On Jupiter and his Galilean satellites: libations of De Sitter's periodic motions

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Abstract

The motion of Jupiter's four Galilean satellites Io–Europa–Ganymedes–Callisto is subjected to an orbital 1:2:4–resonance of the former (and inner) three. Willem de Sitter in the early 20th century gave a mathematical explanation of this in a Newtonian framework. He found a family of stable periodic solutions by using the work of Poincaré. This paper briefly reviews De Sitter's theory, and focuses on the underlying geometry of a suitable covering space, where we develop Kolmogorov–Arnold–Moser theory to find Lagrangean invariant tori excited by the normal modes of the De Sitter periodic orbits. In this way we find many librations near these periodic orbits that may well offer a more realistic explanation of the observations.

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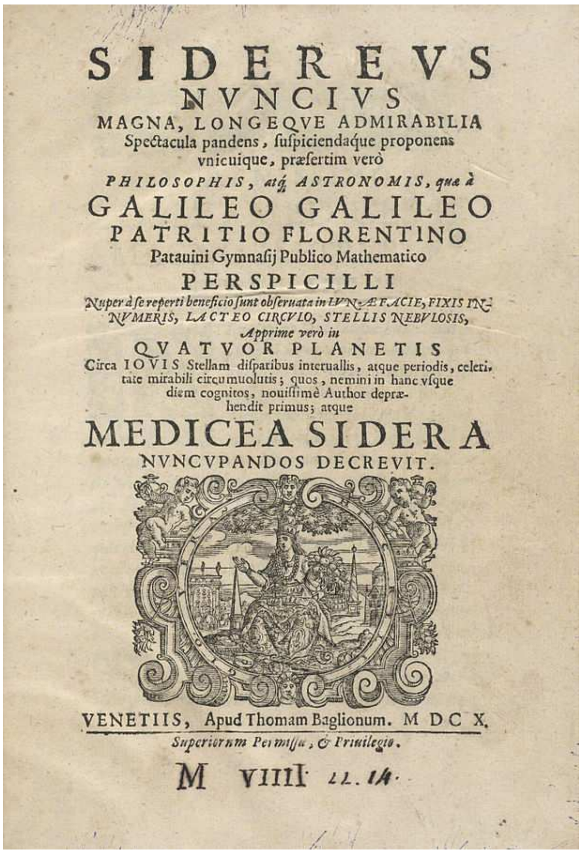
1 Introduction

In 1610 Galileo published his *Siderius Nuncius* (see Figure 1) which forms the beginning of a long history, where celestial observations were performed by a telescope. The message that Jupiter, together with his four satellites Io, Europa, Ganymedes and Callisto, forms a miniature solar system, has had a strong impact on science and society, first of all since it confirms a heliocentric view on the solar system. Later, Newton used this same system to have the validity of Kepler's third law checked for this system, confirmation of which led him to his postulate of universal inverse square gravitation. This is reported in Part III of his celebrated *Philosophiæ Naturalis Principia Mathematicæ*, published in 1687. For a description we refer to, *e.g.*, Westfall [33]. Since then the Galilean system has received quite some attention from celestial mechanics, among others from Laplace. It took until around 1890 that the Astronomer Royal David Gill discovered an (almost) 1:2:4–resonance in the orbital motion of the inner three Galilean satellites Io, Europa and Ganymedes. This resonance is strong enough to have a great influence on the dynamics of the entire Galilean system, including Callisto. During the first decennia of the 20th century, Willem de Sitter [29, 30, 31] developed a mathematical theory that takes this resonance into account and in a Newtonian description he found a family of resonant, linearly stable periodic orbits with a period of about 1 week. We refer to this as the family of *De Sitter periodic orbits*. This work is based on Poincaré [27]. See Guichelaar [19] for a historic description.



SIDEREVS
NVNCIVS
MAGNA, LONGEQVE ADMIRABILIA
Spectacula pandens, suspiciendaque proponens
vnicuique, præsertim verò
PHILOSOPHIS, atq; ASTRONOMIS, quæ à
GALILEO GALILEO
PATRITIO FLORENTINO
Patauini Gymnasij Publico Mathematico
PERSPICILLI
Nuper à se reperti beneficio sunt obseruata in LVNÆ FACIE, FIXIS IN-
NVMERIS, LACTEO CIRCVLO, STELLIS NEBVLOSIS,
Apprime verò in
QVATVOR PLANETIS
Circa IOVIS Stellam disparibus interuallis, atque periodis, celeri-
tate mirabili circumuolutis; quos, nemini in hanc vsque
diem cognitos, nouissimè Author depræ-
hendit primus; atque
MEDICEA SIDERA
NVNCVPANDOS DECREVIT.
VENETIIS, Apud Thomam Baglionum. M D C X.
Superiorum Permissu, & Priuilegio.
M VIIII

Figure 1: Title page of Galileo's *Sidereus Nuncius*.

1.1 Setting of the problem

In a recent paper by Broer and Zhao [35] De Sitter's work was retold and updated regarding the periodic orbit. Also librations of the family of De Sitter periodic orbits were found with help of Kolmogorov-Arnold-Moser (or KAM) theory, mainly based on the work of Arnold, Féjoz and Zhao [1, 5, 15, 16, 17, 34]. It turns out that a nowhere dense, large measure union of Lagrangean invariant tori exists, that carries plenty of such librations.

The present paper gives another existence proof of these Lagrangean tori, as an adaptation of the parametrised KAM theory as developed in Broer-Huitema-Takens [12, 22], based on Moser [25]. Also see Broer-Huitema-Sevryuk [11]. The 1:2:4–resonance leads to a singularity in phase space that can be regularized by passing to co-rotating co-ordinates, which forms a covering map, compare with Arnold [4], Broer-Vegter [14] and Broer-Hanßmann-Jorba-Villanueva-Wagener [9]. On the corresponding covering space we have to keep the associated discrete group of deck transformations into account, which is automatically taken care of in our approach. In this respect the present paper forms both a simplification and an improvement of [35]. The Lagrangean tori arise by the excitation of normal modes of a family of normally elliptic isotropic invariant 2–tori, where the latter arise from superposing the De Sitter periodic orbits with the periodic motion of Callisto.

To incorporate this in our approach we use methods as described in Jorba-Villanueva [23] and [11] . Moreover, several time scales have to be taken into account; here we invoke Han-Li-Yi [20]. In the sequel we prove one parametrised KAM theorem that encompasses all these aspects.

For a more recent treatment of the Galilean satellites, which largely rests on numerical methods, we refer to [24]. Interestingly, for the relativistic correction of the Newtonian equations of motion, an approximation developed by De Sitter has been used.

1.2 Strategy

Since the focus of the present paper lies in the KAM theory and its application to the Galilean system, we only touch on the De Sitter periodic orbits and their normally linear part as far as needed for checking the necessary KAM non-degeneracy conditions. For all computational details in this direction we refer to [35].

We present an overview of the paper, later on filling in details. We start by making the simplifying assumption that Jupiter and his four satellites are all moving in one fixed plane. A second assumption is that the masses m_1, m_2, m_3, m_4 of the succesive satellites are small with respect to the mass m_0 of Jupiter. If the mutual gravitational interactions between the satellites were set to zero, the Newtonian description would lead to Keplerian ellipses, which have small eccentricities. The entire mathematical analysis rests on perturbative properties of these ellipses, using their geometry, their orbital frequencies, etc. Also in normalizing or averaging techniques, as applied below, such Keplerian ellipses act as unperturbed starting point. The occurrences of the term 'small' in the above, in the sequel are made more explicit in terms of perturbation parameters, where also certain scaling arguments play a role.

The planar system formed by Jupiter and the four satellites has 10 degrees of freedom. Following De Sitter [29] we shall deal initially with the 4–body system Jupiter-Io-Europa-Ganymedes, in which case the planar system has 8 degrees of freedom. When adding the 5th body Callisto, which is at a far distance, this increases the number of degrees of freedom by 2. Our first aim is to use the symmetries of the problem to reduce the number of degrees of freedom.

The translational symmetry of the problem allows to reduce by 2 degrees of freedom — fixing the conserved total linear momentum and taking the quotient by the symmetry group $\mathbb{R}^2$ through fixing the centre of mass: we move uniformly with this centre through space.[1] This is the 0th reduction, from 8 to 6 degrees of freedom. Our description of motion of the satellites is always with respect to Jupiter; we are not going to reconstruct the 0th reduction and not describe any motion as superposed with the orbit of Jupiter (or the centre of mass) in space.

The *first* reduction concerns the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry related to the conservation of the total angular momentum. This amounts to fixing the conserved angular momentum and taking the quotient by $\mathrm{SO}(2,\mathbb{R})$ through a passage to co-rotating co-ordinates, making it possible to reduce the system by one degree of freedom, see Arnold [3]. This reduction from 6 to 5 degrees of freedom is presented in § 3.1 of Section 3 and is enabled by a symplectomorphism of the phase space.

[1] Disregarding the influences of Saturn, the Sun, etc.

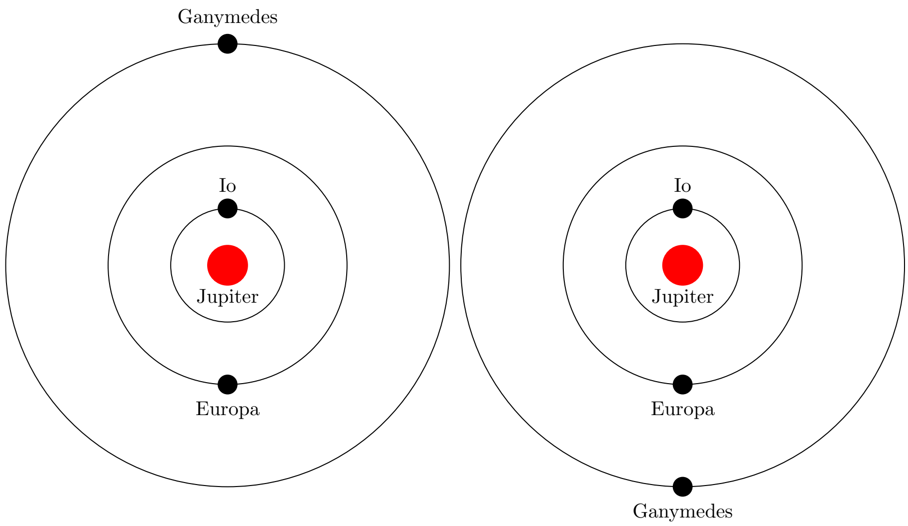


Figure 2: The Jupiter-Io-Europa-Ganymedes system at a collinearity.

For the *second* reduction we shall restrict our analysis to a small neighbourhood of the double resonance of the system: in terms of orbit frequencies this concerns the joint 2:1 frequency resonances of both Io and Europa and of Europa and Ganymedes. These resonances induce an approximate symmetry and the necessary Poincaré–Birkhoff normalization procedure, see e.g. [5, 6] and references therein, is enabled by a transformation to a covering space of the phase space, where an appropriate deck symmetry has to be taken into account. For details see § 3.2 of Section 3. This leads to an extra $SO(2, \mathbb{R})$–symmetry as we shall see now.

The resonant normalisation (or averaging) goes back at least to Poincaré [26], also see, e.g., [3, 4, 14, 28] or [5, 6] and references therein. This amounts to a stepwise normalization process by averaging over the *fast* Keplerian angles of the system, leaving us with the so-called *secular* system, which is related to the the *slow* evolutions of the Keplerian ellipses. The term 'stepwise' refers to a formal series expansion in the small parameters of the system. The normalizing process involves symplectic co-ordinate transformations and also the reduction of the rotational symmetry respects the symplectic structures at hand. After averaging and reduction, certain, so-called *resonant* terms remain, that constitute the secular system. The resulting truncated normal form approximation admits a second $SO(2, \mathbb{R})$–symmetry, where we speak of the *resonant* $SO(2, \mathbb{R})$–symmetry to distinguish it from the rotational $SO(2, \mathbb{R})$–symmetry.

The reduction of the resonant $SO(2, \mathbb{R})$–symmetry is from 5 to 4 degrees of freedom. The desired De Sitter periodic orbit after this reduction becomes a (relative) equilibrium, which turns out to be stable (elliptic). Upon reconstructing to 5 degrees of freedom we indeed obtain a family of periodic orbits, parametrised by the generator of the 1:2:4 resonance. When reconstructing to 6 degrees of freedom the superposition with an over-all rotation makes this a 2–parameter family of conditionally periodic orbits (with 2 frequencies). To overcome this a motion is called *periodic* if it is so in any rotating frame.

1.3 The De Sitter periodic orbits

Following De Sitter we first only consider Jupiter and the inner three satellites Io, Europa and Ganymedes with their orbital 1:2:4 resonance, later adding Callisto, the orbital period of which is not close to any lower resonance of the inner three. It turns out that the inner 1:2:4 resonance greatly affects the dynamics of the entire system. Summarizing from [29, 35] we have the following result.

Theorem 1 (De Sitter's periodic orbits). *In the* 4*–body problem Jupiter-Io-Europa-Ganymedes there exists a family of linearly stable periodic orbits for which the orbital frequencies are in* 1:2:4*–resonance.*

In the sequel the orbits in Theorem 1 are referred to as the *De Sitter periodic orbits*.

