ASYMMETRIC DIFFUSION AND THE ENERGY GAP ABOVE THE 111 GROUND STATE OF THE QUANTUM XXZ MODEL

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Abstract. We consider the anisotropic three dimensional XXZ Heisenberg ferromagnet in a cylinder with axis along the 111 direction and boundary conditions that induce ground states describing an interface orthogonal to the cylinder axis. Let L be the linear size of the basis of the cylinder. Because of the breaking of the continuous symmetry around the $\hat{z}$ axis, the Goldstone theorem implies that the spectral gap above such ground states must tend to zero as $L \to \infty$. In [3] it was proved that, by perturbing in a sub–cylinder with basis of linear size $R \ll L$ the interface ground state, it is possible to construct excited states whose energy gap shrinks as R^{-2}. Here we prove that, uniformly in the height of the cylinder and in the location of the interface, the energy gap above the interface ground state is bounded from below by const.L^{-2}. We prove the result by first mapping the problem into an asymmetric simple exclusion process on $\mathbb{Z}^3$ and then by adapting to the latter the recursive analysis to estimate from below the spectral gap of the associated Markov generator developed in [7]. Along the way we improve some bounds on the equivalence of ensembles already discussed in [3] and we establish an upper bound on the density of states close to the bottom of the spectrum.

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1. Introduction

In recent years there has been a great deal of investigation of the anisotropic spin $\frac{1}{2}$ XXZ Heisenberg model defined by

$$\mathcal{H}_\Lambda = - \sum_{\substack{x,y\in\Lambda:\\ |x-y|=1}} \frac{1}{\Delta}\big(S^1_x S^1_y + S^2_x S^2_y \big) + S^3_x S^3_y \; + \text{boundary conditions} \tag{1.1}$$

where $\Lambda \subset \mathbb{Z}^d$ and $\Delta > 1$ measures the anisotropy. Sometimes the parameter Δ is expressed as $\Delta = (q + q^{-1})/2$, $0 < q < 1$, and the classical Ising model is recovered in the limit $q \to 0$. We refer in particular the reader to [1, 21, 14, 15, 16, 19, 3] and [22].

As it is well known, the XXZ model has two ferromagnetically ordered translation invariant ground states, but also ground states that describe domain walls between regions of opposite sign of the spins. More precisely, for $d \geq 3$ and using a quantum version of the Pirogov–Sinai theory [6], it is possible to prove the existence of low temperature states describing an interface orthogonal to the `001` direction (a kind of Dobrushin state), provided that Δ is large enough. Quite surprisingly, and this is one of the main reasons for the increasing interest in such a model, the anisotropy is able, under certain circumstances, to stabilize a domain wall against quantum fluctuations even when, classically, thermal fluctuations are too strong to allow for a stable interface.

This is indeed the case for the so called `11, 111, ...` diagonal interfaces. The Ising model is not expected to have a Gibbs state describing a diagonal interface at low temperature because the zero temperature configurations compatible with the natural geometry and corresponding boundary conditions are enormously degenerate. A rigorous proof of such a result is available so far only in the solid–on–solid approximation thanks to results of [13]. On the other hand it has been shown independently in [1] and [12] that an appropriate choice of the boundary conditions in (1.1) can lead to ground state selection that favour a diagonal interface. Let us be a little bit more precise. For definiteness we set $d = 3$. We then take as domain Λ a cylinder with basis of linear size L, height H and axis along the `111` direction. The state of the system is described by vectors in the tensor product Hilbert space $\mathfrak{H} = (\mathbb{C}^2)^{\otimes|\Lambda|}$. Fix $\Delta > 1$ and define $A(\Delta) = \frac{1}{2}\sqrt{1 - \Delta^{-2}}$. Boundary condition are then introduced as follows. We let

$$\mathcal{H}_\Lambda = \sum_{b\in\mathcal{B}_\Lambda} \mathcal{H}_b \,, \tag{1.2}$$

where $\mathcal{B}_\Lambda$ is the set of oriented bonds of $\mathbb{Z}^3$ inside Λ, the single bond hamiltonians $\mathcal{H}^q_b$ are given by

$$\mathcal{H}_b = -\Delta^{-1}\left(S^1_{x_b}S^1_{y_b} + S^2_{x_b}S^2_{y_b}\right) - S^3_{x_b}S^3_{y_b} + A(\Delta)\left(S^3_{y_b} - S^3_{x_b}\right) + \frac{1}{4} \,,$$

and we write the bond b as $b = (x_b, y_b)$ if $\ell_{y_b} > \ell_{x_b}$, where $\ell_x = x_1 + x_2 + x_3$, $x = (x_1, x_2, x_3)$ is a signed distance to the origin. Notice that the terms $A(\Delta)\left(S^3_{y_b} - S^3_{x_b}\right)$ cancel everywhere except at the two basis of the cylinder and that the third component of the spin is a conserved quantity. The constant $\frac{1}{4}$ is there in order to have $\mathcal{H}_b \geq 0$.

The reason for the special choice of the coefficient $A(\Delta)$ comes mainly from the one dimensional system (see [14], [15] and [1]). For $d = 1$ ($L = 1$ in our language) and boundary coefficient $A(\Delta)$ the system enjoys a $SU_q(2)$ quantum group symmetry and the ground state degeneracy is equal to $H + 1$. If instead we take the boundary coefficient different from $A(\Delta)$ then the degeneracy is lifted. Moreover, in complete analogy with the exact computation of the ground state wave function of (1.2) in $d = 1$, one can show that, in each sector with $\sum_x S_x^3 = (2n - |\Lambda|)/2$, $n = 0, 1, \ldots, |\Lambda|$, there exists a unique ground state of H_Λ, denoted by ψ_n, with zero energy [1]. More precisely, with the convention that $|1\rangle$ and $|0\rangle$ stand for spin "up" and spin "down" respectively, the ground state ψ_n can be written as

$$\psi_n = \sum_{\substack{\alpha \in \Omega_\Lambda: \\ N_\Lambda(\alpha) = n}} \psi_n(\alpha) \bigotimes_{x \in \Lambda} |\alpha_x\rangle \tag{1.3}$$

where $\Omega_\Lambda := \{0,1\}^\Lambda$, $N_\Lambda(\alpha) := \sum_{x \in \Lambda} \alpha_x$ and

$$\psi_n(\alpha) = \prod_{x \in \Lambda} q^{\ell_x \alpha_x} \tag{1.4}$$

The square of the coefficients $\psi_n(\alpha)$ can be interpreted as the statistical weights of a (non translation invariant) canonical Gibbs measure for a lattice gas with n particles described by the variables $\{\alpha_x\}$. The typical configurations of such a measure form a sharply localized (depending on n) interface orthogonal to the `111` direction, separating a region almost filled with particles ($\alpha_x = 1$) from an almost empty region ($\alpha_x = 0$). That justifies the name "interface ground state" for the vector ψ_n. Because of the degeneracy of the ground states ψ_n, $n = 0, 1, \ldots |\Lambda|$, the continuous symmetry given by rotation around the z–axis is broken and therefore the spectrum above zero energy must be gapless in the thermodynamic limit (see [19]). That makes, in particular, any attempt to go beyond the zero temperature case quite hard. To the best of our knowledge the only model with a state describing a `111`-interface also at positive temperature is the Falicov–Kimball model [10].

The structure of the low-lying excitations above the interface ground states of (1.2) was recently studied in great detail in a series of interesting papers [3, 4, 5]. The main result in the above papers is that one can construct excitations localized in a sub–cylinder of Λ of radius $R \ll L$ such that their energy gap is smaller than $k\,R^{-2}$ for a certain constant $k = k(q)$. Moreover, in an appropriate scaling, the energy spectrum of such low–lying excitations coincides with the spectrum of the $d-1$ Laplacian on a suitable domain. An important ingredient in these works is an equivalence of ensembles result that can be roughly described as follows. If we replace in (1.3) the weights $\psi_n(\alpha)$ by their associated grand canonical weights obtained by adding a suitably chosen constant chemical potential $\lambda := \lambda(\Lambda, n)$ and if we remove the condition $N(\alpha) = n$, we obtain a new vector that we call grand canonical ground state and denote it by ψ^λ. Then, for any local observable X that commutes with the total third component of the spin, the difference between the two averages $\langle \psi_n, X\psi_n \rangle$ and $\langle \psi^\lambda, X\psi^\lambda \rangle$ vanishes as $L \to \infty$.

Let us now discuss our results. As pointed out in [3] it is generally believed that the energy of the lowest excitations in the `111`-cylinder Λ with height H and basis of linear size L, is not only bounded from above but also from below by $O(L^{-2})$, uniformly in H. Our main contribution is a proof of this lower bound on the energy gap, see Theorem 2.2. We also give a proof of the corresponding upper bound by making an ansatz similar to that of [3]. We should emphasize that in contrast to [3] we do not have a detailed control of the q–dependent prefactors in the estimates but rather focus on the uniformity in n (total third component of the spin) and H (height of the cylinder). Another result of this paper concerns an estimate on the density of states. Namely, we consider vectors ψ_n^f of the form

$$\psi_n^f = \sum_{\substack{\alpha\in\Omega_\Lambda:\\ N_\Lambda(\alpha)=n}} f(\alpha)\psi_n(\alpha)\bigotimes_{x\in\Lambda}|\alpha_x\rangle$$

where f is a local bounded function of the variables $\{\alpha_x\}_{x\in\Lambda}$ such that ψ_n^f is orthogonal to ψ_n. Then, using the lower bound on the spectral gap, we prove that the spectral measure $\rho_f(E)$ associated to the vector ψ_n^f satisfies $\rho_f(E)\le k_\varepsilon E^{1-\varepsilon}$ for any $\varepsilon>0$ as $E\to 0$, uniformly in $n\neq 0$, $|\Lambda|$ and in Λ (see Theorem 2.4). We believe that, in the above generality, a linear behaviour near the bottom of the spectrum is the correct one. Along the way we partially improve the equivalence of ensembles results of [3] (see section 3) and we provide a probabilistic proof of the known result ([15]) that the spectral gap for the linear chain XXZ is uniformly positive (but our bound is very rough compared with that of [15]).

