Long-range scattering of three-body quantum systems, I[1]

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1 Introduction and main result

This paper is the first in a series of two papers on asymptotic completeness for (generalized) three-body quantum systems with long-range interaction. The model to be studied is given in terms of a finite family of subspaces $\{X_a | a \in \mathcal{F}\}$ of a finite dimensional Euclidean space X. By definition there exist $a_{\min}, a_{\max} \in \mathcal{F}$ such that $X_{a_{\min}} = X$ and $X_{a_{\max}} = \{0\}$, respectively, and for a and b different from $a_{\min}$ the "three-body" condition $X_a \cap X_b = \{0\}$ is imposed. The position and momentum operators on the basic Hilbert space $\mathcal{H} = L^2(X)$ are denoted by x and p, respectively. The orthogonal complement of X_a in X is denoted by X^a. The corresponding components of x and p are denoted by x_a, p_a and x^a, p^a, respectively.

The basic Hamiltonian on $\mathcal{H}$ is

$$H = \tfrac{1}{2}p^2 + V;\ V(x) = \sum_{a \in \mathcal{F}} V^a(x^a), \tag{1.1}$$

where each "pair potential" V^a is assumed to be a smooth real-valued function on X^a obeying for some $\mu > 0$ (independent of a) and all multiindices β

$$\partial_{x^a}^{\beta} V^a(x^a) = O\big(|x^a|^{-\mu-|\beta|}\big). \tag{1.2}$$

(By definition $V^{a_{\min}} = 0$.)

Asymptotic completeness, henceforth denoted AC, for H is a characterization of the states in the continuous subspace in terms of simplified evolutions (see below for an account of the notion AC). It was proved for three-body systems under the condition (1.2) for $\mu > \sqrt{3} - 1$ first by Enss ([E]) and then by a different method in the context of many-body scattering by Dereziński ([D]), and later for $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$ under more conditions by Gérard and Wang ([G], [W]). The additional conditions imposed by Gérard are essentially spherical symmetry and a global virial condition. The latter implies among other properties a negative upper bound near infinity for each pair potential. The main result by Wang does not require spherical symmetry but essentially positivity near infinity for each pair potential. Finally Yafaev ([Y1]) constructed counter-examples for any $\mu \in \big(0, \frac{1}{2}\big)$ in systems of one-dimensional particles.

One may view the results due to Gérard and Wang as supporting the conjecture that AC for H holds assuming only (1.2) for $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$. We agree with a remark in [G] that indeed it would be very difficult to prove this conjecture. Of course it could be wrong but we see no indication of that. In view of Gérard's result the remark in [D] motivated by examples from classical mechanics with near infinity negative pair potentials, that $\mu > \sqrt{3} - 1$ "seems

to be optimal", is disputable. On the other hand one may argue that the global virial condition of [G] is very restrictive even within classes of such pair potentials.

The aim of this paper is threefold: 1) We give a proof of AC for the regime $\mu \in (\frac{1}{2}, \sqrt{3}-1]$ under weaker conditions than in [G], explicitly without the global virial condition but the negative upper bound near infinity kept. Consequently we provide further support for the above conjecture. 2) We give proofs relying entirely on so-called weak propagation estimates hence providing a considerable simplification compared to the extensive use of so-called strong propagation estimates in [G]. 3) We prove a structure theorem for the regime $\mu \in (0, \frac{1}{2}]$ constituting a basis for an AC result in this case. The latter is finally proved in the other paper of the series [S] by a rather different method compared to the approach for $\mu > \frac{1}{2}$ of this paper. In addition [S] contains a result on AC for (near infinity) positive potentials in the regime $\mu \in (0, \frac{1}{2}]$ (essentially proved by techniques of the present paper).

1.1 Asymptotic completeness

As explained by Dereziński in [D, Section 2] there are various equivalent notions of AC for many-body models and hence in particular for the above three-body model. We shall here recall two of these notions (specialized to our case). The first one relies on the concept of asymptotic velocity of [D], $D^+(H) = s - C_\infty - \lim_{t\to+\infty} e^{itH}t^{-1}xe^{-itH}$. We introduce for $a \neq a_{\max}$ the notation

$$H^a = \frac{1}{2}(p^a)^2 + V^a,\ h_a = \frac{1}{2}p_a^2 + I_a;\ I_a = I_a(x_a) = V(x_a) - V^a(0).$$

The notation $E_\Omega(D)$ is used to denote the spectral projection corresponding to a Borel set Ω for some (possibly) vector-valued operator D of commuting self-adjoint operators. Let $E^{pp}(H^a)$ denote the projection onto the pure point subspace of H^a. Let $Z_a = X_a \backslash (\cup_{b\not\subset a} X_b)$, where by definition $b \subset a \Leftrightarrow X^b \subset X^a$.

Now we can consider the following first statement of existence of wave operators and AC: For all $a \neq a_{\max}$ there exists

$$W_{a1}^+ :\ \mathcal{H}_a = \ \mathrm{ran}E^{pp}(H^a) \otimes \mathrm{ran}E_{Z_a}\big(D^+(h_a)\big) \to \mathrm{ran}E_{Z_a}\big(D^+(H)\big),$$
$$W_{a1}^+ = s - \lim_{t\to+\infty} e^{itH}\big(e^{-itH^a} \otimes e^{-ith_a}\big).$$

Moreover

AC1
$$\mathrm{ran}E^{pp}(H) \oplus \sum_{a\neq a_{\max}} \oplus\, \mathrm{ran}W_{a1}^+ = \mathcal{H}.$$

A more traditional statement of existence of wave operators and AC goes through the existence of a smooth real-valued function $S_a(t, \xi_a)$ on $\mathbf{R}_+ \times Z_a$ which for any compact $\Theta \subset Z_a$ satisfies

$$\sup_{\xi_a\in\Theta} |\partial_{\xi_a}^\beta(\nabla S_a(t,\xi_a) - t\xi_a)| \leq C_\beta t^{1-\mu};\ t \geq 1,$$

$$\partial_t S_a(t,\xi_a) = \frac{1}{2}\xi_a^2 + I_a(\nabla S_a(t,\xi_a));\ \xi_a \in \Theta,\ t \geq t_\Theta.$$

The traditional statement reads: For all $a \neq a_{\max}$ there exists

$$W_{a2}^+ :\ \mathrm{ran}E^{pp}(H^a) \otimes L^2(X_a) \to \mathcal{H},$$
$$W_{a2}^+ = s - \lim_{t\to+\infty} e^{itH}\Big(e^{-itH^a} \otimes e^{-iS_a(t,p_a)}\Big).$$

Moreover

AC2 $$\text{ran}E^{pp}(H) \oplus \sum_{a \neq a_{\max}} \oplus \text{ran}W^+_{a2} = \mathcal{H}.$$

One virtue of AC1 is that this version of AC is invariant, while the virtue of AC2 is its content of simplified evolutions (although not canonical). The latter may serve as a basis for studying the scattering matrix. Henceforth the equivalent statements AC1 and AC2 are referred to as AC.

1.2 Main result

Two of our main results may be combined as follows:

Theorem 1.1 *Suppose in addition to* (1.2) *for some* $\mu \in (0, \sqrt{3}-1]$ *(required for all pair potentials), that for all* $a \neq a_{\max}, a_{\min}$ *the potential* V^a *is spherically symmetric and for two positive numbers* c *and* R

(1.3) $$V^a(x^a) \leq -c|x^a|^{-\mu}, \ |x^a| \geq R.$$

Let $\alpha = 2(2+\mu)^{-1}$. *Then for all such* a *there exists the projection*

(1.4) $$P^+_a = s - \lim_{t \to +\infty} e^{itH} E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(|x^a|) e^{-itH} E_{Z_a}(D^+(H)),$$

which is independent of all small $\epsilon > 0$.

There exist the limits W^+_{a1} *and*

$$\Omega^+_{a1} : \ \big(I - P^+_a\big) E_{Z_a}\big(D^+(H)\big)\mathcal{H} \to \mathcal{H}_a,$$

$$\Omega^+_{a1} = s - \lim_{t \to +\infty} \big(e^{itH^a} \otimes e^{ith_a}\big) e^{-itH}.$$

Furthermore $W^+_{a1}\mathcal{H}_a = (I - P^+_a)E_{Z_a}(D^+(H))\mathcal{H}$, *and* $\Omega^+_{a1}W^+_{a1} = I$ *on* $\mathcal{H}_a$.

Under the further condition $\mu \in (\frac{1}{2}, \sqrt{3}-1]$

(1.5) $$P^+_a = 0.$$

In particular AC holds for this regime.

At a first glance the above result for $\mu \in (0, \frac{1}{2})$ may seem to contradict the counter-examples in [Y1] (for which exceptional states in $\left\{x^a \in X^a \,|\, |x^a| \leq t^{\frac{1}{2}}\right\}$ are constructed), but of course the geometric condition (1.3) is not compatible with these examples. Moreover we remark that (1.3) excludes the existence of zero-energy bound states for subsystems. As a consequence (using decay of subsystem bound states) one can give a simple direct proof of the existence of W^+_{a1}, cf. [D, Theorem 2.6 (b)].

Although we shall not elaborate in this paper the lower bound (1.3) can be slightly relaxed (by introducing a bigger μ in this condition), and one can also add a fast decaying possibly locally singular and non-spherical symmetric perturbation of each pair potential. Also we remark that for one-dimensional pair potentials we dont need spherical symmetry. The core of our methods is essentially one-dimensional, and for this reason we find it convenient to treat this case, or rather a simplified one-dimensional model, in detail while we only sketch the proof of Theorem 1.1 starting in Subsection 1.3 with a discussion of a reduction procedure (see also the outline of the paper at the end of this section). As it follows we proceed by analysing for fixed a; the additional assumptions of Theorem 1.1 are only needed for the a in question.

1.3 A reduction

We shall briefly recall the reduction for AC of [D] (see also [E], for example). It constitutes the first step of the proof of the AC-statement of Theorem 1.1 (and for the general part as well). By an application of the Mourre estimate

$$\mathcal{H} = \mathrm{ran}E^{pp}(H) \oplus \sum_{a \neq a_{\max}} \oplus \, \mathrm{ran}E_{Z_a}\big(D^+(H)\big),$$

cf. [D, (4.17)]. To show AC1 we let any $a \neq a_{\max}, a_{\min}$ and $\phi \in \mathcal{H}$ be given. We need to show the existence of $\lim_{t\to+\infty} \big(e^{itH^a} \otimes e^{ith_a}\big)e^{-itH}E_{\Theta_a}(D^+(H))\phi$ for any compact $\Theta_a \subset Z_a$. Now using [D, Proposition 4.7] there exists $\phi_a^+ = \lim_{t\to+\infty} U_a(t)^* e^{-itH}E_{\Theta_a}(D^+(H))\phi$, where $U_a(t)$ is generated by $H^a + \frac{1}{2}p_a^2 + J\big(\frac{x}{t}\big)I_a(x)$ with J an arbitrary C_0^∞ cutoff function supported in $Y_a = X\backslash(\cup_{b\not\subset a}X_b)$ and equal to one on a neighborhood of Θ_a, and with $I_a(x) = V(x) - V^a(x^a)$. Thus it remains to show that there exists the limit

$$\lim_{t\to+\infty} \big(e^{itH^a} \otimes e^{ith_a}\big)U_a(t)\phi_a^+.$$

For that purpose we notice that there exists the asymptotic energy $H^{a+} = \lim_{t\to+\infty} U_a(t)^* H^a U_a(t)$ (understood in the strong resolvent sense). Next by another application of the Mourre estimate [D, Lemma 4.10] and the exponential decay of negative-energy bound states it suffices to consider $\phi_a^+ \in E_{\{0\}}(H^{a+})\mathcal{H}$. Applying the chain rule again it suffices in fact to show the existence of the limit

$$\tilde{\phi}_a^+ = \lim_{t\to+\infty} \tilde{U}_a(t)^* U_a(t)\phi_a^+, \tag{1.6}$$

where $\tilde{U}_a(t)$ is generated by $H^a + \frac{1}{2}p_a^2 + J\big(\frac{x_a}{t}\big)I_a(x_a)$ and $\phi_a^+ \in E_{\{0\}}(H^{a+})\mathcal{H}$.