Remarks.

- In the $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})$–symmetric approximation, the De Sitter periodic orbits pass through a collinearity, compare with Figures 2 and 4.[2] To establish the existence in the full system the Implicit Function Theorem is used; for more general continuation methods see Poincaré [27].

- To establish the linear stability of the De Sitter periodic orbit also its normal linear part has to be examined.

- It turns out that for a qualitative explanation of De Sitter's theory [29, 30, 31] in the resonant normalization mentioned before only one normalizing step has to be performed. For a discussion and an update of this theory, comparing this with the effects of the 2:5–resonance of Jupiter and Saturn and with the influence of the Sun, see [35].

1.4 Librations

We now sketch our results on the motion of the Galilean satellites. In the 4–body problem Jupiter-Io-Europa-Ganymedes we consider the normal linear part of the family of De Sitter periodic orbits as found in Theorem 1, and the Lagrangean tori that are excited by the corresponding normal modes, compare with [13, 23]. These tori carry the librational motions we are looking for.

Theorem 2 (The 4**–body problem).** *In the* 4*–body problem Jupiter-Io-Europa-Ganymedes, there is a large measure Cantor set of sufficiently small eccentricities and a large measure Cantor set of sufficiently small masses of the satellites for which the following is true. The system admits a large measure Cantor union of Lagrangean invariant tori in a small neighbourhood of the family of De Sitter periodic orbits.*

For the 5–body system also the far-away-moon Callisto is included, which provides a more complete theory of the Galilean satellites. Callisto moves close to a circular orbit, which gives an

[2]This is an Ansatz by Poincaré.

extra period to the system. In fact, this orbit, together with the De Sitter periodic orbit of Theorem 1, gives rise to a normally elliptic, isotropic invariant 2–torus. From observations it is known that there are no low order resonances between the Callisto–frequency and other frequencies. The normal modes of the isotropic 2–tori again excite Lagrangean tori that carry librational motions.

Theorem 3 (The 5–body problem). *In the 5–body problem Jupiter-Io-Europa-Ganymedes-Callisto, there is a large measure Cantor set of sufficiently small eccentricities and a large measure Cantor set of sufficiently small masses of the satellites for which the following is true. The system admits a large 2–dimensional Hausdorff measure Cantor union of isotropic invariant 2–tori superposing the family of the De Sitter periodic orbits with the periodic motion of Callisto. Furthermore the system admits a large measure Cantor union of Lagrangean invariant tori in a small neighbourhood of this family of 2–tori.*

Both Theorems 2 and 3 were proven in [35], based on methods largely inspired by [1, 5, 15, 16, 17, 34]. In this paper we present the proofs of both theorems as an adaptation of one central parametrised isotropic Hamiltonian KAM-theorem that keeps track of the deck group of the covering space, where the latter is an improvement of [35]. This central theorem is a direct application of [12, 22, 25], also see [10, 11]. Moreover we incorporate the excitation theory as described in [13, 23] and we generalize [20] to deal with different time scales.

Remarks.

- The inner three motions in Theorem 3 remain close to the 1:2:4–resonance, while the outermost orbit is almost circular with a frequency that is strongly non-resonant with the inner three.
- The large measure Cantor sets in the above formulations arise from the fact that dense sets of resonances have to be avoided. In fact we impose Diophantine conditions of the internal and normal frequencies. In our approach parameters are needed for the necessary control of these fequencies.
- The parameters used are the masses of the satellites, one of the eccentricities and the[3] action variable conjugate to the periodic De Sitter motion. We note that the latter are in fact distinguished parameters, given by phase space variables.
- The various large measure Cantor sets in the above formulation are all intersections of one large measure Cantor set in the product of phase space and parameter space.

1.5 Outline

The parametrised KAM theory mentioned in § 1.4, which forms the mathematical background of Theorems 2 and 3 is the main result of this paper. Also we have to check all the necessary conditions for application to the librational motions.

[3]For the 5–body problem in Theorem 3 we also use the action variable conjugate to Callsito's periodic motion.

The first sections of the sequel quote from [35] as far as necessary for the application of our KAM theory. First, in § 2 we present a Newtonian model for the 4–body problem, and we give an appropriate perturbative setting for this model. In § 3 we briefly touch on the existence of the De Sitter periodic orbits, and their linear stability. Then, in § 4 the KAM theory is developed, while in § 5 this theory is applied to the librations in the Galilean 4– and 5–body problem.

2 A Newtonian description of the 4–body problem

We consider the 4–body problem of Jupiter–Io–Europa–Ganymedes in the Newtonian setting, giving a Hamiltonian formulation. Only in § 5.2 do we consider the 5–body problem with the far away satellite Callisto added.

2.1 Co-ordinates for the 4–body problem

We consider the 4–body problem with masses m_0, m_1, m_2, m_3. Here the index 0 refers to Jupiter and the indices $1, 2, 3$ to the satellites Io, Europa and Ganymedes, respectively. We follow [35] and replace our 4–body problem by 3 bodies in a central force field, denoting by $\tilde{q}_1$ the position of Io with respect to Jupiter, by $\tilde{q}_2$ the position of Europa with respect to the centre of mass of the former two and by $\tilde{q}_3$ the position of Ganymedes with respect to the centre of mass of the former three. Moreover let $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ be the conjugate momenta.

We split the kinetic energy T into $T = T_0 + T_1$ with

$$T_0 = \frac{1}{2}\sum_{i=1}^{3} \|\tilde{p}_i\|^2 (\frac{1}{m_i} + \frac{1}{m_0}) \quad \text{and} \quad T_1 = \sum_{1 \le i < j \le 3} \frac{\tilde{p}_i \cdot \tilde{p}_j}{m_0}$$

while the potential energy $U = U_0 + U_1$ splits into

$$U_0 = -\sum_{i=1}^{3} \frac{m_0 m_i}{r_{0i}} \quad \text{and} \quad U_1 = -\sum_{1 \le i < j} \frac{m_i m_j}{r_{ij}}$$

where the r_{ij} denote the mutual distances between the 4 bodies. In this way the Hamiltonian F splits as the sum $F = F_{\text{Kep}} + F_{\text{pert}}$ of a Keplerian and a perturbing part

$$\begin{aligned} F_{\text{Kep}} &= T_0 + U_0 \\ F_{\text{pert}} &= T_1 + U_1 \,. \end{aligned}$$

Note that both F_{Kep} and F_{pert} depend on positions and momenta only via inner products and therefore are invariant under the $\mathrm{SO}(2, \mathbb{R})$ action that rotates the plane around the origin (*i.e.* around the centre of mass); we call this the *rotational* $\mathrm{SO}(2, \mathbb{R})$–symmetry of the system. This $\mathrm{SO}(2, \mathbb{R})$ action is generated by the total angular momentum whence Noether's Theorem [3, 5] ensures that the total angular momentum is a conserved quantity.

2.2 Introduction of a small parameter

To account for the fact that the masses m_1, m_2 and m_3 are much smaller than the Jovian mass m_0 we introduce a small parameter μ as the order of the mass ratios and require

$$\frac{m_1}{m_0}, \frac{m_2}{m_0}, \frac{m_3}{m_0} \quad \sim \quad \mu\,,$$

where we use $\sim$ to express that both members of the equation are of the same order. Since the velocities are considered to be of order 1, the same ratios are used for the momenta, resulting in

$$m_1 = \mu\bar{m}_1, m_2 = \mu\bar{m}_2, m_3 = \mu\bar{m}_3, \tilde{p}_1 = \mu\bar{p}_1, \tilde{p}_2 = \mu\bar{p}_2, \tilde{p}_3 = \mu\bar{p}_3.$$

It follows that $F_{\mathrm{Kep}} \sim \mu$ and $F_{\mathrm{pert}} = O(\mu^2)$. This induces a scaling of the symplectic form by taking $(\bar{p}_1, \bar{p}_2, \bar{p}_3, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ as a set of Darboux co-ordinates. To keep the same Hamiltonian vector field we rescale the Hamiltonian function by μ^{-1}. By abuse of notation we now may write $F_{\mathrm{Kep}} \sim 1$ and $F_{\mathrm{pert}} = O(\mu)$.

2.3 First order approximating system

We sketch the set-up of our perturbation problem, for more details referring to § 3. The Hamiltonian system generated by F_{Kep} has 6 degrees of freedom, which leads to 6–dimensional Lagrangean invariant tori. The dynamics consists of three independent elliptic motions of the satellites Io, Europa and Ganymedes. This means that the 6–tori are foliated into invariant 3–tori. Moreover, the 1:2:4–resonance foliates the 3–tori into invariant 2–tori. Recall that we use the name De Sitter periodic orbit for the motion on these 2–tori.

In order to get candidates for the De Sitter periodic orbits from these periodic orbits (also using the Implicit Function Theorem), we average the perturbation F_{pert} along these periodic orbits (or, in other words, over the 1:2:4–resonance), for details see § 3. This procedure leads to a function F_{res} consisting only of resonant terms. We shall truncate at first order in the eccentricities, considering the latter as small parameters. Finally, after performing a symplectic change of co-ordinates, we shall end up with

$$F = F_{\mathrm{Kep}} + F_{\mathrm{res}} + F_{\mathrm{rem}},$$

where $F_{\mathrm{Kep}} \sim 1$, $F_{\mathrm{res}} \sim \mu e$ and the remainder $F_{\mathrm{rem}} = O(\mu e^2) + O(\mu^2)$. Here e is small and dominates the eccentricities e_1, e_2, e_3. For more details see § 3.3.

2.4 The resonant part of the perturbing function

For the three inner satellites Io, Europa and Ganymedes, F_{Kep} gives Keplerian ellipses on which $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ move. Let ℓ_1, ℓ_2, ℓ_3 and g_1, g_2, g_3 be the corresponding mean anomalies and arguments of the pericentres, respectively.

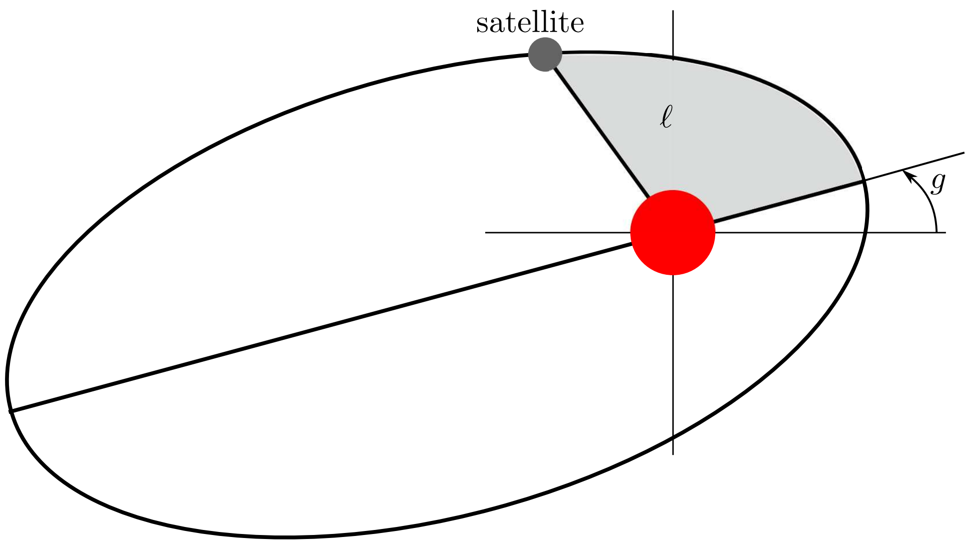


Figure 3: Mean anomaly ℓ and argument g of the pericenter of a Keplerian motion.

Remarks.

- The *mean anomaly* of the particle along the ellipse is the angle $\ell = 2\pi A/B$, where B is the area of the ellipse and A the area swept by the particle, counting from the pericentre. By Kepler's second law, this angle is proportional to the time parametrisation of the motion.
- The *argument of the pericenter* is the angle that the main axis of the Keplerian ellipse makes with a given horizontal direction. See Figure 3.

Note that on a deleted neighbourhood of the circular motions all of this is well-defined. To study things near these cicular motions, in the sequel scalings are used. From Arnold *et al.* [5] we recall that the variables $\ell_1, \ldots, g_3$, together with the semi major axes a_1, a_2, a_3 and the eccentricities e_1, e_2, e_3 constitute a set of regular co-ordinates in such a deleted neighbourhood in the phase space.