We now briefly describe our approach. Let $\mathfrak{H}_n$ denote the sector of the Hilbert space $\mathfrak{H}$ with $\sum_{x\in\Lambda}\alpha_x = n$ and define the normalized states

$$\nu_n(\alpha) = \frac{\psi_n^2(\alpha)}{\sum_\eta \psi_n^2(\eta)}\,.$$

Using the positivity of the ground states ψ_n we may define a unitary transformation between $\mathfrak{H}_n$ and $L^2(\Omega_\Lambda,\nu_n)$ by formally multiplying by ψ_n^{-1}. This transforms $\mathcal{H}_{\Lambda,n}$, the restriction of $\mathcal{H}_\Lambda$ to $\mathfrak{H}_n$, into a new operator $\mathcal{G}_{\Lambda,n}$ on $L^2(\Omega_\Lambda,\nu_n)$. The latter turns out to be nothing but the Markov generator of an asymmetric simple exclusion process in Λ that can be roughly described as follows. We have n particles in Λ and each particle jumps to an empty neighbouring site with rate proportional to q if the signed distance from the origin is increased (by one) and to q^{-1} if it is decreased. The number of particles is a conserved quantity and by construction the measure ν_n is reversible for the process since $\mathcal{G}_{\Lambda,n}$ is self adjoint in $L^2(\Omega_\Lambda,\nu_n)$. The spectral gap of $\mathcal{G}_{\Lambda,n}$ coincides with the spectral gap of $\mathcal{H}_{\Lambda,n}$ and it accounts for the smallest rate of exponential decay to equilibrium for the above process in $L^2(\Omega_\Lambda,\nu_n)$. Note that the isotropic case $q=1$ is the usual symmetric simple exclusion process. Although we discovered such an equivalence independently, we realized later on that it was well known to physicists since some years [2].

Once the problem has become a kind of reversible Kawasaki dynamics for a classical lattice gas, we adapt to it some recent work [7] (see also [18] for a different approach) to bound from below its spectral gap, recursively in L. Although our asymmetric simple exclusion has certain advantages over a high temperature truly interacting lattice gas because its grand canonical measure is product, nevertheless several new problems arise, particularly if one looks for results uniform in n, H, because of the unboundedness of the signed distance ℓ_x entering in the canonical measure ν_n.

As a final remark we observe that all our results are restricted to spin $\frac{1}{2}$. For higher spins one can still compute exactly the ground state (see [1]) for a suitable choice of the boundary conditions and, as described above, it is possible to unitarily transform the Hamiltonian into a Markov generator. The interacting particle process one gets in this way is however more involved than the one considered here. Particles of different kind (namely different spin) appear and, besides the usual asymmetric simple exclusion process, new transitions are allowed in which pairs of particles of opposite spin are created or destroyed with certain rates (see [2]). We plan to analyze this new situation in a near future.

We conclude with a road map of the paper.

- In section 2 we fix the model, define the unitary transformation leading to the Markov generator and state the main results.
- In section 3 we provide a series of technical tools including the results on the equivalence of ensembles.
- In section 4 we describe the recursive approach to prove the lower bound on the spectral gap by assuming a key result that one may call "transport theorem" (see Theorem 4.1). We also prove a lower bound on the gap in one dimension uniformly in the number of "up" spins and in the height H.
- In section 5 we prove the transport theorem.
- Finally in section 6 we prove the upper bound on the spectral gap and the result on the spectral measure of local perturbations of the ground state.

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2. Setup and Main Results

2.1. **Lattice, bonds, 111-planes, sticks and cylinders.** We consider the 3D integer lattice $\mathbb{Z}^3$, and denote e_i, $i = 1, \dots, 3$ the unit vectors in the i-th direction. For any $x \in \mathbb{Z}^3$ we write $x_i = x \cdot e_i$ for the i-th coordinate of x and denote by ℓ_x the signed distance from the origin

$$\ell_x = x_1 + x_2 + x_3 \,.$$

A *bond* in $\mathbb{Z}^3$ is an oriented couple $b = (x, y)$, where $x, y \in \mathbb{Z}^3$ are neighbours, i.e. $\|x-y\|_1 = 1$ with $\|x\|_1 = |x_1| + |x_2| + |x_3|$. We denote $\mathbb{Z}^{3^*}$ the set of all bonds. A given $b \in \mathbb{Z}^{3^*}$ identifies two sites $x_b, y_b \in \mathbb{Z}^3$ such that $b = (x_b, y_b)$. For any subset $\Lambda \subset \mathbb{Z}^d$ we call Λ^* the set of $b \in \mathbb{Z}^{3^*}$, such that $x_b, y_b \in \Lambda$. For any b we have $\ell_{x_b} - \ell_{y_b} = \pm 1$. We choose an orientation according to increasing values of ℓ and denote

$$\mathcal{B}_\Lambda = \big\{ b \in \Lambda^* : \;\; \ell_{y_b} = \ell_{x_b} + 1 \big\} \,.$$

Given $h \in \mathbb{Z}_+$ we call $\mathcal{A}_h$ the 111-*plane* at height h, i.e.

$$\mathcal{A}_h = \big\{ x \in \mathbb{Z}^3 : \;\; \ell_x = h \big\} \,.$$

We define the infinite *stick* Σ_∞ passing through the origin as the doubly infinite sequence

$$\dots, -e_1 - e_2 - e_3, -e_2 - e_3, -e_3, 0, e_1, e_1 + e_2, e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_1, \dots$$

We write $\Sigma_{x,\infty}$ for the infinite stick going through x, i.e. $\Sigma_{x,\infty} = x + \Sigma_\infty$. Note that the union of $\Sigma_{x,\infty}$, $x \in \mathcal{A}_0$ covers all of $\mathbb{Z}^3$. For every positive integer H we define the finite stick

$$\Sigma_H = \big\{ y \in \Sigma_\infty : \quad \ell_y \in \{0, 1, 2, \dots, H-1\} \big\} \,.$$

The finite stick through x is then $\Sigma_{x,H} = x + \Sigma_H$. When no confusion arises we shall simply write Σ_x for a generic finite stick at x. We will often consider cylindrical subsets of $\mathbb{Z}^3$ of the type

$$\Sigma_{\Gamma,H} = \bigcup_{x \in \Gamma} \Sigma_{x,H} \,, \quad \Gamma \subset \mathcal{A}_0 \,,$$

with some finite $\Gamma \subset \mathcal{A}_0$, called the basis. Then $\Sigma_{\Gamma,H}$ contains $H|\Gamma|$ sites, $|\Gamma|$ being the cardinality of Γ. On the plane $\mathcal{A}_0$ it is convenient to parametrize sites as follows. Consider the two vectors $P_u = (1, -1, 0)$ and $P_v = (0, 1, -1)$. Then any $x \in \mathcal{A}_0$ is uniquely determined by a couple of integers (x_u, x_v) with $x = x_u P_u + x_v P_v$. We consider *tilted rectangles*

$$R_{L,M} = \left\{x \in \mathcal{A}_0 : \ x_u \in \{0,1,\dots,L-1\}, x_v \in \{0,1,\dots,M-1\}\right\}.$$

In this way $|R_{L,M}| = LM$. When $L = M$ we call $Q_L = R_{L,L}$ a *tilted square.* Corresponding cylinders $\Sigma_{Q_L,H}$ are denoted $\Sigma_{L,H}$. Note that there are no true neighbours on $\mathcal{A}_0$. We say that two sites x, y are neighbours in $\mathcal{A}_0$ if $x, y \in \mathcal{A}_0$ and $|x_u - y_u| + |x_v - y_v| = 1$.

2.2. **Interface ground states of the XXZ model.** Consider a cylinder $\Lambda = \Sigma_{\Gamma,H}$ for some $\Gamma \subset \mathcal{A}_0$, $H \in \mathbb{Z}_+$. The state of the system is described by vectors in the tensor product Hilbert space $\mathfrak{H} = (\mathbb{C}^2)^{\otimes|\Lambda|}$. Fix $q \in (0,1)$ and define

$$\Delta = \frac{1}{2}(q + q^{-1}), \quad A(\Delta) = \frac{1}{2}\sqrt{1 - \Delta^{-2}}.$$

The Hamiltonian operator is defined by

$$\mathcal{H}_\Lambda = \sum_{b \in \mathcal{B}_\Lambda} \mathcal{H}_b^q, \tag{2.1}$$

where single bond hamiltonians $\mathcal{H}_b^q$, $b = (x_b, y_b)$ are given

$$\mathcal{H}_b^q = -\Delta^{-1}\left(S_{x_b}^1 S_{y_b}^1 + S_{x_b}^2 S_{y_b}^2\right) - S_{x_b}^3 S_{y_b}^3 + A(\Delta)\left(S_{y_b}^3 - S_{x_b}^3\right) + \frac{1}{4}. \tag{2.2}$$

Here the spin operators S_x^i, $i = 1,2,3$, are the Pauli matrices

$$S_x^1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad S_x^2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \quad S_x^3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Expressions (2.1) and (2.2) give the usual XXZ Hamiltonian, with the term proportional to $A(\Delta)$ accounting for boundary conditions which favour 111-interface states. The term $1/4$ has been introduced so that ground states have zero energy, see below.

We choose a basis for $\mathfrak{H}$ labeled by the two states "up" or "down" of the third component of the spin at each site, and write it in terms of configurations $\alpha = \{\alpha_x\}_{x\in\Lambda}$, with $\alpha_x \in \{0,1\}$ with the convention that $\alpha_x = 1$ stands for spin "up" while $\alpha_x = 0$ stands for spin "down". $\Omega_\Lambda = \{0,1\}^\Lambda$ denotes the set of all configurations and $|\alpha\rangle = \prod_{x\in\Lambda} |\alpha_x\rangle$ stands for a generic basis vector. For every $\varphi \in \mathfrak{H}$ we write

$$\varphi(\alpha) = \langle \alpha | \varphi \rangle .$$

Since $\mathcal{H}_b^q$ only acts on the bond b, a simple computation shows that

$$\mathcal{H}_b^q |\alpha\rangle = (q + q^{-1})^{-1}\left\{q^{\alpha_{x_b} - \alpha_{y_b}} |\alpha\rangle - |\alpha^b\rangle\right\}, \tag{2.3}$$

where $\alpha^b := T_{x_b,y_b}\alpha$, and for a generic pair x, y, $T_{x,y}\alpha$ denotes the configuration in which α_x and α_y have been exchanged,

$$(T_{x,y}\alpha)_z = \begin{cases} \alpha_y & z = x \\ \alpha_x & z = y \\ \alpha_z & \text{otherwise} \end{cases}$$

In particular, formula (2.3) shows that if $\alpha = \alpha^b$, then $\mathcal{H}_b^q |\alpha\rangle = 0$. Moreover, $\mathcal{H}_b^q = |\xi\rangle\langle\xi|$ is a projection onto the vector $\xi = \xi_b^q$ with

$$\xi(\alpha) = \frac{1}{\sqrt{1+q^2}}\left\{q\alpha_{x_b}(1 - \alpha_{y_b}) - (1 - \alpha_{x_b})\alpha_{y_b}\right\}.$$

Let $\mathcal{N}_\Lambda$ denote the operator

$$\mathcal{N}_\Lambda \left|\alpha\right\rangle = N_\Lambda(\alpha) \left|\alpha\right\rangle \,, \quad N_\Lambda(\alpha) = \sum_{x\in\Lambda} \alpha_x \,. \tag{2.4}$$