We have completed an outline of the reduction procedure for AC for the three-body model. (A similar procedure would work for the existence of W_{a1}^+.) We notice that the potential $I_a(t,x) = J\big(\frac{x}{t}\big)I_a(x)$ obeys

$$\partial_x^\beta I_a(t,x) = O\big(t^{-\mu-|\beta|}\big) \text{ uniformly in } x. \tag{1.7}$$

Motivated by the above reduction we are going to suppress the variables x_a and p_a, for example only derivatives w.r.t. x^a (up to order one) will matter in (1.7). In fact in the bulk of this paper we shall consider only simplified models given by omitting (or "freezing") the variables x_a and p_a (cf. [D]). This procedure is justified as long as the various constructions to be considered (for example propagation observables) are independent of the omitted variables.

1.4 Ideas and organization of proofs

This paper is to some extent dependent on [G]. However we would like stress that it is self-contained and relies entirely on weak propagation estimates. The latter agrees with most of the known proofs of AC for many-body systems for instance the one of [D] to which we refer the reader for an account of the history of AC (in particular for short-range systems which is not mentioned in this introduction). The weak propagation estimates of this paper are established in the standard fashion by constructing uniformly bounded families of observables whose Heisenberg derivative has a sign up to an integrable error, see for example [D, Lemma A.1, (b)].

Moreover an important intermediate step of our approach is absence of propagation in the region $t^{\alpha^2\mu+\epsilon} \leq |x^a| \leq t^{\alpha-\epsilon}$ (precisely this means that $||E_{[t^{\alpha^2\mu+\epsilon},t^{\alpha-\epsilon}]}(|x^a|)U_a(t)\phi_a^+|| \to 0$ for $\phi_a^+ \in E_{\{0\}}(H^{a+})\mathcal{H}$), which follows from an entirely different method. (In [G] the

global virial condition is used to exclude the entire inner region.) Although this method is somewhat complicated partly due to constructions needed to diminish impact of the uncertainty principle the basic idea is simple: For a state in the indicated region the radial part p_r^a of the momentum p^a cannot be negative (otherwise the wave packet would not propagate as fast as indicated). Moreover by an energy bound the corresponding kinetic energy $\frac{1}{2}(p_r^a)^2$ must be of size $-V^a(x^a)$ (provided that the angular momentum is small). Putting together $\frac{d}{dt}|x^a| = p_r^a \approx \sqrt{-2V^a(x^a)}$ for a state in this region, which classically is impossible since the equation is solved by $|x^a| \approx t^\alpha$. Our approach may be viewed as a quantum version of a similar but simpler approach in the corresponding classical model.

Similarly the absence of propagation in the region $|x^a| \geq t^{\alpha+\epsilon}$ can be understood classically. (We give a different proof than those in [D] and [G].) Thus we are left with the inner region $|x^a| \leq t^{\alpha^2\mu+\epsilon}$ and the "classical region" $t^{\alpha-\epsilon} \leq |x^a| \leq t^{\alpha+\epsilon}$ (the latter supporting the above solution) . For each of these separate regions quantum mechanics enters crucially in terms of certain wave operators (cf. Subsection 1.3). For the classical region the existence of the corresponding wave operator is only shown in this paper for $\mu > \frac{1}{2}$, in which case we conclude (under the conditions of Theorem 1.1) that in fact $E_{\{0\}}(H^{a+}) = 0$.

As indicated above this paper concerns a zero-energy problem for a time-dependent Hamiltonian. We refer to [N] and [Y2] for results on zero-energy problems for two-body systems with stationary long-range potentials.

This paper is organized as follows: We treat the simplified one-dimensional model in detail in Sections 2–6. It is introduced in Section 2, where also various results similar to the statements of Theorem 1.1 are stated and some preliminaries are given. In Sections 3–6 we treat various regions separately, cf. the discussion above: Those regions may be called the exterior, intermediate, classical and the inner region, respectively. The combination of Theorem 6.3 and Corollary 6.4 is the direct analogue of Theorem 1.1. In Section 7 we treat a simplified multi-dimensional model with a spherically symmetric potential. Following the procedure of the previous sections we shall sketch a proof of Theorems 7.1 and 7.2. The combination of these statements is the analogue of Theorem 1.1.

We thank Andreas Hinz and Hubert Kalf for drawing our attention to the reference [O] for a result on absence of a zero-energy bound state for one-dimensional Hamiltonians (which made an appendix in a preliminary draft of this paper superfluous).

2 One-dimensional results and preliminaries

Let $\mu \in (0, \sqrt{3}-1]$ be fixed. On the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_x)$ we consider a time-dependent Hamiltonian $H(t) = H + I(t)$; $H = \frac{1}{2}p^2 + V(x)$, $I(t) = I(t,x)$, $t > 0$, where $p = -i\frac{d}{dx}$ denotes the momentum operator, $I(\cdot) \in C^0\big(\mathbf{R}_+, C^1(\mathbf{R}_x)\big)$ and $V \in C^\infty$ with

$$\begin{aligned} &|\frac{d^k}{dx^k}I(t,x)| \leq C_k t^{-\mu-k};\ k = 0,1, \\ &\frac{d^k}{dx^k}V(x) = O\big(|x|^{-\mu-k}\big). \end{aligned} \tag{2.1}$$

In addition $I(t,x)$ and $V(x)$ are assumed to be real-valued.

We shall impose one or both of the following geometric conditions:

$$V(x) \leq -c|x|^{-\mu};\ x \geq R. \tag{2.2}$$

$$V(x) \leq -c|x|^{-\mu};\ x \leq -R. \tag{2.3}$$

Here c and R are positive constants.

Throughout the paper the letters α and γ are used to denote

$$\alpha \,=\, \tfrac{2}{2+\mu},\ \gamma \,=\, \tfrac{\mu}{2+\mu}. \tag{2.4}$$

Under the condition (2.1) we can consider the propagator $U(t)$ obeying (at least formally)

$$i\partial_t U(t) \,=\, H(t)U(t),\ U(1) \,=\, I. \tag{2.5}$$

We refer to [DG, Appendix B.3] for an elaboration; it is known that $U(t)$ preserves the domain of p. We consider the corresponding asymptotic energy (the limit being understood in the strong resolvent sense)

$$H^+ \,=\, \lim_{t\to+\infty} U(t)^* H U(t). \tag{2.6}$$

Let $\phi^+ \in E_{\{0\}}(H^+)\mathcal{H}$ be arbitrarily given. We aim at completeness under (2.1) and the geometric conditions (2.2) and (2.3), which (by definition) amounts to showing that there exist

$$\tilde{\phi}^+ = \lim_{t\to+\infty} e^{\,i\int_1^t (H+I(s,0))ds}\phi^+(t) = \lim_{t\to+\infty} e^{\,i\int_1^t I(s,0)ds}\phi^+(t);\ \phi^+(t) = U(t)\phi^+. \tag{2.7}$$

Notice that indeed if the first limit of (2.7) exists then the second also exists, and they are given by the same $\tilde{\phi}^+ \in \ker(H)$. Consequently, cf. [O, Theorem 2.2 p. 196], $\phi^+ = 0$.

In this paper the existence of the limits (2.7) will be shown for $\mu \in (\frac{1}{2}, \sqrt{3}-1]$.

In general for $\mu \in (0, \sqrt{3}-1]$ we shall show the following preliminary result: We introduce

$$\begin{aligned} &P^+ = P_r^+ + P_l^+; \\ &P_r^+ = s - \lim_{t\to+\infty} U(t)^* E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(x) U(t) E_{\{0\}}\big(H^+\big), \\ &P_l^+ = s - \lim_{t\to+\infty} U(t)^* E_{[-t^{\alpha+\epsilon}, -t^{\alpha-\epsilon}]}(x) U(t) E_{\{0\}}\big(H^+\big), \end{aligned} \tag{2.8}$$

where the limits will be shown to exist and be independent of (small) $\epsilon > 0$. Then we shall show the existence of

$$\tilde{\phi}^+ \,=\, \lim_{t\to+\infty} e^{\,i\int_1^t I(s,0)ds} U(t)(I-P^+)\phi^+, \tag{2.9}$$

yielding $\phi^+ \,=\, P^+\phi^+$.

We note that without the conditions (2.2) and (2.3) there are counter-examples for any $\mu \in (0, \frac{1}{2})$ for which the limit of (2.9) doesn't exist, see [DG, Section 3.8.3].

Clearly the existence of (2.7) follows from the existence of (2.9) if $P^+ = 0$. The latter property will be proved for $\mu \in (\frac{1}{2}, \sqrt{3}-1]$ in this paper.

Imposing only the additional condition (2.2) (not (2.3) here) we shall prove that for any $\tilde{\alpha} \,>\, \alpha_{cr} \,:=\, \alpha^2\mu$

$$\lim_{t\to+\infty} ||F\big(t^{-\tilde{\alpha}}x > 1\big)U(t)(I-P_r^+)\phi^+|| \,=\, 0. \tag{2.10}$$

Here we have used notation from the following list.

Definitions 2.1 *For any $C > 0$ the notation $F(\cdot > C)$, $F(\cdot < C)$ and $F(\cdot \approx C)$ is used to denote smooth functions with $x^k \frac{d^k}{dx^k} F(x)$ bounded for all k and supported in $\big(\frac{1}{2}C, \infty\big)$, $(-\infty, 2C)$ and $\big(\frac{1}{2}C, 2C\big)$, respectively.*

The notation $F(C_1 \cdot > 1, C_2 \cdot < 1)$ is used to symbolize the product function $F(C_1 \cdot > 1)F(C_2 \cdot < 1)$ (for $0 < C_2 < C_1$).

Let $\mathcal{F}_+$ denote the largest set of $F = F_+ = F_+(\cdot > 1) = F(\cdot > 1)$, such that (in addition) $0 \leq F \leq 1$, $F' = \frac{d}{dx}F \geq 0, F' \in C_0^\infty\big(\big(\frac{1}{2}, 2\big)\big)$,$F\big(\frac{1}{2}\big) = 0$, $F(2) = 1$ and $\sqrt{1-F}$, $\sqrt{F}$, $\sqrt{F'} \in C^\infty$, which is stable under the maps $F \to F^m$ and $F \to 1 - (1-F)^m$; $m \in \mathbf{N}$. Let $\mathcal{F}_-$ denote the set of functions $F_- = 1 - F_+$ where $F_+ \in \mathcal{F}_+$.

The proof of (2.10) is divided into two parts: We show in Sections 3 and 4 that for all $\epsilon > 0$

$$\lim_{t \to +\infty} ||F\big(\tfrac{x}{t^{\alpha+\epsilon}} > 1\big)\phi^+(t)|| = 0, \tag{2.11}$$

and

$$\lim_{t \to +\infty} ||F\big(\tfrac{x}{t^{\alpha_{cr}+\epsilon}} > 1, \tfrac{x}{t^{\alpha-\epsilon}} < 1\big)\big)\phi^+(t)|| = 0, \tag{2.12}$$

respectively.

The existence of P_r^+ is a by-product of the proofs of (2.11) and (2.12). Obviously (2.10) follows.

For $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$ the statement $P_r^+ = 0$ follows from

$$\lim_{t \to +\infty} ||F\big(\tfrac{x}{t^{\alpha-\epsilon}} > 1, \tfrac{x}{t^{\alpha+\epsilon}} < 1\big)\phi^+(t)|| = 0; \tag{2.13}$$

the latter will be proved in Section 5.

The existence of (2.9) will be shown in Section 6 under the conditions (2.2) and (2.3).

In the remaining part of this section we don't use (2.2) and (2.3).

We state the following elementary preliminary result.

Lemma 2.2 *Let $\tilde{\alpha} \geq 0$ and $\kappa, t \geq 1$. Then for all $w \in \mathbf{C}$*

$$||F\big(t^{-\tilde{\alpha}}x > 1\big)p(\kappa H - w)^{-1}|| \leq C\frac{\langle w\rangle^{\frac{1}{2}}}{|Imw|}\big(t^{-\mu\tilde{\alpha}} + \kappa^{-1}\big)^{\frac{1}{2}};\ \langle w\rangle = \big(1 + |w|^2\big)^{\frac{1}{2}}. \tag{2.14}$$

Proof We use below and henceforth the notation $\langle A\rangle_\psi$ to denote the expectation value of an observable A in a state ψ. Let $\psi \in \mathcal{H}$ with $||\psi|| = 1$, and $\tilde{\psi} = (\kappa H - w)^{-1}\psi$.