As announced in § 2.3, in order to find candidates for the De Sitter periodic solution, we average the function F_{pert} over the fast Keplerian angle and truncate at first order in the (small) eccentricities e_1, e_2, e_3. Following [29, 30, 31, 35] we so obtain

$$\begin{aligned} F_{\mathrm{res}} = & \frac{m_1 m_2}{a_2} \left\{ \bar{A}\, e_1 \cos(\ell_1 - 2\ell_2 + 2g_1 - 2g_2) - \bar{B}\, e_2 \cos(\ell_1 - 2\ell_2 + g_1 - g_2) \right\} \\ & + \frac{m_2 m_3}{a_3} \left\{ \bar{A}\, e_2 \cos(\ell_2 - 2\ell_3 + 2g_2 - 2g_3) - \bar{B}\, e_3 \cos(\ell_2 - 2\ell_3 + g_2 - g_3) \right\} + \bar{C}, \end{aligned}$$

for the resulting normalized part of F_{pert}. Here $\bar{A}$ and $\bar{B}$ are two functions of the semi major axes ratio $\alpha = \dfrac{a_1}{a_2} = \dfrac{a_2}{a_3}$ given by

$$\begin{aligned} \bar{A} &= \tfrac{3}{4}\alpha\, b^1_{3/2}(\alpha) - \tfrac{1}{2}\alpha\, b^3_{3/2}(\alpha) - \alpha^2\, b^2_{3/2}(\alpha) \\ \bar{B} &= \tfrac{3}{4}\alpha\, b^1_{1/2}(\alpha) + \tfrac{3}{2}\alpha\, b^0_{3/2}(\alpha) - \alpha^2\, b^1_{3/2}(\alpha) - \tfrac{1}{2}\alpha\, b^2_{3/2}(\alpha)\,. \end{aligned}$$

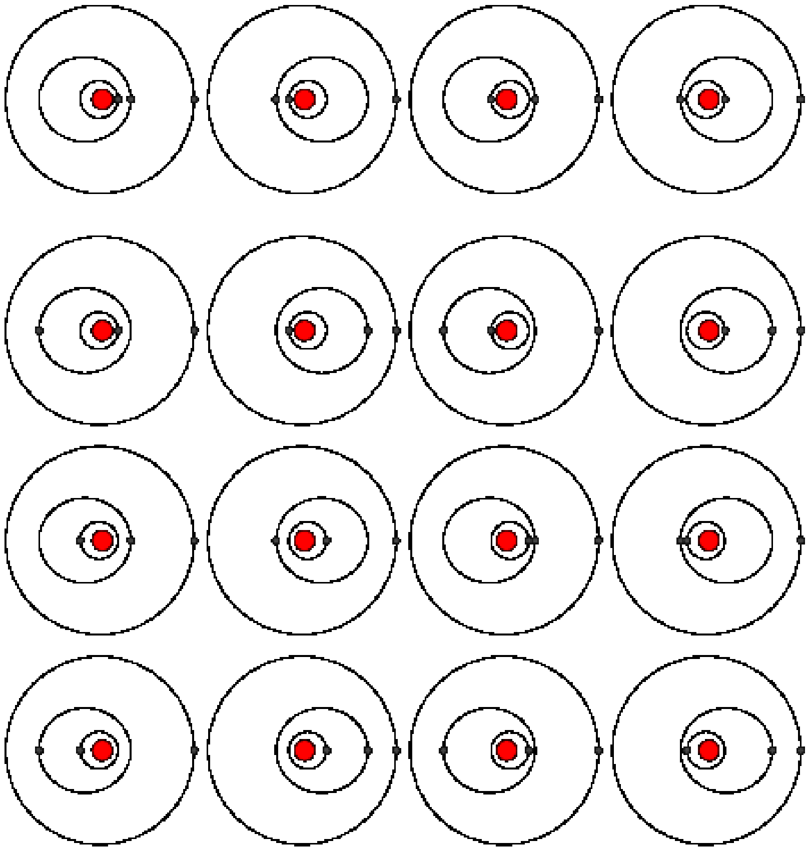


Figure 4: The 16 possible collinear configurations of the periodic orbits. The outer ellipse (of Ganymedes) and its starting point are kept fixed, the changes are both apocentres and starting points of both inner ellipses (of Europa and Io).

Indeed, the two semi major axes ratios are necessarily equal at the resonance we are now considering: as Io versus Europa and Europa versus Ganymedes are both in 1:2 resonance, the two semi major axes ratios have to be the same by Kepler's third law. Moreover

$$\bar{C} = \bar{C}(m_0, m_1, m_2, m_3, a_1, a_2, a_3)$$

is a certain constant (depending only on the masses and the semi major axes, *i.e.* independent of ℓ and g) which therefore is ignored in the sequel. For the Laplace coefficients $b_s^{(k)}(\alpha)$, $s = \frac{1}{2}, \frac{3}{2}$ we refer to [35]. In the sequel, we shall restrict to $\bar{A} > 0, \bar{B} > 0$.

3 Resonant periodic orbits and a covering space

Recall that $F = F_{\mathrm{Kep}} + F_{\mathrm{res}} + F_{\mathrm{rem}}$, where $F_{\mathrm{rem}} = O(\mu e^2 + \mu^2)$, is invariant under the rotational $\mathrm{SO}(2, \mathbb{R})$ action whence X_F can be reduced from 6 to 5 degrees of freedom. One-parameter families of periodic orbits of the truncated normal form $F_{\text{Kep}} + F_{\text{res}}$ can be continued to X_F using the Implicit Function Theorem. Normalization introduces the resonant $\mathrm{SO}(2,\mathbb{R})$–symmetry whence the truncated system can be further reduced to 4 degrees of freedom where the periodic orbits are determined by critical points of $F_{\text{Kep}} + F_{\text{res}}$ and ultimately by critical points of F_{res}, see [35].

For studying critical points of F_{res}, it is of importance that the Fourier series of F_{res} only contains cosines, which suggest to search for collinearities of the three satellites when simultaneously passing through their peri- or apocentres.[4] By the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry we only have to consider orbits of $X_{F_{\text{res}}}$ passing through the points with

$$(\ell_1, \ell_2, \ell_3, g_1, g_2, g_3) = (0 \text{ or } \pi, 0 \text{ or } \pi, 0 \text{ or } \pi, g_3 \text{ or } g_3 + \pi, g_3 \text{ or } g_3 + \pi, g_3)\,,$$

see Figure 2. Introducing the resonant angles $\delta_1 = \ell_1 - 2\ell_2, \delta_2 = \ell_2 - 2\ell_3$, we find the 16 different cases

$$(\delta_1, \delta_2, \eta_1 := g_1 - g_2, \eta_2 := g_2 - g_3) = (\frac{\pi}{2} \pm \frac{\pi}{2}, \frac{\pi}{2} \pm \frac{\pi}{2}, \frac{\pi}{2} \pm \frac{\pi}{2}, \frac{\pi}{2} \pm \frac{\pi}{2}) \tag{1}$$

to consider. We code these cases by $\mathcal{D}_{+,+,+,+}, \mathcal{D}_{+,+,+,-} \cdots$ etc., where the subscript exactly denotes the $\pm$–signs in (1), compare with Figure 4.

We conclude that if for the arguments of the pericenters we have $g_1 = g_2 = g_3$ the function F_{res} has a critical point. This implies that for sufficiently small μ the system generated by $F = F_{\text{Kep}} + F_{\text{res}} + F_{\text{rem}}$ has a periodic orbit passing close to a collinearity. Moreover the normal dynamics of such a solution is determined by the corresponding Hessian of F_{res}.

3.1 Reduction of the rotational symmetry

We recall that the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry is related to the conservation of the total angular momentum. Our aim is to reduce this symmetry. To this purpose we use symplectic Delaunay co-ordinates (L_i, ℓ_i, G_i, g_i), $i = 1, 2, 3$, defined by

$$\begin{cases} L_i = \mu_i \sqrt{M_i} \sqrt{a_i} & \text{circular angular momentum} \\ \ell_i & \text{mean anomaly} \\ G_i = L_i \sqrt{1 - e_i^2} & \text{angular momentum} \\ g_i & \text{argument of pericentre}\,. \end{cases} \tag{2}$$

Here the lower-case letters stand for angles, while

$$\mu_i = \frac{m_0 \bar{m}_i}{m_0 + \mu \bar{m}_i}\,, \quad M_i = m_0 + \mu \bar{m}_i,$$

compare with Figure 3.

One of the most important tools now consists of the transformation that co-rotates with the pure 1:2:4–resonance. In fact, we transform the system (2) symplectically to the co-ordinate system

[4] Earlier in this paper, we referred to this assumption as Poincaré's Ansatz.

$(D_1, D_2, D_3, \delta_1, \delta_2, \delta_3, Z_1, Z_2, Z_3, \eta_1, \eta_2, \eta_3)$ by

$$\left\{\begin{array}{ll} D_1 = L_1, & \delta_1 = \ell_1 - 2\ell_2, \\ D_2 = 2L_1 + L_2, & \delta_2 = \ell_2 - 2\ell_3, \\ D_3 = 4L_1 + 2L_2 + L_3, & \delta_3 = \ell_3, \\ Z_1 = G_1, & \eta_1 = g_1 - g_2, \\ Z_2 = G_1 + G_2, & \eta_2 = g_2 - g_3, \\ Z_3 = G_1 + G_2 + G_3, & \eta_3 = g_3 \,. \end{array}\right. \tag{3}$$

This transformation, in fact, consists of two independent parts. The upper three lines correspond to co-rotation with the purely resonant part and are dealt with in § 3.2. The lower three lines allow the reduction from 6 to 5 degrees of freedom since the angle $\eta_3 = g_3$ becomes cyclic [3], due to the conservation of total angular momentum Z_3. The corresponding symplectic reduction of the rotational $\mathrm{SO}(2, \mathbb{R})$–symmetry is thus achieved by fixing Z_3 and ignoring the variable η_3.

3.2 Covering and deck symmetry

The change of variables (3) corresponds to a covering mapping $\Pi : \mathbb{T}^6 \times \mathbb{R}^6 \longrightarrow \mathbb{T}^6 \times \mathbb{R}^6$, that is built up as follows. We emphasize the covering that takes place in the first three lines of (3).

First, in the angular directions (3) gives rise to the mappings

$$\begin{array}{rcl} \Pi_1 : (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z}) & \longrightarrow & (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})\,, \\ (\delta_3, \delta_2, \delta_1) & \mapsto & (\ell_3, \ell_2, \ell_1) := (\delta_3, \delta_2 + 2\delta_3, \delta_1 + 2\delta_2 + 4\delta_3) \\ & & \bmod (2\pi\mathbb{Z}) \end{array} \tag{4}$$

which is multiple-to-one, and

$$\begin{array}{rcl} \Pi_2 : (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) & \longrightarrow & (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})\,, \\ (\eta_3, \eta_2, \eta_1) & \mapsto & (g_3, g_2, g_1) := (\eta_3, \eta_2 + \eta_3, \eta_1 + \eta_2 + \eta_3) \\ & & \bmod (2\pi\mathbb{Z}) \end{array} \tag{5}$$

which is an automorphism of the 3–torus.

For the corresponding actions we simply get the linear automorphisms

$$\begin{array}{rcl} \Pi_3 : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (D_1, D_2, D_3) & \mapsto & (L_1, L_2, L_3) := (D_1, D_2 - 2D_1, D_3 - 2D_2) \end{array} \tag{6}$$

and

$$\begin{array}{rcl} \Pi_4 : \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (Z_1, Z_2, Z_3) & \mapsto & (G_1, G_2, G_3) := (Z_1, Z_2 - Z_1, Z_3 - Z_2)\,. \end{array} \tag{7}$$

Remark. The total transformation then is $\Pi = (\Pi_1 \times \Pi_2) \times (\Pi_3 \times \Pi_4)$. Note that this transformation combines two independent transformations $(\mathrm{Id} \times \Pi_2) \times (\mathrm{Id} \times \Pi_4)$ and $(\Pi_1 \times \mathrm{Id}) \times (\Pi_3 \times \mathrm{Id})$. The first of these has been detailed at the end of § 3.1. As said before, our focus now is on the second of these.

To understand the covering mapping Π_1 better, we consider the deck transformations

$$\Delta_{1,2} : (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z}) \longrightarrow (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z})$$

defined by

$$\Delta_1(\delta_3, \delta_2, \delta_1) = (\delta_3, \delta_2 - 2\pi, \delta_1) \quad \text{and} \quad \Delta_2(\delta_3, \delta_2, \delta_1) = (\delta_3, \delta_2, \delta_1 - 2\pi)\,. \tag{8}$$

The following then is immediate.

Proposition 4 (Deck group). *With the above definitions we have*

- $\Pi_1 \circ \Delta_j = \Pi_1$, *for* $j = 1, 2$, *which expresses that the* Δ_j *are deck transformations,*
- *Moreover*
$$\Lambda = \langle \Delta_1, \Delta_2 \mid \Delta_1^2 = \mathrm{Id}, \Delta_2^4 = \mathrm{Id},\ \Delta_1 \circ \Delta_2 = \Delta_2 \circ \Delta_1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$
is the deck group of the covering Π_1 .

The mapping Π is a covering, which only in the δ–direction is multi-to-one. Therefore the deck group also in all the other directions is trivial. By abuse of notation we indicate the total deck group also by Λ, and similarly the deck transformations $\Delta \in \Lambda$.

Referring to *e.g.* [4, 9, 14], we recall that functions H, vector fields X, mappings Φ, etc. on the base space $M = \{\ell, g, L, G\}$ are lifted to invariant functions $\tilde{H}$, equivariant vector fields $\tilde{X}$, equivariant mappings $\tilde{\Phi}$, etc. on the covering space $\tilde{M} = \{\delta, \eta, D, Z\}$, *i.e.* with

$$\tilde{H} \circ \Delta = \tilde{H}, \quad \Delta_* \tilde{X} = \tilde{X}, \quad \Delta \circ \tilde{\Phi} = \tilde{\Phi} \circ \Delta \tag{9}$$

for all $\Delta \in \Lambda$. Conversely, any function, vector field, mapping, etc. on $\tilde{M}$ only projects down to a similar object on M when the equivariance relation (9) holds.

