a way that each cube removal takes place on a bar of minimal length is in $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$. Thus for each polyomino c in $\mathcal{M}(j_1 \times j_2 \times j_3, -r)$, there exists a polyomino d in $\mathcal{M}(j_1 \times j_2 \times j_3, -r+1)$ such that $q(d,c) > 0$ i.e. c may be obtained by removing a unit cube from d.*

Theorem 7.9. *Let j_1, j_2, j_3, r be positive integers such that $r < \min(j_1^2, j_2^2, j_3^2)$. The best way to add r cubes to the parallelepiped $j_1 \times j_2 \times j_3$ is to add a minimal two dimensional polyomino of M_r on one side of the parallelepiped. Equivalently, we have*

$$\mathcal{M}(j_1 \times j_2 \times j_3, +r) = \overline{\{\, j_1 \times j_2 \times j_3 \oplus_i d,\ 1 \le i \le 3,\, d \in M_r \,\}}^{123}.$$

In particular, $j_1 \times j_2 \times j_3 \oplus m_r \subset \mathcal{M}(j_1 \times j_2 \times j_3, +r)$.

Application to the three dimensional Ising model.

We first express the energy of the Ising model with the help of the area and the volume of the polyomino associated to the configuration.

Lemma 7.10. *For any configuration σ in X, we have*

$$E(\sigma) \;=\; -\frac{1}{2}\sum_{\{x,y\}:x\sim y}\sigma(x)\sigma(y) \;-\; \frac{h}{2}\sum_{x\in\Lambda}\sigma(x) \;=\; a(\sigma) \;-\; hv(\sigma) \;+\; N^3(h-3)/2\,.$$

We do not change the dynamics nor the cycle decomposition by adding a constant to the energy E. In dimension three, we will work with the energy $E(\sigma) = a(\sigma) - hv(\sigma)$. We denote by $E(n)$ the minimal energy of a configuration of $\mathcal{C}_n$ (so that $E(n) = E(\sigma) = a(n) - hn$ for any σ in $\mathcal{M}_n$).

Hypothesis on the magnetic field h and the size N. We suppose that h is small compared to the unity and that for any configurations η, σ, the equality $E(\eta) = E(\sigma)$ implies $v(\eta) = v(\sigma)$ and $a(\eta) = a(\sigma)$. Whenever we take the integral part of a quantity involving h, we assume that this quantity is not an integer. For instance $4/h$ is not an integer. Finally, N^3 is large enough to ensure that the combinatorial results proved on the infinite lattice $\mathbb{Z}^3$ remain valid on the torus until the critical volume $4/h(4/h+1)^2 + 2/h(2/h+1)$. This is obviously true if $h^3N > 106$.

Remark. We believe that a careful study would yield the weaker condition $h^3N^3 > K$ for some constant K. However this seems to require lengthy extensions of the techniques used in [1].

As we did in dimension two, we follow the general strategy outlined in section 4. Our first aim is to determine the communication altitude between $\underline{-1}$ and $\underline{+1}$ (Corollary 7.12) and to compute it explicitly (Proposition 7.13).

Proposition 7.11. *(communication altitude between configurations of different volumes) Let $m \le n$ be two integers. The communication altitude between the sets $\mathcal{C}_m$ and $\mathcal{C}_n$ (or $\mathcal{S}_m$ and $\mathcal{S}_n$) is*

$$E(\mathcal{C}_m, \mathcal{C}_n) \;=\; E(\mathcal{S}_m, \mathcal{S}_n) \;=\; \max\{\, E(r) : m \le r \le n \,\} \;=\; E(\mathfrak{m}_m, \mathfrak{m}_n).$$

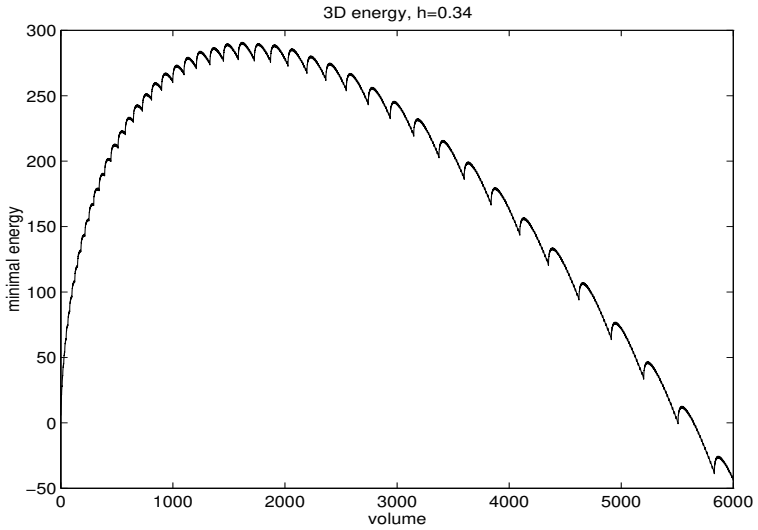
Proof. This result might be proven in exactly the same way as in dimension two (see propositions 6.11 and 6.12, corollary 6.13). □

Corollary 7.12. $E(\underline{-1}, \underline{+1}) \;=\; \max\{\, E(n) : 0 \le n \le N^3 \,\}$.

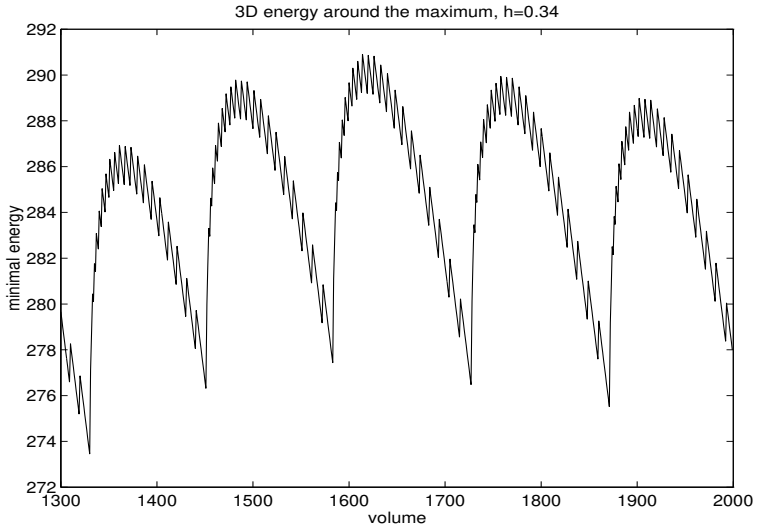
To compute $E(n)$ we apply theorem 7.2. For each integer n there exists a unique 6–uple $(j, l, k, \delta, \theta, \epsilon)$ such that $\delta, \theta, \epsilon \in \{0, 1\}$, $\delta \le \theta$, $k < l + \epsilon, l(l+\epsilon) + k < (j+\delta)(j+\theta)$ and $n = j(j+\delta)(j+\theta) + l(l+\epsilon) + k$. The minimal area $a(n)$ of a polyomino of volume n is then

$$a(n) = 2(j(j+\delta) + j(j+\theta) + (j+\delta)(j+\theta)) + 2(2l+\epsilon) + 2{\times}1_{\{k>0\}}.$$

The next figure shows a graph of $E(n)$ for $h = 0.34$.



Let us point out that each arch corresponds to a two dimensional energy curve. Below is a narrow view of the graph around its maximum.



We denote by E_2 the two dimensional energy, so that $E(n) = E(j(j+\delta)(j+\theta)) + E_2(l(l+\epsilon)+k)$. We first compute the energy barrier between two consecutive quasicubes:

$$\begin{aligned}
\max_{j^3 \le n \le j^2(j+1)} E(n) &= E(j^3) + \max_{0 \le n \le j^2} E_2(n)\,, \\
\max_{j^2(j+1) \le n \le j(j+1)^2} E(n) &= E(j^2(j+1)) + \max_{0 \le n \le j(j+1)} E_2(n)\,, \\
\max_{j(j+1)^2 \le n \le (j+1)^3} E(n) &= E(j(j+1)^2) + \max_{0 \le n \le (j+1)^2} E_2(n)\,.
\end{aligned}$$

We next compare these quantities. Let

$$\Delta_1 = \max_{j^3 \le n \le j^2(j+1)} E(n) - \max_{(j-1)j^2 \le n \le j^3} E(n) = j(4-hj)\,,$$

$$\Delta_2 = \max_{j^2(j+1) \le n \le j(j+1)^2} E(n) - \max_{j^3 \le n \le j^2(j+1)} E(n) = j(4-hj) + \max_{0 \le n \le j(j+1)} E_2(n) - \max_{0 \le n \le j^2} E_2(n)\,,$$

$$\Delta_3 = \max_{j(j+1)^2 \le n \le (j+1)^3} E(n) - \max_{j^2(j+1) \le n \le j(j+1)^2} E(n) = j(4-hj) + 2 - hj + \max_{0 \le n \le (j+1)^2} E_2(n) - \max_{0 \le n \le j(j+1)} E_2(n)\,.$$