From (2.3) we see that $\mathcal{H}_\Lambda$ commutes with $\mathcal{N}_\Lambda$. We divide $\mathfrak{H}$ in $|\Lambda|+1$ sectors corresponding to all possible values of N_Λ. Namely, given $\varphi \in \mathfrak{H}$ we write

$$|\varphi\rangle = \sum_{n=0}^{|\Lambda|} |\varphi_n\rangle \,, \quad |\varphi_n\rangle = \sum_{\substack{\alpha\in\Omega_\Lambda:\\ N_\Lambda(\alpha)=n}} \varphi(\alpha)\,|\alpha\rangle \,.$$

In this way $\mathfrak{H}$ is unitarily equivalent to the direct sum $\oplus_n \mathfrak{H}_n$, where $\mathfrak{H}_n$ is the closed subspace of $\mathfrak{H}$ spanned by all vectors $|\alpha\rangle$ with $N_\Lambda(\alpha) = n$. Now, ground states for the Hamiltonian (2.1) are vectors ψ in $\mathfrak{H}$ such that $\mathcal{H}_\Lambda |\psi\rangle = 0$. As in [1], [3] and [4], in each sector $\mathfrak{H}_n$, $n = 0, 1, \ldots, |\Lambda|$, there is a unique ground state ψ_n given by

$$\psi_n(\alpha) = \begin{cases} \prod_{x\in\Lambda} q^{\ell_x \alpha_x} & N_\Lambda(\alpha) = n \\ 0 & N_\Lambda(\alpha) \neq n \end{cases} \tag{2.5}$$

We shall interpret ψ_n^2 as the weights of a *canonical* probability distribution ν_n on Ω_Λ, by writing

$$\nu_n(f) = \sum_{\alpha\in\Omega_\Lambda} \nu_n(\alpha) f(\alpha)\,, \quad f : \Omega_\Lambda \to \mathbb{R}\,,$$

with

$$\nu_n(\alpha) = \frac{\psi_n^2(\alpha)}{\sum_{\eta\in\Omega_\Lambda} \psi_n^2(\eta)}\,. \tag{2.6}$$

It is convenient to introduce the corresponding *grand canonical* distributions. For every $\lambda \in \mathbb{R}$ we define the product measure μ^λ on Ω_Λ given by

$$\mu^\lambda(f) = \sum_{\alpha\in\Omega_\Lambda} \mu^\lambda(\alpha) f(\alpha)\,, \quad \mu^\lambda(\alpha) = \prod_{x\in\Lambda} \frac{q^{2(\ell_x-\lambda)\alpha_x}}{1+q^{2(\ell_x-\lambda)}}\,. \tag{2.7}$$

For every $\lambda \in \mathbb{R}$, ν_n can be obtained from μ^λ by conditioning on $N_\Lambda(\alpha) = n$, i.e.

$$\nu_n = \mu^\lambda(\,\cdot\,|N_\Lambda(\alpha) = n)\,. \tag{2.8}$$

To avoid confusion we sometimes write explicitly the region Λ we are considering and use the notations $\nu_{\Lambda,n}$ and μ_Λ^λ instead of ν_n and μ^λ. We shall adopt the standard notation for the variance and covariances w.r.t. a measure μ:

$$\mathrm{Var}_\mu(f) = \mu(f,f) = \mu\big((f-\mu(f))^2\big)\,, \quad \mu(f,g) = \mu\big((f-\mu(f))(g-\mu(g))\big)\,. \tag{2.9}$$

2.3. **The spectral gap.** We call $\mathrm{gap}(\mathcal{H}_\Lambda)$ the energy of the first excited state of $\mathcal{H}_\Lambda$. Let us write $\mathcal{H}_{\Lambda,n}$ for the restriction of $\mathcal{H}_\Lambda$ to the sector $\mathfrak{H}_n$. For each n we define the gap

$$\mathrm{gap}(\mathcal{H}_{\Lambda,n}) = \inf_{\substack{0\neq\varphi\in\mathfrak{H}_n:\\ \langle\varphi|\psi_n\rangle=0}} \frac{\langle\varphi|\,\mathcal{H}_{\Lambda,n}\,|\varphi\rangle}{\langle\varphi|\varphi\rangle}\,. \tag{2.10}$$

We then have

$$\mathrm{gap}(\mathcal{H}_\Lambda) = \min_n \mathrm{gap}(\mathcal{H}_{\Lambda,n})\,. \tag{2.11}$$

2.4. **Ground state transformation.** For each n we consider now the Hilbert space $\widetilde{\mathfrak{H}}_n := L^2(\Omega_\Lambda, \nu_n)$ with scalar product

$$\langle \varphi, \psi \rangle_{\nu_n} = \sum_{\alpha \in \Omega_\Lambda} \nu_n(\alpha) \overline{\varphi}(\alpha) \psi(\alpha) \,. \tag{2.12}$$

The ground state transformation is defined by the unitary map

$$\mathcal{U}_n : \mathfrak{H}_n \to \widetilde{\mathfrak{H}}_n \,, \quad \varphi \to \mathcal{U}_n \varphi \,,$$

where, for every $\alpha \in \Omega_\Lambda$ with $N_\Lambda(\alpha) = n$,

$$\big(\mathcal{U}_n \varphi\big)(\alpha) = (\nu_n(\alpha))^{-1/2} \varphi(\alpha) \,. \tag{2.13}$$

Let us define the operator $\mathcal{G}_{\Lambda,n}$ on $\widetilde{\mathfrak{H}}_n$ given by

$$\mathcal{G}_{\Lambda,n} f(\alpha) = \frac{1}{(q+q^{-1})} \sum_{b \in \mathcal{B}_\Lambda} \frac{\psi_n(\alpha^b)}{\psi_n(\alpha)} \Big[f(\alpha^b) - f(\alpha) \Big] \,. \tag{2.14}$$

A simple computation shows that $-\mathcal{G}_{\Lambda,n}$ is a symmetric, non-negative operator with

$$\langle f, (-\mathcal{G}_{\Lambda,n}) f \rangle_{\nu_n} = \frac{1}{2(q+q^{-1})} \sum_{\alpha \in \Omega_\Lambda} \sum_{b \in \mathcal{B}_\Lambda} \nu_n(\alpha) \frac{\psi_n(\alpha^b)}{\psi_n(\alpha)} \Big| f(\alpha^b) - f(\alpha) \Big|^2 \,. \tag{2.15}$$

Moreover, $\mathcal{G}_{\Lambda,n}\mathbf{1}= 0$ with $\mathbf{1}$ denoting the constant $f \equiv 1$. We may define the gap in the spectrum of $-\mathcal{G}_{\Lambda,n}$ as

$$\mathrm{gap}(\mathcal{G}_{\Lambda,n}) = \inf_{0 \neq f \perp \mathbf{1}} \frac{\langle f, (-\mathcal{G}_{\Lambda,n}) f \rangle_{\nu_n}}{\langle f, f \rangle_{\nu_n}} \,. \tag{2.16}$$

Here the orthogonality $f \perp \mathbf{1}$ means $\nu_n(f) = 0$. The next proposition motivates the introduction at this stage of the operator $\mathcal{G}_{\Lambda,n}$ and of its spectral gap.

Proposition 2.1. *For every finite $\Lambda \subset \mathbb{Z}^3$, for every $n = 0, 1, \dots, |\Lambda|$, we have the identity*

$$\mathcal{H}_{\Lambda,n} = \mathcal{U}_n^{-1} (-\mathcal{G}_{\Lambda,n}) \mathcal{U}_n \,. \tag{2.17}$$

In particular, $\mathrm{gap}(\mathcal{G}_{\Lambda,n}) = \mathrm{gap}(\mathcal{H}_{\Lambda,n})$.

Proof. If (2.17) holds we see that for any $\varphi \in \mathfrak{H}_n$,

$$\langle \varphi | \, \mathcal{H}_{\Lambda,n} \, | \varphi \rangle = \langle \mathcal{U}_n \varphi, (-\mathcal{G}_{\Lambda,n}) \mathcal{U}_n \varphi \rangle_{\nu_n} \,. \tag{2.18}$$

From this $\mathrm{gap}(\mathcal{G}_{\Lambda,n}) = \mathrm{gap}(\mathcal{H}_{\Lambda,n})$ follows since

$$\langle \varphi | \psi_n \rangle = 0 \iff \nu_n(\mathcal{U}_n \varphi) = 0 \,.$$

We turn to the proof of (2.17). Let $\widetilde{\psi}_n(\alpha) = \sqrt{\nu_n(\alpha)}$, so that $\varphi / \widetilde{\psi}_n = \mathcal{U}_n \varphi$. Observe that for every $b \in \mathcal{B}_\Lambda$ we have $\ell_{y_b} = \ell_{x_b} + 1$ and therefore

$$\widetilde{\psi}_n(\alpha^b) = q^{\alpha_{x_b} - \alpha_{y_b}} \widetilde{\psi}_n(\alpha) \,. \tag{2.19}$$

From (2.1) and (2.3) we see that

$$\begin{aligned} \langle \alpha | \, \mathcal{H}_\Lambda \, | \varphi \rangle &= \frac{1}{(q+q^{-1})} \sum_{b \in \mathcal{B}_\Lambda} \Big[q^{\alpha_{x_b} - \alpha_{y_b}} \varphi(\alpha) - \varphi(\alpha^b) \Big] \\ &= \frac{1}{(q+q^{-1})} \sum_{b \in \mathcal{B}_\Lambda} \widetilde{\psi}_n(\alpha^b) \Big[\big(\mathcal{U}_n \varphi\big)(\alpha) - \big(\mathcal{U}_n \varphi\big)(\alpha^b) \Big] \\ &= \widetilde{\psi}_n(\alpha) \big[(-\mathcal{G}_{\Lambda,n}) \mathcal{U}_n \varphi \big](\alpha) = [\mathcal{U}_n^{-1} (-\mathcal{G}_{\Lambda,n}) \mathcal{U}_n] \varphi(\alpha) \,. \end{aligned} \tag{2.20}$$

Then (2.20) proves the claim. ☐

2.5. **Asymmetric exclusion process.** The operator $\mathcal{G}_{\Lambda,n}$ in (2.14) can be interpreted as the generator of an interacting particle system (see e.g. [17] for a general reference). We define

$$\nabla_{xy} f(\alpha) := f(T_{x,y}\alpha) - f(\alpha)\,, \qquad \nabla_b f(\alpha) := \nabla_{x_b y_b} f\,. \tag{2.21}$$

Let also

$$c_b(\alpha) = \frac{q^{\alpha_{x_b}-\alpha_{y_b}}}{q+q^{-1}}\,, \quad b = (x_b, y_b)\,. \tag{2.22}$$

Then (2.14) may be rewritten

$$\mathcal{G}_{\Lambda,n} f(\alpha) = \sum_{b\in\mathcal{B}_\Lambda} c_b(\alpha)\nabla_b f(\alpha)\,. \tag{2.23}$$

For every n, this defines a Markov Process with n particles in Λ jumping to empty neighbouring sites. The rate of a jump is proportional to q if a particle moves from ℓ_x to $\ell_y = \ell_x + 1$, and to q^{-1} if it moves from ℓ_y to ℓ_x. The number of particles is conserved and the measure ν_n is reversible for the process since $\mathcal{G}_{\Lambda,n}$ is self adjoint in $L^2(\Omega_\Lambda, \nu_n)$.