Then we use the formula

$$F2HF = HF^2 + F^2H + t^{-2\breve{\alpha}}F'^2;\ F = F\big(t^{-\tilde{\alpha}}x > 1\big),$$

to estimate

$$\begin{aligned} &||F\big(t^{-\tilde{\alpha}}x > 1\big)p\tilde{\psi}||^2 \leq C_1 t^{-2\bar{\alpha}}|\mathrm{Im}w|^{-2} + 4\langle H - V\rangle_{F(\cdot)\tilde{\psi}} \\ &\leq C_2 t^{-\mu\tilde{\alpha}}|\mathrm{Im}w|^{-2} + C_3|\mathrm{Im}w|^{-1}||H\tilde{\psi}|| \\ &\leq C_2 t^{-\mu\tilde{\alpha}}|\mathrm{Im}w|^{-2} + C_3\kappa^{-1}|\mathrm{Im}w|^{-1}\Big(1 + |\frac{w}{\mathrm{Im}w}|\Big). \end{aligned} \tag{2.15}$$

□

For our problems we may assume that

(2.16) $I(t,x) = I(t,0)$ for $|x| < t^{\breve{\alpha}}$

for any given non–negative $\breve{\alpha} < \mu$ (cf. [E]). In Sections 3 and 4 we may take $\breve{\alpha} = 0$; in Section 5 we shall need a positive $\breve{\alpha} < \alpha$.

Under the condition (2.16) for some fixed $\breve{\alpha} \in [0, \alpha)$ we get using Lemma 2.2 (cf. [E]):

Lemma 2.3 *Suppose for this* $\breve{\alpha}$

(2.17) $\mu\breve{\alpha} < \delta < \mu\big(1 + \frac{\breve{\alpha}}{2}\big)$.

Then with $\phi^+(t) = U(t)\phi^+$ *(and* $\phi^+ \in E_{\{0\}}(H^+)\mathcal{H}$*)*

(2.18) $\limsup_{t\to+\infty} t^{\frac{1}{2}\left(\mu+\frac{1}{2}\mu\breve{\alpha}-\delta\right)} ||F\big(|t^\delta H| > 1\big)\phi^+(t)|| < \infty$.

Proof We pick an almost analytic extension $\tilde{F}$ of $F(|\cdot| > 1)$ with

(2.19) $|\Big(\bar{\partial}\tilde{F}\Big)(w)| \leq C_k\langle w\rangle^{-1-k}|\mathrm{Im}w|^k;\ k \in \mathbf{N}$,

yielding the representation (cf. [DG, Appendix C.3])

$$
\begin{aligned}
&i[H(t), F(|\kappa H| > 1)] \\
&= -\frac{1}{\pi}\int\limits_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}\Big)(w)(\kappa H - w)^{-1} i\Big[I(t), \frac{\kappa}{2}p^2\Big](\kappa H - w)^{-1} du dv; \\
&w = u + iv,\ \kappa \geq 1.
\end{aligned} \tag{2.20}
$$

We compute

$$
i\Big[I(t), \frac{\kappa}{2}p^2\Big] = \kappa O\Big(t^{-(1+\mu)}\Big) F\Big(\frac{x}{t^{\breve{\alpha}}} > 1\Big)p + h.c.,
$$

which combined with (2.20), (2.14) and (2.19) yields to

$$
\begin{aligned}
&||i[H(t), F(|\kappa H| > 1)]|| \\
&\leq \kappa C_1 t^{-(1+\mu)} \int\limits_{\mathbf{C}} |\Big(\bar{\partial}\tilde{F}\Big)(w)|\, |\mathrm{Im}w|^{-2}\langle w\rangle^{\frac{1}{2}}\big(t^{-\mu\breve{\alpha}} + \kappa^{-1}\big)^{\frac{1}{2}} du dv \\
&\leq \kappa C_2 t^{-(1+\mu)}\Big(t^{-\mu\breve{\alpha}/2} + \kappa^{-\frac{1}{2}}\Big).
\end{aligned} \tag{2.21}
$$

Since in the state $\phi^+(t)$

$$
\langle F(|\kappa H| > 1)\rangle_{\phi^+(t)} = -\int\limits_t^\infty ds \langle i[H(s), F(|\kappa H| > 1)]\rangle_{\phi^+(s)} ds
$$

we obtain from (2.21) that

$$\langle F(|\kappa H|>1)\rangle_{\phi^+(t)} \leq \int_t^\infty \kappa C_1 s^{-(1+\mu)}\Big(s^{-\mu\breve{\alpha}/2}+\kappa^{-\frac{1}{2}}\Big)ds$$

$$= C_2\kappa t^{-(\mu+\mu\breve{\alpha}/2)} + C_3\kappa^{\frac{1}{2}}t^{-\mu}.$$

Upon inserting $\kappa = t^\delta$ with δ obeying (2.17) on the right hand side the resulting expression is $O\big(t^{\delta-\mu-\mu\breve{\alpha}/2}\big)$. □

Lemma 2.4 *Suppose the conditions of Lemma 2.3. Then with the notation* $\phi(t) = U(t)\phi$ *for* $\phi \in \mathcal{H}$

$$\int_1^\infty t^{-1}|\big\langle F\big(|t^\delta H|\approx 1\big)\big\rangle_{\phi(t)}|dt \leq C||\phi||^2. \tag{2.22}$$

Proof Consider the uniformly bounded family of observables

$$\Phi(t) = F_-\big(|t^\delta H|\big);\ t\geq 1.$$

Similarly to (2.20) we compute, using for the last equality (2.16) and Lemma 2.2, the Heisenberg derivative

$$\mathbf{D}\Phi(t) := \frac{d}{dt}\Phi(t) + i[H(t),\Phi(t)]$$

$$= -\frac{1}{\pi}\int_{\mathbf{C}}\Big(\bar\partial\tilde F\Big)(w)\big(t^\delta H - w\big)^{-1}\bigg(\delta t^{\delta-1}H + i\Big[I(t), t^\delta\frac{1}{2}p^2\Big]\bigg)\big(t^\delta H - w\big)^{-1}dudv \tag{2.23}$$

$$= \delta t^{-1}t^\delta H G\big(t^\delta H\big) + O\Big(t^{-(\mu+1)+\delta-\frac{1}{2}\mu\breve{\alpha}}\Big);\ G(E) = \frac{d}{dE}F_-(|E|).$$

The second term on the right hand side is integrable. Consequently

$$-\int_1^\infty t^{-1}\big\langle t^\delta H G\big(t^\delta H\big)\big\rangle_{\phi(t)}dt \leq C||\phi||^2,$$

from which we obtain the lemma by using the freedom in chosing F_-. □

3 Exterior bound

In this section we don't need (2.2) and (2.3). The notation α_0 is used to denote a real number with

$$\alpha_0 > \alpha. \tag{3.1}$$

Clearly (2.11) follows from Corollary 3.2 stated below by choosing $\alpha_0 \approx \alpha$. We also introduce the conditions

$$\mu\alpha_0 < \delta < \big(1+\tfrac{\mu}{2}\big)\alpha_0. \tag{3.2}$$

Lemma 3.1 *Suppose (2.17), (3.1) and (3.2). Then with $\phi(t) = U(t)\phi$ for any $\phi \in \mathcal{H}$*

$$\lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_0}} > 1\big)F_-\big(|t^\delta H|\big)\phi(t)|| \,=\, 0. \tag{3.3}$$

Proof We need to show that

$$\begin{aligned} &\lim_{t\to+\infty} ||B(t,t^{\alpha_0})\phi(t)|| = 0; \\ &B(t,\kappa) = F_1F_2,\ F_1 = F_+\big(\kappa^{-1}x\big),\ ,\ F_2 = F_-\big(|t^\delta H|\big). \end{aligned} \tag{3.4}$$

For that we are going to adapt a technique of [D] computing with $A(t,\kappa) = B(t,\kappa)^*B(t,\kappa)$, $t \geq t_0$ and $\kappa = t^{\alpha_0}$ (the latter considered as fixed)

$$\langle A(t,\kappa)\rangle_{\phi(t)} \,=\, \langle A(t_0,\kappa)\rangle_{\phi(t_0)} + \int_{t_0}^{t} \langle \mathbf{D}A(s,\kappa)\rangle_{\phi(s)} ds, \tag{3.5}$$

where the Heisenberg derivative is decomposed as

$$\begin{aligned} &\mathbf{D}A(s,\kappa) = T_1 + T_2, \\ &T_1 = F_2\big(\mathbf{D}F_1^2\big)F_2, \\ &T_2 = (\mathbf{D}F_2)F_1^2F_2 + h.c.. \end{aligned} \tag{3.6}$$

To handle the contribution from the term T_1 we estimate under use of (3.2) and Lemma 2.2

$$F_2\big(\mathbf{D}F_1^2\big)F_2 = 2^{-1}\kappa^{-1}F_2\big(pF_+^{2\prime}\big(\kappa^{-1}x\big) + F_+^{2\prime}\big(\kappa^{-1}x\big)p\big)F_2 = \kappa^{-1}O\big(s^{-\frac{\mu}{2}\alpha_0}\big);\ s \leq t. \tag{3.7}$$

Upon inserting (3.7) into the integrand it follows that

$$\int_{t_0}^{t} \langle T_1\rangle_{\phi(s)} ds \,\leq\, C\kappa^{-1}t^{1-\frac{\mu}{2}\alpha_0}. \tag{3.8}$$

For the contribution from the term T_2 we use (2.23) and Lemma 2.4. The contribution from the error term in (2.23) is estimated as

$$\int_{t_0}^{t} |\big\langle O\big(s^{-(\mu+1)+\delta-\frac{\mu}{2}\check{\alpha}}\big)\big\rangle_{\phi(s)}| ds \leq \int_{t_0}^{\infty} |\langle\cdot\rangle_{\phi(s)}| ds \leq Ct_0^{-\epsilon}||\phi||^2 \text{ for some } \epsilon > 0, \tag{3.9}$$

while the first term on the right hand side of (2.23) needs to be symmetrized before Lemma 2.4 can be applied: We write $s^\delta HG\big(s^\delta H\big)F_3 = -\tilde{G}^2\big(s^\delta H\big)$ and

$$s^{-1}s^\delta HG\big(s^\delta H\big)F_1^2F_2 + h.c. = -2s^{-1}\tilde{G}\big(s^\delta H\big)F_1^2\tilde{G}\big(s^\delta H\big) + R, \tag{3.10}$$

where by commutation and use of Lemma 2.2

$$R \,=\, \kappa^{-1}O\big(s^{\delta-\frac{\mu}{2}\alpha_0-1}\big). \tag{3.11}$$

The contribution from the first term on the right hand side of (3.10) is by Lemma 2.4 of the form $o\big(t_0^0\big)$ for $t_0 \to +\infty$, uniformly in $t \geq t_0$.

Clearly by (3.11)

$$\int_{t_0}^{t} R ds \,=\, \int_{t_0}^{t} \kappa^{-1}O\big(s^{\delta-\frac{\mu}{2}\alpha_0-1}\big) ds \,\leq\, C\kappa^{-1}t^{\delta-\frac{\mu}{2}\alpha_0}. \tag{3.12}$$

Now using that $\kappa = t^{\alpha_0}$ in (3.8) and (3.12) we finally conclude from (3.5) that

$$\langle A(t,t^{\alpha_0})\rangle_{\phi(t)} \leq \langle A(t_0,t^{\alpha_0})\rangle_{\phi(t_0)} + o\big(t_0^0\big) + o\big(t^0\big), \tag{3.13}$$

where the middle term on the right hand side is estimated uniformly w.r.t. to $t \geq t_0$.

Obviously we get from (3.13) that

$$\langle A(t,t^{\alpha_0})\rangle_{\phi(t)} = o\big(t^0\big), \tag{3.14}$$

by first chosing a large t_0 to bound the middle term and noticing that indeed the first term has the form of the last term (for the fixed t_0).