The unique positive root of $4j - hj^2 + 2 - hj$ is $j_c^1 = 2/h(1 + \sqrt{1 + h^2/16}) - 1/2$ which is less than $j_c^2 = 4/h$, but greater than l_c. For $j < j_c^2$, Δ_1 is positive, and for $j > j_c^2$, Δ_1 is negative. For $j < j_c^2$, Δ_2 is positive. For $j > j_c^2$, we have $j > l_c$ (the two dimensional critical length) so that the two dimensional maxima are equal and $\Delta_2 = j(4-hj)$ which is negative. Finally, Δ_3 is positive for $j < j_c^1$ and negative for $j > j_c^1$.
Let D be the set of the quasisquares $\{\,(j,\delta,\theta) : j \in \mathbb{N}, 0 \le \delta \le \theta \le 1\,\}$. We endow D with the lexicographical order, and for each (j,δ,θ) in D, we denote by $\mathrm{pred}\,(j,\delta,\theta)$ (respectively $\mathrm{succ}\,(j,\delta,\theta)$) the greatest element of D (resp. the smallest) which is smaller (resp. greater) than (j,δ,θ) To an element (j,δ,θ) of D we associate the integer $j(j+\delta)(j+\theta)$. The inequality $n \le (j,\delta,\theta)$ then means $n \le j(j+\delta)(j+\theta)$. We consider the function ϕ defined on the set D by

$$\phi(j,\delta,\theta) = \max\{\,E(n) : (j,\delta,\theta) \le n \le \mathrm{succ}\,(j,\delta,\theta)\,\} - \max\{\,E(n) : \mathrm{pred}\,(j,\delta,\theta) \le n \le (j,\delta,\theta)\,\}.$$

We have shown that ϕ is positive for

- $\delta = 0$, $\theta = 0$, $j < j_c^2$ (case of Δ_1)
- $\delta = 0$, $\theta = 1$, $j < j_c^2$ (case of Δ_2)
- $\delta = 1$, $\theta = 1$, $j < j_c^1$ (case of Δ_3)

and negative for

- $\delta = 0$, $\theta = 0$, $j > j_c^2$ (case of Δ_1)
- $\delta = 0$, $\theta = 1$, $j > j_c^2$ (case of Δ_2)
- $\delta = 1$, $\theta = 1$, $j > j_c^1$ (case of Δ_3)

We now define the critical quasicube.

- If $\lfloor j_c^1 \rfloor = \lfloor j_c^2 \rfloor$ we put $j_c = \lfloor j_c^1 \rfloor$ and $\delta_c = 1$.
- If $\lfloor j_c^1 \rfloor < \lfloor j_c^2 \rfloor$ we put $j_c = \lfloor j_c^2 \rfloor$ and $\delta_c = 0$.

The critical quasicube is $j_c(j_c + \delta_c)(j_c + 1)$. We can now compute the energy barrier between two remote quasicubes.

$$\max\{\,E(n) : (j_1,\delta_1,\theta_1) \le n \le (j_2,\delta_2,\theta_2)\,\} = \begin{cases} E((j_c,\delta_c,1)) + E_2(l_c(l_c+1)+1) & \text{if } (j_1,\delta_1,\theta_1) \le (j_c,\delta_c,1) < (j_2,\delta_2,\theta_2) \\ E(\mathrm{pred}\,(j_2,\delta_2,\theta_2)) + E_2(l_c(l_c+1)+1) & \text{if } (j_2,\delta_2,\theta_2) \le (j_c,\delta_c,1),\ j_2 > l_c \\ E((j_1,\delta_1,\theta_1)) + E_2(l_c(l_c+1)+1) & \text{if } (j_c,\delta_c,1) < (j_1,\delta_1,\theta_1) \end{cases}$$

We finally obtain the global energy barrier.

Proposition 7.13. *We set $\delta_c = 0$ if $4+\sqrt{16+h^2} < h(2\lfloor 4/h\rfloor + 1)$ and $\delta_c = 1$ otherwise. The communication altitude between $\underline{-1}$ and $\underline{+1}$ is*

$$E(\underline{-1},\underline{+1}) \;=\; E(n_c) \;= E((j_c,\delta_c,1)) + E_2(l_c(l_c+1)+1)$$

where $l_c = \lfloor 2/h\rfloor$ is the two dimensional critical length, $j_c = \lfloor 4/h\rfloor$ is the three dimensional critical length, $n_c = j_c(j_c+\delta_c)(j_c+1) + l_c(l_c+1)+1$ is the critical volume.

Remark. We have $E(\underline{-1},\underline{+1}) \sim 32/h^2$ as $h \to 0$: the global energy barrier goes to infinity like h^{-2} as h goes to zero.

From this we deduce the level of the greatest cycle containing $\underline{-1}$ and not $\underline{+1}$.

Corollary 7.14. *The level of the cycle $\pi(\underline{-1},\underline{+1}^c)$ is $pred E(n_c)$.*

Notation 7.15. If Y is a subset of X, its minimal and maximal volumes $\underline{v}(Y)$ and $\overline{v}(Y)$ are

$$\underline{v}(Y) \;=\; \min\{\, v(\sigma) : \sigma \in Y \,\}, \quad \overline{v}(Y) \;=\; \max\{\, v(\sigma) : \sigma \in Y \,\}.$$

Since $E(\underline{-1},\mathcal{C}_n) \geq E(n_c)$ for any $n \geq n_c$ (by proposition 7.11), then all the configurations of the cycle $\pi(\underline{-1},\underline{+1}^c)$ have a volume less then $n_c - 1$ i.e. $\overline{v}(\pi(\underline{-1},\underline{+1}^c)) \leq n_c - 1$. To complete step i) of the general strategy, we determine the configurations of the principal boundary of the cycle $\pi(\underline{-1},\underline{+1}^c)$.

Theorem 7.16. *The principal boundary $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$ of the cycle $\pi(\underline{-1},\underline{+1}^c)$ is the set $\widetilde{\mathcal{M}}_{n_c}$ of the principal configurations of volume n_c. For σ in $\widetilde{\mathcal{M}}_{n_c}$ there exists a unique configuration in $\pi(\underline{-1},\underline{+1}^c)$ communicating with σ, which is the quasicube plus a quasisquare $j_c \times (j_c+\delta_c) \times (j_c+1) \oplus l_c \times (l_c+1)$ or $(l_c+1)\times l_c$ included in σ.*

Proof. Let σ belong to $\widetilde{\mathcal{M}}_{n_c}$. Then $E(\sigma) = E(n_c)$ and theorem 7.4 yields the existence of a sequence $\sigma_0, \cdots, \sigma_{n_c}$ such that

$$\sigma_0 = \underline{-1}, \quad \sigma_{n_c} = \sigma, \quad \forall j \in \{0 \cdots n_c - 1\} \quad \sigma_j \in \mathcal{M}_j, \quad q^+(\sigma_j,\sigma_{j+1}) = 1.$$

In particular, we have $\max\{\, E(\sigma_n) \,:\, 0 \leq n < n_c \,\} < E(n_c)$ so that σ_{n_c-1} belongs to $\pi(\underline{-1},\underline{+1}^c)$. Since $q(\sigma_{n_c-1},\sigma_{n_c}) > 0$, the configuration σ_{n_c} is in the principal boundary of $\pi(\underline{-1},\underline{+1}^c)$. Thus $\widetilde{\mathcal{M}}_{n_c} \subset \widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$.
Conversely, let σ belong to $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$. Necessarily $E(\sigma) = E(n_c)$ so that σ is of volume n_c and it is a minimal configuration. In addition there must exist η in $\pi(\underline{-1},\underline{+1}^c)$ communicating with σ. This η is in $\mathcal{C}_{n_c-1}$ and satisfies $E(\eta) < E(\sigma)$. Thus $a(\eta) \leq a(\sigma) - h < a(n_c) = a(n_c-1)+2$ and $a(\eta) \geq a(n_c-1)$. However the area is an even integer. The only

possibility is $a(\eta) = a(n_c - 1)$ whence η is minimal. Yet $n_c - 1 = j_c(j_c+\delta_c)(j_c+1)+l_c(l_c+1)$ is a quasicube plus a quasisquare; by theorem 7.3, $\mathcal{M}_{n_c-1} = \widetilde{\mathcal{M}}_{n_c-1}$ so that η belongs to $\widetilde{\mathcal{M}}_{n_c-1}$ and it is a quasicube plus a quasisquare $j\times(j+\delta_c)\times j+1)\oplus l_c\times(l_c+1)$. Lemma 7.6 shows that the only points of $\mathcal{M}_{n_c}$ which communicate with $\mathcal{M}_{n_c-1}$ are the configurations of $\widetilde{\mathcal{M}}_{n_c}$. Thus σ is a principal configuration of volume n_c and $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c)) \subset \widetilde{\mathcal{M}}_{n_c}$. $\square$

We now proceed to steps *ii)* and *iii)* of the general strategy. On the set of the cycles included in $\{\underline{-1},\underline{+1}\}^c$, we consider the graph $\mathcal{G}$ defined by

$$(\pi_1 \to \pi_2) \in \mathcal{G} \qquad \Longleftrightarrow \qquad \widetilde{B}(\pi_1) \cap \pi_2 \neq \emptyset.$$

We denote by $\mathcal{G}^+$ the restriction of $\mathcal{G}$ to $\pi(\underline{-1},\underline{+1}^c)^c$ (obtained by deleting all the arrows whose starting cycle is included in $\pi(\underline{-1},\underline{+1}^c)$). According to the general method for determining the saddle paths of minimal cost exiting from $\underline{-1}$ and reaching $\underline{+1}$, the next step is to find, for each point σ' in the principal boundary of $\pi(\underline{-1},\underline{+1}^c)$, the minimal stable subgraph of $\mathcal{G}^+$ containing $\{\sigma'\}$. If there does not exist a path in this graph leading from $\{\sigma'\}$ to $\underline{+1}$, then σ' is a dead–end and the last exit from $\pi(\underline{-1},\underline{+1}^c)$ before reaching $\underline{+1}$ won't take place through σ'. Otherwise, σ' is a possible global saddle between $\underline{-1}$ and $\underline{+1}$ and we have to determine all the paths in $\mathcal{G}^+$ leading from $\{\sigma'\}$ to $\underline{+1}$; then we must search for all the cycle paths in $\{\underline{-1},\underline{+1}\}^c$ of null cost realizing the exit of $\pi(\underline{-1},\underline{+1}^c)$ at σ'. More precisely, if we denote by $\mathcal{G}^-$ the restriction of $\mathcal{G}$ to $\pi(\underline{+1},\underline{-1}^c)^c$ (obtained by deleting all the arrows whose starting cycle is in $\pi(\underline{+1},\underline{-1}^c)$), we have to determine the minimal stable subgraph of $\mathcal{G}^-$ containing σ' and all the paths in this graph starting at $\{\sigma'\}$ and ending at $\underline{-1}$. To achieve these goals, we will next describe a list of relevant cycles, which are included in $\{\underline{-1},\underline{+1}\}^c$ and we will precise their principal boundaries. Notice that we don't know *a priori* that these cycles are in $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$. Lemma 6.19 will yield "instantaneously" the list of the relevant cycles and will designate those cycles of the list which are in $\mathcal{M}(\{\underline{-1},\underline{+1}\}^c)$. Our list of cycles will be large enough to ensure that for each configuration σ' in $\widetilde{B}(\pi(\underline{-1},\underline{+1}^c))$, it includes all the vertices of the minimal stable subgraph of $\mathcal{G}$ containing $\{\sigma'\}$. Moreover, we will handle separately the case of the configurations in $\mathcal{S}_{n_c}$ and in $\widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}$: it turns out that step *ii)* succeeds for the standard configurations and fails for the principal non standard configurations. The latter are dead–ends.