Consider a cylinder $\Lambda := \Sigma_{L,H}$ of height H and whose 111-section $\Sigma_{L,H} \cap \mathcal{A}_0$ is a tilted square Q_L containing L^2 sites. Since the degenerate cases $n = 0$ and $n = HL^2$ are trivial (ν_n is simply a delta on the empty/full configuration), the variable n will be assumed to range from 1 and $HL^2 - 1$ in all statements below. Our main results can be stated as follows.

Theorem 2.2. *For any $q \in (0,1)$ there exists a constant $k \in (0,\infty)$ such that for every positive integer L of the form $L = 2^j$ for some $j \in \mathbb{N}$*

$$\inf_{H,n} \operatorname{gap}(\mathcal{G}_{\Sigma_{L,H},n}) \geqslant k^{-1} L^{-2}\,, \tag{2.24}$$

$$\sup_{H,n} \operatorname{gap}(\mathcal{G}_{\Sigma_{L,H},n}) \leqslant k L^{-2}\,. \tag{2.25}$$

Remark 2.3. The proof of the lower bound (2.24) is based on a recursive analysis very similar to that of [7]. The problem here is in many aspects simpler in view of the product structure of the underlying grand canonical measure. On the other hand the approach of [7] has to be modified in several important points because of the asymmetry in the 111 direction. In the next subsection we will reformulate the problem in a more convenient fashion.

The second result concerns the behaviour of the spectral measure associated to suitable local functions near the bottom of the spectrum.

Theorem 2.4. *In the same setting of Theorem 2.2, let f be a bounded function of zero mean w.r.t. ν_n and such that its support is contained in a sub-cylinder $\Lambda_0 := \Sigma_{L_0,H}$. Let E_s denote the spectral projection of the operator $-\mathcal{G}_{\Lambda,n}$ associated to the interval $[0,s]$. Then, for any $q \in (0,1)$ and any $\epsilon > 0$ there exists a constant k_ϵ depending on ϵ, L_0 and $\|f\|_\infty$ such that*

$$\sup_{\Lambda,n} \langle f, E_s f\rangle_{\nu_{\Lambda,n}} \leq k_\epsilon\, s^{1-\epsilon} \tag{2.26}$$

Remark 2.5. Theorem 2.4 can be obviously formulated also for the quantum Hamiltonian $\mathcal{H}_n$ thanks to the unitary equivalence stated in Proposition 2.1. In this context the result is as follows. Consider the vector ψ_n^f in the Hilbert space $\mathfrak{H}_n$ defined by $\psi_n^f(\alpha) = f(\alpha)\psi_n(\alpha)$, with f as in the theorem. Then the spectral measure of ψ_n^f has an almost linear bound close to the bottom of the spectrum of $\mathcal{H}_n$

2.6. **From tilted to straight shapes.** In order to avoid unnecessary complications coming from the tilted geometry of our setting we shall make the following simple transformation which allows us to go from the 111-cylinders described above to more familiar straight cylinders in $\mathbb{Z}^3$, with axis along one coordinate axis. Recall that any point $x \in \mathbb{Z}^3$ is identified by the triple (x_u, x_v, ℓ_x), where $\ell_x = x_1 + x_2 + x_3$ and the pair (x_u, x_v) specifies the projection of x onto the $\mathcal{A}_0$ plane, obtained as the intersection $\Sigma_x \cap \mathcal{A}_0$. We then have an isomorphism $\Phi : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by

$$(\Phi x)_1 = x_u\,, \quad (\Phi x)_2 = x_v\,, \quad (\Phi x)_3 = \ell_x \tag{2.27}$$

The map Φ brings the 111-cylinder $\Sigma_{L,H}$ into the straight cylinder

$$\Phi\Sigma_{L,H} = \{0, 1, \dots, L-1\}^2 \times \{0, 1, \dots, H-1\}\,.$$

We now introduce a new exclusion process, with n particles in a given $\Lambda \subset \mathbb{Z}^3$, jumping to empty neighbouring sites. A jump in the horizontal direction occurs with rate 1 while in the vertical direction it occurs with rate q or q^{-1} if the particle is going upwards or downwards, respectively. The asymmetry of the original process along the 111 direction becomes here an asymmetry along the 001 direction (the third axis). Consider the set of oriented bonds Λ^*. We choose an arbitrary orientation for the horizontal bonds, which we denote $\mathcal{O}_\Lambda$. $\mathcal{O}_\Lambda$ can be taken to be the set of couples $b = (x, y) \in \Lambda^*$ such that $x_3 = y_3$, $y_1 \geqslant x_1$ and $y_2 \geqslant x_2$. For vertical bonds, which we denote $\mathcal{V}_\Lambda$, we choose the orientation according to increasing values of the third component. Thus

$$\mathcal{V}_\Lambda = \big\{ b = (x, y) \in \Lambda^* : \quad y_3 = x_3 + 1 \big\}\,.$$

The generator of our new process can be written

$$\mathcal{L}_{\Lambda,n} f(\alpha) = \sum_{b \in \mathcal{O}_\Lambda} \nabla_b f(\alpha) + \sum_{b \in \mathcal{V}_\Lambda} q^{\alpha_{x_b} - \alpha_{y_b}} \nabla_b f(\alpha)\,. \tag{2.28}$$

The generator $\mathcal{L}_{\Lambda,n}$ is symmetric in $L^2(\Omega_\Lambda, \tilde{\nu}_{\Lambda,n})$, where now $\tilde{\nu}_{\Lambda,n}$ is again given by (2.5) and (2.6) but we interpret ℓ_x as the third coordinate x_3. The Dirichlet form associated to this process is defined by

$$\mathcal{E}_{\Lambda,n}(f, f) = \mathcal{E}_n^{\mathcal{O}_\Lambda}(f, f) + \mathcal{E}_n^{\mathcal{V}_\Lambda}(f, f)\,, \tag{2.29}$$

where $f : \Omega_\Lambda \to \mathbb{R}$ and

$$\mathcal{E}_n^{\mathcal{O}_\Lambda}(f, f) = \frac{1}{2} \sum_{\alpha \in \Omega_\Lambda} \tilde{\nu}_{\Lambda,n}(\alpha) \sum_{b \in \mathcal{O}_\Lambda} \big(\nabla_b f(\alpha)\big)^2\,, \tag{2.30}$$

$$\mathcal{E}_n^{\mathcal{V}_\Lambda}(f, f) = \frac{1}{2} \sum_{\alpha \in \Omega_\Lambda} \tilde{\nu}_{\Lambda,n}(\alpha) \sum_{b \in \mathcal{V}_\Lambda} q^{\alpha_{x_b} - \alpha_{y_b}} \big(\nabla_b f(\alpha)\big)^2\,. \tag{2.31}$$

We have the following simple relation between the old process on Λ and the new process on $\Phi\Lambda$. Given $\Lambda \subset \mathbb{Z}^3$ and $f : \Omega_\Lambda \to \mathbb{R}$, set $\tilde{\Lambda} = \Phi\Lambda$ and $\tilde{f}(\alpha) = f(\alpha \circ \Phi^{-1})$.

Lemma 2.6. *For every $q \in (0,1]$ there exists $k < \infty$ such that for every Λ and $n = 0,1,\dots,|\Lambda|$*

$$k^{-1}\langle f,(-\mathcal{G}_{\Lambda,n})f\rangle_{\nu_n} \leqslant \mathcal{E}_{\tilde\Lambda,n}(\tilde f,\tilde f) \leqslant k\,\langle f,(-\mathcal{G}_{\Lambda,n})f\rangle_{\nu_n} \tag{2.32}$$

Proof. To prove the second estimate observe that by (2.15)

$$\mathcal{E}_n^{\mathcal{V}_{\tilde\Lambda}}(\tilde f,\tilde f) \leqslant (q+q^{-1})\langle f,(-\mathcal{G}_{\Lambda,n})f\rangle_{\nu_n}\,,$$

since for any $\tilde b \in \mathcal{V}_{\tilde\Lambda}$, $\tilde b = (x,y)$, we have $b := (\Phi^{-1}x,\Phi^{-1}y) \in \mathcal{B}_\Lambda$ and the rates coincide apart from the factor $(q+q^{-1})$. Therefore we only have to control the horizontal part (2.30). Let us fix an horizontal bond $\tilde b \in \mathcal{O}_{\tilde\Lambda}$, $\tilde b = (x,y)$. The point is that $b := (\Phi^{-1}x,\Phi^{-1}y)$ is not a true bond in $\mathcal{B}_\Lambda$ (since $\ell_{\Phi^{-1}x} = \ell_{\Phi^{-1}y}$), but we can find $b_1, b_2 \in \mathcal{B}_\Lambda$ such that, with the notation $T_b\alpha = \alpha^b$ one has

$$\alpha^b = T_{b_1}T_{b_2}T_{b_1}\alpha\,,\quad \nabla_b f(\alpha) = \nabla_{b_1} f(T_{b_2}T_{b_1}\alpha) + \nabla_{b_2} f(T_{b_1}\alpha) + \nabla_{b_1} f(\alpha)\,.$$

Observing that $c_b \geqslant q/(q+q^{-1})$ and that changes of measures give at most an additional factor q^{-2}, e.g. $\nu_n(\alpha) \leqslant q^{-2}\nu_n(T_{b_1}\alpha)$ for any α, we can estimate

$$\begin{aligned}
\sum_{\alpha\in\Omega_{\tilde\Lambda}} \tilde\nu_{\tilde\Lambda,n}(\alpha)\big(\nabla_{\tilde b}\tilde f(\alpha)\big)^2 &= \sum_{\alpha\in\Omega_\Lambda} \nu_{\Lambda,n}(\alpha)(\nabla_b f(\alpha))^2\\
&\leqslant 3\sum_{\alpha\in\Omega_\Lambda} \nu_{\Lambda,n}(\alpha)\big[(\nabla_{b_1} f(T_{b_2}T_{b_1}\alpha))^2 + (\nabla_{b_2} f(T_{b_1}\alpha))^2 + (\nabla_{b_1} f(\alpha))^2\big]\\
&\leqslant 3q^{-3}(q+q^{-1})\sum_{\alpha\in\Omega_\Lambda} \nu_{\Lambda,n}(\alpha)\big[2c_{b_1}(\alpha)(\nabla_{b_1} f(\alpha))^2 + c_{b_2}(\alpha)(\nabla_{b_2} f(\alpha))^2\big]\,.
\end{aligned}$$