We have proved (3.4). □

Clearly by combining Lemmas 2.3 and 3.1 we obtain:

Corollary 3.2 *Under the conditions of Lemma 3.1 and with $\phi^+(t)$ given as in Lemma 2.3*

$$\lim_{t\to+\infty} ||\phi^+(t) - F_-\big(\tfrac{x}{t^{\alpha_0}}\big)F_-\big(|t^\delta H|\big)\phi^+(t)|| = 0. \tag{3.15}$$

Lemma 3.3 *Under the conditions of Lemma 3.1 and with the notation $\phi(t) = U(t)\phi$ for $\phi \in \mathcal{H}$,*

$$\int_1^\infty t^{-1}|\big\langle F\big(\tfrac{x}{t^{\alpha_0}} \approx 1\big)\big\rangle_{F_-(|t^\delta H|)\phi(t)}|dt \leq C||\phi||^2. \tag{3.16}$$

Proof Consider the uniformly bounded family of observables

$$\Phi(t) = F_-F_+\Big(\frac{x}{t^{\alpha_0}}\Big)F_-;\ F_- = F_-\big(|t^\delta H|\big).$$

We compute the Heisenberg derivative. The contribution from the two factors F_- is treated by (2.23) and Lemma 2.4 (by symmetrizing cf. (3.10) and (3.11)). So we need only to consider the contribution from

$$\mathbf{D}F_+\big(\tfrac{x}{t^{\alpha_0}}\big) = \tfrac{1}{2}\big(\tfrac{p}{t^{\alpha_0}} - \alpha_0\tfrac{x}{t^{\alpha_0+1}}\big)F_+'\big(\tfrac{x}{t^{\alpha_0}}\big) + F_+'\big(\tfrac{x}{t^{\alpha_0}}\big)\tfrac{1}{2}\big(\tfrac{p}{t^{\alpha_0}} - \alpha_0\tfrac{x}{t^{\alpha_0+1}}\big). \tag{3.17}$$

Using Lemma 2.2 we estimate

$$\begin{aligned} &F_-\bigg(\frac{1}{2}\Big(\frac{p}{t^{\alpha_0}} - \alpha_0\frac{x}{t^{\alpha_0+1}}\Big)F_+'\Big(\frac{x}{t^{\alpha_0}}\Big) + h.c.\bigg)F_- \\ &\leq -\alpha_0 F_-\frac{x}{t^{\alpha_0+1}}F_+'\Big(\frac{x}{t^{\alpha_0}}\Big)F_- + Ct^{-\big(1+\frac{\mu}{2}\big)\alpha_0}. \end{aligned} \tag{3.18}$$

The second term on the right hand side of (3.18) is integrable, whence it follows that

$$\int_1^\infty t^{-1}\Big\langle F_+'\Big(\frac{x}{t^{\alpha_0}}\Big)\Big\rangle_{F_-\phi(t)}dt \leq C||\phi||^2.$$

□

Remark 3.4 Completeness in the regime $\mu > \sqrt{3} - 1$, although not considered in this paper, follows readily from various bounds that are completely similar to those derived so far. Moreover one can give a proof of Wangs result in [W] along the same line; see [S, Section 8].

4 Intermediate bound

In this and the next section we impose (2.2). We shall consider two strictly decreasing sequences of positive numbers $(\gamma_j)_1^\infty$ and $(\alpha_j)_0^\infty$ assumed to satisfy:

(4.1) $\quad \delta + \gamma_j < (\mu+1)\alpha_j.$

(4.2) $\quad \frac{\mu}{2}\alpha_{j-1} < \gamma_j < \mu.$

(4.3) $\quad \mu\alpha_{j-1} < \delta.$

(4.4) $\quad \alpha_j + \gamma_j < 1.$

Here α_0 and δ are chosen in agreement with the conditions (2.17), (3.1) and (3.2). Since $(\alpha_j)_0^\infty$ is decreasing obviously (4.3) follows from (3.2).

We shall be interested in the "limiting regions": $\alpha_0 \approx \alpha$, $\delta \approx \mu\alpha$, $\gamma_j \approx \frac{\mu}{2}\alpha_{j-1}$ and $\alpha_j \approx \frac{\delta+\gamma_j}{\mu+1}$. (See the proof of Proposition 4.6 below for precise requirements.)

For later applications we notice that (4.1), (4.2) and (4.4) yield

(4.5) $\quad \delta < \big(1+\frac{\mu}{2}\big)\alpha_j < \min(2\alpha_j, 1),$

while (4.1) and (4.3) yield

$$\mu\alpha_j + \gamma_j < \mu\alpha_{j-1} + \gamma_j < \delta + \gamma_j < (\mu+1)\alpha_j,$$

and therefore by subtraction

(4.6) $\quad \gamma_j < \alpha_j.$

Lemma 4.1 *Suppose (4.1)–(4.3). Then*

(4.7) $\quad ||t^{2\gamma_j}p^2 F\big(t^{2\gamma_j}p^2 > 4\big)F\big(\frac{x}{t^{\alpha_j}} > 1, \frac{x}{t^{\alpha_{j-1}}} < 1\big)F\big(t^{2\gamma_j}p^2 < 1\big)|| = O(t^{-\infty}),$

(4.8) $\quad ||F\big(\frac{x}{t^{\alpha_j}} > 1\big)F\big(t^{2\gamma_j}p^2 < 1\big)F\big(\frac{x}{t^{\alpha_j}} < \frac{1}{4}\big)|| = O(t^{-\infty}),$

(4.9) $\quad ||F\big(\frac{x}{t^{\alpha_j}} > 1, \frac{x}{t^{\alpha_{j-1}}} < 1\big)F\big(t^{2\gamma_j}p^2 < 1\big)F\big(|t^\delta H| < 1\big)|| = O(t^{-\infty}).$

Proof We use (4.6) and the calculus of pseudodifferential operators, cf. [H, Sections 18.5–6] and [DG, Appendix D.4–5], to conclude (4.7) and (4.8). (Notice that (4.6) leads to a calculus with "the Planck constant $= t^{\gamma_j-\alpha_j}$".)

As for (4.9), suppose

(4.10) $\quad ||F\big(\frac{x}{t^{\alpha_j}} > 1, \frac{x}{t^{\alpha_{j-1}}} < 1\big)F\big(t^{2\gamma_j}p^2 < 1\big)F\big(|t^\delta H| < 1\big)|| = O(t^{-s}),$

for some $s \geq 0$ and all such F's. Then we shall show the better bound $O(t^{-s-\epsilon})$ with $\epsilon = 2^{-1}\delta - 2^{-1}\mu\alpha_{j-1}$, which is positive by (4.3). For that we compute with $\psi \in \mathcal{H}$ given with $||\psi|| = 1$ and

$$\tilde{\psi} = F_1F_2F_3\psi;$$

$$F_1 = F\Big(\frac{x}{t^{\alpha_j}} > 1, \frac{x}{t^{\alpha_{j-1}}} < 1\Big),\ F_2 = F\big(t^{2\gamma_j}p^2 < 1\big),\ F_3 = F\big(|t^\delta H| < 1\big),$$

using the calculus

$$
\begin{aligned}
&-\langle 2H\rangle_{\tilde\psi} = -\left\langle HF_1^2 + F_1^2H + \left(\frac{d}{dx}F_1\right)^2\right\rangle_{F_2F_3\psi}\\
&\le 2\Big(t^{-\delta}||F_1F_2\big(t^\delta HF_3\big)\psi|| + t^{\gamma_j-(1+\mu)\alpha_j}||\tilde F_1\tilde F_2F_3\psi||\Big)\,||\tilde\psi||\\
&+t^{-2\alpha_j}||F\Big(\frac{x}{t^{\alpha_j}}\approx 1\Big)F_2F_3\psi||^2 + t^{-2\alpha_{j-1}}||F\Big(\frac{x}{t^{\alpha_{j-1}}}\approx 1\Big)F_2F_3\psi||^2 + O\big(t^{-\infty}\big).
\end{aligned}
\tag{4.11}
$$

Here the last term $O(t^{-\infty})$ is estimated uniformly w.r.t. ψ, and $\tilde F_1 = \tilde F_1(x) = \tilde F_1\big(\frac{x}{t^{\alpha j}} > 1, \frac{x}{t^{\alpha j-1}} < 1\big)$ with $\tilde F_1(\cdot)$ equal to one on a neighborhood of the support of $F_1(\cdot)$; and similarly for $\tilde F_2$.

On the other hand by (4.7) and (4.2)

$$
\begin{aligned}
&\langle -2H\rangle_{\tilde\psi} = \langle -2V\rangle_{\tilde\psi} - \big\langle p^2\big\rangle_{\tilde\psi}\\
&\ge C_1||\tilde\psi||^2t^{-\mu\alpha_{j-1}} - C_2||\tilde\psi||^2t^{-2\gamma_j} + O\big(t^{-\infty}\big)\\
&\ge C_3||\tilde\psi||^2t^{-\mu\alpha_{j-1}} + O\big(t^{-\infty}\big).
\end{aligned}
\tag{4.12}
$$

Combining (4.11), (4.12), (4.1) and (4.5) gives by reduction

$$
\begin{aligned}
&\Big(||F_1F_2\big(t^\delta HF_3\big)\psi|| + ||\tilde F_1\tilde F_2F_3\psi||\Big)\,||\tilde\psi||\\
&+||F\Big(\frac{x}{t^{\alpha_j}}\approx 1\Big)F_2F_3\psi||^2 + ||F\Big(\frac{x}{t^{\alpha_{j-1}}}\approx 1\Big)F_2F_3\psi||^2\\
&\ge C^{-1}t^{\delta-\mu\alpha_{j-1}}||\tilde\psi||^2 + O\big(t^{-\infty}\big),
\end{aligned}
$$

whence by the induction hypothesis (4.10), $||\tilde\psi|| \le Ct^{-s-\epsilon}||\psi|| = Ct^{-s-\epsilon}$. □

Lemma 4.2 *Under the conditions (2.17), (3.1), (3.2) and (4.1)–(4.4), for all $\phi(t) = U(t)\phi$ and $j \ge 1$*

$$
H(1,j) \qquad \int_1^\infty t^{-\alpha_j}\big\langle pF_+^2(t^{\gamma_j}p)\big\rangle_{\tilde\phi_j(t)}dt \le C||\phi||^2,
$$

$$
H(2,j) \qquad \int_1^\infty t^{-\alpha_j}\big\langle -pF_+^2(-t^{\gamma_j}p)\big\rangle_{\tilde\phi_j(t)}dt \le C||\phi||^2,
$$

where

$$
\tilde\phi_j(t) = D_j(t)\phi(t);\ D_j(t) = \big(F_+^{2\prime}\big)^{\frac12}\Big(\frac{x}{t^{\alpha_j}}\Big)F_-\big(|t^\delta H|\big).
$$

Proof Let

$$
\begin{aligned}
&\Phi_j(t) = B_j(t)^*B_j(t);\ B_j(t) = F_1F_2F_3F_4,\\
&F_1 = F_+\Big(\frac{x}{t^{\alpha_j}}\Big),\ F_2 = F_-\Big(\frac{x}{t^{\alpha_{j-1}}}\Big),\ F_3 = F_+(t^{\gamma_j}p),\ F_4 = F_-\big(|t^\delta H|\big).
\end{aligned}
\tag{4.13}
$$

We start by showing $H(1,1)$:

Consider the uniformly bounded family of observables $\Phi_1(t)$. The Heisenberg derivative is decomposed as

$$
\begin{aligned}
&\mathbf{D}\Phi_1(t) = T_1 + T_2 + T_3 + T_4;\\
&T_1 = F_4F_3F_2\mathbf{D}\big(F_1^2\big)F_2F_3F_4,\\
&T_2 = F_4F_3(\mathbf{D}F_2)F_1B_1(t) + h.c.,\\
&T_3 = F_4(\mathbf{D}F_3)F_2F_1B_1(t) + h.c.,\\
&T_4 = (\mathbf{D}F_4)F_3F_2F_1B_1(t) + h.c..
\end{aligned}
\tag{4.14}
$$

We claim that use of Lemmas 3.3, 4.1 and 2.4, respectively, and commutation (the latter producing integrable errors) yield

$$
\int_1^\infty |\langle T_k\rangle_{\phi(t)}|dt \le C||\phi||^2;\ k = 2,3,4. \tag{4.15}
$$

Clearly the bound for $k = 2$ follows from Lemma 3.3, its proof and commutation.