The standard configurations. We start by describing the relevant list of cycles for determining the minimal stable subgraph of $\mathcal{G}$ containing the standard configurations of volume n_c. These are the cycles around a parallelepiped $j_1 \times j_2 \times j_3$, a parallelepiped plus a rectangle $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2$ and finally a parallelepiped plus a rectangle plus a bar $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2 \oplus k$, where each configuration contains strictly a critical configuration.

Theorem 7.17. *Let j_1, j_2, j_3 be three integers such that the parallelepiped $j_1 \times j_2 \times j_3$*

contains a configuration of $\mathcal{S}_{n_c}$. *The cycle*

$$\pi = \pi(j_1 \times j_2 \times j_3, \mathrm{pred}\, E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c+1) \oplus 1))$$

does not contain $\underline{-1}$ *and* $\underline{+1}$. *Moreover*

$$\begin{aligned} \underline{v}(\pi) \;&>\; j_1 j_2 j_3 - \min(j_1 j_2, j_1 j_3, j_2 j_3) + l_c(l_c+1) + 1\,, \\ \overline{v}(\pi) \;&=\; j_1 j_2 j_3 + l_c(l_c+1)\,. \end{aligned}$$

The bottom of this cycle is $\{j_1 \times j_2 \times j_3\}$ *and its principal boundary is*

$$\widetilde{B}(\pi) \;=\; \overline{j_1 \times j_2 \times j_3 \oplus \overline{l_c \times (l_c+1) \oplus 1}}^{123}\,.$$

Proof. We check that the parallelepiped $j_1 \times j_2 \times j_3$ and the cycle π satisfy the hypothesis of theorem 5.5. Let $x_1, \cdots, x_r$ be a sequence of sites such that $F(x_1, \cdots, x_s)(j_1 \times j_2 \times j_3)$ is in π for s in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\mathrm{pred}\, E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c+1) \oplus 1))$. We put $\eta_s = F(x_1, \cdots, x_s)(j_1 \times j_2 \times j_3)$ for s in $\{0 \cdots r\}$.

- First case: all the sites are outside $j_1 \times j_2 \times j_3$. We have that $j_1 \times j_2 \times j_3 \subset \eta_s$ whence $E(\eta_s) \geq E(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3)$ and

$$\mathrm{pred}\, E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c+1) \oplus 1)) \;\geq\; \max_{0 \leq s \leq r} E(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|v(\eta_{s+1}) - v(\eta_s)| \leq 1$ and $(v(\eta_s),\, 0 \leq s \leq r)$ takes all the values between $j_1 j_2 j_3$ and $v(\eta_r)$. Henceforth

$$\max_{0 \leq s \leq r} E(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3) \;\geq\; \max\{\, E(j_1 \times j_2 \times j_3, k) : 0 \leq k \leq v(\eta_r) - j_1 j_2 j_3 \,\}$$

and the volume of η_r must satisfy

$$\max\{\, E(j_1 \times j_2 \times j_3, k) : 0 \leq k \leq v(\eta_r) - j_1 j_2 j_3 \,\} \;\leq\; \mathrm{pred}\, E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c+1) \oplus 1)).$$

By theorem 7.9, we have

$$E(j_1 \times j_2 \times j_3, l_c(l_c+1)+1) = E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c+1) \oplus 1))$$

so that $v(\eta_r) < j_1 j_2 j_3 + l_c(l_c+1) + 1$.

- Second case: all the sites are inside $j_1 \times j_2 \times j_3$. Now $\eta_s \subset j_1 \times j_2 \times j_3$ so that

$$\begin{aligned} \mathrm{pred}\, E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c+1) \oplus 1)) \;&\geq\; \max_{0 \leq s \leq r} E(j_1 \times j_2 \times j_3, v(\eta_s) - j_1 j_2 j_3) \\ &\geq\; \max\{\, E(j_1 \times j_2 \times j_3, k) : v(\eta_r) - j_1 j_2 j_3 \leq k \leq 0 \,\}. \end{aligned}$$

Suppose for instance that $j_1 j_2$ is the smallest side of the parallelepiped. Theorem 7.8 shows that for $j_1(j_2 - j_1) \leq k < j_1 j_2$,

$$E(j_1 \times j_2 \times j_3, -k) = E(j_1 \times j_2 \times (j_3 - 1) \oplus m_{j_1 j_2 - k})$$

However, $E(j_1 \times j_2 \times (j_3 - 1) \oplus l_c \times (l_c + 1) \oplus_1 1) > E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c + 1) \oplus 1))$ whence necessarily $j_1 j_2 - k > l_c(l_c + 1) + 1$ and $v(\eta_r) > j_1 j_2 j_3 - j_1 j_2 + l_c(l_c + 1) + 1$. In addition, we have that $E(j_1 \times j_2 \times j_3, k) > E(j_1 \times j_2 \times j_3)$ for all k such that

$$-j_1 j_2 + l_c(l_c + 1) + 1 < k \leq l_c(l_c + 1),\ k \neq 0.$$

We have thus proved that $j_1 \times j_2 \times j_3$ and the cycle π satisfy the hypothesis of theorem 5.5. Thus the bottom of the cycle is $\{j_1 \times j_2 \times j_3\}$. We have also proved the inequalities on $\overline{v}$ and $\underline{v}$. That $\mathcal{M}(j_1 \times j_2 \times j_3, k)$ is included in the cycle for $0 \leq k \leq l_c(l_c + 1)$ is obvious: each configuration of this set communicates with $j_1 \times j_2 \times j_3$ under the level $\mathrm{pred}\, E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c + 1) \oplus 1))$ (cf theorem 7.9). Finally a configuration ξ of the principal boundary of this cycle is of energy $E(\xi) = E(j_1 \times j_2 \times j_3 \oplus l_c \times (l_c + 1) \oplus 1))$ so that its volume is equal to $j_1 j_2 j_3 + l_c(l_c + 1) + 1$, and its area is $2(j_1 j_2 + j_2 j_3 + j_1 j_3) + 2(2l_c + 1) + 2$. Let η be a configuration of the cycle such that $q(\eta, \xi) > 0$. Necessarily, the volume of η is $j_1 j_2 j_3 + l_c(l_c + 1)$. Thus η is a configuration of maximal volume of the cycle π and as such it is a maximal configuration of the cycle for the inclusion relation. By theorem 5.3 the parallelepiped $j_1 \times j_2 \times j_3$ is included in η. In addition $E(\eta) < E(\xi)$ implies that $a(\eta) \leq 2(j_1 j_2 + j_2 j_3 + j_1 j_3) + 2(2l_c + 1)$ whence in fact η belongs to $\mathcal{M}(j_1 \times j_2 \times j_3, l_c(l_c + 1))$. Thus η is the sum of $j_1 \times j_2 \times j_3$ and a quasisquare $l_c \times (l_c + 1)$ (theorem 7.9). Also ξ belongs to $\mathcal{M}(j_1 \times j_2 \times j_3, l_c(l_c + 1) + 1)$. The only configurations of this set which communicate with $\mathcal{M}(j_1 \times j_2 \times j_3, l_c(l_c + 1))$ are the configurations of

$$\overline{\overline{j_1 \times j_2 \times j_3} \oplus \overline{l_c \times (l_c + 1)} \oplus 1}^{1\,2\,3}.$$

Conversely, it is clear that all these configurations belong to the principal boundary of the cycle. □

Theorem 7.18. *Let j_1, j_2, j_3, l_1, l_2 be integers such that the parallelepiped $j_1 \times j_2 \times j_3$ contains a critical quasicube and the rectangle $l_1 \times l_2$ contains a critical two dimensional configuration. The cycle*

$$\pi = \pi(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2, \mathrm{pred}\, E(j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 \oplus 1))$$

does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\underline{v}(\pi) > j_1 j_2 j_3 + l_1 l_2 - l_c + 1\,, \quad \overline{v}(\pi) = j_1 j_2 j_3 + l_1 l_2\,.$$

The bottom of this cycle is $\{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2\}$ and its principal boundary is

$$\widetilde{B}(\pi) = j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 \oplus 1.$$

Remark. Obviously, similar results hold for any parallelepiped plus rectangle $j_1 \times j_2 \times j_3 \oplus l_1 \times l_2$ satisfying the requirements of the theorem.