Summing over $\tilde b \in \mathcal{O}_{\tilde\Lambda}$ we obtain

$$\mathcal{E}_n^{\mathcal{O}_{\tilde\Lambda}}(\tilde f,\tilde f) \leqslant 6q^{-3}(q+q^{-1})\langle f,(-\mathcal{G}_{\Lambda,n})f\rangle_{\nu_n}\,.$$

To prove the first inequality in (2.32) we repeat the same reasoning, observing that for every bond in $b \in \mathcal{B}_\Lambda$ either b is along a single stick in which case the bound is straightforward since $\tilde b \in \mathcal{V}_{\tilde\Lambda}$ ($\tilde b$ is the image of b under Φ), or b connects two different sticks. In the latter case there are $\tilde b_1 \in \mathcal{O}_{\tilde\Lambda}$ and $\tilde b_2 \in \mathcal{V}_{\tilde\Lambda}$ such that the exchange across b can be realized by successive exchanges across $\tilde b_1$ and $\tilde b_2$ and the above arguments apply. ∎

Thanks to Lemma 2.6 we will obtain Theorem 2.2 as a consequence of the following

Theorem 2.7. *For any $q \in (0,1)$ there exists a constant $k \in (0,\infty)$ such that for every positive integer L of the form $L = 2^j$, $j \in \mathbb{N}$*

$$\inf_{H,n} \operatorname{gap}(\mathcal{L}_{\Sigma_{L,H},n}) \geqslant k^{-1}L^{-2}\,, \tag{2.33}$$

$$\sup_{H,n} \operatorname{gap}(\mathcal{L}_{\Sigma_{L,H},n}) \leqslant kL^{-2}\,. \tag{2.34}$$

Remark 2.8. Since there is complete symmetry between particles ($\alpha_x = 1$) and holes ($\alpha_x = 0$), for any Λ and any $n = 0,1,\dots,|\Lambda|$ we have

$$\operatorname{gap}(\mathcal{L}_{\Lambda,n}) = \operatorname{gap}(\mathcal{L}_{\Lambda,|\Lambda|-n})\,. \tag{2.35}$$

Convention. In the rest of the paper we work in the straight geometrical setting described above. With some abuse we keep all the notations unchanged and write ℓ_x for the third coordinate x_3. In this way sets $\mathcal{A}_h$ are now horizontal planes, Σ_x denotes a vertical stick, $\Sigma_{\Gamma,H}$ denotes a straight cylinder, $R_{L,M}$ denotes a rectangle on the plane $\mathcal{A}_0$ and so on. Moreover, the probability measure $\tilde{\nu}_n$ will be simply written ν_n, so that ν_n and μ^λ are defined as in (2.6) and (2.7) provided ℓ_x stands for x_3.

3. Preliminary Results

In this section we collect several preliminary technical results that will enter at different stages in the proof of our two main results. As a rule, in what follows k denotes a positive finite constant depending only on q, whose value may change from line to line.

3.1. **Mean and variance of the number of particles.** In this first paragraph we give some elementary bounds on the statistics of the number of particles in a stick and on the chemical potential as a function of the mean number of particles. Part of the results discussed below have already been derived with more accurate constants in [3] in the case of an interface sitting roughly in the middle of the cylinder. Here we need results that are uniform in the location of the interface.

Let us consider a single stick of height H, $\Lambda := \Sigma_H$, and the grand canonical measure μ^λ on Λ, $\lambda \in \mathbb{R}$. We have the following simple relations between λ and $m(\lambda) := \mu^\lambda(N_\Lambda)$, the mean number of particles in Λ.

Lemma 3.1. *For each $q \in (0,1)$ there exists $k < \infty$ such that for every $H \geqslant 1$*

$$(\lambda - k) \vee \frac{\lambda}{2} \leqslant m(\lambda) \leqslant k + \lambda \qquad \text{if } \lambda \in [0, H-1] \tag{3.1}$$

$$\frac{1}{k}\, q^{2|\lambda|} \leqslant m(\lambda) \leqslant k\, q^{2|\lambda|} \qquad \text{if } \lambda < 0 \tag{3.2}$$

$$H - k\, q^{2(\lambda - H)} \leqslant m(\lambda) \leqslant H \qquad \text{if } \lambda \geqslant H \tag{3.3}$$

Proof. From (2.7) we have the identity

$$m(\lambda) = \sum_{j=0}^{H-1} \frac{q^{2(j-\lambda)}}{1 + q^{2(j-\lambda)}}\,. \tag{3.4}$$

If $\lambda > 0$, the summand in (3.4) is bounded below by $1/2$ for all $j \leqslant [\lambda]$ and the bound $m(\lambda) \geqslant \lambda/2$ is straightforward. When $\lambda \in [0, H-1]$, writing $\lambda = [\lambda] + \{\lambda\}$ we have

$$m(\lambda) = [\lambda] + 1 - \sum_{k=0}^{[\lambda]} \frac{q^{2(k+\{\lambda\})}}{1 + q^{2(k+\{\lambda\})}} + \sum_{l=1}^{H-1-[\lambda]} \frac{q^{2(l-\{\lambda\})}}{1 + q^{2(l-\{\lambda\})}}\,. \tag{3.5}$$

The estimate $|m(\lambda) - \lambda| \leqslant k$ then follows easily from (3.5). This proves (3.1). To prove (3.2) observe that if $\lambda < 0$

$$m(\lambda) = q^{2|\lambda|} \sum_{j=0}^{H-1} \frac{q^{2j}}{1 + q^{2(j-\lambda)}}\,, \tag{3.6}$$

and therefore

$$q^{2|\lambda|} \frac{1 - q^{2H}}{2(1-q^2)} \leqslant m(\lambda) \leqslant q^{2|\lambda|} \frac{1 - q^{2H}}{1 - q^2}\,.$$

Finally (3.3) follows from

$$m(\lambda)=\sum_{j=0}^{H-1}\frac{1}{q^{2(\lambda-j)}+1}=H-q^{2(\lambda-H)}\sum_{k=0}^{H-1}\frac{q^{2(k+1)}}{1+q^{2(k+1+\lambda-H)}}\,. \tag{3.7}$$

□

Next we consider $\sigma^2(\lambda):=\mathrm{Var}_{\mu^\lambda}(N_\Lambda)$, the variance of the number of particles in Λ.

Lemma 3.2. *For each $q\in(0,1)$ there exists $k<\infty$ such that for every $H\geqslant 1$, $\lambda\in\mathbb{R}$*

$$\frac{1}{k}\leqslant q^{2[\lambda\wedge 0+(H-\lambda)\wedge 0]}\,\sigma^2(\lambda)\leqslant k\,. \tag{3.8}$$

Proof. Recall that

$$\sigma^2(\lambda)=\sum_{j=0}^{H-1}\frac{q^{2(j-\lambda)}}{(1+q^{2(j-\lambda)})^2}=\sum_{j=0}^{H-1}\left(q^{-(j-\lambda)}+q^{(j-\lambda)}\right)^{-2}\,. \tag{3.9}$$

By symmetry (particle-hole duality), it is sufficient to consider the range $\lambda\leqslant H/2$. For the upper bound observe that (3.9) is bounded above by

$$\sum_{j=0}^{H-1}q^{-2|j-\lambda|}\leqslant\begin{cases}\frac{1+q^2}{1-q^2} & 0\leqslant\lambda\leqslant H/2\\ q^{2|\lambda|}\frac{1}{1-q^2} & \lambda<0\end{cases} \tag{3.10}$$

For the lower bound we estimate (3.9) below by

$$\frac{1}{4}\sum_{j=0}^{H-1}q^{-2|j-\lambda|}\geqslant\frac{1}{4}\begin{cases}q^2 & 0\leqslant\lambda\leqslant H/2\\ q^{2(|\lambda|+1)} & \lambda<0\end{cases} \tag{3.11}$$

□

Remark 3.3. A simple consequence of the bounds in Lemma 3.1 and Lemma 3.2 is the following. Let $\Lambda_L=\Sigma_{L,H}$ be the usual cylinder containing HL^2 sites and let $\sigma^2_L(\lambda)$ denote the variance of N_{Λ_L} w.r.t. μ^λ. Let also $m_L(\lambda)=\mu^\lambda(N_{\Lambda_L})$. Since $m_L(\lambda)=L^2m(\lambda)$ and $\sigma^2_L(\lambda)=L^2\sigma^2(\lambda)$ the estimates of the two lemmas can be combined to obtain that there exists $k=k(q)<\infty$ such that for any $\lambda\in\mathbb{R}$ and any $H\geqslant 1$

$$\frac{1}{k}(m_L(\lambda)\wedge L^2)\leqslant\sigma^2_L(\lambda)\leqslant k\,(m_L(\lambda)\wedge L^2)\,. \tag{3.12}$$

3.2. **Comparison of canonical and grand canonical measures.** In this second paragraph we will discuss some simple but important results on the equivalence and comparison between the finite volume canonical measure $\nu_{\Lambda,n}$ and the grand canonical one μ^λ_Λ, where $\Lambda:=\Sigma_{L,H}$ and the chemical potential $\lambda=\lambda(\Lambda,n)$ is chosen in such a way that $\mu^\lambda_\Lambda(N_\Lambda)=n$. Although some of the results discussed below have already been discussed in the seminal paper [3], for later purposes we need to improve the estimates obtained in [3] to get bounds similar to those established in [8] for general lattice gases.
For notation convenience we drop all the super/sub scripts in the measures.

Theorem 3.4. *For any $q\in(0,1)$ there exists a constant $k\in(0,\infty)$ such that for every positive integers H,L, $L_0\leqslant L$, for every $n=0,1,\ldots,HL^2$ and for any bounded function f such that its support is contained in the sub-cylinder $\Sigma_{L_0,H}$ we have*

$$|\nu(f)-\mu(f)|\leqslant k\|f\|_\infty\frac{L_0^2}{L^2} \tag{3.13}$$

Remark 3.5. For the above result it is completely irrelevant whether we are in a tilted or straight geometry. Also, because of horizontal translation invariance, all what matters is that the support of f is contained in some cylinder with basis of linear size L_0.

Remark 3.6. It is interesting to compare our result with that of [3]. There the dependence on L_0, L is worst because the leading term is of the order of $\frac{L_0^2}{L-L_0}$ but the coefficient in front of it, $k\|f\|_\infty$ in our case, is better since it is proportional to $\sup_n |\nu_{\Lambda,n}(f)|$. On the other hand the proof given below works in any dimension so that the bound (3.13) is valid for all $d \geq 2$ with $(L_0/L)^2$ replaced by $(L_0/L)^{d-1}$.