In the sequel we shall apply both averaging and KAM Theory on the covering space $\tilde{M} = \{\delta, \eta, D, Z\}$; in both cases we have to take these equivariance considerations into account.

3.3 Normalization

On the covering space $\mathbb{T}^6 \times \mathbb{R}^6$ we now perform a standard normalization process, based on the infinitesimal action of the Keplerian vector field

$$X_{F_{\text{kep}}} = \nu_3 \partial_{\delta_3}, \tag{10}$$

for $\mu = 0$. The corresponding transformations iteratively normalize the power series in μ and e, where normal means independence of the fast angle $\delta_3 = \ell_3$. Here e measures the size of the eccentricities e_1, e_2, e_3. This induces the resonant $\mathrm{SO}(2, \mathbb{R})$–symmetry of the truncated normal form, where δ_3 becomes a cyclic variable. The normalizing transformations respect both the sympletic form and the deck group Λ as well as the scaling that are introduced later. Such considerations go back at least to Poincaré [26], also compare with [5, 6].

Remarks.

- In another language we speak of averaging over the fast angle δ_3, where δ_1 and δ_2 are semi-fast. Near the 1:2:4–resonance $\delta_3 = \ell_3$ is the only fast angle.

- For the present purposes it is sufficient to carry out only the first step of the normal form iteration, but for application of KAM theory later on we use N normalizing steps.

- In terms of Poisson brackets the infinitesimal action (10) is generated by the function F_{kep} for $\mu = 0$. The action leads to an adjoint operator Ad : $H \mapsto \{F_{\text{Kep}}, H\}$, which gives a direct sum splitting
$$\ker \mathrm{Ad} \oplus \mathrm{im}\, \mathrm{Ad}$$
of the space. Normalization then amounts to removing all terms in the image, where the remaining terms in the kernel exactly are the resonant symmetric ones.

Here we only formulate the results of this procedure, for details referring to [29, 35].

Domains in the covering space. We assume that $0 < a_1 < a_2 < a_3, 0 < e_1, e_2, e_3 \sim e \ll 1$, where e is a positive constant considered as small, and such that

$$\frac{a_1(1+e)}{a_2(1-e)} < 1, \quad \frac{a_2(1+e)}{a_3(1-e)} < 1.$$

guaranteeing that the three elliptic orbits are bounded away from each other.

This gives an open subset $\tilde{\mathcal{P}} \subseteq \tilde{M}$. In terms of the co-ordinates (δ, η, D, Z), we now have that the components of D and Z are positive.

We choose $(D_1^0, D_2^0, D_3^0) \in \mathbb{R}_+^3$ such that the 9–dimensional set

$$\tilde{\mathcal{D}} = \{(\delta, \eta, D, Z) \in \tilde{\mathcal{P}} : D_1 = D_1^0, D_2 = D_2^0, D_3 = D_3^0\}$$

contains all 1:2:4–resonant Keplerian motions in $\tilde{\mathcal{P}}$. Consider a neighbourhood $\tilde{N} \subseteq \tilde{\mathcal{P}}$ of $\tilde{\mathcal{D}}$, which is sufficiently small in terms of $|\mu|$. We now have the following result proven in [35].

Theorem 5 (Normal form). *Assume that all functions are real analytic in their arguments. Then there exists a Λ–equivariant, near-identity, real-analytic symplectic transformation $\Phi : \tilde{N} \to \Phi(\tilde{N})$, such that*

$$F \circ \Phi = F_{\text{Kep}} + F_{\text{res}} + F_{\text{rem}}.$$

On $\tilde{N}$ the functions $F_{\text{res}} \in \ker \mathrm{Ad}$ and F_{rem} have the following properties.

- $F_{\text{res}} = \frac{1}{2\pi} \int_0^{2\pi} F_{\text{pert}}\, d\,\delta_3$ *is the truncation at* 1*st order of the eccentricities, and*

- $F_{\mathrm{rem}} = O(\mu e^2) + O(\mu^2)$.

Remarks.

- In a complex extension of $\tilde{N}$ in the supremum norm we have $|\Phi - \mathrm{Id}| = O(\mu)$.
- The normalized Hamiltonian function now reads

$$F_{\mathrm{Kep}}(D_1, D_2, D_3) + F_{\mathrm{res}}(D_1, D_2, D_3, \delta_1, \delta_2, Z_1, Z_2, \eta_1, \eta_2; Z_3) + O\Big(\mu e^2\Big) + O(\mu^2),$$

  defined on a subset of $\mathbb{T}^5 \times \mathbb{R}^5$ where, due to the rotational symmetry, Z_3 acts as a (distinguished) parameter. Note that truncating to $F_{\mathrm{Kep}} + F_{\mathrm{res}}$ turns δ_3 into a cyclic variable and thus D_3 into a (distinguished) parameter as well.

3.4 Relative frequencies of the pericentres

In the sequel we need several details on the relative frequencies as these appear in the reductions. From [29, 35] we quote that the angles $\eta_1 = g_1 - g_2$ and $\eta_2 = g_2 - g_3$ have frequencies

$$\begin{aligned}
\nu_1 &= \frac{\partial F_{\mathrm{res}}}{\partial Z_1} = -\frac{2\sqrt[3]{2}\bar{A}\cos(\delta_1 + 2\eta_1)e_2 m_2 + 2\bar{B}\cos(\delta_1 + \eta_1)e_1 m_1 - \sqrt[3]{2}\bar{A}\cos(\delta_2 + 2\eta_2)e_1 m_3}{\sqrt{m_0}e_1 e_2}, \\
\nu_2 &= \frac{\partial F_{\mathrm{res}}}{\partial Z_2} = \frac{2\bar{B}\cos(\delta_1 + \eta_1)e_3 m_1 - 2\sqrt[3]{2}\bar{A}\cos(\delta_2 + 2\eta_2)e_3 m_3 - 2\bar{B}\cos(\delta_2 + \eta_2)e_2 m_2}{\sqrt{m_0}e_2 e_3},
\end{aligned}$$

called *relative frequencies* of the pericentres in $X_{F_{\mathrm{Kep}}+F_{\mathrm{res}}}$. They are differences of the frequencies of g_1, g_2 and g_3

$$\begin{aligned}
\nu_{g_1} &= -\frac{2\sqrt[3]{2}\bar{A}\cos(\delta_1 + 2\eta_1)m_2}{\sqrt{m_0}e_1}, \\
\nu_{g_2} &= \frac{2\bar{B}\cos(\delta_1 + \eta_1)m_1 - 2\sqrt[3]{2}\bar{A}\cos(\delta_2 + 2\eta_2)m_3}{\sqrt{m_0}e_2}, \\
\nu_{g_3} &= \frac{2\bar{B}\cos(\delta_2 + \eta_2)m_2}{\sqrt{m_0}e_3}.
\end{aligned}$$

As explained in § 1.2, after reduction of both the rotational and the resonant $\mathrm{SO}(2,\mathbb{R})$–symmetry, we end up in 4 degrees of freedom, where we have to look for relative equilibria in order to find periodic orbits of the 4-body problem at hand.

Equilibria. These equilibria occur as the critical points of the function F_{res} as obtained in the Normal Form Theorem 5. Following [29, 35] we put

$$\nu_1 = \nu_2 = 0, \tag{11}$$

Case	Sign of $\bar{Q}$	Sign of Hessian determinant
$E_{-,-,+,+}$	irrelevant	positive
$E_{+,+,+,+}$	irrelevant	positive
$E_{-,-,-,+}$	positive	negative
$E_{+,+,-,+}$	positive	negative
$E_{-,+,+,-}$	negative	negative
$E_{+,-,+,-}$	negative	negative

Table 1: The 6 families of relative equilibria, the sign of $\bar{Q} = \sqrt[3]{2}\bar{A}\bar{m}_3 - 2\bar{B}\bar{m}_1$ related to their existence and $\operatorname{sgn}\det D^2F_{\mathrm{res}}$ related to application of the Implicit Function Theorem.

in order to find the 'simple' solutions under Poincaré's Ansatz. According to the considerations of [29, pp. 10-12] regarding the signs of $\nu_{g_1}, \nu_{g_2}, \nu_{g_3}$, this leads only to the two families

$$E_{-,-,+,+} \ \ (i.e.\ [29], \text{Case } (6)) \text{ and } \ E_{+,+,+,+} \ \ (i.e.\ [29], \text{Case } (16))$$

if no further conditions are put on the masses. For the remaining cases we introduce the quantity

$$\bar{Q} = \sqrt[3]{2}\bar{A}\bar{m}_3 - 2\bar{B}\bar{m}_1,$$

which has to be non-zero since ν_{g_2}, ν_{g_1} and ν_{g_3} are non-zero. It follows that for $\bar{Q} > 0$, also for

$$E_{-,-,-,+}(i.e.\ [29],\ \text{Case } (2)) \ \text{ and } \ E_{+,+,-,+}(i.e.\ [29],\ \text{Case } (12)),$$

equation (11) is satsified. Moreover, for the case where $\bar{Q} < 0$, for

$$E_{-,+,+,-}(i.e.\ [29],\ \text{Case } (7)) \ \text{ and } \ E_{+,-,+,-}(i.e.\ [29],\ \text{Case } (13)\ ),$$

equation (11) is satisfied. See [29, p.10]. In all of these cases, this gives a one-parameter family of eccentricities (e_1, e_2, e_3) (parametrised by one of the eccentricities, *e.g.* by e_2), such that the two relative frequencies ν_1, ν_2 are zero.

Periodic orbits. In Table 1 this information is summarized: Given the appropriate signs of $\bar{Q}$ there exist 6 families of relative equilibria of F_{res} with corresponding periodic orbits

$$D_{-,-,+,+}, D_{+,+,+,+}, D_{-,-,-,+}, D_{+,+,-,+}, D_{-,+,+,-}, D_{+,-,+,-}$$

of $F_{\mathrm{Kep}} + F_{\mathrm{res}}$, which are candidates for De Sitter periodic orbits of $F = F_{\mathrm{Kep}} + F_{\mathrm{res}} + F_{\mathrm{rem}}$. Indeed, for all candidates the continuation from $F_{\mathrm{Kep}} + F_{\mathrm{res}}$ to $F = F_{\mathrm{Kep}} + F_{\mathrm{res}} + F_{\mathrm{rem}}$ can be carried out with help of the Implicit Function Theorem for sufficiently small μ and e, since by the Normal Form Theorem 5

$$F_{\mathrm{rem}} = O(\mu e^2) + O(\mu^2).$$

This is due to the fact that the Hessian matrices D^2F_{res} have non-zero determinants. To gain more insight in the normal linear dynamics we have to extend this study further.

3.5 Normal linear dynamics of the periodic orbits

Note that for the study of the normal linear dynamics of the periodic orbits $D_{-,-,+,+}$, $D_{+,+,+,+}$, $D_{-,-,-,+}$, $D_{+,+,-,+}$, $D_{-,+,+,-}$ and $D_{+,-,+,-}$ we only have to consider the system generated by $F_{\text{Kep}} + F_{\text{res}}$. In that case we may again reduce the resonant $\mathrm{SO}(2,\mathbb{R})$–symmetry to 4 degrees of freedom and direct our attention to the (relative) equilibria $E_{-,-,+,+}$ and $E_{+,+,+,+}$, etc. Recall that the corresponding 8–dimensional phase space has co-ordinates $(D_1, D_2, \delta_1, \delta_2, Z_1, Z_2, \eta_1, \eta_2)$ with small μ.

Remark. Note that F_{Kep} and F_{res} do not appear in the same magnitude of μ. This is the reason for the proper degeneracy of the system. Application of the Implicit Function Theorem in fact becomes simpler since the different time scales make the Hessian effectively a block-diagonal matrix. This simplification carries over when replacing the Implicit Function Theorem by KAM theory.

The normal linear behaviour is determined by the matrix $\mathcal{L}$, obtained from the Hessian of F_{Kep} and the Hessian of F_{res} by

$$\mathcal{L} = J \cdot (D^2 F_{\text{Kep}} + D^2 F_{\text{res}})$$

where the matrix

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

corresponds to the symplectic structure. It follows that $\mathcal{L}$ has the form

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & \mu^2 e & 0 & 0 & 0 & \mu^2 e & 0 \\ 0 & 0 & 0 & \mu^2 e & 0 & 0 & 0 & \mu^2 e \\ \mu^{-1} & \mu^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu^{-1} & \mu^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu^2 e & 0 & 0 & 0 & \mu^2 e & 0 \\ 0 & 0 & 0 & \mu^2 e & 0 & 0 & 0 & \mu^2 e \\ 0 & 0 & 0 & 0 & e^{-3} & e^{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-3} & e^{-3} & 0 & 0 \end{pmatrix}, \tag{12}$$

where we indicate the entries only by their orders in μ and e, for details see [35]. A direct observation teaches us that $\mathcal{L}$ has eigenvalues of the orders $\sqrt{\mu e}$ and μ/e. For the following refinement of Theorem 1 also see [35].

Theorem 6 (De Sitter [29]). *In the above circumstances the following is true.*

1. $D_{-,-,+,+}$ *is the only stable family of periodic orbits in the system generated by* $F_{\text{Kep}} + F_{\text{res}}$, *for sufficiently small values of* μ *and* e.