To treat the term T_3 we notice (cf. (2.23))

$$
\begin{aligned}
&\mathbf{D}F_3 = i[V,F_3]\\
&-\frac{1}{\pi}\int_{\mathbf{C}}\Big(\bar\partial\tilde F_+\Big)(w)(t^{\gamma_1}p-w)^{-1}\big(\gamma_1t^{\gamma_1-1}p + i[I(t),t^{\gamma_1}p]\big)(t^{\gamma_1}p-w)^{-1}dudv\\
&= T + \gamma_1t^{-1}H(t^{\gamma_1}p) + O\Big(t^{\gamma_1-(\mu+1)}\Big);\\
&T = i[V,F_3],\ H(\xi) = \xi\frac{d}{d\xi}F_+(\xi).
\end{aligned}
\tag{4.16}
$$

Since the form of the function H in (4.16) is $H(\xi) = F\big(\xi^2 < 1\big)$, we can use Lemma 4.1 to conclude that

$$
F_4\gamma_1t^{-1}H(t^{\gamma_1}p)F_2F_1B_1(t) + h.c. = O(t^{-\infty}). \tag{4.17}
$$

As for the term T in (4.16) we use Lemma 4.1 again, and the calculus of pseudodifferential operators, to write

$$
F_4TF_2F_1B_1(t) + h.c. = O(t^{-\infty}). \tag{4.18}
$$

Using finally the second inequality of (4.2) to treat the third term on the right hand side of (4.16), we conclude the bound (4.15) for $k = 3$.

To treat the term T_4 we use (2.23). The contribution from the first term on the right hand side is symmetrized with an integrable error. This can be shown by Lemma 2.2, (4.1), (4.5) and (4.6), cf. (3.10) and (3.11).

As for the term T_1 we compute, cf. (3.17),

$$
\begin{aligned}
&\mathbf{D}F_1^2 = \frac{1}{2}\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}\Big)F_+^{2\prime}\Big(\frac{x}{t^{\alpha_1}}\Big) + h.c.\\
&= \big(F_+^{2\prime}\big)^{\frac12}\Big(\frac{x}{t^{\alpha_1}}\Big)\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}\Big)\big(F_+^{2\prime}\big)^{\frac12}\Big(\frac{x}{t^{\alpha_1}}\Big).
\end{aligned}
\tag{4.19}
$$

By commutation under use of the calculus of pseudodifferential operators, Lemma 4.1, (4.4) and (4.6) we obtain

$$
\begin{aligned}
&T_1 = D_1(t)^\star F_3\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}F_-\Big(4^{-1}\frac{x}{t^{\alpha_1}}\Big)\Big)F_3D_1(t) + O\big(t^{-\infty}\big)\\
&\geq \frac{1}{2}D_1(t)^\star\frac{p}{t^{\alpha_1}}F_3^2D_1(t) + O\big(t^{-\infty}\big).
\end{aligned}
\tag{4.20}
$$

From (4.20) it follows that

$$
\int_1^T 2^{-1}t^{-\alpha_1}\big\langle pF_+^2(t^{\gamma_1}p)\big\rangle_{\tilde{\phi}_1(t)}dt \leq \int_1^T \langle T_1\rangle_{\phi(t)}dt + C||\phi||^2 \tag{4.21}
$$

with a positive constant C independent of $T > 1$.

Combining (4.15) and (4.21) yields $H(1,1)$.

Suppose now $H(1,j)$. We need to show $H(1,j+1)$:

We proceed by considering $\Phi_{j+1}(t)$ given in (4.13). Its Heisenberg derivative is decomposed as in (4.14). We claim that the analogue of (4.15) holds true: For the term T_2 we use $H(1,j)$ and the fact that $\gamma_{j+1} < \gamma_j$. (The latter allows for replacing $F_+(t^{\gamma_{j+1}}p)$ by $F_+(t^{\gamma_j}p)$.) For the terms T_3 and T_4 we use (as before) Lemmas 4.1 and 2.4, respectively.

By the same arguments used for (4.20) we get a completely analogous statement and therefore also the following analogue of (4.21),

$$
\int_1^T 2^{-1}t^{-\alpha_{j+1}}\big\langle pF_+^2(t^{\gamma_{j+1}}p)\big\rangle_{\tilde{\phi}_{j+1}(t)}dt \leq \int_1^T \langle T_1\rangle_{\phi(t)}dt + C||\phi||^2, \tag{4.22}
$$

yielding $H(1,j+1)$.

To show $H(2,j)$ we define $\Phi_j(t)$ by (4.13) except that F_3 given there is replaced by $F_3 = F_+(-t^{\gamma_1}p)$. We show $H(2,1)$:

Computing $\mathbf{D}\Phi_1(t)$ as in (4.14) and arguing as before we obtain the complete analogue of (4.15). The analogue of (4.20) is

$$
\begin{aligned}
&T_1 = D_1(t)^* F_3\Big(\frac{p}{t^{\alpha_1}} - \alpha_1\frac{x}{t^{\alpha_1+1}}F_+\Big(4\frac{x}{t^{\alpha_1}}\Big)\Big)F_3D_1(t) + O\big(t^{-\infty}\big)\\
&\leq D_1(t)^\star\frac{p}{t^{\alpha_1}}F_3^2D_1(t) + O\big(t^{-\infty}\big).
\end{aligned}
\tag{4.23}
$$

Upon integrating (4.23)

$$
\int_1^T t^{-\alpha_1}\big\langle -pF_+^2(t^{\gamma_1}p)\big\rangle_{\tilde{\phi}_1(t)}dt \leq -\int_1^T \langle T_1\rangle_{\phi(t)}dt + C||\phi||^2. \tag{4.24}
$$

Clearly $H(2,1)$ follows from (4.24) and the analogue of (4.15).

The verification of $H(2,j+1)$ given $H(2,j)$ follows the same pattern using the hypothesis $H(2,j)$ for the term T_2 and for the term T_1 a treatment similar to (4.23). □

Lemma 4.3 *Under the conditions of Lemma 4.2*

$$
\lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_j}} > 1, \tfrac{x}{t^{\alpha_{j-1}}} < 1\big)F_-(t^{\gamma_j}p)F_-\big(|t^\delta H|\big)\phi(t)|| = 0. \tag{4.25}
$$

Proof By Lemma 4.1 it suffices to show that

$$
\lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_j}} > 1, \tfrac{x}{t^{\alpha_{j-1}}} < 1\big)F_+(-t^{\gamma_j}p)F_-\big(|t^\delta H|\big)\phi(t)|| = 0. \tag{4.26}
$$

We prove (4.26) by using the scheme of the proof of Lemma 3.1:

We introduce for $\kappa > 1$

$$
\begin{aligned}
&B(t,\kappa) = F_1F_2F_3F_4,\ A(t,\kappa) = B(t,\kappa)^*B(t,\kappa);\\
&F_1 = F_+\big(\kappa^{-1}x\big),\ F_2 = F_-\Big(\frac{x}{t^{\alpha_{j-1}}}\Big),\ F_3 = F_+(-t^{\gamma_j}p),\ F_4 = F_-\big(|t^\delta H|\big).
\end{aligned}
\tag{4.27}
$$

Then (3.5) holds with

$$
\begin{aligned}
&\mathbf{D}A(t,\kappa) = T_1 + T_2 + T_3 + T_4;\\
&T_1 = F_4F_3F_2\mathbf{D}\big(F_1^2\big)F_2F_3F_4,\\
&T_2 = F_4F_3(\mathbf{D}F_2)F_1B(t,\kappa) + h.c.,\\
&T_3 = F_4(\mathbf{D}F_3)F_2F_1B(t,\kappa) + h.c.,\\
&T_4 = (\mathbf{D}F_4)F_3F_2F_1B(t,\kappa) + h.c..
\end{aligned}
\tag{4.28}
$$

We let $\kappa = t^{\alpha_j}$ and need to show the analogue of (3.13). The contributions to the integral from the terms T_3 and T_4 are handled using Lemmas 4.1 and 2.4, respectively, and the one from T_2 using $H(2,j-1)$ of Lemma 4.2 (for $j>1$) or Lemma 3.3 (for $j=1$), cf. the proof of Lemma 4.2. All contributions are $o\big(t_0^0\big)$ uniformly in $t \geq t_0$.

As for T_1, (4.23) and its analogue for $j \geq 2$ show that

$$
T_1 \leq D_j(t,s)^*F_+(-s^{\gamma_j}p)t^{-\alpha_j}pF_+(-s^{\gamma_j}p)D_j(t,s) + O(s^{-\infty}) \leq O(s^{-\infty}),
\tag{4.29}
$$

where we have used the notation

$$
D_j(t,s) = \big(F_+^{2\prime}\big)^{\frac{1}{2}}\Big(\frac{x}{t^{\alpha_j}}\Big)F_-\Big(\frac{x}{s^{\alpha_{j-1}}}\Big)F_-\big(|s^\delta H|\big),\ t \geq s \geq 1.
$$

Clearly (4.29) implies the estimate

$$
\int_{t_0}^{t} \langle T_1\rangle_{\phi(s)}ds \ \leq\ O\big(t_0^{-\infty}\big).
\tag{4.30}
$$

The bound of the contributions from the terms T_2, T_3 and T_4 combined with (4.30) yields the analogue of (3.13) and therefore (4.26). □

Lemma 4.4 *Under the conditions of Lemma 4.2 there exists*

$$
P_j^+\phi = \lim_{t\to+\infty} U(t)^*F_-\big(|t^\delta H|\big)F_+^2\big(\tfrac{x}{t^{\alpha_j}}\big)F_-^2\big(\tfrac{x}{t^{\alpha_{j-1}}}\big)F_-\big(|t^\delta H|\big)\phi(t).
\tag{4.31}
$$

Proof By Lemma 4.3 it suffices to show the existence of

$$
\lim_{t\to+\infty} U(t)^*\Phi_j(t)\phi(t),
$$

where $\Phi_j(t)$ is given in (4.13). This is done by using $H(1,j)$ and $H(1,j-1)$ (the latter for $j>1$; otherwise by Lemma 3.3) as well as the proof of these statements together with a standard argument of scattering theory, see [D, Lemma A.2]. □

Lemma 4.5 *Under the conditions of Lemma 4.2*

$$
P_j^+ \ =\ 0;\ j \geq 2.
\tag{4.32}
$$

Proof Define

$$K(x) = \begin{cases} \int_R^x (-2V(y))^{-\frac{1}{2}} dy;\ x \geq R, \\ 0;\ x < R, \end{cases} \tag{4.33}$$

with R given by the assumption (2.2). Obviously

$$K(x) \leq C|x|^{1+\frac{\mu}{2}}. \tag{4.34}$$

Let for $\kappa > 1$

$$\begin{aligned} &A(t,\kappa) = B(t,\kappa)^\star B(t,\kappa),\ B(t,\kappa) = F_0F_1F_2F_3F_4; \\ &F_0 = F_+\big(\kappa^{-1}|t - K(x)|\big), \\ &F_1 = F_+\Big(\frac{x}{t^{\alpha_j}}\Big),\ F_2 = F_-\Big(\frac{x}{t^{\alpha_{j-1}}}\Big),\ F_3 = F_+(t^{\gamma_j}p),\ F_4 = F_-\big(|t^\delta H|\big). \end{aligned} \tag{4.35}$$

We pick $\beta_j \in (0,1)$ such that

$$1 < \beta_j - \mu\alpha_{j-1} + \min\big(\delta, (\mu+1)\alpha_j - \tfrac{\mu}{2}\alpha_{j-1}\big) = \beta_j - \mu\alpha_{j-1} + \delta. \tag{4.36}$$

(Here we used (4.1) and (4.2).) By (4.3) indeed such β_j can be chosen.

We shall again follow the scheme of the proof of Lemma 3.1. We use (3.5) for the above $A(t,\kappa)$ with $\kappa = t^{\beta_j}$.