Proof. We begin by proving the result for $f(\alpha) = \alpha_x$, $x \in \Sigma_{L,H}$. In what follows k will denote a generic constant depending only on q whose value may change from time to time. It will first be convenient to fix some additional handy notation. We let

$$\sigma_y^2 := \mu(\alpha_y, \alpha_y)\ ; \qquad \sigma^2 := L^2 \sum_{y \in \Sigma_0} \sigma_y^2\ ; \qquad \rho_x := \mu(\alpha_x)$$

$$\bar\alpha_x := \alpha_x - \rho_x\ ; \qquad \beta_{xy} := \frac{\sigma_y^2}{\sigma_x^2}\ ; \qquad \phi_x(t) := \mu(e^{i\frac{t}{\sigma}\bar\alpha_x})$$

Notice that $\sigma^2 = \mu(N_\Lambda, N_\Lambda)$ because of the product structure of the measure μ.

Following [8] we begin by proving that for any $x, y \in \Lambda$

$$|\nu(\alpha_y - \beta_{xy}\alpha_x - \mu(\alpha_y - \beta_{xy}\alpha_x)| \leqslant k\frac{\sigma_y^2}{\sigma^2} \tag{3.14}$$

for some constant $k = k(q)$. Once (3.14) holds then, a summation over y together with the identity $\sum_y [\nu(\alpha_y) - \mu(\alpha_y)] = 0$ and the definition of β_{xy} yields

$$|\nu(\alpha_x) - \mu(\alpha_x)| \leqslant k\frac{\sigma_x^2}{\sigma^2}\,, \tag{3.15}$$

which is a slightly stronger result than the sought bound $\frac{k}{L^2}$ because

$$\sigma_x^2/\sigma^2 \leqslant \sum_{z \in \Sigma_x} \sigma_z^2/\sigma^2 = \frac{1}{L^2}\,. \tag{3.16}$$

In order to prove (3.14) we write

$$\nu(\alpha_y - \beta_{xy}\alpha_x) - \mu(\alpha_y - \beta_{xy}\alpha_x) = \frac{1}{2\pi\sigma\mu(N_\Lambda = n)} \int_{-\pi\sigma}^{\pi\sigma} \mathrm{d}t\, \mu(e^{i\frac{t}{\sigma}(N_\Lambda - n)}, \alpha_y - \beta_{xy}\alpha_x)\,, \tag{3.17}$$

where

$$\mu(N_\Lambda = n) = \frac{1}{2\pi\sigma} \int_{-\pi\sigma}^{\pi\sigma} \mathrm{d}t\, \mu\big(e^{i\frac{t}{\sigma}(N_\Lambda - n)}\big)$$

denotes the μ-probability of having n particles in Λ. Since μ is a product measure, the absolute value of the numerator in the integrand is bounded from above by

$$\prod_{z \neq x,y} |\phi_z(t)|\, |\mu(e^{i\frac{t}{\sigma}(\bar\alpha_x + \bar\alpha_y)}[\bar\alpha_y - \beta_{xy}\bar\alpha_x])| \tag{3.18}$$

It is quite easy to check that

$$|\phi_z(t)| \leqslant e^{-k\frac{\sigma_z^2}{\sigma^2}t^2} \qquad \forall\ |t| \leqslant \pi\sigma \tag{3.19}$$

for some constant $k = k(q)$, so that $\prod_{z\neq x,y} |\phi_z(t)| \leqslant e^{-k\,t^2}$. Moreover the identity $e^{i\vartheta} = 1 + i\vartheta + R(\vartheta)$ with $|R(\vartheta)| \leqslant \frac{\vartheta^2}{2}$ together with the definition of the coefficient β_{xy} gives

$$
\begin{aligned}
&| \mu(e^{i\frac{t}{\sigma}(\bar\alpha_x+\bar\alpha_y)}[\bar\alpha_y - \beta_{xy}\bar\alpha_x])| \\
&\qquad = | \phi_x(t)\mu(e^{i\frac{t}{\sigma}\bar\alpha_y}\bar\alpha_y) - \beta_{xy}\phi_y(t)\mu(e^{i\frac{t}{\sigma}\bar\alpha_x}\bar\alpha_x)| \leqslant k\frac{t^2}{\sigma^2}\sigma_y^2
\end{aligned}
\tag{3.20}
$$

In conclusion, putting together (3.19) and (3.20), the numerator of (3.17) is bounded from above by $k\frac{\sigma_y^2}{\sigma^2}$.

We are left with the analysis of the denominator in (3.17). We will show that

$$
\sigma\mu(N_\Lambda = n) \geqslant k^{-1} \tag{3.21}
$$

uniformly in n, L, H. We have to distinguish between the case in which σ^2 is "large" and the case in which σ^2 is "small". To be more specific we fix a large number A and start to analyze the case $\sigma \geq A^5$. Again using Fourier analysis we write

$$
\begin{aligned}
2\pi\sigma\mu(N_\Lambda = n) &= \int_{-\pi\sigma}^{\pi\sigma} \mathrm{d}t\,\mu\big(e^{i\frac{t}{\sigma}(N_\Lambda - n)}\big) \\
&= \int_{A\,\leqslant\,|t|\,\leqslant\,\pi\sigma} \mathrm{d}t\,\mu\big(e^{i\frac{t}{\sigma}(N_\Lambda - n)}\big) + \int_{-A}^{A}\mathrm{d}t\prod_{x\in\Lambda}\phi_x(t) \\
&= \int_{A\,\leqslant\,|t|\,\leqslant\,\pi\sigma} \mathrm{d}t\,\mu\big(e^{i\frac{t}{\sigma}(N_\Lambda - n)}\big) + \int_{-A}^{A}\mathrm{d}t\prod_{x\in\Lambda}[1 - \frac{t^2}{2}\frac{\sigma_x^2}{\sigma^2} + R_x(t)] \\
&= \int_{A\,\leqslant\,|t|\,\leqslant\,\pi\sigma} \mathrm{d}t\,\mu\big(e^{i\frac{t}{\sigma}(N_\Lambda - n)}\big) + \int_{-A}^{A}\mathrm{d}t\prod_{x\in\Lambda}[1 - \frac{t^2}{2}\frac{\sigma_x^2}{\sigma^2}] + \int_{-A}^{A}\mathrm{d}t\,R(t)
\end{aligned}
\tag{3.22}
$$

where $R_x(t) = \phi_x(t) - 1 + \frac{t^2}{2}\frac{\sigma_x^2}{\sigma^2}$ and a simple expansion gives

$$
R(t) = \sum_{j=1}^{|\Lambda|}\frac{1}{j!}\sum_{\substack{x_1\ldots x_j\in\Lambda:\\ x_1\neq x_2\neq\cdots\neq x_j}}\prod_{i=1}^{j}R_{x_i}(t)\prod_{z\in\Lambda\setminus\{x_1,\ldots x_j\}}[1 - \frac{t^2}{2}\frac{\sigma_z^2}{\sigma^2}]\,. \tag{3.23}
$$

Let us examine the three terms in the r.h.s. of (3.22). The first one is smaller in absolute value than $ke^{-k'A^2}$ because of the gaussian bound (3.19). Observing that

$$
\prod_{x\in\Lambda}[1 - \frac{t^2}{2}\frac{\sigma_x^2}{\sigma^2}] \geqslant e^{-\frac{t^2}{4\sigma^2}\sum_{x\in\Lambda}\sigma_x^2} = e^{-\frac{t^2}{4}}
$$

we see that the second one is greater than $1/2$ provided that A is large enough, uniformly in all the parameters. Finally, using

$$
|R_x(t)| \leqslant k\frac{|t|^3}{\sigma^3}\sigma_x^2
$$

and (3.23), the absolute value of the third one is smaller than

$$
2A\sup_{|t|\,\leqslant\,A}\sum_{j=1}^{|\Lambda|}\frac{1}{j!}\Big(\sum_{x\in\Lambda}|R_x(t)|\Big)^j \leqslant 2A\sup_{|t|\,\leqslant\,A}\sum_{j=1}^{|\Lambda|}\frac{1}{j!}\Big(\frac{|t|^3}{\sigma}\Big)^j \leqslant k\,A^{-1}\,.
$$

In conclusion, if $\sigma \geq A^5$ and A is large enough (but independent of L, H, n) (3.21) holds true.

Let us now examine the case $\sigma \leqslant A^5$ which, for large values of L and H, corresponds to an extremely low density of particles (cf. (3.12)) In this case we bound $\mu(N_\Lambda = n)$ from below as follows. If $L^2 \leqslant 2n$ then we impose that all the particles in the cylinder Λ are packed starting from the bottom and according to an arbitrary ordering of the sites on each horizontal square $Q_L + (0,0,\ell)$. It is not difficult to check that the probability of such event is bounded below by $\exp(-\gamma L^2)$ for some $\gamma = \gamma(q) > 0$, uniformly in the height of the cylinder. But since $L^2 \leqslant 2n \leqslant k\sigma^2$ (by (3.12)) we have a lower bound $\exp(-k\gamma A^{10})$. If instead $L^2 \geq 2n$ then we impose that all the particles are at height $\ell = 0$. The probability of this last event is equal to

$$\binom{L^2}{n} p_0^n (1-p_0)^{L^2-n} \prod_{\ell=1}^{H-1} (1-p_\ell)^{L^2} \tag{3.24}$$

where $p_\ell = \frac{q^{2(\ell-\lambda)}}{1+q^{2(\ell-\lambda)}}$ is the probability that there is a particle at a site x with $\ell_x = \ell$. Notice that $p_\ell \leqslant \frac{1}{2}$ for any $\ell \geq 0$ since $L^2 \sum_{\ell \geq 0} p_\ell = n$. In particular

$$\prod_{\ell=1}^{H-1} (1-p_\ell)^{L^2} \geqslant \exp\bigl(-4L^2 \sum_{\ell=0}^{H-1} (1-p_\ell)p_\ell\bigr) = \exp(-4\sigma^2) \geqslant \exp(-4A^{10}). \tag{3.25}$$

Finally, since $p_0 L^2 \leqslant n$ and $n \leqslant k\sigma^2 \leqslant kA^{10}$, also the factor $\binom{L^2}{n} p_0^n (1-p_0)^{L^2-n}$ is bounded away from zero uniformly in L, H.

We are now in a position to give the result in its full generality.

For any $x \in \Lambda_0 = \Sigma_{L_0,H}$, set $\beta_{x,f} := \mu(f, N_0)/\sigma_x^2$, where N_0 denotes the number of particles in Λ_0. Observing that

$$\mathrm{Var}_\mu(f) \leqslant 2\|f\|_\infty^2 \mathrm{Var}_\mu(N_0),$$

and using Schwarz' inequality, it is not difficult to check that

$$|\beta_{x,f}| \leqslant k\|f\|_\infty \frac{\mathrm{Var}_\mu(N_0)}{\sigma_x^2}. \tag{3.26}$$

Thanks to (3.15),

$$|\beta_{x,f}|\,|\nu(\alpha_x) - \mu(\alpha_x)| \leqslant k\|f\|_\infty \frac{\mathrm{Var}_\mu(N_0)}{\sigma^2} = k\|f\|_\infty \frac{L_0^2}{L^2}$$

We can thus safely replace f by $f - \beta_{x,f}\alpha_x$.