2. *This family can be continued as an elliptic periodic solution to the full system generated by* $F_{\mathrm{Kep}} + F_{\mathrm{pert}}$.

Recall that in (11) we had put $\nu_1 = 0$ and $\nu_2 = 0$ to find $D_{-,-,+,+}$. From these equations we furthermore obtain

$$e_1 = \frac{2 \cdot 2^{5/6} \bar{A} \bar{m}_2 e_2}{2\sqrt{2}\bar{B}\bar{m}_1 + 2^{5/6}\bar{A}\bar{m}_3} \quad \text{and} \quad e_3 = \frac{\sqrt{2}\bar{B}\bar{m}_2 e_2}{2\sqrt{2}\bar{B}\bar{m}_1 + 2^{5/6}\bar{A}\bar{m}_3}.$$

From this it follows that in Theorem 6 we may well replace the small constant e by the small parameter e_2.

Remarks.

- The continued stable periodic orbits thus provide a possible explanation of the real evolution of the system Jupiter-Io-Europa-Ganymedes, which was the base of De Sitter's theory of the Galilean satellites [30, 31].
- One question is whether for $\bar{Q} = 0$ the families undergo bifurcations. This value is outside the relevant domain of parameters, but still may provide us with an organising centre, visible in the dynamics. Compare with Broer-Hanßmann-Hoo-Naudot [7, 10] for examples of such bifurcations.
- Is the family $D_{-,-,+,+}$ also stable in the spatial problem? This is an open research question.

4 On Kolmogorov Arnold Moser theory

In this section we develop the Kolmogorov-Arnold-Moser or KAM theory for the investigation of invariant tori near the family of De Sitter periodic orbits $D_{-,-,+,+}$ found before. This amounts to the formulation of a general theorem on the persistence of isotropic invariant tori in nearly integrable Hamiltonian systems, where a discrete group symmetry is preserved. This result goes back to Huitema [22], also see [12], also see [8]. Similar results were found by Herman (unpublished) and developed further by *e.g.* Féjoz [15], also see [16, 17] and Zhao [34]; for historical remarks compare with [13], p. 312. The entire development heavily rests on Moser [25], particularly on the Lie algebra aspects therein.

We note that an extra complication in our setting is the presence of three different time scales, which is characteristic for proper degeneracy. First of all the motion in the De Sitter periodic orbit is fast. The normal dynamics has two more time scales corresponding to the block structure in the matrix (12). Moreover the motion of Callisto is fast again. A similar KAM setting was encountered in the theory of Arnold [1, 15]. We here have to extend this approach to isotropic lower dimensional tori.

Lagrangean invariant tori are typically parametrised by the actions conjugate to the toral angles. The Kolmogorov condition stipulates that the actions map diffeomorphically to the frequencies, thereby ensuring persistence of most of these tori. For the invariant tori near the De Sitter periodic orbit, however, it will be important that other parameters of the system, to wit the masses, can be used to control the frequencies. Fortunately, a general parametrised KAM theory was developed by Broer-Huitema-Sevryuk-Takens and Moser [11, 12, 22, 25], also see Broer-Hanßmann-Sevryuk-Wagener [8, 13, 21] and references therein. In the sequel we shall apply this parameter dependent approach to the properly degenerate system at hand.

Recall that a *properly degenerate system* is a superintegrable system which has more independent first integrals than required for integrability. In the case of proper degeneracy the flow is never ergodic on Lagrangean tori but can only be so on lower dimensional isotropic tori. One consequence of this degeneracy is that part of the perturbation is needed to fulfill the KAM non-degeneracy conditions at hand. An advantage, however, is that this leads to multiple time scales, which makes the check of these non-degeneracy conditions easier, compare with [1, 15, 20].

4.1 Parametrised KAM theory

Stepping to a somewhat more abstract setting we consider a real analytic system of the form

$$\begin{aligned} \dot{x} &= \omega(\lambda) + f(x, y, z, \lambda) \\ \dot{y} &= g(x, y, z, \lambda) \\ \dot{z} &= \Omega(\lambda)\, z + h(x, y, z, \lambda)\,, \end{aligned} \tag{13}$$

with $x \in \mathbb{T}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^{2p}$ and $\lambda \in \mathbb{R}^s$, which is Hamiltonian with respect to the symplectic form

$$\mathrm{d}\, x \wedge \mathrm{d}\, y + \mathrm{d}\, z^2 = \sum_{i=1}^{n} \mathrm{d}\, x_i \wedge \mathrm{d}\, y_i + \sum_{j=1}^{p} \mathrm{d}\, z_{2j-1} \wedge \mathrm{d}\, z_{2j}\,.$$

Here λ is as an external parameter. In applications the variables $y = (y_1, y_2, \ldots, y_n)$ can act as (distinguished) parameters, but also masses or eccentricities can take the role of parameters. In vector field form the system (13) is called X and its integrable approximation $\tilde{X}$ then has the form

$$\tilde{X} = \omega(\lambda)\frac{\partial}{\partial x} + \Omega(\lambda)\, z \frac{\partial}{\partial z}\,, \tag{14}$$

where the terms f, g and h are considered small in the compact-open topology. The latter is the topology of local uniform convergence of complex analytic extensions. Note that the integrable family $\tilde{X}$ has isotropic invariant tori given by the equation $z = 0$, this is a family of tori parametrised by y and λ. The aim of this section is to study their persistence as $\tilde{X}$ is perturbed to the vector field X.

The present properly degenerate situation allows to split up the vector field (14) according to the time scales. Since the motion of the satellites is fast, the frequency vector ω is of order 1. Recall that the components of this vector describe the 1:2:4–resonant periodic motion of the inner satellites Io, Europa, Ganymedes and, where applicable, also the periodic motion of Callisto. Therefore either with $n = 1$ or $n = 2$ and thus always $p = 4$. The matrix $\Omega \in \mathrm{SP}(8, \mathbb{R})$ is given by (12) with its 2 time scales, of order $\sqrt{\mu e}$ and μ/e. We now shall formulate the relevant KAM theorem for the general case of multiple time scales.

4.2 The perturbation problem, non-degeneracy and time scales

To study persistence we single out one of the isotropic[5] tori, say for the values $y = y_0$ and $\lambda = \lambda_0$. For simplicity we translate to $y_0 = 0$ and $\lambda_0 = 0$.

We assume the matrix $\Omega(0)$ to be simple, in particular with no eigenvalue 0. The real analytic mapping

$$\mathcal{F} : \lambda \in \mathbb{R}^s \mapsto (\omega(\lambda), \Omega(\lambda)) \in \mathbb{R}^n \times \mathrm{sp}(2p, \mathbb{R}) \tag{15}$$

is called the (generalized) *frequency mapping*. We assume this mapping to be *transversal* to the Cartesian product

$$\omega(0) \times \mathcal{O}(\Omega(0)) \subset \mathbb{R}^n \times \mathrm{sp}(2p, \mathbb{R}),$$

where $\mathcal{O}(\Omega(0))$ denotes the orbit of $\Omega(0)$ under the adjoint action of the symplectic group $\mathrm{SP}(2p, \mathbb{R})$ on $\mathrm{sp}(2p, \mathbb{R})$. For this terminology also compare with Arnold [2]. We then may assume that $\Omega(\lambda)$ has a Williamson diagonal form [18], with eigenvalues of type

$$\pm \mathrm{i}\beta, \quad \alpha \pm \mathrm{i}\beta \quad \text{and} \quad \pm\alpha,$$

of respective numbers n_E (purely elliptic), n_C (complex hyperbolic) and n_R (real hyperbolic), where necessarily $n_E + 2n_C + n_R = p$. Let β be the $n_E + n_C$ vector containing all the purely imaginary (parts of the) eigenvalues, the so-called normal frequencies of the isotropic torus. We abbreviate $r = n_E + n_C$. The above non-degeneracy condition often is referred to as the Broer-Huitema-Takens (BHT)-condition. We assemble the data of internal and normal frequencies in the mapping

$$\tilde{\mathcal{F}} : \lambda \in \mathbb{R}^s \mapsto (\omega(\lambda), \beta(\lambda)) \in \mathbb{R}^n \times \mathbb{R}^r. \tag{16}$$

Note that the BHT-condition amounts to $\tilde{\mathcal{F}}$ being a submersion.

Remarks.

- Since we have n internal frequencies it follows that the problem needs $s = n + n_E + 2n_C + n_R = n + p$ parameters for versal unfolding. So even if we let the y–variables act as n (distinguished) parameters, we still need p further parameters.
- The BHT-non-degeneracy for $p = 0$ boils down to the standard Kolmogorov non-degeneracy condition for the case of Lagrangean invariant tori.
- In the present case the Lagrangean tori are obtained by excitation of normal modes and this necessitates the use of external parameters like the masses.

In the present situation of multiple time scales we split the frequency mapping (16) into

$$\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2, \ldots, \tilde{\mathcal{F}}_m) : \mathbb{R}^s \longrightarrow \prod_{j=1}^{m} (\mathbb{R}^{n_j} \times \mathbb{R}^{r_j}) \,. \tag{17}$$

[5] Recall that in the 4–body problem this torus is a periodic orbit and in the 5–body problem the dimension of the torus is 2.

In this way the values of $\tilde{\mathcal{F}}$ turn into

$$(\omega,\beta) = \left((\omega^1,\beta^1),(\omega^2,\beta^2),\cdots,(\omega^m,\beta^m)\right) ,$$

where we have n_j internal frequencies ω^j of order ε_j and r_j normal frequencies β^j of the same order ε_j. The orders of magnitudes of time scales are $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_m > 0$. For the 4–body problem we have $m = 3$ with $n = (1,0,0)$ and $r = (0,2,2)$ while for the 5–body problem we have $m = 4$ with $n = (2,0,0,0)$ and $r = (0,2,2,1)$

Let us consider the BHT-condition and what it means for the splitting (17). First of all when $\tilde{\mathcal{F}}$ is submersive, then so are its components $\tilde{\mathcal{F}}_j$. The other implication is more involved and we return to it later, also splitting the parameter space $\mathbb{R}^s$.

4.3 Diophanticity

It is well-known that in the present perturbation problem a dense set of normal-internal resonances

$$\langle k,\omega\rangle + \langle \ell,\beta\rangle = 0$$

shows up that leads to the notorious small divisor problem. A usual way to overcome this is by introducing the following Diophantine property. Let $\tau > n-1$ and $\gamma > 0$. We say that the pair of vectors $(\omega,\beta) \in \mathbb{R}^n \times \mathbb{R}^r$ is (τ,γ)*–Diophantine* if for all $k \in \mathbb{Z}^n \setminus \{0\}$ and all $\ell \in \mathbb{Z}^r$ with $|\ell| \leq 2$ one has

$$|\langle k,\omega\rangle + \langle \ell,\beta\rangle| \geq \frac{\gamma}{|k|^\tau}. \tag{18}$$

The set of all (ω,β) satisfying (18) is denoted by $(\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma}$. Note that for $(\omega,\beta) \in (\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma}$ and $\sigma > 0$ also the scalar multiple $\sigma\cdot(\omega,\beta)$ is contained in $(\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma}$, which means that $(\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma}$ is the union of closed half lines. The intersection of $(\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma}$ with the unit sphere $\mathbb{S}^{n+r-1}$ moreover is a Cantor set and the measure of $\mathbb{S}^{n+r-1} \setminus (\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma}$ is of order $O(\gamma)$ as $\gamma \to 0$. *E.g.* compare with [12].

In the present situation of multiple time scales there are no low-order resonances between frequencies of different time scales. However, the Diophantine conditions (18) still remain necessary to exclude high-order resonances.

4.4 KAM persistence

For a precise formulation of the persistence result we need some further specifications. Let $\Gamma \subset \mathbb{R}^n\times\mathbb{R}^r$ be a box with boundaries parallel to the co-ordinate hyperplanes and define

$$\Gamma_{\tau,\gamma} = \Gamma \cap (\mathbb{R}^n\times\mathbb{R}^r)_{\tau,\gamma} .$$

Finally define

$$\Gamma^\gamma_{\tau,\gamma} = \{(\omega,\alpha) \in \Gamma_{\tau,\gamma} \mid \operatorname{dist}((\omega,\alpha),\partial(\Gamma)) \geq \gamma\} ,$$

observing that the measure of $\Gamma^\gamma_{\tau,\gamma}$ again is full up to order $O(\gamma)$ as $\gamma \to 0$.

In the present case of multiple time scales it is helpful to also split the parameter space into

$$\Gamma^{\gamma}_{\tau,\gamma} = \Gamma^{\gamma,1}_{\tau,\gamma} \times \Gamma^{\gamma,2}_{\tau,\gamma} \times \cdots \times \Gamma^{\gamma,m}_{\tau,\gamma} . \tag{19}$$

Here we require that the component $\tilde{\mathcal{F}}_j : \Gamma^{\gamma}_{\tau,\gamma} \longrightarrow \mathbb{R}^{n_j} \times \mathbb{R}^{r_j}$ is not only a submersion, but that the derivative has maximal rank, already with respect to the variables in $\Gamma^{\gamma,j}_{\tau,\gamma}$. If this property holds for $j = 1, \ldots, m$, we say that the unperturbed vector field $\tilde{X}$ satisfies the *scaled* BHT non-degeneracy condition. This condition is sufficient for $\tilde{\mathcal{F}}$ to be submersive. Furthermore, this condition is easy to check in our applications: here multiple time scales actually help to solve our problems.