The Heisenberg derivative is decomposed into five terms. Only the contribution from the term

$$T_0 = S^*\big(\mathbf{D}F_0^2\big)S;\ S = S(t) = F_1F_2F_3F_4,$$

will be given careful elaboration. The remaining terms can be treated by methods used before in this section. However we notice that symmetrizing the contribution from the first term on the right hand side of (2.23) in treating the Heisenberg derivative of F_4 leads to the requirement

$$\delta < \beta_j - \tfrac{\mu}{2}(\alpha_{j-1} - \alpha_j), \tag{4.37}$$

while the ones from the terms F_1 and F_2 lead to

$$\gamma_j + \tfrac{\mu}{2}\alpha_{j-1} < \beta_j; \tag{4.38}$$

in all cases to commute through F_0.

The condition (4.37) is fulfilled for β_j close to one since by (4.2), (4.4) and (4.5) (used in the indicated order)

$$1 - \frac{\mu}{2}(\alpha_{j-1} - \alpha_j) > 1 - \gamma_j + \frac{\mu}{2}\alpha_j > \Big(1 + \frac{\mu}{2}\Big)\alpha_j > \delta,$$

while the condition (4.38) is fulfilled for β_j close to one since by (4.3), (4.1) and (4.4)

$$\gamma_j + \frac{\mu}{2}\alpha_{j-1} < \gamma_j + \frac{\delta}{2} < \frac{\gamma_j}{2} + \frac{\mu+1}{2}\alpha_j < \frac{\mu+1}{2} < 1.$$

We compute

$$\begin{aligned} &\mathbf{D}F_0^2 = \frac{\kappa^{-1}}{2}\Big\{\Big(I - p(-2V)^{-\frac{1}{2}}\Big)\tilde{F} + h.c.\Big\}; \\ &\tilde{F} = F\big(\kappa^{-1}(s - K(x))\big),\ F(r) = \frac{d}{dr}F_+^2(|r|). \end{aligned} \tag{4.39}$$

Moreover, putting

$$G = pF_+(4s^{\gamma_j}p)$$
$$+ (-2V)^{\frac{1}{2}}F_+\Big(4\frac{x}{s^{\alpha_j}}\Big)F_-\Big(4^{-1}\frac{x}{s^{\alpha_{j-1}}}\Big) + I - F_+\Big(4\frac{x}{s^{\alpha_j}}\Big)F_-\Big(4^{-1}\frac{x}{s^{\alpha_{j-1}}}\Big),$$

by a commutation (cf. the proof of (4.7))

$$\begin{aligned} &S^*\Big(I - p(-2V)^{-\frac{1}{2}}\Big)\tilde{F}S \\ &= -S^*\Big(p - (-2V)^{\frac{1}{2}}\Big)GG^{-1}(-2V)^{-\frac{1}{2}}\tilde{F}S \\ &= -S^*\Big(p - (-2V)^{\frac{1}{2}}\Big)\Big(p + (-2V)^{\frac{1}{2}}\Big)G^{-1}(-2V)^{-\frac{1}{2}}\tilde{F}S + O\big(s^{-\infty}\big) \\ &= -S^*\Big(2H + i(-2V)^{-\frac{1}{2}}V'\Big)G^{-1}\tilde{F}(-2V)^{-\frac{1}{2}}S + O\big(s^{-\infty}\big). \end{aligned} \tag{4.40}$$

We notice

$$\begin{aligned} &||(-2V)^{-\frac{1}{2}}S|| = O\Big(s^{\frac{\mu}{2}\alpha_{j-1}}\Big), \\ &||G^{-1}|| = O\Big(s^{\frac{\mu}{2}\alpha_{j-1}}\Big), \\ &||S^*(-2V)^{-\frac{1}{2}}V'|| = O\Big(s^{\frac{\mu}{2}\alpha_{j-1}-(\mu+1)\alpha_j}\Big)\,. \end{aligned} \tag{4.41}$$

Combining (4.39)–(4.41) and Lemmas 2.2 and 4.1 (the lemmas to commute H in (4.40) to the left) yields

$$\int_{t_0}^{t} \langle T_0\rangle_{\phi(s)}ds \,=\, O\Big(t^{1-\beta_j+\mu\alpha_{j-1}-\min\left(\delta,(\mu+1)\alpha_j-\frac{\mu}{2}\alpha_{j-1}\right)}\Big), \tag{4.42}$$

which by (4.36) is $o\big(t^0\big)$.

In conclusion

$$||B\big(t,t^{\beta_j}\big)\phi(t)|| = o\big(t^0\big).$$

Moreover since by (4.5)

$$\alpha_j \,\leq\, \alpha_1 \,<\, \alpha, \tag{4.43}$$

we readily obtain in combination with (4.34) that for $j \geq 2$

$$F_-\big(t^{-\beta_j}|t - K(x)|\big)F_2 = 0 \text{ for large } t.$$

Putting together it follows that for $j \geq 2$

$$\big\langle P_j^+\big\rangle_\phi = \lim_{t\to+\infty} ||S(t)\phi(t)||^2 = 0.$$

□

We can now prove (2.12).

Proposition 4.6 *For all* $\epsilon > 0$

$$\lim_{t\to+\infty} ||F\big(\tfrac{x}{t^{\alpha_{cr}+\epsilon}} > 1, \tfrac{x}{t^{\alpha-\epsilon}} < 1)\big)\phi^+(t)|| \,=\, 0. \tag{4.44}$$

Proof We shall use Lemmas 2.3 and 4.5 and the freedom in chosing parameters obeying (2.17), (3.1), (3.2) and (4.1)–(4.4):

As for the latter we define for sufficiently small $\sigma > 0$:

$$\begin{aligned}
&\alpha_0 = \alpha + \frac{\sigma}{\mu},\\
&\delta = \mu\alpha + 2\sigma,\\
&\gamma_j = \frac{\mu}{2}\alpha_{j-1} + \sigma,\\
&\alpha_1 = 1 - \gamma_1 - \sigma,\\
&\alpha_j = (\mu+1)^{-1}(\delta + \gamma_j + \sigma);\ j \geq 2.
\end{aligned} \tag{4.45}$$

Then clearly the relations (2.17) (for any $\breve{\alpha} \in [0, \mu)$), (3.1) and (3.2) hold. By definition, (4.2) is true. Obviously (4.1) holds for $j \geq 2$. Since $(\mu+1)^{-1}(\delta + \gamma_1 + \sigma) < \alpha_1$ (again for small σ), (4.1) holds for $j = 1$ as well.

For $j \geq 2$

$$\alpha_j = (\mu+1)^{-1}\big(\delta + \tfrac{\mu}{2}\alpha_{j-1} + 2\sigma\big), \tag{4.46}$$

which implies the existence of $\alpha_\infty = \lim_{n\to\infty} \alpha_n$. Upon substituting in (4.46) we obtain

$$\alpha_\infty = (\mu+1)^{-1}\Big(\delta + \frac{\mu}{2}\alpha_\infty + 2\sigma\Big),$$

which is solved by

$$\alpha_\infty = \alpha(\mu\alpha + 4\sigma).$$

Since $\alpha_\infty < \alpha_1 < \alpha$ it follows that indeed the sequence $(\alpha_j)_0^\infty$ is strictly decreasing. Then by the definition the same holds for $(\gamma_j)_1^\infty$. In particular since (4.4) holds for $j = 1$, this statement holds for $j \geq 2$ as well.

Given $\epsilon > 0$ we fix $\sigma > 0$ so small that in addition to the above requirements

$$\alpha_{cr} + \epsilon > \alpha_\infty \text{ and } \alpha - \epsilon \leq \alpha_1. \tag{4.47}$$

Then for $j = J$ large enough

$$\alpha_{cr} + \epsilon \geq \alpha_J. \tag{4.48}$$

Now using Lemma 2.3, (4.47) and (4.48) the proposition obviously follows if we can prove

$$\lim_{t\to+\infty} ||F_+\Big(\frac{x}{t^{\alpha_J}}\Big)F_-\Big(\frac{x}{t^{\alpha_1}}\Big)F_-\big(t^\delta|H|\big)\phi^+(t)|| = 0.$$

For that we insert repeatedly

$$I = F_+\Big(\frac{x}{t^{\alpha_j}}\Big) + F_-\Big(\frac{x}{t^{\alpha_j}}\Big);\ 2 \leq j \leq J-1,$$

to the left and invoke Lemma 4.5. □

For application in Section 6 we state the following result which readily follows from the methods of this section.

Lemma 4.7 *Under the conditions of Lemma 4.2 there exists for* $j \geq 1$

$$Q_j^+\phi = \lim_{t\to+\infty} U(t)^* F_-\big(|t^\delta H|\big)F_-^2\big(\tfrac{x}{t^{\alpha_j}}\big)F_-\big(|t^\delta H|\big)\phi(t). \tag{4.49}$$

Proof Notice the following consequence of Lemmas 4.1 and 4.2:

$$\int_1^\infty t^{-\alpha_j-\gamma_j}\big\langle \big(F_+^{2\prime}\big)\big(\tfrac{x}{t^{\alpha_j}}\big)\big\rangle_{F_-(|t^\delta H|)\phi}dt \le C||\phi||^2. \tag{4.50}$$

The contribution to the Heisenberg derivative of $F_-\big(|t^\delta H|\big)F_-^2\big(\frac{x}{t^{\alpha_j}}\big)F_-\big(|t^\delta H|\big)$ from each of the factors $F_-\big(|t^\delta H|\big)$ is treated by Lemma 2.4.

We compute

$$\mathbf{D}F_-^2\big(\tfrac{x}{t^{\alpha_j}}\big) = \tfrac{1}{2}\Big(\tfrac{p}{t^{\alpha_j}} - \alpha_j\tfrac{x}{t^{\alpha_j+1}}\Big)F_-^{2\prime}\big(\tfrac{x}{t^{\alpha_j}}\big) + h.c. \tag{4.51}$$

Using (4.4) and (4.50) it remains to treat the term

$$F_-\big(|t^\delta H|\big)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(\frac{x}{t^{\alpha_j}}\Big)\frac{p}{t^{\alpha_j}}\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(\frac{x}{t^{\alpha_j}}\Big)F_-\big(|t^\delta H|\big).$$

For that we substitute

$$I = F_+(t^{\gamma_j}p) + F_-\big(t^{2\gamma_j}p^2\big) + F_+(-t^{\gamma_j}p);$$

each term is integrable (with the right type of bounding constant) due to Lemmas 4.1 and 4.2. □

5 Bound for the classical region

In this section we shall show that for $\mu > \frac{1}{2}$, $P_1^+ = 0$ with P_1^+ been defined in Lemma 4.4. Notice that it is independent of the particular choice of parameters obeying the conditions of Lemma 4.2 (by the results of Sections 3 and 4). In agreement with Section 2 we may use the notation $P_r^+ = P_1^+$.

Clearly, by symmetry, we may show under the condition (2.3) that $P_l^+ = 0$ if $\mu > \frac{1}{2}$. So we shall here only assume the condition (2.2) (as in the previous section).