Theorem 7.19. *Let $j_1 \times j_2 \times j_3$ be a parallelepiped containing strictly a three dimensional critical configuration. The cycle*

$$\pi = \pi(j_1 \times j_2 \times j_3 + l_c \times (l_c + 2), pred\, E(j_1 \times j_2 \times j_3 + l_c \times (l_c + 1) +_2 1))$$

is included in $\{\underline{-1}, \underline{+1}\}^c$. Moreover,

$$\underline{v}(\pi) \;=\; j_1 j_2 j_3 + l_c(l_c+1) + 2\,, \quad \overline{v}(\pi) \;=\; j_1 j_2 j_3 + l_c(l_c+2)\,.$$
$$\forall k \in \{1 \cdots l_c - 2\} \qquad \mathcal{M}(j_1 \times j_2 \times j_3 + l_c \times (l_c+2), -k) \;\subset\; \pi\,,$$

The bottom of this cycle is $\{j_1 \times j_2 \times j_3 + l_c \times (l_c + 2)\}$; its principal boundary is

$$\widetilde{B}(\pi) \;=\; \mathcal{M}(j_1 \times j_2 \times j_3 + l_c \times (l_c+2), -(l_c - 1))$$

and thus contains $j_1 \times j_2 \times j_3 + \overline{l_c \times (l_c+1) \oplus_2 1}^{12}$.

Proof. Using theorem 7.8, the proof is similar to the proof of theorem 6.25. □

Remark. Obviously, similar statements are true for the configurations in $j_1 \times j_2 \times j_3 \oplus (l_c + 2) \times l_c$ and $j_1 \times j_2 \times j_3 \oplus l_c \times (l_c + 2)$.

Lemma 7.20. *Let j_1, j_2, j_3, l_1, l_2 be positive integers. Let k, i be such that $k > 1$, $i \geq 0$, $l_1 > k + i$. The cycle $\{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i k\}$ has one or two configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i (k+1)\}$,*
- *if $i = l_1 - 1 - k$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} (k+1)\}$,*
- *if $0 < i < l_1 - 1 - k$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} (k+1),\ l_1 \times l_2 +_2^i (k+1)\}$.*

Lemma 7.21. *Let j_1, j_2, j_3, l_1, l_2 be positive integers with $l_1 \times l_2 \subset j_1 \times j_2$ and let i be such that $i \geq 0$, $l_1 > i + 1$. The cycle $\{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i 1\}$ has two or three configurations in its principal boundary $\widetilde{B}$:*

- *if $i = 0$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2,\ j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i 2\}$,*
- *if $i = l_1 - 1 - k$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2,\ j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} 2\}$,*
- *if $0 < i < l_1 - 1 - k$ then $\widetilde{B} = \{j_1 \times j_2 \times j_3 +_1 l_1 \times l_2,\ j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^{i-1} 2,$ $j_1 \times j_2 \times j_3 +_1 l_1 \times l_2 +_2^i 2\}$.*

Remark. Results similar to those stated in lemmas 7.20 and 7.21 hold for the configurations in $\overline{j_1 \times j_2 \times j_3 \oplus l_1 \times l_2 \oplus 1}$.

Corollary 7.22. *The following cycles are maximal cycles of* $\{\underline{-1},\underline{+1}\}^c$*, where* $j_1\times j_2\times j_3$ *contains a configuration of* $\mathcal{S}_{n_c}$*,* $l_1\times l_2$ *contains a critical two dimensional configuration,*

1) $\pi(j_1\times j_2\times j_3, pred\,E(j_1\times j_2\times j_3 +_1 l_c\times(l_c+1)+_2 1))$,
2) $\pi(j_1\times j_2\times j_3 +_1 l_c\times(l_c+2), pred\,E(j_1\times j_2\times j_3 +_1 l_c\times(l_c+1)+_2 1))$,
3) $\pi(j_1\times j_2\times j_3 +_1 l_1\times l_2, pred\,E(j_1\times j_2\times j_3 +_1 l_1\times l_2\oplus 1))$,
   and this cycle is not included in a cycle of type 1),
4) $\pi(j_c\times(j_c+\delta_c)\times(j_c+1) +_1 l_1\times l_2, pred\,E(j_c\times(j_c+\delta_c)\times(j_c+1)+l_1\times l_2\oplus 1))$,
   and this cycle is not included in a cycle of type 1),
5) $\{\eta\}$, $\eta\in\mathcal{M}(j_1\times j_2\times j_3 +_1 l_c\times(l_c+2),-(l_c-1))$,
6) $\{\eta\}$, $\eta\in\overline{j'_1\times j'_2\times j'_3\oplus\overline{l'_1\times l'_2\oplus k}}$ *where* $j'_1\times j'_2\times j'_3$ *contains a quasicube* $j_c\times(j_c+\delta_c)\times(j_c+1)$*,* $l'_1\times l'_2$ *contains a quasisquare* $l_c\times(l_c+1)$*,* $k>0$*,*
   and this cycle is not included in a cycle of type $1),2),3),4),5)$.

The list should be completed with all the isometric cycles (obtained by applying an isometry) as well as all the cycles whose bottom is obtained by translating the rectangle $l_1\times l_2$ *along the side of the parallelepiped (i.e. the configurations in* $j_1\times j_2\times j_3\oplus l_1\times l_2$*).*

Proof. This corollary is a consequence of lemma 6.19 together with theorems 7.17, 7.18, 7.19 and lemmas 7.20 and 7.21. Notice that we have to put together the descriptions of the cycles of theorems 7.17, 7.18, 7.19, lemmas 7.20, 7.21 in order to check that for each cycle π in the above list, there is a sequence of cycles $\pi_0,\cdots,\pi_r$ such that $\pi_0=\pi$, $\widetilde{B}(\pi_i)\cap\pi_{i+1}\neq\emptyset$, $0\le i<r$ and $\underline{+1}\in\widetilde{B}(\pi_r)$. □

Corollary 7.23. *Let* σ *belong to* $\mathcal{S}_{n_c+1}$*. The minimal stable subgraph* $\mathcal{G}^+(\sigma)$ *of* $\mathcal{G}^+$ *containing* σ *is the restriction of* $\mathcal{G}$ *to the vertices listed in corollary* 7.22*. The arrows of* $\mathcal{G}^+(\sigma)$ *are (where* η *is the parallelepiped* $j_1\times j_2\times j_3$ *and the cycles which are the starting point of the arrows belong to the list of cycles of corollary* 7.22*):*

1) around the parallelepipeds:

$\pi(\eta, pred\,E(\eta\oplus l_c\times(l_c+1)\oplus 1))\leftrightarrow\{\xi\}$, $\xi\in\overline{\overline{\eta\oplus\overline{l_c\times(l_c+1)\oplus 1}}}^{123}$,

2) growing in the direction of e_1*:*

$\pi(\eta +_1 l_c\times(l_c+2), pred\,E(\eta +_1 l_c\times(l_c+1)\oplus 1))\leftrightarrow\{\xi\}$,
$\quad\xi\in\mathcal{M}(\eta +_1 l_c\times(l_c+2),-(l_c-1))$,
$\pi(\eta +_1 l_1\times l_2, pred\,E(\eta +_1 l_1\times l_2\oplus 1))\leftrightarrow\{\xi\}$, $\xi\in\eta +_1 l_1\times l_2\oplus 1$
$\{\eta +_1 l_1\times l_2 +_1 k\}\to\{\eta +_1 l_1\times l_2 +_1 (k+1)\}$, $0<k<l_2-l_c$,
$\{\eta +_1 l_1\times l_2 +_2 k\}\to\{\eta +_1 l_1\times l_2 +_2 (k+1)\}$, $0<k<l_1-l_c$,
$\{\eta +_1 l_1\times l_2 +_2 l_1-l_c\}\to\pi(\eta +_1 l_1\times(l_2+1), pred\,E(\eta +_1 l_1\times(l_2+1)+1))$,

$\{\eta +_1 l_1 \times l_2 +_1 l_2 - l_c\} \to \pi(\eta +_1 (l_1+1) \times l_2, pred E(\eta +_1 (l_1+1) \times l_2 + 1))\,,$
$\{\eta +_1 l_1 \times l_2 +_2 l_1 - l_c\} \to \pi(\eta +_1 j_2 \times j_3, pred E(\eta +_1 j_2 \times j_3 +_1 l_c \times (l_c+1) +_1 1))\,,$
$\{\eta +_1 l_1 \times l_2 +_1 l_2 - l_c\} \to \pi(\eta +_1 j_2 \times j_3, pred E(\eta +_1 j_2 \times j_3 +_1 l_c \times (l_c+1) +_1 1))\,,$

3) growing in the direction of e_2:

$\pi(\eta +_2 l_c \times (l_c+2), pred E(\eta +_2 l_c \times (l_c+1) \oplus 1)) \leftrightarrow \{\xi\},$
$\quad \xi \in \mathcal{M}(\eta +_2 l_c \times (l_c+2), -(l_c-1))\,,$
$\pi(\eta +_2 l_1 \times l_2, pred E(\eta +_2 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_2 l_1 \times l_2 \oplus 1$
$\{\eta +_2 l_1 \times l_2 +_1 k\} \to \{\eta +_2 l_1 \times l_2 +_1 (k+1)\},\ 0 < k < l_2 - l_c\,,$
$\{\eta +_2 l_1 \times l_2 +_2 k\} \to \{\eta +_2 l_1 \times l_2 +_2 (k+1)\},\ 0 < k < l_1 - l_c\,,$
$\{\eta +_2 l_1 \times l_2 +_2 l_1 - l_c\} \to \pi(\eta +_2 l_1 \times (l_2+1), pred E(\eta +_2 l_1 \times (l_2+1) + 1))\,,$
$\{\eta +_2 l_1 \times l_2 +_1 l_2 - l_c\} \to \pi(\eta +_2 (l_1+1) \times l_2, pred E(\eta +_2 (l_1+1) \times l_2 + 1))\,,$
$\{\eta +_2 l_1 \times l_2 +_2 l_1 - l_c\} \to \pi(\eta +_2 j_1 \times j_3, pred E(\eta +_2 j_1 \times j_3 +_2 l_c \times (l_c+1) +_1 1))\,,$
$\{\eta +_2 l_1 \times l_2 +_1 l_2 - l_c\} \to \pi(\eta +_2 j_1 \times j_3, pred E(\eta +_2 j_1 \times j_3 +_2 l_c \times (l_c+1) +_1 1))\,,$