We proceed at this point exactly as in (3.17)$\cdots$(3.20) and we observe that, because of the very definition of $\beta_{x,f}$ the analogous of (3.20) holds, namely

$$|\,\mu\bigl(e^{i\frac{t}{\sigma}(\bar\alpha_x + \bar N_0)}[f - \beta_{x,f}\alpha_x]\bigr)| \leqslant k\|f\|_\infty \frac{L_0^2}{L^2}. \tag{3.27}$$

The theorem then follows because of the bound (3.21). □

Our second result provides a simple upper bound on the canonical expectation of a rather general, non negative function f in terms of its grand canonical mean. The relevance of such bounds is that they permit to control the canonical large deviations in terms of the grand canonical ones (see [7] and especially [9]). We begin with the very simple one dimensional case $\Lambda = \Sigma_H$.

Proposition 3.7. *For any $q \in (0,1)$ there exists a constant $k \in (0,\infty)$ such that for any one dimensional stick $\Lambda = \Sigma_H$, for every $n = 0,\dots,H$ and for every $f \geqslant 0$*

$$\nu(f) \leqslant k\mu(f) \tag{3.28}$$

Proof. It suffices to observe that the probability $\mu(N_\Lambda = n)$ can be bounded from below by

$$\mu(N_\Lambda = n) \geqslant \prod_{x\in\Lambda:\,\ell_x \leqslant n-1} \mu(\alpha_x) \prod_{y\in\Lambda:\,\ell_y \geqslant n} \mu(1-\alpha_y) \geqslant k'(q) > 0\,, \tag{3.29}$$

since, by Lemma 3.1, there exists a constant $k = k(q)$ such that $|\lambda(n) - n| \leqslant k$. □

Next we turn to the genuine three dimensional case $\Lambda = \Sigma_{L,H}$.

Proposition 3.8. *For any $q \in (0,1)$ and $\delta \in (0,1)$ there exists a constant $k \in (0,\infty)$ such that for every positive integers H and L, for every $n = 0,1,\dots,HL^2$ and for every non–negative function f whose support does not intersect more than $(1-\delta)L^2$ sticks*

$$\nu(f) \leqslant k\mu(f) \tag{3.30}$$

In particular,

$$\nu(f,f) \leqslant k\mu(f,f)\,. \tag{3.31}$$

Proof. Let us denote by Δ_f the support of f. Then, using (3.21) together with the gaussian upper bound on the absolute value of the characteristic function (3.19)

$$\begin{aligned}\nu(f) &= \frac{1}{2\pi\sigma\mu(N_\Lambda = n)} \int_{-\pi\sigma}^{\pi\sigma} \mathrm{d}t\, \mu(e^{i\frac{t}{\sigma}(N_\Lambda - n)} f) \\ &\leqslant k\mu(f) \int_{-\pi\sigma}^{\pi\sigma} \mathrm{d}t \prod_{x\in\Lambda\setminus\Delta_f} |\phi_x(t)| \\ &\leqslant k\mu(f) \int_{-\pi\sigma}^{\pi\sigma} \mathrm{d}t\, e^{-k\delta t^2} \leqslant k'\mu(f)\,.\end{aligned} \tag{3.32}$$

The result for the variance follows at once from

$$\nu(f,f) \leqslant \nu\left((f-\mu(f)^2\right) \leqslant k\mu(f,f)\,.$$

□

3.3. **An estimate on covariances.** An important ingredient of our approach is the following version of a well known estimate due to [18] (see also [7]). Set $\Lambda = \Sigma_{L,H}$, and let ν denote the canonical measure with n particles in Λ. The result given below will be used in the recursive estimate of section 4 in the regime of large L, see Theorem 4.1. On the other hand its proof uses an estimate for small values of L (see (4.4) and (4.5)) that will be proven independently later on.

In the following B denotes a planar section of Λ, i.e. $B = \Lambda \cap \mathcal{A}_h$ for some integer $h \leqslant H-1$, and N_B is the number of particles in B.

Proposition 3.9. *For every $q \in (0,1)$ and for every $\epsilon > 0$ there exists $C_\epsilon = C_\epsilon(q) < \infty$ such that for any function f, any height $H \geqslant 1$ and for all $n = 0,\dots|\Lambda|$ we have*

$$\nu(f,N_B)^2 \leqslant \left(L^2 \wedge n\right)\Big[\, C_\epsilon\, \mathcal{E}_\nu(f,f) + \epsilon\, \mathrm{Var}_\nu(f)\,\Big] \tag{3.33}$$

Proof. We take $R \in \mathbb{Z}_+$ and write the square Q_L as the disjoint union of smaller squares Q^i_R of side $R \ll L$. This is no real loss since, in view of the horizontal exchangeability of variables under ν, the geometry of the basis does not play any role and we can always assume Q_L to be given by the union of $\cup_i Q^i_R$ and $\bar{Q}$, where $\bar{Q}$ is a small region contained in a square of side R which is inessential in the argument below. Let then $\Lambda = \cup_i \Sigma_i$, $\Sigma_i := \Sigma_{Q^i_R,H}$, $N_i := N_{\Sigma_i}$ and let $\mathcal{F}$ be the σ-algebra generated by the random variables $\{N_i\}$. For any pair of functions f, g we have

$$\nu(f,g) = \nu\big(\nu(f,g\,|\,\mathcal{F})\big) + \nu\big(f,\nu(g\,|\,\mathcal{F})\big)\,. \tag{3.34}$$

Simple estimates then allow to write

$$\nu(f,g)^2 \leqslant 2\nu\big(\mathrm{Var}_\nu(g\,|\,\mathcal{F})\big)\nu\big(\mathrm{Var}_\nu(f\,|\,\mathcal{F})\big) + 2\mathrm{Var}_\nu(f)\mathrm{Var}_\nu\big(\nu(g\,|\,\mathcal{F})\big)\,. \tag{3.35}$$

Now define the function $g = \sum_i g_i$, with $g_i = N_{B_i} - \beta N_i$, where $B_i = B \cap \Sigma_i$ and β is a parameter to be fixed later on. Observe that with this choice $\nu(f, N_B) = \nu(f,g)$ and

$$\mathrm{Var}_\nu(g\,|\,\mathcal{F}) = \sum_i \mathrm{Var}_\nu(N_{B_i}\,|\,\mathcal{F}_i)\,, \tag{3.36}$$

where we used $\mathcal{F}_i$ to denote the σ-algebra generated by N_i. We fix now a value n_i for N_i and write $\lambda(n_i)$ for the corresponding chemical potential, i.e.

$$n_i = \mu^{\lambda(n_i)}(N_i) = |B_i| \sum_{\ell=0}^{H-1} p_\ell^{\lambda(n_i)}\,, \quad p_\ell^{\lambda(n_i)} = \frac{q^{2(\ell-\lambda(n_i))}}{1+q^{2(\ell-\lambda(n_i))}}\,. \tag{3.37}$$

Using Proposition 3.8 we have (recall that h is the level of every B_i)

$$\mathrm{Var}_\nu(N_{B_i}|N_i = n_i) \leqslant k\mathrm{Var}_{\mu^{\lambda(n_i)}}(N_{B_i}) = k|B_i|p_h^{\lambda(n_i)}(1-p_h^{\lambda(n_i)})\,, \tag{3.38}$$

and by Lemma 3.2 and Lemma 3.1 we have

$$\mathrm{Var}_\nu(N_{B_i}|N_i = n_i) \leqslant k\big(|B_i| \wedge n_i\big)\,. \tag{3.39}$$

In particular, together with (3.36) this implies

$$\max_{\{n_i\}:\,\sum_i n_i = n} \mathrm{Var}_\nu(g\,|\,\mathcal{F}) \leqslant k\big(L^2 \wedge n\big)\,. \tag{3.40}$$

Since the measure $\nu(\cdot\,|\,\mathcal{F})$ is a product $\otimes_i \nu(\cdot\,|\,\mathcal{F}_i)$ and each factor $\nu(\cdot\,|\,\mathcal{F}_i)$ satisfies a Poincaré inequality with a constant $W(R)$ uniform in the conditioning field (see (4.4) and (4.5)), we have

$$\begin{aligned}\nu\left(\mathrm{Var}_\nu(f\,|\,\mathcal{F})\right) &\leqslant \sum_i \nu\Big[\mathrm{Var}_\nu(f\,|\,\mathcal{F}_i)\Big] \\ &\leqslant W(R)\sum_i \nu\Big[\sum_{b\in\Sigma_i^*} \nu\big((\nabla_b f)^2\,|\,\mathcal{F}_i\big)\Big] \\ &\leqslant W(R)\,\mathcal{E}_\nu(f,f)\end{aligned} \tag{3.41}$$

Plugging (3.40) and (3.41) in (3.35) we obtain

$$\nu(f,g)^2 \leqslant 2kW(R)\left(L^2 \wedge n\right)\mathcal{E}_\nu(f,f) + 2kL^2R^{-2}\,\mathrm{Var}_\nu(f)\mathrm{Var}_\mu\left(\nu(g_1\,|\,\mathcal{F}_1)\right)\,, \tag{3.42}$$

where we have used again Proposition 3.8 to bound $\mathrm{Var}_\nu(\nu(g\,|\,\mathcal{F}))$ in terms of

$$\mathrm{Var}_\mu(\nu(g\,|\,\mathcal{F})) = L^2R^{-2}\mathrm{Var}_\mu\left(\nu(g_1\,|\,\mathcal{F}_1)\right)$$

with $\mu := \mu^\lambda$ the grand canonical measure on Λ, and $\lambda := \lambda(n)$ such that $\mu(N_\Lambda) = n$. At this point we consider separately two cases corresponding to "many" and "few" particles respectively.