We define

$$I_0(t,x) = I(t,x)F_+\big(4\tfrac{x}{t^{\alpha_1}}\big) + I(t,0)F_-\big(4\tfrac{x}{t^{\alpha_1}}\big), \tag{5.1}$$

$$H_0(t) = H + I_0(t,x), \tag{5.2}$$

and corresponding propagator $U_0(t)$

$$i\partial_t U_0(t) = H_0(t)U_0(t),\ U_0(1) = I. \tag{5.3}$$

Lemma 5.1 *Under the conditions of Lemma 4.2 there exists*

$$P_{1,0}^+\phi = \lim_{t\to+\infty} U_0(t)^* F_-\big(|t^\delta H|\big)F_+^2\big(\tfrac{x}{t^{\alpha_1}}\big)F_-^2\big(\tfrac{x}{t^{\alpha_0}}\big)F_-\big(|t^\delta H|\big)\phi(t). \tag{5.4}$$

Proof Since

$$I_0(t,x)F_+^2\Big(\frac{x}{t^{\alpha_1}}\Big) - F_+^2\Big(\frac{x}{t^{\alpha_1}}\Big)I(t,x) = 0,$$

the proof of Lemma 4.7 carries over; for the contribution from the Heisenberg derivative of $F_-^2\big(\frac{x}{t^{\alpha_0}}\big)$ we use Lemma 3.4 (and its proof). □

A consequence of Lemma 5.1 is that we can assume that P_1^+ is defined by (4.31) with $U(t)$ replaced by $U_0(t)$. In other words, keeping the original notation, we can proceed by

showing $P_1^+ = 0$ under the assumption (2.16) for any given $\breve{\alpha} < \alpha_1$ (or $\breve{\alpha} < \alpha$ by adjusting α_1, cf. the proof of Proposition 4.6). Moreover, corresponding to the new $\breve{\alpha}$, we can assume δ is arbitrarily close to $\mu\big(1 + 2^{-1}\breve{\alpha}\big)$, cf. Lemma 2.3. In this section we need the requirements (2.17), (3.1), (3.2) and (4.1)–(4.4) but the latter only for $j = 1$. To be specific we define for sufficiently small $\sigma > 0$ (cf. the proof of Proposition 4.6):

$$
\begin{aligned}
&\alpha_0 = \alpha + \frac{\sigma}{\mu},\\
&\gamma_1 = \frac{\mu}{2}\alpha_0 + \sigma = \gamma + \frac{3}{2}\sigma,\\
&\alpha_1 = 1 - \gamma_1 - \sigma = \alpha - \frac{5}{2}\sigma,\\
&\delta = \frac{1}{2} + \gamma,\\
&\beta_1 = \frac{1}{2} + \gamma + 2\sigma.
\end{aligned}
\tag{5.5}
$$

Then all of the above requirements as well as (4.36)-(4.38) are fulfilled.

We shall proceed from here by using the proof of Lemma 4.5 (with $j = 1$) to obtain the representation

$$
\begin{aligned}
&P_1^+ = s - \lim_{t\to+\infty} U(t)^*\Phi(t)U(t),\\
&\Phi(t) = S^*F_-^2\big(t^{-\beta_1}|t - K(x)|\big)S,
\end{aligned}
\tag{5.6}
$$

where $S = S(t)$ and $K = K(x)$ are given as in the proof.

We notice the bound

$$
\int_1^\infty t^{-1}|\big\langle F\big(t^{-\beta_1}|t - K(x)| \approx 1\big)\big\rangle_{S(t)\phi(t)}|dt \leq C||\phi||^2.
$$

The inverse of K is denoted by $L = L_r$. It is characterized as the solution of the initial value problem

$$
\tfrac{d}{dt}L(t) = \sqrt{-2V(L(t))},\ L(0) = R;\ t \geq 0. \tag{5.7}
$$

Putting $x = L(t)$ in the bound (4.34) leads to

$$
t^\alpha \leq CL(t);\ t \geq 0. \tag{5.8}
$$

We define the propagator

$$
U_r(t) = \exp\left(-i\int_1^t (H + I(s, L(s)))ds\right).
$$

We want to prove that there exists

$$
P_{1,r}^+ = s - \lim_{t\to+\infty} U_r(t)^*\Phi(t)U(t), \tag{5.9}
$$

where $\Phi(t)$ is specified in (5.6). Since obviously the x–independent potential $I(t, L(t))$ fulfills (2.1) we can use the previously established estimates for the full propagator also for the above auxiliary propagator. Therefore for proving the integrability after differentiating w.r.t. t, it suffices to verify that

$$||(I(t, L(t)) - I(t, x))F_-^2\big(t^{-\beta_1}|t - K(x)|\big)||$$

is integrable:

By (2.1), (5.7) and (5.8)

$$I(t, L(t)) - I(t, x) = O\big(t^{-\mu-1}\big)|L(t) - L(K(x))|$$
$$\leq O\big(t^{-\mu-1}\big)|t - K(x)| \sup_{s\in[t,K(x)]} \sqrt{-2V(L(s))}$$
$$\leq O\big(t^{-\mu-1+\beta_1}\big) \sup_{s\in[t,K(x)]} s^{-\gamma} \leq Ct^{-\mu-1+\beta_1-\gamma},$$

where we in the last step used that $\beta_1 < 1$. Since $\mu > \frac{1}{2}$ clearly $\beta_1 < \mu + \gamma$, and therefore we have verified the existence of (5.9).

From the existence of $P_{1,r}^+$ it follows readily that indeed $P_{1,r}^+ = 0$. As a consequence $P_1^+ = 0$, completing the proof.

In conclusion:

Theorem 5.2 *Under the conditions (2.1) and (2.2) for* $\mu > \frac{1}{2}$

$$P_1^+ = 0.$$

6 Simplification for the inner region

We impose the conditions (2.2) and (2.3). We shall show the existence of (2.9). Under the conditions of Lemma 4.2 the statement (2.10) and Lemma 2.3 lead to

(6.1) $\quad Q_j^+\phi^+ = (I - P_r^+)\phi^+;\ j \geq 1,$

with Q_j^+ given in Lemma 4.7 (and $\phi^+ \in E_{\{0\}}(H^+)\mathcal{H}$ arbitrary).

We define

(6.2) $\quad I_j(t, x) = I(t, x)F_-\big(\frac{x}{t^{\alpha_{j-1}}}\big) + I(t, 0)F_+\big(\frac{x}{t^{\alpha_{j-1}}}\big),$

(6.3) $\quad H_j(t) = H + I_j(t, x),$

and corresponding propagator $U_j(t)$

(6.4) $\quad i\partial_t U_j(t) = H_j(t)U_j(t),\ U_j(1) = I.$

Lemma 6.1 *Under the conditions of Lemma 4.2 there exists for* $j \geq 1$

(6.5) $\quad Q_{j,j}^+\phi = \lim_{t\to+\infty} U_j(t)^* F_-\big(|t^\delta H|\big)F_-^2\big(\frac{x}{t^{\alpha_j}}\big)F_-\big(|t^\delta H|\big)\phi(t).$

Proof We mimic the proof of Lemma 5.1. □

We shall be interested in the sequence $(\alpha_j)_1^\infty$ constructed in the proof of Proposition 4.6 with input $\epsilon > 0$ such that $\alpha_{cr} + \epsilon < \mu$. We fix $j = J$ such that (4.48) holds.

We define

$$I^l(t,x) = I(t,x)F_+\big(-\tfrac{x}{t^{\alpha_J}}\big) + I(t,0)F_-\big(-\tfrac{x}{t^{\alpha_J}}\big), \tag{6.6}$$

$$H^l(t) = H + I^l(t,x), \tag{6.7}$$

and corresponding propagator $U^l(t)$

$$i\partial_t U^l(t) = H^l(t)U^l(t),\ U^l(1) = I. \tag{6.8}$$

Since $\alpha_J < \mu$ and

$$|I^l(t,x) - I_{J+1}(t,x)| \leq Ct^{-(\mu+1)+\alpha_J},$$

it follows that there exists the limit

$$s - \lim_{t\to+\infty} U^l(t)^* U_{J+1}(t).$$

Combining this fact and (6.1) and (6.5) for $j = J+1$, we obtain the following result.

Proposition 6.2 *Under the condition (2.2) there exists the limit*

$$\phi^l = \lim_{t\to+\infty} U^l(t)^* U(t)(I - P_r^+)\phi^+. \tag{6.9}$$

If we also impose the condition (2.3) the previous analysis can be repeated to the left (we may argue by a symmetry argument). The analogous construction of (6.2) leads to

$$I_j(t,x) = I^l(t,x)F_-\big(-\tfrac{x}{t^{\alpha_{j-1}}}\big) + I^l(t,0)F_+\big(-\tfrac{x}{t^{\alpha_{j-1}}}\big) = I(t,0), \tag{6.10}$$

provided $j \geq J+2$.

Combining Proposition 6.2, (6.10) and an analogue of Lemma 6.1 leads to the existence of (2.9) (for the last conclusion we use [O, Theorem 2.2 p. 196]):

Theorem 6.3 *Under the conditions (2.1)–(2.3) there exist the limits*

$$\begin{aligned}\tilde{\phi}^+ &= \lim_{t\to+\infty} e^{i\int_1^t (H+I(s,0))ds} U(t)\big(I - P^+\big)\phi^+ \\ &= \lim_{t\to+\infty} e^{i\int_1^t I(s,0)ds} U(t)\big(I - P^+\big)\phi^+\end{aligned} \tag{6.11}$$

(with $P^+ = P_l^+ + P_r^+$). In particular $\tilde{\phi}^+ = 0$.

In combination with Section 5 we obtain:

Corollary 6.4 *Under the conditions (2.1)–(2.3) with $\mu > \frac{1}{2}$*

$$E_{\{0\}}(H^+) = 0. \tag{6.12}$$

7 Multi-dimensional spherically symmetric case

In this section we shall consider the n–dimensional case assuming that the potential $V(x)$ is a function of $|x|$ only (and $n \geq 2$). We shall impose the condition (2.1) with derivatives replaced by partial derivatives; for $I(t)$ up to order one and for V to all orders. Again μ is a given fixed number in $(0, \sqrt{3}-1]$.

We shall impose the following version of (2.2) and (2.3):

$$V(x) \leq -c|x|^{-\mu}, \ |x| \geq R. \tag{7.1}$$

We define H, $H(t)$, $U(t)$ and the asymptotic energy H^+ as in Section 2 except that the underlying Hilbert space now is $\mathcal{H} = L^2(\mathbf{R}^n_x)$.

We introduce

$$P^+ = s - \lim_{t \to +\infty} U(t)^* E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(|x|) U(t) E_{\{0\}}(H^+), \tag{7.2}$$

which as in the one-dimensional case can be shown to be well-defined and independent of small $\epsilon > 0$.

The analogues of Theorem 6.3 and Corollary 6.4 are the following results.

Theorem 7.1 *Under the conditions (2.1) (modified) and (7.1) there exist the limits*

$$\begin{aligned} &s - \lim_{t \to +\infty} e^{i\int_0^t (H + I(s,0))ds} U(t)\big(I - P^+\big) E_{\{0\}}\big(H^+\big) \\ &= s - \lim_{t \to +\infty} e^{i\int_0^t I(s,0)ds} U(t)\big(I - P^+\big) E_{\{0\}}\big(H^+\big) = 0. \end{aligned} \tag{7.3}$$

Theorem 7.2 *Under the conditions (2.1) (modified) and (7.1) for $\mu > \frac{1}{2}$*

$$E_{\{0\}}\big(H^+\big) = P^+ = 0.$$

We shall embark upon sketching the proofs of the above statements following closely the procedure of the previous sections for the one-dimensional case. At most points only minor modifications are needed.

We write the (minus) Laplacian on $\mathcal{H}$ as

$$p^2 = 4^{-1}\Big(\tfrac{x}{|x|} \cdot p + p \cdot \tfrac{x}{|x|}\Big)^2 + |x|^{-2}L^2 + \tfrac{n-1}{2}\tfrac{n-3}{2}|x|^{-2}, \tag{7.4}$$

where the Laplace–Beltrami operator $L^2 = \sum_{i>j} L_{ij}^2$, $L_{ij} = x_i p_j - x_j p_i$.

Motivated by (7.4) we introduce the self-adjoint operator

$$p_r = \frac{1}{2}(\hat{x} \cdot p + p \cdot \hat{x}), \ \hat{x} = \nabla r, \ r = F_+(|x|)|x|.$$

Due to the spherical symmetry the operator L^2 tends to be "preserved". Hence we may write $H \approx 2^{-1}p_r^2 + V(x)$ in various spherical shells which are analogous of the intervals considered in Section 4. Heuristically, this indicates why the one-dimensional procedure to a large extent works.

We pick a state $\phi^+ \in E_{\{0\}}(H^+)\mathcal{H}$. The results Lemmas 2.2– 2.4 carry over with almost identical proofs; in Lemma 2.2 x needs to be replaced by r. The same can be said about all results in Section 3; the rule for transcribing the statements is the same.

To generalize the results of Section 4 we need more refined modifications. In addition to the sequences $(\gamma_j)_1^\infty$ and $(\alpha_j)_0^\infty$ of Section 4 we shall consider a decreasing sequence $(\lambda_j)_1^\infty$ of positive numbers satisfying

$$\lambda_j \,<\, \alpha_j - \tfrac{\mu}{2}\alpha_{j-1}, \tag{7.5}$$

and

$$\lambda_j \,>\, \alpha_{j-1} - \mu. \tag{7.6}$$

(The positivity of the right hand side of (7.5) follows readily from (4.1)-(4.3.)