4) growing in the direction of e_3:

$\pi(\eta +_3 l_c \times (l_c+2), pred E(\eta +_3 l_c \times (l_c+1) \oplus 1)) \leftrightarrow \{\xi\},$
$\quad \xi \in \mathcal{M}(\eta +_3 l_c \times (l_c+2), -(l_c-1))\,,$
$\pi(\eta +_3 l_1 \times l_2, pred E(\eta +_3 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_3 l_1 \times l_2 \oplus 1$
$\{\eta +_3 l_1 \times l_2 +_1 k\} \to \{\eta +_3 l_1 \times l_2 +_1 (k+1)\},\ 0 < k < l_2 - l_c\,,$
$\{\eta +_3 l_1 \times l_2 +_2 k\} \to \{\eta +_3 l_1 \times l_2 +_2 (k+1)\},\ 0 < k < l_1 - l_c\,,$
$\{\eta +_3 l_1 \times l_2 +_2 l_1 - l_c\} \to \pi(\eta +_3 l_1 \times (l_2+1), pred E(\eta +_3 l_1 \times (l_2+1) + 1))\,,$
$\{\eta +_3 l_1 \times l_2 +_1 l_2 - l_c\} \to \pi(\eta +_3 (l_1+1) \times l_2, pred E(\eta +_3 (l_1+1) \times l_2 + 1))\,,$
$\{\eta +_3 l_1 \times l_2 +_2 l_1 - l_c\} \to \pi(\eta +_3 j_1 \times j_2, pred E(\eta +_3 j_1 \times j_2 +_3 l_c \times (l_c+1) +_1 1))\,,$
$\{\eta +_3 l_1 \times l_2 +_1 l_2 - l_c\} \to \pi(\eta +_3 j_1 \times j_2, pred E(\eta +_3 j_1 \times j_2 +_3 l_c \times (l_c+1) +_1 1))\,.$

The symbol $\leftrightarrow$ means that both arrows $\to$ and $\leftarrow$ are present. The above list should be completed with all the isometric arrows (obtained by applying the same isometry to both ends of an arrow), as well as all the arrows obtained by sliding either the bar along the rectangle (i.e. the configurations $l_1 \times l_2 +_1^i k$, $0 \le i < l_2 - k$) or by translating the rectangle $l_1 \times l_2$ on the side of the parallelepiped η (i.e. the configurations in $\eta \oplus l_1 \times l_2$).

The loops in the graph $\mathcal{G}^+(\sigma)$ are

$$\pi(\eta, \mathrm{pred}\, E(\eta \oplus l_c \times (l_c+1) \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \overline{\overline{\overline{\eta \oplus l_c \times (l_c+1) \oplus 1}}}^{123}\,,$$
$$\pi(\eta +_1 l_c \times (l_c+2), \mathrm{pred}\, E(\eta +_1 l_c \times (l_c+1) \oplus 1)) \leftrightarrow \{\xi\}\,,\ \xi \in \mathcal{M}(\eta +_1 l_c \times (l_c+2), -(l_c-1))\,,$$
$$\pi(\eta +_1 l_1 \times l_2, \mathrm{pred}\, E(\eta +_1 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_1 l_1 \times l_2 \oplus 1,$$
$$\pi(\eta +_2 l_c \times (l_c+2), \mathrm{pred}\, E(\eta +_2 l_c \times (l_c+1) \oplus 1)) \leftrightarrow \{\xi\}\,,\ \xi \in \mathcal{M}(\eta +_2 l_c \times (l_c+2), -(l_c-1))\,,$$
$$\pi(\eta +_2 l_1 \times l_2, \mathrm{pred}\, E(\eta +_2 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_2 l_1 \times l_2 \oplus 1,$$
$$\pi(\eta +_3 l_c \times (l_c+2), \mathrm{pred}\, E(\eta +_3 l_c \times (l_c+1) \oplus 1)) \leftrightarrow \{\xi\}\,,\ \xi \in \mathcal{M}(\eta +_3 l_c \times (l_c+2), -(l_c-1))\,,$$
$$\pi(\eta +_3 l_1 \times l_2, \mathrm{pred}\, E(\eta +_3 l_1 \times l_2 \oplus 1)) \leftrightarrow \{\xi\},\ \xi \in \eta +_3 l_1 \times l_2 \oplus 1.$$

All these loops are of length two. Any other arrow $\pi_1 \to \pi_2$ of $\mathcal{G}^+(\sigma)$ satisfies $\overline{v}(\pi_1) < \underline{v}(\pi_2)$. As a consequence a path in $\mathcal{G}^+(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{+1}\}$.

We are done with the standard configurations. We have to examine the remaining configurations of the principal boundary of the cycle $\pi(\underline{-1}, \underline{+1}^c)$ i.e. the principal non standard configurations.

The principal non standard configurations. We now do the same work for the configurations in $\widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}$. The relevant cycles are the cycles around the parallelepipeds $j_c \times (j_c + \delta_c) \times (j_c + 2)$ and around the configurations of $\widetilde{\mathcal{M}}_{n_c+1} \setminus \mathcal{S}_{n_c+1}$.

Theorem 7.24. *The cycle $\pi = \pi(j_c \times (j_c + \delta_c) \times (j_c + 2), \operatorname{pred} E(n_c))$ is the greatest cycle containing $j_c \times (j_c + \delta_c) \times (j_c + 2)$ included in $\{\underline{-1}, \underline{+1}\}^c$. Moreover,*

$$\underline{v}(\pi) \;=\; n_c + 1\,, \quad \overline{v}(\pi) \;<\; j_c(j_c + \delta_c)(j_c + 2) + l_c(l_c + 1) + 1\,.$$

The bottom of this cycle is $\{j_c \times (j_c + \delta_c) \times (j_c + 2)\}$; its principal boundary contains

$$\{\, \xi : \xi \subset j_c \times (j_c + \delta_c) \times (j_c + 2),\, \xi \in \widetilde{\mathcal{M}}_{n_c} \,\}$$

and is included in $\mathcal{M}(j_c \times (j_c + \delta_c) \times (j_c + 2), n_c - j_c(j_c + \delta_c)(j_c + 1))$.

Remark. Obviously, similar statements are true for the parallelepipeds isometric to $j_c \times (j_c + \delta_c) \times (j_c + 2)$.

Proof. We check that the parallelepiped $j_c \times (j_c + \delta_c) \times (j_c + 2)$ and the cycle π satisfy the hypothesis of theorem 5.5. Let $x_1, \cdots, x_r$ be a sequence of sites such that $F(x_1, \cdots, x_s)(j_c \times (j_c + \delta_c) \times (j_c + 2)$ is in π for s in $\{1 \cdots r\}$ (i.e. these configurations have an energy less or equal than $\operatorname{pred} E(n_c)$). We put $\eta_s = F(x_1, \cdots, x_s)(j_c \times (j_c + \delta_c) \times (j_c + 2))$ for s in $\{0 \cdots r\}$.

- First case: all the sites are outside $j_c \times (j_c + \delta_c) \times (j_c + 2)$. We have that $j_c \times (j_c + \delta_c) \times (j_c + 2) \subset \eta_s$ whence $E(\eta_s) \;\geq\; E(j_c \times (j_c + \delta_c) \times (j_c + 2), v(\eta_s) - j_c(j_c + \delta_c)(j_c + 2))$ and

$$\operatorname{pred} E(n_c) \;\geq\; \max_{0 \leq s \leq r} E(j_c \times (j_c + \delta_c) \times (j_c + 2), v(\eta_s) - j_c(j_c + \delta_c)(j_c + 2)).$$

Since the sequence $\eta_0, \cdots, \eta_r$ is a sequence of spin flips we have $|v(\eta_{s+1}) - v(\eta_s)| \leq 1$ and $(v(\eta_s),\, 0 \leq s \leq r)$ takes all the values between $j_c(j_c + \delta_c)(j_c + 2)$ and $v(\eta_r)$. Henceforth

$$\max_{0 \leq s \leq r} E(j_c \times (j_c + \delta_c) \times (j_c + 2), v(\eta_s) - j_c(j_c + \delta_c)(j_c + 2)) \;\geq$$
$$\max\{\, E(j_c \times (j_c + \delta_c) \times (j_c + 2), k) : 0 \leq k \leq v(\eta_r) - j_c(j_c + \delta_c)(j_c + 2) \,\}$$

and the volume of η_r must satisfy

$$\max\{\, E(j_c \times (j_c+\delta_c) \times (j_c+2), k) : 0 \le k \le v(\eta_r) - j_c(j_c+\delta_c)(j_c+2) \,\} \;\le\; \operatorname{pred} E(n_c).$$

By theorem 7.9, we have $E(j_c \times (j_c+\delta_c) \times (j_c+2), l_c(l_c+1)+1) = E(j_c \times (j_c+\delta_c) \times (j_c+2) \oplus l_c \times (l_c+1)+1)$ which is strictly greater than the level of π. Thus $v(\eta_r)$ is strictly less than $j_c \times (j_c+\delta_c) \times (j_c+2) + l_c(l_c+1)+1$.