Theorem 7 (Main Result). *Let X and $\tilde{X}$ be real analytic families of Hamiltonian vector fields as described above on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma$, so where $\tilde{X}$ is integrable and where $\Gamma = \Gamma^1 \times \Gamma^2 \times \cdots \times \Gamma^m \subset \mathbb{R}^n \times \mathbb{R}^r$ is an open box. Assume that Ω has only simple eigenvalues and that $\tilde{X}$ is scaled* BHT *non-degenerate. Fixing constants $\tau > n + n_E - 1$ and $\gamma > 0$, where γ is sufficiently small, also take $X - \tilde{X}$ sufficiently small in the compact-open topology. Then there exists a C^∞–diffeomorphism (onto its image)*

$$\Phi : \mathbb{T}^n \times U \times V \times \Gamma \longrightarrow \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma, \tag{20}$$

for neighbourhoods U of $0 \in \mathbb{R}^n$ and V of $0 \in \mathbb{R}^{2p}$, such that

1. *The mapping Φ is skew in the sense that the following diagram commutes.*

$$\begin{array}{ccc}
\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma & \xrightarrow{\Phi} & \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2p} \times \Gamma \\
\downarrow & & \downarrow \\
\mathbb{T}^n \times \mathbb{R}^n \times \Gamma & \longrightarrow & \mathbb{T}^n \times \mathbb{R}^n \times \Gamma \\
\downarrow & & \downarrow \\
\mathbb{T}^n \times \Gamma & \longrightarrow & \mathbb{T}^n \times \Gamma \\
\downarrow & & \downarrow \\
\Gamma & \longrightarrow & \Gamma \, ;
\end{array}$$

   where vertical arrows denote natural projections and where the horizontal arrow indicate the relevant components of Φ. Moreover Φ is real-analytic in the x variables and maps fibers in the $\mathbb{R}^n$– and $\mathbb{R}^{2p}$–directions affinely onto fibers;

2. *Φ is near the identity in the C^∞–topology and preserves the symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z^2$. Moreover, if the vector fields X and $\tilde{X}$ are invariant under a discrete symmetry group Λ, compatible with the symplectic structure $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z^2$, then the diffeomorphism* (20) *is equivariant with respect to Λ ;*

3. *Restricted to* $\mathbb{T}^n \times \{0\} \times \{0\} \times \Gamma^{\gamma}_{\tau,\gamma}$ *the mapping* Φ *conjugates* $\tilde{X}$ *to* X*, i.e.,*

$$\Phi_*(\tilde{X}) = X. \tag{21}$$

   The restriction $\Phi_*|_{\mathbb{T}^n\times\{0\}\times\{0\}\times\Gamma^{\gamma}_{\tau,\gamma}}$ *also preserves the normal linear dynamics of these invariant tori.*

4. *The* n_E *elliptic normal modes give rise to a Cantor family of invariant* $(n+n_E)$*–tori perturbed from* $\mathbb{T}^{n+n_E} \times \{0\} \times \{0\} \times \Gamma \subseteq \mathbb{T}^{n+n_E} \times \mathbb{R}^{n+n_E} \times \mathbb{R}^{2(p-n_E)} \times \Gamma$ *where the Cantorisation results from Diophantine conditions of the form* (18) *with* $\ell \in \mathbb{Z}^r$ *restricted to* $|\ell_{n_E+1}| + \ldots + |\ell_r| \leq 2$ *and* $|k|^\tau$ *in the denominator of the right hand side replaced by* $(|k| + |\ell_1| + \ldots + |\ell_{n_E}|)^\tau$.

Proof. The only problem is to show that the scaled BHT-condition implies the BHT-condition. This follows directly from the definitions, and the conclusions 1–3 of the theorem follow directly from the results in [12, 22]. For conclusion 4 rewrite (13) in the elliptic z–directions in symplectic polar co-ordinates and apply the first part of the theorem with n replaced by $n + n_E$ and p replaced by $p - n_E$. □

Remarks.

- Observe that the mapping Φ conjugates the unperturbed restriction $\tilde{X}|_{\mathbb{T}^n\times\{0\}\times\{0\}\times\Gamma^{\gamma}_{\tau,\gamma}}$ to a quasi-periodic subsystem of the perturbation X. Therefore the perturbed tori have the same frequencies as the unperturbed ones. Moreover these tori are close together, since Φ is near the identity mapping.

- For the case that $m = 1$ (and $\Lambda = \{\mathrm{Id}\}$), the conclusions 1–3 of Theorem 7 reduce to Theorem 6.1 in [12, 22] and Theorem 2.6 in [11]. Note that the extension to Λ–equivariance in conclusion 2 is not an extra complication as it already follows from Theorem 8.1 in [12, 22], which uses the Lie algebra approach that goes back to Moser.

- The excitation of normal modes expressed in conclusion 4 can also be applied separately to any selection of $\nu \leq n_E$ elliptic normal modes, compare with [13, 23]. The main 'complication' lies in properly assigning the ν modes that have been chosen among the n_E elliptic modes; for instance, the choice to simply excite the first ν modes leads to Diophantine conditions of the form (18) with $\ell \in \mathbb{Z}^r$ restricted to $|\ell_{\nu+1}| + \cdots + |\ell_r| \leq 2$ and $|k|^\tau$ in the denominator of the right hand side replaced by $(|k| + |\ell_1| + \cdots + |\ell_\nu|)^\tau$. This leads to $\binom{n_E}{\nu}$ families of $(n+\nu)$–tori, i.e. the whole ramified torus bundle near the invariant n–tori gets Cantorized. The proof remains the same.

- Excitation of normal modes behaves well with respect to different time scales as normal frequencies β_j of time scale ε_j are merely turned into internal frequencies of that same time scale.

- Theorem 7 can also be formulated and proven for C^∞–vector fields or even for vector fileds of finite (but sufficiently high) differentiability. In fact, real analyticity allows to improve Theorem 7 by replacing the Diophantine conditions by Bryuno conditions and by Nekhoroshev-like estimates on the remainder of the Birkhoff normal form. The latter allows for estimates that are exponentially small in the distance to the initial n–tori for both the portion of nearby invariant tori of dimensions $n + \nu$ in the perturbed system and, in the normally elliptic case $p = r = n_E$, for the inverse of the (Arnold) diffiusion time for initial conditions not on invariant tori, see [23].

- The formulation of Theorem 7 is kept in the style of [11, 12, 22], also see [8], but its formulation can be easily adapted to the case with multiple eigenvalues [10]. In particular the linear theory of the quasi-periodic Hamiltonian Hopf bifurcation can be covered, for the non-linear theory see [7, 21].

- For the case that $p = 0$ (and $\Lambda = \{\mathrm{Id}\}$), Theorem 7 reduces to Theorem (Main Result) in [20], also see [34, Example-Condition 5.3]. If moreover $m = 2$, then this is Arnold's Theorem [1, 15].

4.5 Comments on estimates

We describe the smallness condition on $X - \tilde{X}$ in Theorem 7. A neighbourhood $\mathcal{U}$ in the compact-open topology can be expressed in terms of the supremum norm on compacta of holomorphic extensions of the real analytic components of the vector fields X and $\tilde{X}$, extended into a complex domain $O \subseteq (\mathbb{C}^n/\mathbb{Z}^n) \times \mathbb{C}^n \times \mathbb{C}^{2p} \times \mathbb{C}^s$. Indeed, there exists $\delta > 0$ such that $X - \tilde{X} \in \mathcal{U}$ if and only if the estimates

$$|f|_O < \gamma\delta, \quad |g|_O < \gamma\delta^2 \quad \text{and} \quad |h|_O < \gamma\delta^2 \tag{22}$$

hold. Here f, g and h are the component functions of $X - \tilde{X}$, compare with (13), and $\gamma > 0$ is the Diophantine constant in (18). Also, δ is independent of γ. As a consequence, if $|f|_O$, $|g|_O$ and $|h|_O$ are of order ε, we can take $\gamma \sim \varepsilon$. Moreover, if after normalisation the sizes $|f|_O$, $|g|_O$ and $|h|_O$ of the nonlinear terms are even smaller, we can choose γ smaller as well. This is good for the measure theoretic estimates.

Remark. Since Φ is a near-identity diffeomorphism, the measure theoretic estimates of the unperturbed situation are largely maintained. To be precise, the union of the perturbed, X–invariant n–tori project down to Γ onto a set of full measure up to order γ as $\gamma \to 0$. In particular, in phase space the quasi-periodic $(n+n_E)$–tori acummulate with Hausdorff density 1 on the invariant n–tori.

The present situation of multiple time scales allows to relax the smallness condition on $X - \tilde{X}$ a bit, not letting the smallest part of the unperturbed $\tilde{X}$ dictate the required smallness of the perturbation. Indeed, splitting (13) according to the time scales $1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_m > 0$ we may replace (22) by

$$|f_j|_O < \varepsilon_j\gamma\delta, \quad |g_j|_O < \varepsilon_j\gamma\delta^2 \quad \text{and} \quad |h_j|_O < \varepsilon_j\gamma\delta^2 \tag{23}$$

for $j = 1, \ldots, m$.

4.6 Comments on parameter dependence

We formulated Theorem 7 for vector fields that explicitly depend on external parameters. These parameters are used to control both the internal and normal frequencies of the invariant tori. If only internal frequencies have to be controlled, one can always resort to the actions y conjugate to the toral angles x. This is taken care of by localisation $y = y_{\mathrm{loc}} + \kappa$, for details see [11, 12]. In this way the actions y turn into distinguished parameters κ. However, when also normal frequencies have to be controlled one meets the so-called 'lack-of-parameter' problem.

One possible solution for the 'lack-of-parameter' problem would be to invoke Rüssmann–like conditions involving higher derivatives of the frequencies with respect to the distinguished parameters, see [11] for details. This is not what we plan to do in this paper.

Instead, in the application to the 4– and 5–body problems we use the masses of the Galilean satellites as external parameters. This not only solves the 'lack-of-parameter' problem for the invariant 2–tori in the 5–body problem, but also serves for persistence of the Lagrangean tori. In this way we avoid the (heavy) computation of a nonlinear Birkhoff normal form around the De Sitter periodic orbits and the 2–tori, respectively.

Remarks.

- In the present approach, the Diophantine conditions (18) single out a large measure Cantor set in the product of phase space and parameter space. For this reason persistence only holds on a Cantor set in the space of masses. We note that the projection of the former Cantor set to phase space leads to persistent libration occurring on a set of positive measure.

- When performing higher order Birkhoff normalisation we would expect to obtain a result similar to Theorem 3, but for *all* (sufficiently small) masses.

5 Application of KAM theory

In the remainder we aim to apply Theorem 7 to the 4–body problem Jupiter-Io-Europa-Ganymedes first and then to the 5–body problem Jupiter-Io-Europa-Ganymedes-Callisto. In both cases the rotational $SO(2,\mathbb{R})$–symmetry not only applies to the unperturbed system (the normal form, which in addition has the resonant $SO(2,\mathbb{R})$–symmetry), but also to the perturbed system (the 'original' system describing the 4–body resp. 5–body problem). For this reason we do not (yet) reconstruct to 6 resp. 8 degrees of freedom, but let the perturbation analysis take place in 5 resp. 7 degrees of freedom. We start giving a sketch of the approach, for details see the next two subsections.

Remarks.

- Both in the 4– and in the 5–body setting we aim to find Lagrangean tori that are excited by normally elliptic modes: these Lagrangean tori carry the librating motion. In the 4–body case the excitation occurs from the (normally) elliptic periodic motion $D_{-,-,+,+}$. In the 5–body case this role is taken by the normally elliptic isotropic invariant 2–tori. Such situations are usually referred to as 'local' Lagrangean KAM Theory, for a description and further references, see [13], § 8.4. Note that the internal frequencies of the Lagrangean tori are approximated by the internal and normal frequencies of the excited periodic orbit resp. invariant 2–torus.

- All Lagrangean tori as well as the isotropic 2–tori persist by Theorem 7. Note that the persistence of the periodic orbits $D_{-,-,+,+}$ is established by the Implicit Function Theorem.
- For the 4–body problem a list of the corresponding orders of the frequencies in $F_{\text{Kep}} + F^N_{\text{res}} + F^N_{\text{rem}}$ is itemized in § 5.1, while their non-degeneracy is checked in the pages immediately thereafter. Here N denotes an appropriate truncation.
- For the 5–body problem similar issues are addressed at the end of § 5.2.