Now, (4.9) needs to be replaced by the following result.

Lemma 7.3 *Suppose (4.1)-(4.3) and (7.5). Then*

$$||F\big(\tfrac{r}{t^{\alpha_j}} > 1, \tfrac{r}{t^{\alpha_{j-1}}} < 1\big)F\big(t^{2\gamma_j}p_r^2 < 1\big)F_-\big(t^{-2\lambda_j}L^2\big)F\big(|t^\delta H| < 1\big)|| = O(t^{-\infty}). \tag{7.7}$$

This result can be proved along the line of that of Lemma 4.1. We notice that the commutator expansion lemma, cf. [DG, Lemma C.3.1], constitutes a useful substitute for the calculus of pseudodifferential operators.

We shall need the following two results (with no parallels in Section 4).

Lemma 7.4 *Suppose (7.6), and the following condition on the potential $I(t,x)$ for some $J \in \mathbf{N}$*

$$I(t,x) \,=\, I(t,0) \text{ for } r \,\geq\, 2t^{\alpha_{J-1}}. \tag{7.8}$$

Then for any $\phi(t) \,=\, U(t)\phi$

$$\lim_{t\to+\infty} ||F_+\big(t^{-2\lambda_J}L^2\big)\phi(t)|| \,=\, 0. \tag{7.9}$$

Proof We use the familiar scheme of the proof of Lemma 3.2 under use of

$$[I(t,x), L_{ij}] \,=\, O\big(t^{-\mu-1+\alpha_{J-1}}\big), \tag{7.10}$$

and (7.6) for $j = J$. □

Lemma 7.5 *Under the conditions of Lemma 7.4, for any $j \leq J$ and $\phi(t) = U(t)\phi$*

$$\int_1^\infty t^{-1}|\big\langle F\big(t^{-2\lambda_j}L^2 \approx 1\big)\big\rangle_{\phi(t)}|dt \,\leq\, C||\phi||^2. \tag{7.11}$$

Proof We mimic the proof of Lemma 2.4 under use of (7.10). □

The analogue of Lemma 4.2 reads:

Lemma 7.6 *Under the conditions (2.17) (with $\breve{\alpha} = 0$), (3.1), (3.2), (4.1)–(4.4), (7.5), (7.6) and (7.8), for any $\phi(t) = U(t)\phi$ and $j \leq J$*

$$H(1,j) \qquad \int_1^\infty t^{-\alpha_j} \big\langle p_r F_+^2(t^{\gamma_j} p_r) \big\rangle_{\tilde{\phi}_j(t)} dt \leq C||\phi||^2,$$

$$H(2,j) \qquad \int_1^\infty t^{-\alpha_j} \big\langle -p_r F_+^2(-t^{\gamma_j} p_r) \big\rangle_{\tilde{\phi}_j(t)} dt \leq C||\phi||^2,$$

where

$$\tilde{\phi}_j(t) = B_j(t)\phi(t);\ B_j(t) = \big(F_+^{2\prime}\big)^{\frac{1}{2}} \Big(\frac{r}{t^{\alpha_j}}\Big) F_-\big(t^{-2\lambda_j} L^2\big) F_-\big(|t^\delta H|\big).$$

Proof We mimic the proof of Lemma 4.2 proceeding by induction from $j = 1$ to $j = J$. The propagation observable in (4.13) needs to be modified in agreement with the rule mentioned above and in addition be multiplied twice by the factor $F_-\big(t^{-2\lambda_j} L^2\big)$, one from the left and one from the right (for example). The contribution from its Heisenberg derivative is handled by Lemma 7.5, and the substitute for Lemma 4.1 is Lemma 7.3. □

The analogue of Lemma 4.3 reads:

Lemma 7.7 *Under the conditions of Lemma 7.6*

$$\lim_{t\to+\infty} ||F\big(\tfrac{r}{t^{\alpha_j}} > 1, \tfrac{r}{t^{\alpha_{j-1}}} < 1\big) F_-(t^{\gamma_j} p_r) F_-\big(|t^\delta H|\big)\phi(t)|| = 0;\ j \leq J. \tag{7.12}$$

Proof Using Lemmas 7.4 and 7.3 it suffices to show that

$$\lim_{t\to+\infty} ||F\Big(\frac{r}{t^{\alpha_j}} > 1, \frac{r}{t^{\alpha_{j-1}}} < 1\Big) F_+(-t^{\gamma_j} p_r) F_-\big(t^{-2\lambda_j} L^2\big) F_-\big(|t^\delta H|\big)\phi(t)|| = 0,$$

for which we mimic the proof of Lemma 4.3. □

The next result in Section 4, Lemma 4.4, is modified similarly (i.e. the analogous limits are only defined for $j \leq J$) with obvious proof.

As for Lemma 4.5 the corresponding analogue holds true, but the proof needs a comment: Of course we need to multiply by the factor $F_-\big(t^{-2\lambda_j} L^2\big)$ to get the correct object $B(t,\kappa)$ in (4.35). The proof relies on the factorization $2H \approx p_r^2 + 2V(r) \approx \Big(p_r - (-2V)^{\frac{1}{2}}\Big)\Big(p_r + (-2V)^{\frac{1}{2}}\Big)$, cf. (4.40). Here the last two terms on the right hand side of (7.4) are treated as errors, which is justified if their contributions are integrable. By inspection we need for the latter the condition

$$\beta_j - \mu\alpha_{j-1} + 2(\alpha_j - \lambda_j) > 1, \tag{7.13}$$

in addition to (4.36).

By (7.5) the condition (7.13) is fulfilled for β_j close to one (but smaller). The rest of the proof of Lemma 4.5 carries over to the present context.

The last result in Section 4, Lemma 4.7, has an obvious analogue with obvious proof.

It remains to prove a statement similar to Proposition 4.6. (In the process of doing that we prove Theorem 7.1.) We shall proceed differently verifying inductively the condition (7.8)

for a sequence of modified propagators, the latter defined as in Section 6. Precisely we define I_j, H_j and U_j by (6.2)-(6.4), respectively (with x replaced by r in (6.2)).

Then by the analogues of Corollary 3.2 and Lemmas 2.4 and 3.3 there exists the limit

$$\phi_1^+ = \lim_{t\to+\infty} U_1(t)^* U(t)\phi^+.$$

(Remember $H^+\phi^+ = 0$.)

For U_1 the statement (7.8) holds for $J = 1$. Consequently we can define the corresponding operator P_1^+. Next we put $\psi_1^+ = \big(I - P_1^+\big)\phi_1^+$. By the definition of P_1^+ we can write $\psi_1^+ = Q_{1,1}^+\phi_1^+$, where $Q_{1,1}^+$ is given as in (6.5). Equivalently we can write

$$\psi_1^+(t) = U_1(t)\psi_1^+ \approx F_-^2\big(\tfrac{r}{t^{\alpha_1}}\big)\psi_1^+(t). \tag{7.14}$$

Obviously we may assume that $(I_2(t,x) - I_1(t,x))F_-^2\big(\frac{r}{t^{\alpha_1}}\big) = 0$ (by using the freedom in defining I_2). Using (7.14) and the modified results of Section 4 discussed above we then get the existence of

$$\psi_2^+ = \lim_{t\to+\infty} U_2(t)^*\psi_1^+(t).$$

Since the statement (7.8) for U_2 holds for $J = 2$ we obtain for the corresponding operator P_2^+ that it is zero (by the analogue of Lemma 4.5). But this implies that

$$\psi_1^+(t) \approx \psi_2^+(t) = U_2(t)\psi_2^+ \approx F_-^2\Big(\frac{r}{t^{\alpha_2}}\Big)\psi_2^+(t).$$

Repeating the procedure yields

$$\psi_1^+(t) \approx \psi_j^+(t) = U_j(t)\psi_j^+ \approx F_-^2\big(\tfrac{r}{t^{\alpha_j}}\big)\psi_j^+(t), \tag{7.15}$$

where by definition

$$\psi_j^+ = \lim_{t\to+\infty} U_j(t)^*\psi_{j-1}^+(t);\ j \geq 2.$$

For $\alpha_j < \mu$ we conclude from (7.15) and by an argument in Section 6 that

$$\psi_1^+(t) \approx \psi_j^+(t) \approx e^{-i\int_1^t (H+I(s,0))ds}\tilde{\psi}^+.$$

By [O, Theorem 2.2 p. 196] it follows that $\tilde{\psi}^+ = 0$, and then in turn that indeed $\psi_1^+ = 0$.

It remains to show that indeed we can choose our parameters to obey (2.17) (with $\breve{\alpha} = 0$), (3.1), (3.2), (4.1)–(4.4) (7.5) and (7.6) (and the used condition $\alpha_j < \mu$ for j large enough). For that we mimic the construction in the proof of Proposition 4.6: We keep the first four definitions in (4.45), while the last one is replaced by

$$\alpha_j = \max\Big((\mu+1)^{-1}(\delta+\gamma_j+\sigma), \big(1+\tfrac{\mu}{2}\big)\alpha_{j-1} - \mu + \sigma\Big);\ j \geq 2. \tag{7.16}$$

Then by the proof of Proposition 4.6 the conditions (2.17), (3.1), (3.2) and (4.1)–(4.4) are fulfilled. Notice that $\alpha_2 < \alpha_1$, and therefore that

$$\alpha_j = \max\Big((\mu+1)^{-1}\big(\delta+\tfrac{\mu}{2}\alpha_{j-1}+2\sigma\big), \big(1+\tfrac{\mu}{2}\big)\alpha_{j-1} - \mu + \sigma\Big);\ j \geq 2, \tag{7.17}$$

defines a decreasing sequence with the same limit $\alpha_\infty < \mu$ as before.

Next we need to choose λ_j: By the definition (7.17) (for $j \geq 2$ and (4.45) for $j = 1$)

(7.18) $\alpha_j - \frac{\mu}{2}\alpha_{j-1} > \alpha_{j-1} - \mu$.

We define J as the largest natural number for which $\alpha_{J-1} - \mu \geq 0$. Then it follows from (7.18) that (7.5) and (7.6) are fulfilled for $j \leq J$ by the construction

$$\lambda_j = \alpha_{j-1} - \mu + \sigma';\ j \leq J,\ \sigma' > 0 \text{ small.}$$

Notice that the finite sequence it positive and decreasing. We need to supplement by a decreasing sequence of positive λ's with index larger than J. Since $0 < \alpha_j - \frac{\mu}{2}\alpha_{j-1}$ clearly (7.5) can be fulfilled for a decreasing sequence of positive λ's smaller than λ_J. Since (7.6) is trivial for $j > J$ for the (combined) constructed infinite sequence we finally conclude (7.5) and (7.6).

This completes the outline of a proof of Theorem 7.1.

As for Theorem 7.2 we mimic Section 5: In addition to (5.5) we define

$$\lambda_1 = \alpha - \frac{1}{2} - 3\sigma.$$

Then (7.5), (7.6) and (7.13) (and previous conditions) are fulfilled for $j = 1$. Moreover with

$$W(t) = \exp\left(-i\int_1^t \left(\frac{1}{2}p_r^2 + V(|x|) + I\Big(s, L(s)\frac{x}{|x|}\Big)\right)ds\right),$$

where $L(t)$ is given by (5.7), one can define (cf. (5.9))

$$P^+_{1,W} = s - \lim_{t\to+\infty} W(t)^*\Phi(t)U(t),$$

where $\Phi(t)$ is given by a similar construction as in (5.6) (including two factors of $F_-\big(t^{-2\lambda_1}L^2\big)$). Here we use various propagation estimates and that the last two terms on the right hand side of (7.4) are integrable, the latter due to the localization properties of $\Phi(t)$ and the fact that $2(\lambda_1 - \alpha_1) < -1$. Next we notice that $\big(\frac{1}{2}p_r^2 + V(|x|)\big)P^+_{1,W} = 0$ yielding the conclusion that indeed $P^+_{1,W} = 0$.

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