- Second case: all the sites are inside $j_c \times (j_c+\delta_c) \times (j_c+2)$. Now $\eta_s \subset j_c \times (j_c+\delta_c) \times (j_c+2)$ so that

$$\begin{aligned} \operatorname{pred} E(n_c) &\ge \max_{0 \le s \le r} E(j_c \times (j_c+\delta_c) \times (j_c+2), v(\eta_s) - j_c(j_c+\delta_c)(j_c+2)) \\ &\ge \max\{\, E(j_c \times (j_c+\delta_c) \times (j_c+2), k) : v(\eta_r) - j_c(j_c+\delta_c)(j_c+2)) \le k \le 0 \,\}. \end{aligned}$$

Theorem 7.8 shows that $E(j_c \times (j_c+\delta_c) \times (j_c+2), -j_c(j_c+\delta_c) + l_c(l_c+1)+1)$ is equal to $E(n_c)$. Necessarily, $v(\eta_r)$ is strictly greater than n_c.
In addition, we have that $E(j_c \times (j_c+\delta_c) \times (j_c+2), k) > E(j_c \times (j_c+\delta_c) \times (j_c+2))$ for all k such that

$$\underline{v}(\pi) - j_c(j_c+\delta_c)(j_c+2) \le k \le \overline{v}(\pi) - j_c(j_c+\delta_c)(j_c+2),\ k \neq 0.$$

We have thus proved that $j_c \times (j_c+\delta_c) \times (j_c+2)$ and the cycle π satisfy the hypothesis of theorem 5.5. It follows that the bottom of the cycle is $\{j_c \times (j_c+\delta_c) \times (j_c+2)\}$. The inequalities on $\overline{v}$ and $\underline{v}$ have also been proved. Moreover the set $\{\, \xi : \xi \subset j_c \times (j_c+\delta_c) \times (j_c+2),\ \xi \in \widetilde{\mathcal{M}}_{n_c+1} \,\}$ is included in the cycle: each configuration of this set communicates with $j_c \times (j_c+\delta_c) \times (j_c+2)$ under the level $\operatorname{pred} E(n_c)$ (this is a consequence of theorem 7.8). Thus $\{\, \xi : \xi \subset j_c \times (j_c+\delta_c) \times (j_c+2),\ \xi \in \widetilde{\mathcal{M}}_{n_c} \,\}$ is included in the principal boundary of π. Finally a configuration ξ of the principal boundary of this cycle is of energy $E(\xi) = E(n_c)$ so that its volume is equal to n_c, and its area to $a(n_c)$. Let η be a configuration of the cycle such that $q(\eta, \xi) > 0$. Necessarily, the volume of η is $n_c + 1$. Thus η is a configuration of minimal volume of the cycle and as such it is a minimal configuration of the cycle for the inclusion relation. By theorem 5.3 it is included in the parallelepiped $j_c \times (j_c+\delta_c) \times (j_c+2)$. Thus ξ belongs to $\mathcal{M}(j_c \times (j_c+\delta_c) \times (j_c+2), n_c - j_c(j_c+\delta_c)(j_c+2))$. $\square$

Corollary 7.25. *Let σ belong to $\widetilde{\mathcal{M}}_{n_c+1} \setminus \mathcal{S}_{n_c+1}$. Suppose $j_c \times (j_c+\delta_c) \times (j_c+1) +_3 l_c \times (l_c+1) \subset \sigma$. Let $\mathcal{G}^+(\sigma)$ be the minimal stable subgraph of $\mathcal{G}^+$ containing $\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c)$. The only arrows of $\mathcal{G}^+(\sigma)$ entering $\pi(\underline{-1}, \underline{+1}^c)$ are*

$$\begin{aligned} &\{\xi\} \to \pi(j_c \times (j_c+\delta_c) \times (j_c+1) +_3 l_c \times (l_c+1), \{\underline{-1}, \underline{+1}\}^c), \\ &\xi \in \widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}, \quad j_c \times (j_c+\delta_c) \times (j_c+1) +_3 l_c \times (l_c+1) \subset \xi. \end{aligned}$$

The remaining arrows of $\mathcal{G}^+(\sigma)$ are

$$\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c) \leftrightarrow \{\eta\}, \quad \eta \in \widetilde{B}(\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c)).$$

There is no arrow in $\mathcal{G}^+(\sigma)$ ending at $\{\underline{+1}\}$.

Remark. We have proved in theorem 7.24 that

$$\widetilde{B}(\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c)) \subset \mathcal{M}(j_c \times (j_c + \delta_c) \times (j_c + 2), n_c - j_c(j_c + \delta_c)(j_c + 2)).$$

Proof. That these arrows belong to $\mathcal{G}^+(\sigma)$ is a straightforward consequence of theorem 7.24 which implies in particular that $\pi(\sigma, \{\underline{-1}, \underline{+1}\}^c) = \pi(j_c \times (j_c + \delta_c) \times (j_c + 2), \{\underline{-1}, \underline{+1}\}^c)$. We have to check that there is no other arrow. Let η belong to the principal boundary of $\pi(j_c \times (j_c + \delta_c) \times (j_c + 2), \operatorname{pred} E(n_c))$. If η is in $\widetilde{\mathcal{M}}_{n_c}$, then all arrows of G starting at the cycle $\{\eta\}$ are present in the above list (lemma 7.21). If η is not in $\widetilde{\mathcal{M}}_{n_c}$, we claim that

$$\{\eta\} \to \pi(j_c \times (j_c + \delta_c) \times (j_c + 2), \operatorname{pred} E(n_c))$$

is the unique arrow of $\mathcal{G}$ starting at $\{\eta\}$. Let ξ be a point such that $q(\eta, \xi) > 0$ and $E(\xi) \leq E(\eta)$. Since $v(\eta) = n_c$, then $v(\xi)$ is equal to $n_c - 1$ or $n_c + 1$. Moreover η is minimal, and the inequality $E(\xi) \leq E(\eta)$ implies in both cases that ξ is also minimal. Thus ξ cannot be of volume $n_c - 1$ (by lemma 7.6, the only configurations of $\mathcal{M}_{n_c}$ communicating with $\mathcal{M}_{n_c-1}$ are the principal configurations $\widetilde{\mathcal{M}}_{n_c}$). Thus $v(\xi) = n_c + 1$. We next show that ξ belongs to $\pi(j_c \times (j_c + \delta_c) \times (j_c + 2), \operatorname{pred} E(n_c))$. Let $x_1, \cdots, x_r$ be a sequence of sites inside σ such that $F(x_1, \cdots, x_s)(j_c \times (j_c + \delta_c) \times (j_c + 2))$ is in $\pi(j_c \times (j_c + \delta_c) \times (j_c + 2), \operatorname{pred} E(n_c))$ for s in $\{1 \cdots r - 1\}$ and $\eta = F(x_1, \cdots, x_r)(j_c \times (j_c + \delta_c) \times (j_c + 2))$. We put $\eta_s = F(x_1, \cdots, x_j)(j_c \times (j_c + \delta_c) \times (j_c + 2))$, $0 \leq s \leq r$. Since η is in the boundary of the cycle, we have $E(\eta_{r-1}) < E(\eta_r)$. Moreover η_{r-1} is a minimal configuration of the cycle, of volume $n_c + 1$, so that the last spin flip at site x_r has decreased the volume. Let x_{r+1} be the unique site such that $\xi = F(x_{r+1})(\eta)$. We have $\xi = F(x_{r+1})(\eta_r) = F(x_{r+1}, x_r)(\eta_{r-1}) = F(x_r, x_{r+1})(\eta_{r-1}) = F(x_r)(\eta'_r)$ where $\eta'_r = F(x_{r+1})(\eta_{r-1})$. The energy inequality 5.1 yields $E(\eta'_r) - E(\eta_{r-1}) \leq E(\xi) - E(\eta)$ whence $E(\eta'_r) \leq E(\eta_{r-1})$. It follows that η'_r is in the cycle, as well as ξ (their energies are less or equal than $E(n_c)$ and they both communicate with a configuration of the cycle). □

Corollary 7.26. *The principal non standard configurations of volume n_c (i.e. the set $\widetilde{\mathcal{M}}_{n_c} \setminus \mathcal{S}_{n_c}$) are dead–ends: there is no saddle path of null cost between $\underline{-1}$ and $\underline{+1}$ passing through them.*

Corollary 7.27. *The set of the global saddle points between $\underline{-1}$ and $\underline{+1}$ is exactly $\mathcal{S}_{n_c}$. These configurations are the critical three dimensional configurations.*

Steps *ii*) and *iii*) are now completed and we proceed to steps *iv*) and *v*).

The ascending part. We now proceed to the last part of the program: for each configuration σ' in $\mathcal{S}_{n_c}$, we must determine the minimal stable subgraph of $\mathcal{G}^-$ containing σ' and all the paths in this graph starting at $\{\sigma'\}$ and ending at $\underline{-1}$. Our exposition is similar as before: we first list the set of the relevant cycles and we use lemma 6.19 to find those belonging to $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$. We finally check that we have in hand all the vertices of $\mathcal{G}^-$.

Theorem 7.28. *Let j be an integer less or equal than j_c and greater than l_c.*
The cycle $\pi = \pi(j \times (j+\delta) \times (j+\theta), pred\, E(j(j+\delta)(j+\theta-1) + l_c(l_c+1) + 1)$ does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\underline{v}(\pi) = j(j+\delta)(j+\theta-1) + l_c(l_c+1) + 2\,, \quad \overline{v}(\pi) < j(j+\delta)(j+\theta) + l_c(l_c+1) + 1\,.$$

The bottom of this cycle is $\{j \times (j+\delta) \times (j+\theta)\}$; its principal boundary contains the set

$$\{\, \xi : \xi \subset j \times (j+\delta) \times (j+\theta),\, \xi \in \mathcal{S}_{j(j+\delta)(j+\theta-1)+l_c(l_c+1)+1} \,\}$$

and is included in $\mathcal{M}(j \times (j+\delta) \times (j+\theta), l_c(l_c+1) + 1 - j(j+\delta))$.