The 4–body system. As we have seen in § 3, for the 4–body model, after reduction of the planar rotational symmetry in the plane, the family $D_{-,-,+,+}$ of De Sitter's periodic orbits is normally elliptic. The periodic orbits in the integrable approximation are accumulated by 5–dimensional Lagrangean tori that provide the librating orbits. These are the local Lagrangean tori as obtained by excitation of normal modes; we apply Theorem 7 with $n = 1$ internal frequencies and $n_E = r = p = 4$ excited frequencies. For the scaled BHT non-degeneracy we choose $m = 3$ and subdivide $n_1 = n = 1$, $r_1 = 0$, $n_2 = n_3 = 0$ and $r_2 = r_3 = 2$. The time scales are $\varepsilon_1 = 1$, $\varepsilon_2 = \sqrt{\mu e}$ and $\varepsilon_3 = \mu/e$. Note that $n = 1$ just means that we are exciting the normal modes of a periodic orbit, in particular we only have to deal with the normal frequencies of § 3.5, *i.e.* the eigenvalues the matrix $\mathcal{L}$ in (12), and their dependence on the parameters $\bar{m}_1, \bar{m}_2, \bar{m}_3$ and e_2.

The 5–body system. Following the same line of thought, the role of the De Sitter family of periodic orbits of the inner three satellites $D_{-,-,+,+}$ is replaced by normally elliptic isotropic 2–tori, obtained by taking these together with the (almost) circular motion of Callisto. This yields the integrable appromation $\tilde{X}$ to which we apply Theorem 7 with $n = 2$ internal frequencies and $n_E = r = p = 5$ excited frequencies. For the scaled BHT non-degeneracy we choose $m = 4$ and subdivide $n_1 = n = 2$, $r_1 = 0$, $n_2 = n_3 = n_4 = 0$, $r_2 = r_3 = 2$ and $r_4 = 1$. The time scales are $\varepsilon_1 = 1$, $\varepsilon_2 = \sqrt{\mu e}$, $\varepsilon_3 = \mu/e$ and $\varepsilon_4 = \mu e^2$. As a result we obtain both persistence of the normally elliptic 2–tori themselves and the the local Lagrangean tori as obtained by excitation of normal modes, that provide the desired libration for this case.

5.1 Librations in the 4–body setting

As announced above, we now apply Theorem 7 to the 4–body problem Jupiter-Io-Europa-Ganymedes to detect many librations of the De Sitter periodic orbit $D_{-,-,+,+}$. These librational motions are perturbed families of Lagrangean tori $\mathbb{T}^5 \times U$, $U \subseteq \mathbb{R}^5$ open, which arise by excitation of the normal modes of $D_{-,-,+,+}$. To use conclusion 4 of Theorem 7, we have to verify the scaled BHT-condition.

We have shown that for $0 < \mu \ll 1$ the elliptic equilibrium $E_{-,-,+,+}$ of the $\mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$–reduced system $F_{\text{Kep}} + F_{\text{res}}$ is non-degenerate, with normal frequencies appearing at different orders in the small quantity μ. Recall that the orders of frequencies are as follows. The internal frequency ν_{per} of δ_3 is of order 1. The normal frequencies $\nu_{n,1}$ and $\nu_{n,2}$ are of order

$$\varepsilon_2 = \sqrt{\mu e}$$

and the normal frequencies $\nu_{n,3}$ and $\nu_{n,4}$ are of order

$$\varepsilon_3 = \frac{\mu}{e}.$$

In order to control the smallness conditions required for Theorem 7 we consider an appropriate N-th order normal form approximation of $F_{\text{Kep}} + F_{\text{pert}}$ by eliminating the dependence on the angle δ_3. This leads to $F_{\text{Kep}} + F^N_{\text{res}} + F^N_{\text{rem}}$ in which

- for $N = 2$, $F^2_{\text{res}} = F_{\text{res}}$,
- $F^N_{\text{res}} - F_{\text{res}} = O(\mu e^2 + \mu^2)$,
- F^N_{res} is independent of δ_3,
- F^N_{rem} is of order $O(\mu e^N) + O(\mu^N)$,

where we recall that e measures the size of the eccentricities e_1, e_2 and e_3 of the three internal satellites. Note that next to $O(\mu^N)$ we only achieve $O(\mu e^N)$. For this reason we also need to consider e as a small parameter. To ensure that $\varepsilon_1 = 1 \gg \varepsilon_2 \gg \varepsilon_3 > 0$ we require that $0 < \mu \ll e^3 \ll 1$. Note that this implies that for $N \geq 2$ we have $\mu^N \ll \mu e^N$.

We now analyse the dynamics of $F_{\text{Kep}} + F^N_{\text{res}}$. We note that also the higher normal forms have the $\mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$–symmetry, consisting of rotations over the angles $\eta_3 = g_3$ and $\delta_3 = \ell_3$. Reducing this symmetry turns the De Sitter periodic orbit into the equilibrium point $E_{-,-,+,+}$. By the Implicit Function Theorem the non-degeneracy of $E_{-,-,+,+}$ allows for continuation to a non-degenerate elliptic equilibrium $E^N_{-,-,+,+}$ of the reduced system with Hamiltonian $F_{\text{Kep}} + F^N_{\text{res}}$, provided that μ and e are sufficiently small. Reconstructing the resonant $\mathrm{SO}(2,\mathbb{R})$–symmetry leads to a family of normally elliptic periodic orbits $D^N_{-,-,+,+}$ of $F_{\text{Kep}} + F^N_{\text{res}}$, where only the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry is reduced.

Using the 8–dimensional local co-ordinate z around the equilibrium $E^N_{-,-,+,+}$, the Hamiltonian function $F_{\text{Kep}} + F^N_{\text{res}}$ gives rise to

$$\dot{z} = \Omega z + h(z) \tag{24}$$

where Ω has the form (12). We recall that Ω is elliptic, *i.e.*, has only simple, purely imaginary eigenvalues. The equation (24) corresponds to the third equation in (13), the first and second equation of which have disappeared by the reduction.

The perturbation problem takes place in 10 dimensions where we have reconstructed the resonant $\mathrm{SO}(2,\mathbb{R})$–symmetry by adding the first and second equation of (13). This turns the equilibrium $E_{-,-,+,+}$ into the periodic orbit $D_{-,-,+,+}$. While the persistence of $D_{-,-,+,+}$ follows from the Implicit Function Theorem, we use Theorem 7 to obtain Lagrangean tori by the excitation of normal modes.

It is thus sufficient to verify the non-degeneracy condition for the frequencies $\nu_{per}, \nu_{n,1}, \nu_{n,2}, \nu_{n,3}, \nu_{n,4}$ of $D_{-,-,+,+}$. Moreover, according to Theorem 7, it is enough to verify the non-degeneracy conditions separately for frequencies in the different time scales. We choose as parameters

$$\lambda = (\bar{m}_1, \bar{m}_2, \bar{m}_3, e_2) \tag{25}$$

the rescaled masses and one of the eccentricities. While e_2 is a distinguished parameter, the masses are external parameters. We refrain from using only distinguished paramer, in this way avoiding the computation of a nonlinear Birkhoff normal form around $D_{-,-,+,+}$.

The non-degeneracy conditions on the frequencies $\nu_{per}, \nu_{n,1}, \nu_{n,2}, \nu_{n,3}, \nu_{n,4}$ are as follows.

1. The frequency of the periodic orbit ν_{per}. Here the non-degeneracy condition is nothing more than
$$\frac{\partial F_{\text{Kep}}}{\partial D_3} \neq 0\,. \tag{26}$$
2. The normal frequencies $\nu_{n,j},\ j = 1, 2, 3, 4$. Here the check boils down to the dependence of the coefficients of the monic quadratic equations (4) and (5) of [35] with respect to the parameters. We abbreviate these equations to
$$x^2 + b_1 x + c_1 = 0 \tag{27}$$
and
$$x^2 + b_2 x + c_2 = 0 \tag{28}$$
respectively.

Lemma 8 (Non-degeneracy). *The Jacobians of* (b_1, c_1) *in* (27) *with respect to* $(\bar{m}_1, e_2)$ *and of* (b_2, c_2) *in* (28) *with respect to* $(\bar{m}_2, \bar{m}_3)$ *are both non-degenerate almost everywhere.*

Proof. Assisted by MAPLE 16, we find that

- $\det\left(\dfrac{\partial(b_1, c_1)}{\partial(m_1, e_2)}\right)$ evaluated at $(m_1 = 1, m_2 = m_3 = 0)$ equals $8\bar{B}^3\bar{A}$;
- $\det\left(\dfrac{\partial(b_2, c_2)}{\partial(m_2, m_3)}\right)$ evaluated at $(m_1 = 1, m_2 = m_3 = 0)$ equals $-16 \cdot 2^{5/6}\bar{B}^{10}$.

These expressions are analytically dependent on the variables e_2 and $m_1 = \mu\bar{m}_1$ and on $m_2 = \mu\bar{m}_2$ and $m_3 = \mu\bar{m}_3$ respectively. Since these expressions are not identically zero, the zeroes are confined to subsets of lower dimension. $\square$

We aim to apply Theorem 7 to obtain 5–dimensional (Lagrangean) tori that carry the librational motions of the De Sitter periodic orbit $D^N_{-,-,+,+}$, by the excitation of normal modes. First, for the distinguished parameters e_2 and D_3 we have to introduce localized variables

$$\begin{aligned} e_2^{\text{loc}} &= e_2 - \kappa_1, \\ D_3^{\text{loc}} &= D_3 - \kappa_2 \end{aligned}$$

which yields external parameters κ_1 and κ_2, compare with [11, 12]. By abuse of notation we denote the parameter by

$$\lambda = (\bar{m}_1, \bar{m}_2, \bar{m}_3, \kappa_1, \kappa_2),$$

compare with (25). This allows to specify a box $\Gamma \subseteq \mathbb{R}^5$ in the parameter space. In the range of the frequency mapping

$$\mathcal{F} : \Gamma \longrightarrow \mathbb{R}^5, \quad \lambda \mapsto (\nu_{per}, \nu_{n,1}, \nu_{n,2}, \nu_{n,3}, \nu_{n,4})$$

Diophantine conditions (18) apply. The splitting (19) amounts to considering the parameters κ_2, $(\bar{m}_1, \kappa_1)$ and $(\bar{m}_2, \bar{m}_3)$ separately. The corresponding time scales are $\varepsilon_1 = 1$, $\varepsilon_2 = \sqrt{\mu e}$ and $\varepsilon_3 = \mu/e$, where we recall that $0 < \mu \ll e^3 \ll 1$. This also ensures that the smallness condition on the perturbation as expressed in (23) is satisfied.

Theorem 9 (Librational motions). *In the Diophantine conditions* (18) *we take* $n = 5$, $r = 0$, $\tau > 4$ *and* $\gamma > 0$ *sufficiently small. Then there exists a bound* $e_0 > 0$ *on the eccentricities such that for* $\mu \ll e^3$ *and* $e \le e_0$ *there exists a* C^∞*–diffeomorphism*

$$\Phi : \mathbb{T}^5 \times U \times \Gamma \longrightarrow \mathbb{T}^5 \times \mathbb{R}^5 \times \Gamma$$

onto its image, where U *is a neighbourhood of* 0 *in* $\mathbb{R}^5$*, with the following properties.*

1. *The mapping* Φ *is skew in the sense that the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{T}^5 \times \mathbb{R}^5 \times \Gamma & \xrightarrow{\Phi} & \mathbb{T}^5 \times \mathbb{R}^5 \times \Gamma \\ \downarrow & & \downarrow \\ \mathbb{T}^5 \times \Gamma & \longrightarrow & \mathbb{T}^5 \times \Gamma \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \Gamma\ ; \end{array}$$

   where vertical arrows denote natural projections and where the horizontal arrows indicate the relevant components of Φ*. Moreover* Φ *is real-analytic in the toral variables and maps fibers in the* $\mathbb{R}^5$*–direction affinely onto fibers;*

2. Φ *is near the identity in the* C^∞*–topology and preserves the symplectic form. Moreover, the diffeomorphism* (20) *is equivariant with respect to the deck group* Λ *as given in Proposition 4;*

3. *Restricted to* $\mathbb{T}^5 \times \{0\} \times \Gamma^\gamma_{\tau,\gamma}$ *the mapping* Φ *conjugates the unperturbed system to the perturbed system.*

Proof. The constant δ in the specification (23) of the neighbourhood $\mathcal{U}$ is determined by Theorem 7. The constants γ and ε_j, $j = 1, 2, 3$ have been specified before. To achieve the smallness conditions (23) it suffices to take the normalization order N sufficiently large. Indeed, then the order $O(\mu e^N) + O(\mu^N)$ of the remainder $F^N_{\rm rem}$ in the Hamiltonian function, as specified at the beginning of the section, ensures that the corresponding vector field satisfies the smallness condition (23). The necessary scaled BHT non-degeneracy condition follows from Lemma 8 and (26).

From conclusion 4 of Theorem 7 we obtain the desired Cantor family of Lagrangean invariant tori. □

Remarks.

- In summary Theorem 9 roughly implies the following. There is a Cantor set of sufficiently small e, such that for $\mu \ll e^3$ there exists a set of masses m_1, m_2, m_3 of large relative measure, satisfying $\max\{m_1, m_2, m_3\} \leq \mu m_0$, for which the following is true. There is a union of Lagrangean invariant tori of the Hamiltonian vector field X_F, of large relative measure in a small neighbourhood of the continued family of stable periodic orbits $D_{-,-,+,+}$.
- When reconstructing the rotational $\mathrm{SO}(2, \mathbb{R})$–symmetry, returning from 5 to 6 degrees of freedom, an extra angle $\eta_3 = g_3$ is restored. As a consequence, the De Sitter periodic orbits lift to conditionally periodic orbits on 2–tori and the librating Lagrangean 5–tori lift to librating Lagrangean 6–tori.
- By choosing γ as a suitable power of e, related to the bound $O(\mu e^N)$, we achieve that the measure of the complement of the persisting Lagrangean invariant tori scales with the size of the perturbation; compare with [5, 11].

5.2 Invariant 2–tori and their librations in the 5–body setting

Callisto is not at resonance with Io, Europa and Ganymedes. Thus, in the 5–body problem obtained by adding Callisto to our model, De Sitter's periodic orbits have to be replaced by invariant 2–tori while the librating Lagrangean tori become of dimension 7 (with Callisto we add 2 degrees of freedom). The small mass m_4 of Callisto is of the same order μ as m_1, m_2, m_3 and correspondingly we rescale $m_4 = \mu\bar{m}_4$. With $\tilde{p}_4 = \mu\bar{p}_4$ and $\tilde{q}_4$ denoting linear momentum and relative position of the fourth satellite with respect to the common centre of mass of Jupiter and the other three satellites the Hamiltonian $\widetilde{F}$ (rescaled by μ^{-1}) of the system becomes $\widetilde{F} = \widetilde{F}_{\rm Kep} + \widetilde{F}_{\rm pert}$ with $\widetilde{F}_{\rm Kep} = F_{\rm Kep} + F_{\rm Kep,4}$ and $\widetilde{F}_{\rm pert} = F_{\rm pert} + F_{\rm pert,4}$. Here

$$F_{\rm Kep,4} = \frac{\|\bar{p}_4\|^2}{\mu_4} - \frac{\mu_4 M_4}{r_{04}} \quad \text{and} \quad F_{\rm pert,4} = \mu \sum_{i=1}^{3} \frac{\bar{p}_i \bar{p}_4}{m_0} - \mu \sum_{i=1}^{3} \frac{\bar{m}_i \bar{m}_4}{r_{i4}}$$

while $M_4 = m_0 + \mu\bar{m}_4$ and $\mu_4 = m_0\bar{m}_4/M_4$.

In the Darboux co-ordinates (3) on our covering space we replace Z_3 by

$$Z_3' = G_1 + G_2 + G_3 + G_4$$

and furthermore add

$$\begin{aligned} D_4 &= L_4 \\ \delta_4 &= \ell_4 + g_4 - g_3 \end{aligned}$$

together with

$$\xi_4 + i\eta_4 = \sqrt{2(L_4 - G_4)}e^{-i(g_4-g_3)}. \tag{29}$$

Note that the latter choice (going back to Poincaré) allows the orbit of the last body to be circular while the other orbits have to be ellipses with non-zero eccentricity.

Recall that ν_{per} is the frequency of the De Sitter periodic orbit (co-inciding with the Keplerian frequency of the third satellite), and denote by $\nu_{\mathrm{Kep},4}$ the Keplerian frequency of the fourth satellite. Extending the hypothesis made in § 3.3 by $a_3 < a_4, 0 \leq e_4 < e < 1$ such that

$$\frac{a_3(1+e)}{a_4(1-e)} < 1$$

we ensure that the four elliptic orbits are bounded away from each other. These conditions extend $\tilde{\mathcal{P}} \subseteq \tilde{M}$ to $\tilde{\mathcal{Q}}$.

The covering mapping (3) thereby is replaced by $\widetilde{\Pi} : \mathbb{T}^7 \times \mathbb{R}^9 \longrightarrow \mathbb{T}^8 \times \mathbb{R}^8$, consisting of

$$\begin{aligned} \widetilde{\Pi}_1 : (\mathbb{R}/2\pi\mathbb{Z})^5 \times (\mathbb{R}/4\pi\mathbb{Z}) \times (\mathbb{R}/8\pi\mathbb{Z}) &\longrightarrow (\mathbb{R}/2\pi\mathbb{Z})^5 \times (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}), \\ (\eta_3, \eta_2, \eta_1, \delta_4, \delta_3, \delta_2, \delta_1) &\mapsto (g_3, g_2, g_1, \ell_4, \ell_3, \ell_2, \ell_1) \end{aligned}$$

(which is multiple-to-one), $\widetilde{\Pi}_2$ assigning (ξ_4, η_4) to (g_4, G_4) as in (29) and the linear automorphism

$$\begin{aligned} \widetilde{\Pi}_3 : \mathbb{R}^7 &\longrightarrow \mathbb{R}^7 \\ (Z_1, Z_2, Z_3, D_1, D_2, D_3, D_4) &\mapsto (G_1, G_2, G_3, L_1, L_2, L_3, L_4) \end{aligned}$$

The total transformation then is $\widetilde{\Pi} = \widetilde{\Pi}_1 \times \widetilde{\Pi}_2 \times \widetilde{\Pi}_3$. The deck transformation Δ of this covering is still given by (8).

Let $D_4^0 \in \mathbb{R}_+$ be chosen such that for $D_3 = D_3^0, D_4 = D_4^0$, the vector $(\nu_{per}, \nu_{\mathrm{Kep},4})$ is (γ, τ)–Diophantine for some $\gamma > 0$, $\tau > 1$ and extend $\tilde{\mathcal{D}} \subseteq \tilde{\mathcal{P}}$ to

$$\tilde{\mathcal{E}} = \{(\delta, \eta, D, Z') \in \tilde{\mathcal{Q}} : D_1 = D_1^0, D_2 = D_2^0, D_3 = D_3^0, D_4 = D_4^0\}$$

containing the 1:2:4–resonant Keplerian motions. Consider a neighbourhood $\tilde{\mathcal{N}} \subseteq \tilde{\mathcal{Q}}$ of $\tilde{\mathcal{E}}$, which is sufficiently small in terms of $|\mu|$. Then we can extend Proposition 5 to obtain for any integer N a Λ–equivariant analytic symplectic transformation $\psi : \tilde{\mathcal{N}} \longrightarrow \phi(\tilde{\mathcal{N}})$ that is $O(\mu)$–close to the identity such that

$$\widetilde{F} \circ \psi = \widetilde{F}_{\mathrm{Kep}} + \widetilde{F}_{\mathrm{res}} + \widetilde{F}_{\mathrm{rem}} ,$$

with in $\tilde{\mathcal{N}}$ analytic functions

- $\widetilde{F}_{\mathrm{res}} = \int_{\mathbb{T}^2} \widetilde{F}_{\mathrm{res}} \, d\delta_3 d\delta_4$,

- $\widetilde{F}_{\text{rem}} = O(\mu e^N + \mu^N)$,

compare with Proposition 7.1 of [35]. From $\widetilde{F}$ the truncated normal form

$$\widetilde{F}_{\text{Kep}} + \widetilde{F}_{\text{res}} \tag{30}$$

inherits the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry, which can be reduced by fixing the value of the total angular momentum Z_3' and ignoring the cyclic angle $\eta_3 = g_3$. In fact, normalization could as well have taken place in 7 degrees of freedom and below the perturbation analysis does take place on the reduced phase space.

For $N = 2$ the normalized $\widetilde{F}_{\text{res}}$ consists of the average F_{res} over δ_3 as in Theorem 5 (see also § 2.4) and of the average of $F_{\text{pert},4}$ over both fast angles δ_3 and δ_4. Since the first term $\mu \sum_{i=1}^{3} \bar{p}_i \bar{p}_4 / m_0$ in $F_{\text{pert},4}$ does not contribute to the secular system, see Lemma 64 of [15], the latter average is

$$\bar{F}_{\text{sec},4} = \int_{\mathbb{T}^2} F_{\text{pert},4}\, d\delta_3 d\delta_4 = -\mu \sum_{i=1}^{3} \int_{\mathbb{T}^2} \frac{\bar{m}_i \bar{m}_4}{r_{i4}}\, d\delta_3 d\delta_4\,;$$

a function of order $O(\mu e^2)$, see [32, p. 405], and thus dominated by F_{res}. This allows us to evaluate $\bar{F}_{\text{sec},4}$ at the corresponding circular orbits of the inner three satellites, obtaining a function $\widehat{F}_{\text{sec},4}$ and arriving at an approximating system with Hamiltonian function

$$\widehat{F} \;=\; \widetilde{F}_{\text{Kep}} \;+\; F_{\text{res}} \;+\; \widehat{F}_{\text{sec},4}\,.$$

The point $(0,0)$ is an elliptic equilibrium of $\widehat{F}_{\text{sec},4}(\xi_4, \eta_4)$ with normal frequency

$$\nu_{n,5} = \mu \bar{m}_4 f(\bar{m}, a) \tag{31}$$

with f a function of the masses $\bar{m}_1, \bar{m}_2, \bar{m}_3$ and the semi-major axes a_1, a_2, a_3, a_4 ; see Claim 7.1 in [35] for the explicit expression. The periodic solution of the three inner satellites superposed with a circular orbit of the fourth satellite descends to a non-degenerate equilibrium of the reduced system obtained by dividing out the $\mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$–symmetry of shifting the angles δ_3, δ_4 (with the reduction of the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry of shifting η_3 already in place). For any N and small e, μ the non-degeneracy allows to continue the equilibrium from $\widehat{F}$ to the truncated normal form (30). When reconstructing this $\mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$–symmetry, the equilibrium corresponds to normally elliptic invariant 2-tori of (30) with a neighbourhood consisting of nearby librating Lagrangean tori of dimension 7.

In this situation the full strength of Theorem 7 can be applied — showing persistence of the invariant isotropic 2–tori (under small perturbation $O(\mu e^N + \mu^N)$ for sufficiently large N) and obtaining librational Lagrangean tori by excitation of normal modes. Next to the parameters for the inner three satellites as in § 5.1 we take $\bar{m}_4$ and D_4 as additional parameters, the latter distinguished with respect to the former. While the orbital motion of Callisto is fast, with frequency $\nu_{\text{Kep},4}$ of the same order as ν_{per}, the time scale of the normal frequency is of order μe^2. We use $\bar{m}_4$ to control the latter, see (31), and use for control of the former that $F_{\text{Kep},4}$ is non-degenerate with respect to D_4.

Theorem 10 (Invariant 2–tori and their librations). *In the Diophantine conditions* (18) *we take* $n = 2$*,* $r = 5$*,* $\tau > 6$ *and* $\gamma > 0$ *sufficiently small. Then there exists a bound* $e_0 > 0$ *on the eccentricities such that for* $\mu \ll e^3$ *and* $e \leq e_0$ *the following holds true.*

1. *Persistence of the invariant* 2*–tori: there exists a* C^∞*–diffeomorphism*

$$\Phi : \mathbb{T}^2 \times U \times V \times \Gamma \longrightarrow \mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}^{10} \times \Gamma$$

onto its image, where U *is a neighbourhood of* 0 *in* $\mathbb{R}^2$ *and* V *is a neighbourhood of* 0 *in* $\mathbb{R}^{10}$*, with the properties* 1–3 *of Theorem 7.*

2. *The* 5 *elliptic normal modes of these* 2*–tori give rise to a Cantor family of invariant Lagrangean* 7*–tori perturbed from* $\mathbb{T}^7 \times \{0\} \times \Gamma \subseteq \mathbb{T}^7 \times \mathbb{R}^7 \times \Gamma$ *where the Cantorisation results from the Diophantine conditions*

$$|\langle k, \omega\rangle + \langle \ell, \beta\rangle| \geq \frac{\gamma}{(|k| + |\ell|)^\tau},$$

for all $k \in \mathbb{Z}^2$ *and all* $\ell \in \mathbb{Z}^5$*.*

Proof. Again we apply Theorem 7. The constant δ in the specification (23) of the neighbourhood $\mathcal{U}$ is determined by Theorem 7. The constant γ has been specified before while the $m = 4$ time scales are $\varepsilon_1 = 1$, $\varepsilon_2 = \sqrt{\mu e}$, $\varepsilon_3 = \mu/e$ and $\varepsilon_4 = \mu e^2$. To achieve the smallness conditions (23) it again suffices to take the normalization order N sufficiently large. The necessary scaled BHT non-degeneracy condition follows from Lemma 8 together with (31) and

$$\det\left(\frac{\partial(\nu_{per}, \nu_{\mathrm{Kep},4})}{\partial(D_3, D_4)}\right) \neq 0$$

where the latter is a consequence of Kepler's third law. □

Remarks.

- In the 5–body setting the De Sitter periodic orbits are replaced by isotropic invariant 2–tori. By this we mean that the periodic orbits as found by De Sitter are close to the projection $(\delta_3, \delta_4) \mapsto \delta_3$ of these 2–tori.
- The quasi-periodic orbits with seven frequencies (and five rather small amplitudes) seem to provide a more realistic approximation of the actual motion of the Galilean satellites.
- When reconstructing the rotational $\mathrm{SO}(2,\mathbb{R})$–symmetry, returning from 7 to 8 degrees of freedom, an extra angle $\eta_3 = g_3$ is restored. As a consequence, the quasi-periodic 2–tori lift to conditionally periodic 3–tori and the librating Lagrangean 7–tori lift to librating Lagrangean 8–tori.
- By choosing γ as a suitable power of e, related to the bound $O(\mu e^N)$, we achieve that the measure of the complement of the persisting Lagrangean invariant tori scales with the size of the perturbation; compare with [5, 11].

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