Theorem 7.29. *Let j be an integer less or equal than l_c. Let $\delta \leq \theta$ belong to $\{0,1\}$. The cycle*

$$\pi = \pi(j \times (j+\delta) \times (j+\theta), pred\, E(j(j+\delta)(j+\theta) - j + 1)$$

does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\underline{v}(\pi) = j(j+\delta)(j+\theta) - j + 2\,, \quad \overline{v}(\pi) = j(j+\delta)(j+\theta)\,,$$
$$\forall k \in \{1 \cdots j-2\} \qquad \mathcal{M}(j \times (j+\delta) \times (j+\theta), -k) \subset \pi\,.$$

The bottom of this cycle is $\{j \times (j+\delta) \times (j+\theta)\}$; its principal boundary is

$$\widetilde{B}(\pi) = \mathcal{M}(j \times (j+\delta) \times (j+\theta), -j+1)$$

and thus contains $\{\, \xi : \xi \subset j \times (j+\delta) \times (j+\theta),\, \xi \in \mathcal{S}_{j(j+\delta)(j+\theta)-j+1} \,\}$.

Theorem 7.30. *Let j be an integer less or equal than j_c. Let $\delta \leq \theta$ belong to $\{0,1\}$. Let $l \times (l+\epsilon)$ be a quasisquare strictly included in $j \times (j+\delta)$. The cycle*

$$\pi = \pi(j \times (j+\delta) \times (j+\theta) +_3 l \times (l+\epsilon), pred\, E(j(j+\delta)(j+\theta) + l(l+\epsilon) - l + 1)$$

does not contain $\underline{-1}$ and $\underline{+1}$. Moreover

$$\underline{v}(\pi) = j(j+\delta)(j+\theta) + l(l+\epsilon) - l + 2\,, \quad \overline{v}(\pi) = j(j+\delta)(j+\theta) + l(l+\epsilon)\,,$$
$$\forall k \in \{1 \cdots l-2\} \qquad \mathcal{M}(j \times (j+\delta) \times (j+\theta) +_3 l \times (l+\epsilon), -k) \subset \pi\,.$$

The bottom of this cycle is $\{j \times (j+\delta) \times (j+\theta) +_3 l \times (l+\epsilon)\}$; its principal boundary is

$$\widetilde{B}(\pi) = \mathcal{M}(j \times (j+\delta) \times (j+\theta) +_3 l \times (l+\epsilon), -l+1)$$

and thus contains $\{\, \xi : \xi \subset j \times (j+\delta) \times (j+\theta),\, \xi \in \mathcal{S}_{j(j+\delta)(j+\theta)+l(l+\epsilon)-l+1} \,\}$.

Remark. This theorem cover the cases $j_c \leq l_c$ and $j_c > l_c$, which are of slightly different natures.

Corollary 7.31. *The following cycles are maximal cycles of* $\{\underline{-1},\underline{+1}\}^c$ *(where* $j \le j_c$*):*

$\pi(j\times(j+\delta)\times(j+\theta), pred\, E(j(j+\delta)(j+\theta-1)+l_c(l_c+1)+1))$, $l_c<j$,

$\{\eta\},\quad \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\theta), l_c(l_c+1)+1-j(j+\delta))$, $l_c<j$,

$\pi(j\times(j+\delta)\times(j+\theta), pred\, E(j(j+\delta)(j+\theta)-j+1))$, $j\le l_c$,

$\{\eta\},\quad \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\theta), -j+1)$, $j\le l_c$,

$\pi(j\times(j+\delta)\times(j+\theta)+_3 l\times(l+\epsilon), pred\, E(j(j+\delta)(j+\theta)+l(l+\epsilon)-l+1))$, $l\le l_c$,

$\{\eta\},\quad \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\theta)+_3 l\times(l+\epsilon), -l+1)$, $l\le l_c$.

Proof. This corollary is a consequence of lemma 6.19 together with theorems 7.28, 7.29, 7.30. Notice that we have to put together the descriptions of the cycles of these theorems in order to check that for each cycle π in the above list, there is a sequence of cycles $\pi_0,\cdots,\pi_r$ such that $\pi_0=\pi$, $\widetilde{B}(\pi_i)\cap\pi_{i+1}\neq\emptyset$, $0\le i<r$ and $\underline{-1}\in\widetilde{B}(\pi_r)$. □

Corollary 7.32. *Let* σ *belong to* $\mathcal{S}_{n_c-1}$*. The minimal stable subgraph* $\mathcal{G}^-(\sigma)$ *of* $\mathcal{G}^-$ *containing* σ *is the restriction of* $\mathcal{G}$ *to the vertices listed in corollary* 7.31*. The arrows of* $\mathcal{G}^-(\sigma)$ *are (where the starting cycles belong to the list given in corollary* 7.31*):*

$$\pi(j\times(j+\delta)\times(j+\theta), pred\, E(j(j+\delta)(j+\theta-1)+l_c(l_c+1)+1))\leftrightarrow\{\eta\},\quad \eta\in\widetilde{B}(\pi),\quad l_c<j\le j_c,$$

$$\{j\times(j+\delta)\times(j+\theta-1)+_3 l_c\times(l_c+1)+_1 1\}\to\pi(j\times(j+\delta)\times(j+\theta-1)+_3 l_c\times(l_c+1),\{\underline{-1},\underline{+1}\}^c),\ l_c<j\le j_c,$$

$$\pi(j\times(j+\delta)\times(j+\theta), pred\, E(j(j+\delta)(j+\theta)-j+1))\leftrightarrow\{\eta\},\quad \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\theta),-j+1),\quad j\le l_c,$$

$$\{j\times(j+\delta)\times(j+\theta-1)+_3 j\times(j+\delta-1)+_1 1\}\to\pi(j\times(j+\delta)\times(j+\theta-1)+_3 j\times(j+\delta-1),\{\underline{-1},\underline{+1}\}^c),\ j\le l_c,$$

$$\pi(j\times(j+\delta)\times(j+\theta)+_3 l\times(l+\epsilon), pred\, E(j(j+\delta)(j+\theta)+l(l+\epsilon)-l+1))\leftrightarrow\{\eta\},\quad \eta\in\mathcal{M}(j\times(j+\delta)\times(j+\theta)+_3 l\times(l+\epsilon),-l+1),\quad l\le l_c,$$

$$\{j\times(j+\delta)\times(j+\theta)+_3 l\times(l+\epsilon-1)+_1 1\}\to\pi(j\times(j+\delta)\times(j+\theta)+_3 l\times(l+\epsilon-1),\{\underline{-1},\underline{+1}\}^c),\ l\le l_c.$$

As usual, this list should be completed with all the isometric arrows as well as with the arrows between configurations where the quasisquares $l\times(l+\epsilon)$ *are slid against the side of the quasicubes, and the unit cube* 1 *is slid against the side of the quasisquare (i.e. the configurations in* $j\times(j+\delta)\times(j+\theta)\oplus_3 l\times(l+\epsilon)\oplus 1$*).*

The only loops in the graph $\mathcal{G}^-(\sigma)$ are loops around two cycles. Any other arrow $\pi_1\to\pi_2$ of $\mathcal{G}^-(\sigma)$ satisfies $\underline{v}(\pi_1)>\overline{v}(\pi_2)$. As a consequence a path in $\mathcal{G}^-(\sigma)$ starting at $\{\sigma\}$ with no loop ends in $\{\underline{-1}\}$.

The exit path. We have finally reached the last step $vi)$ which consists in doing the synthesis of all the results gathered in the previous steps $i) - v)$. The set of all the cycle paths of null cost is obtained by glueing together an ascending part and a descending part. We define a graph $\mathcal{H}$ over the cycles of $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ by

$$\pi_1 \to \pi_2 \in \mathcal{H} \qquad \Longleftrightarrow \qquad \exists\, \sigma \in \mathcal{S}_{n_c} \quad \pi_1 \to \pi_2 \in \mathcal{G}^+(\sigma) \quad \text{or} \quad \pi_2 \to \pi_1 \in \mathcal{G}^-(\sigma)$$

i.e. the graph $\mathcal{H}$ is the union for σ in $\mathcal{S}_{n_c}$ of the graphs $\mathcal{G}^+(\sigma)$ (described in corollary 7.23) and the reversed graphs $\mathcal{G}^-(\sigma)$ (described in corollary 7.32). The set of the cycle paths of null cost is the set of all the paths in the graph $\mathcal{H}$ joining $\{\underline{-1}\}$ to $\{\underline{+1}\}$. In addition, there exists only one optimal saddle path between two cycles associated to each arrow of the graph $\mathcal{H}$. Thus the graph $\mathcal{H}$ contains all the information necessary to obtain the set of the saddle paths of null cost between $\underline{-1}$ and $\underline{+1}$. We describe for instance the canonical saddle path, which follows the sequence of the canonical configurations:

$$\begin{gathered}
\underline{-1} \to \mathfrak{m}_1,\ \mathfrak{m}_1 \to \mathfrak{m}_2,\ \mathfrak{m}_2 \to \mathfrak{m}_3,\ \mathfrak{m}_3 \to \mathfrak{m}_4,\ \mathfrak{m}_4 \to \mathfrak{m}_5,\ \mathfrak{m}_5 \to \mathfrak{m}_6,\ \mathfrak{m}_6 \to \mathfrak{m}_7,\ \mathfrak{m}_7 \to \mathfrak{m}_8, \\
\mathfrak{m}_8 \to \mathfrak{m}_9,\ \mathfrak{m}_9 \to \mathfrak{m}_{10},\ \mathfrak{m}_{10} \to \mathfrak{m}_{11},\ \mathfrak{m}_{12} \to \mathfrak{m}_{13},\ \mathfrak{m}_{13} \to \mathfrak{m}_{14},\ \mathfrak{m}_{14} \to \mathfrak{m}_{15},\ \mathfrak{m}_{16} \to \mathfrak{m}_{17}, \\
\mathfrak{m}_{18} \to \mathfrak{m}_{19},\ \mathfrak{m}_{19} \to \mathfrak{m}_{20},\ \mathfrak{m}_{20} \to \mathfrak{m}_{21},\ \mathfrak{m}_{22} \to \mathfrak{m}_{23},\ \mathfrak{m}_{24} \to \mathfrak{m}_{25},\ \mathfrak{m}_{27} \to \mathfrak{m}_{28},\ \cdots, \\
\mathfrak{m}_{j^3+l_c(l_c+1)} \to \mathfrak{m}_{j^3+l_c(l_c+1)+1},\ \mathfrak{m}_{j^3+l_c(l_c+1)+1} \to \mathfrak{m}_{j^3+l_c(l_c+1)+2}, \\
\mathfrak{m}_{j^2(j+1)+l_c(l_c+1)} \to \mathfrak{m}_{j^2(j+1)+l_c(l_c+1)+1},\ \mathfrak{m}_{j^2(j+1)+l_c(l_c+1)+1} \to \mathfrak{m}_{j^2(j+1)+l_c(l_c+1)+2}, \\
\mathfrak{m}_{j(j+1)^2+l_c(l_c+1)} \to \mathfrak{m}_{j(j+1)^2+l_c(l_c+1)+1},\ \mathfrak{m}_{j(j+1)^2+l_c(l_c+1)+1} \to \mathfrak{m}_{j(j+1)^2+l_c(l_c+1)+2}, \\
\mathfrak{m}_{(j+1)^3+l_c(l_c+1)} \to \mathfrak{m}_{(j+1)^3+l_c(l_c+1)+1},\ \cdots, \\
\mathfrak{m}_{j_c(j_c+\delta_c)(j_c+1)+l_c(l_c+1)} \to \mathfrak{m}_{j_c(j_c+\delta_c)(j_c+1)+l_c(l_c+1)+1}, \\
\mathfrak{m}_{j_c(j_c+\delta_c)(j_c+1)+l_c(l_c+1)+1} \to \mathfrak{m}_{j_c(j_c+\delta_c)(j_c+1)+l_c(l_c+1)+2},\ \cdots, \\
\mathfrak{m}_{(N-1)N^2+l_c(l_c+1)} \to \mathfrak{m}_{(N-1)N^2+l_c(l_c+1)+1}, \\
\mathfrak{m}_{(N-1)N^2+l_c(l_c+1)+1} \to \mathfrak{m}_{(N-1)N^2+l_c(l_c+1)+2},\ \cdots, \\
\mathfrak{m}_{(N-1)N^2+(N-1)N} \to \mathfrak{m}_{(N-1)N^2+(N-1)N+1},\ \cdots \mathfrak{m}_{N^3-1} \to \mathfrak{m}_{N^3}.
\end{gathered}$$

We state here some consequences of the information provided by the graph $\mathcal{H}$ which describes the set of all the saddle paths of null cost. We stress once more that the graph $\mathcal{H}$ provides the most complete information available on the limiting dynamics. Once we know this graph, the results obtained in a general framework [7] may be applied in a systematic fashion to obtain various of estimates. We let the process $(\sigma_n)_{n\in\mathbb{N}}$ start from $\underline{-1}$. We recall that $\tau(\underline{+1}^c)$ is the hitting time of the ground state $\underline{+1}$ and $\theta(\underline{-1}, \tau(\underline{+1}^c))$ is the last visit to the metastable state $\underline{-1}$ before reaching $\underline{+1}$.

Theorem 7.33. *(the exit path)*
For any positive ϵ, the following events take place with probability converging to one exponentially fast as β goes to infinity:

- $\exp\beta(E(n_c)-\epsilon) \le \tau(\underline{+1}^c) \le \exp\beta(E(n_c)+\epsilon)$;
- $\exp\beta(E_{2c}-\epsilon) \le \tau(\underline{+1}^c) - \theta(\underline{-1}, \tau(\underline{+1}^c)) \le \exp\beta(E_{2c}+\epsilon)$, *where* E_{2c} *is the two dimensional global energy barrier given in proposition* 6.15*;*
- *during the exit path* $\sigma_n, \theta \le n \le \tau$, *the process crosses the set of critical configurations* $\mathcal{S}_{n_c}$ *at exactly one point* σ_c*; it does not cross* $\mathcal{C}_{n_c} \setminus \mathcal{S}_{n_c}$*;*
- *if we let* $n_* = \min\{n \ge \theta : v(\sigma_n) = n_c\}$, $n^* = \max\{n \le \tau : v(\sigma_n) = n_c\}$, *we have that* $\sigma_n = \sigma_{n_*} = \sigma_{n^*}$ *for all* n *in* $\{n_* \cdots n^*\}$ *and* $n^* - n_* \le \exp(\beta\epsilon)$*;*
- *all the configurations of the exit path before time* n_* *are of volume less than* n_c*, all configurations after* n^* *are of volume greater than* n_c*;*
- *during the ascending part, the process goes through an increasing sequence of quasicubes* $(j \times (j+\delta) \times (j+\theta),\ j \le j_c)$*. It visits a quasicube* $j \times (j+\delta) \times (j+\theta)$ *approximately* $\exp\beta(j-1)h$ *times if* $j \le l_c$ *and* $\exp\beta(E_2(l_c(l_c+1)+1) - E_2(j(j+\delta)))$ *times if* $j > l_c$*. When it definitely leaves a quasicube, it fills the right side of the quasicube in order to obtain the next larger quasicube, this filling occurs by following the two dimensional nucleation mechanism; in particular, it does not make a parallelepiped which is not a quasicube (there is no growth mistake);*
- *during the descending part, the process goes through an increasing sequence of parallelepipeds* $(j_1 \times j_2 \times j_3,\ \min(j_1, j_2, j_3) > j_c)$*. It visits a parallelepiped* $j_1 \times j_2 \times j_3$ *approximately* $\exp\beta E_{2c}$ *times. When it definitely leaves a quasicube, it fills a side of the parallelepiped chosen at random in order to obtain a larger parallelepiped, this filling occurs by following the two dimensional nucleation mechanism;*
- *the process reaches the thermal equilibrium within each cycle of* $\mathcal{M}(\{\underline{-1}, \underline{+1}\}^c)$ *it crosses: the distribution of the process before the exit of the cycle is very close to the Gibbs distribution at inverse temperature* β *restricted to the cycle.*

References

1. L. Alonso–R. Cerf, *The three dimensional polyominoes of minimal area*, preprint (1995).
2. M. Cassandro–A. Galves–E. Olivieri–M.E. Vares, *Metastable behaviour of stochastic dynamics: a pathwise approach*, Jour. Stat. Phys. **35 nos. 5/6** (1984), 603–634.
3. O. Catoni, *Large deviations for annealing*, PhD Thesis, University Paris XI (1990).
4. O. Catoni, *Notes de cours de DEA* (1995), Université d'Orsay.
5. O. Catoni, *Rough large deviations estimates for simulated annealing. Application to exponential schedules*, The Annals of Probability **20** (1992), 1109–1146.
6. O. Catoni, *Sharp large deviations estimates for simulated annealing algorithms*, Ann. Inst. Henri Poincaré **Vol. 27 no. 3** (1991), 291–383.
7. O. Catoni–R. Cerf, *The exit path of a Markov chain with rare transitions*, preprint (1995).
8. R. Cerf, *Une théorie asymptotique des algorithmes génétiques*, Thèse, Université Montpellier II (1994).
9. M.I. Freidlin–A.D. Wentzell, *Random perturbations of dynamical systems*, Springer–Verlag, New York, 1984.
10. R. Kotecky–E. Olivieri, *Droplet dynamics for asymmetric Ising model*, Jour. Stat. Phys. **70 nos. 5/6** (1993), 1121–1148.

11. R. Kotecky–E. Olivieri, *Shapes of growing droplets – a model of escape from a metastable phase*, Jour. Stat. Phys. **75 nos 3/4** (1994), 409–506.
12. E.J. Neves–R.H. Schonmann, *Critical droplets and metastability for a Glauber dynamics at very low temperatures*, Commun. Math. Phys. **137** (1991), 209–230.
13. E.J. Neves–R.H. Schonmann, *Behaviour of droplets for a class of Glauber dynamics at very low temperatures*, Prob. Th. Related Fields **91** (1992), 331–354.
14. O. Penrose–J.L. Lebowitz, *Towards a rigorous molecular theory of metastability, Fluctuation phenomena (second edition)*, E.W. Montroll and J.L. Lebowitz Eds., North Holland Physics publishing, 1987.
15. R.H. Schonmann, *The pattern of escape from metastability of a stochastic Ising model*, Commun. Math. Phys **147** (1992), 231–240.
16. A. Trouvé, *Convergence optimale pour les algorithmes de recuits généralisés*, C. R. Acad. Sci. Paris **t.315 Série I** (1992), 1197–1202.
17. A. Trouvé, *Cycle decompositions and simulated annealing*, SIAM J. Control Optimization, (1995) (to appear).
18. A. Trouvé, *Parallélisation massive du recuit simulé*, PhD Thesis, University Paris XI (1993).
19. A. Trouvé, *Rough large deviation estimates for the optimal convergence speed exponent of generalized simulated annealing algorithms*, Ann. Inst. H. Poincaré, (1995) (to appear).

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