We start with the case of many particles. Suppose $n > \epsilon^2 L^2$. Here the claim (3.33) will follow from

$$\mathrm{Var}_\mu(\nu(g_1 \,|\, \mathcal{F}_1)) \leqslant k\,, \tag{3.43}$$

by taking R sufficiently large in (3.42). To prove (3.43) we begin by observing that by Theorem 3.4

$$\sup_{n_1} \left|\nu(g_1 \,|\, N_1 = n_1) - \mu_{\Sigma_1}^{\lambda(n_1)}(g_1)\right| \leqslant k\,,$$

and therefore it is sufficient to show

$$\mathrm{Var}_\mu\left[\mu_{\Sigma_1}^{\lambda(N_1)}(g_1)\right] \leqslant k\,. \tag{3.44}$$

An estimate on the variance of $\varphi(N_1) := \mu_{\Sigma_1}^{\lambda(N_1)}(g_1)$ can be obtained as follows. Since μ is a product $\otimes_{x\in\Sigma_i}\mu_x$, one has

$$\mathrm{Var}_\mu(\varphi) \leqslant \sum_{x\in\Sigma_1} \mu\left(\mathrm{Var}_{\mu_x}(\varphi)\right)\,.$$

But

$$\mu\big(\mathrm{Var}_{\mu_x}(\varphi)\big) = \sigma_x^2\,\mu\Big(\big[\varphi(1+\sum_{y\neq x}\alpha_y) - \varphi(\sum_{y\neq x}\alpha_y)\big]^2\Big)\,, \quad \sigma_x^2 = p_{\ell_x}^\lambda(1-p_{\ell_x}^\lambda)\,.$$

It is not difficult now to deduce

$$\mathrm{Var}_\mu(\varphi) \leqslant k\Big[\sum_{x\in\Sigma_1}\sigma_x^2\Big]\,\mu\left([\varphi(N_1+1) - \varphi(N_1)]^2\right)\,. \tag{3.45}$$

By Remark 3.12 we know that $\sum_{x\in\Sigma_1}\sigma_x^2 \leqslant k(R^2 \wedge \mu(N_1))$. Since $\mu(N_1) = nR^2/L^2$ we have

$$\mathrm{Var}_\mu\left(\varphi(N_1)\right) \leqslant k\,R^2\mu\left[\left(\varphi(N_1+1) - \varphi(N_1)\right)^2\right]\,. \tag{3.46}$$

Now we can choose β so that

$$\mu\left(\mu_{\Sigma_1}^{\lambda(N_1+1)}(N_{B_1}) - \mu_{\Sigma_1}^{\lambda(N_1)}(N_{B_1})\right) = \beta\,.$$

In this way the right hand side of (3.46) is again a variance and a new application of (3.45) gives

$$\mathrm{Var}_\mu(\varphi) \leqslant k\,R^4\mu\left[(\Delta\varphi(N_1+1))^2\right]\,, \tag{3.47}$$
$$\Delta\varphi(m) := \varphi(m+1) + \varphi(m-1) - 2\varphi(m)\,.$$

We are going to show that

$$\sup_m |\Delta\varphi(m)|^2 \leqslant kR^{-4}\,. \tag{3.48}$$

We have

$$\Delta\varphi(m) = \int_0^1\int_0^1 \left[\partial_t\partial_s\mu_{\Sigma_1}^{\lambda(m+s+t)}(N_{B_1})\right]\,\mathrm{d}t\,\mathrm{d}s \tag{3.49}$$

Set $\lambda_{s,t} = \lambda(m+s+t)$. Using the identities

$$\partial_t \mu_{\Sigma_1}^{\lambda_{s,t}}(N_{B_1}) = \partial_s \mu_{\Sigma_1}^{\lambda_{s,t}}(N_{B_1}) = \frac{\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_{B_1})}{\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_1)}\,, \tag{3.50}$$

we have

$$\partial_t \partial_s \mu_{\Sigma_1}^{\lambda_{s,t}}(N_{B_1}) = \frac{\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_1, N_{B_1})}{\left(\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_1)\right)^2} - \frac{\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_1, N_1)\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_{B_1})}{\left(\mu_{\Sigma_1}^{\lambda_{s,t}}(N_1, N_1)\right)^3}\,. \tag{3.51}$$

Here we use the standard notation

$$\mu(f,g,h) = \mu((f-\mu(f))(g-\mu(g))(h-\mu(h)))\,.$$

Direct computations show that for any λ

$$\begin{aligned}
\mu_{\Sigma_1}^{\lambda}(N_1, N_1) &= |B_1| \sum_{j=0}^{H-1} p_j^{\lambda}(1-p_j^{\lambda}) \\
\mu_{\Sigma_1}^{\lambda}(N_1, N_{B_1}) &= |B_1| p_h^{\lambda}(1-p_h^{\lambda}) \\
\mu_{\Sigma_1}^{\lambda}(N_1, N_1, N_1) &= |B_1| \sum_{j=0}^{H-1} p_j^{\lambda}(1-p_j^{\lambda})(1-2p_j^{\lambda}) \\
\mu_{\Sigma_1}^{\lambda}(N_1, N_1, N_{B_1}) &= |B_1| p_h^{\lambda}(1-p_h^{\lambda})(1-2p_h^{\lambda})
\end{aligned}$$

From (3.51), using $|B_1| = R^2$ we have

$$\begin{aligned}
\partial_t \partial_s \mu_{\Sigma_1}^{\lambda_{s,t}}(N_{B_1}) &= R^{-2}\mathcal{C}(h, H, \lambda_{s,t}) \qquad (3.52)\\
\mathcal{C}(h,H,\lambda) &:= \frac{2p_h^{\lambda}(1-p_h^{\lambda})\sum_{j=0}^{H-1} p_j^{\lambda}(1-p_j^{\lambda})[p_j^{\lambda}-p_h^{\lambda}]}{\left(\sum_{j=0}^{H-1} p_j^{\lambda}(1-p_j^{\lambda})\right)^3}\,.
\end{aligned}$$

From (3.52) and (3.49) we see that (3.48) will follow from

$$\sup_{\substack{h,H\in\mathbb{N}\\ \lambda\in\mathbb{R}}} \left|\mathcal{C}(h,H,\lambda)\right| < \infty\,. \tag{3.53}$$

A first estimate gives

$$\left|\mathcal{C}(h,H,\lambda)\right| \leqslant \frac{2\sum_{j=0}^{H-1} p_j^{\lambda}(1-p_j^{\lambda})|p_h^{\lambda}-p_j^{\lambda}|}{\left(\sum_{j=0}^{H-1} p_j^{\lambda}(1-p_j^{\lambda})\right)^2}\,.$$

Then Lemma 3.2 shows that $|\mathcal{C}(h,H,\lambda)|$ is bounded whenever $\lambda \in [0, H-1]$. On the other hand if $\lambda \leqslant 0$ then $1 - p_j^{\lambda} \geqslant 1/2$, whereas if $\lambda \geqslant H-1$ then $p_j^{\lambda} \geqslant 1/2$. In any case for $0 \leqslant j, h \leqslant H-1$

$$|p_h^{\lambda} - p_j^{\lambda}| \leqslant 2\sum_{\ell=0}^{H-1} p_\ell^{\lambda}(1-p_\ell^{\lambda})$$

and (3.53) follows.

We turn to analyze the case of few particles: $n \leqslant \epsilon^2 L^2$. In this case we simply take $R = 1$ and call $\psi(N_1) := \nu(g_1 \,|\, N_1)$. We may assume that $\epsilon^2 k \leqslant \epsilon$. Thus looking back at

(3.42) we see that it will be sufficient to show

$$\mathrm{Var}_\mu(\psi) \leqslant k\epsilon^2 \frac{n}{L^2}\,. \tag{3.54}$$

Since now $\mu(N_1) = n/L^2$, (3.45) gives

$$\mathrm{Var}_\mu(\psi) \leqslant k\frac{n}{L^2}\mu\left([\psi(N_1+1) - \psi(N_1)]^2\right)$$

Choosing $\beta = \mu\left(\nu(N_{B_1} \mid N_1+1) - \nu(N_{B_1} \mid N_1)\right)$, (3.47) becomes

$$\mathrm{Var}_\mu(\psi) \leqslant k\frac{n^2}{L^4}\mu\left[(\Delta\psi(N_1+1))^2\right] \leqslant k\epsilon^2\frac{n}{L^2}\mu\left[(\Delta\psi(N_1+1))^2\right].$$

On the other hand a trivial bound (remember that now B_1 is just a single site) gives $|\Delta\psi(m)| \leqslant 8$ and (3.54) follows immediately. ☐

3.4. **Glauber bound for the number of particles in half volume.** Consider the cylinder $\Sigma_{2L,H}$ and divide it into two parts

$$\Lambda = \Sigma_{2L,H} = \Lambda_1 \cup \Lambda_2\,, \quad \Lambda_1 = \Sigma_{R_{L,2L},H}\,,\ \Lambda_2 = (L,0,0) + \Sigma_{R_{L,2L},H}\,. \tag{3.55}$$

Fix $n \in \{1,\dots,2L^2H = |\Lambda|/2\}$, let $\nu := \nu_{\Lambda,n}$ denote the canonical measure on $\Lambda = \Sigma_{2L,H}$ with total particle number n and let $p_n(m) = \nu(N_{\Lambda_1} = m)$. We begin by establishing upper and lower bounds on the ratio $\frac{p_n(m+1)}{p_n(m)}$ for $m \geq \frac{n}{2}$. In what follows λ_s will denote the chemical potential such that $\mu^{\lambda_s}(N_{\Lambda_1}) = s$.

Lemma 3.10. *For any $q \in (0,1)$, there exists $k < \infty$ such that, uniformly in L, H, n and $m \in [\frac{n}{2}, n]$*

$$k^{-1}q^{2(\lambda_{m+1}-\lambda_{n-m-1})} \leqslant \frac{p_n(m+1)}{p_n(m)} \leqslant k\,q^{2(\lambda_m-\lambda_{n-m})}\,. \tag{3.56}$$

Moreover, for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\frac{p_n(m+1)}{p_n(m)} \leqslant \epsilon \tag{3.57}$$

whenever $m \in [\delta n, n]$, uniformly in all other parameters.

Proof. We write

$$\begin{aligned}
p_n(m+1) &= \sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2}} \frac{\nu\big(\alpha_x(1-\alpha_y)\,\mathbb{1}_{N_{\Lambda_1}=m+1}\big)}{(m+1)(|\Lambda_1|-n-m-1)} \\
&= \sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2}} \frac{q^{2(\ell_x-\ell_y)}\nu\big(\alpha_y(1-\alpha_x)\mid N_{\Lambda_1}=m\big)p_n(m)}{(m+1)(|\Lambda_1|-n-m-1)} \\
&\leqslant k\sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2}} \frac{q^{2(\ell_x-\ell_y)}\mu^{\lambda_m}\big((1-\alpha_x)\big)\mu^{\lambda_{n-m}}(\alpha_y)\,p_n(m)}{(m+1)(|\Lambda_1|-n-m-1)} \\
&= k\Big[\frac{m}{m+1}\Big]\Big[\frac{|\Lambda_1|-n-m}{|\Lambda_1|-n-m-1}\Big]q^{2(\lambda_m-\lambda_{n-m})}\,p_n(m) \\
&\leqslant k' q^{2(\lambda_m-\lambda_{n-m})}\,p_n(m)
\end{aligned} \tag{3.58}$$

where we used Proposition 3.7 and the fact that $\nu(\cdot\,|\,m)$ is the product of $\nu_{\Lambda_1,m}$ and $\nu_{\Lambda_2,n-m}$. The lower bound in (3.56) can be obtained in a similar way if we write $p_n(m)$ as

$$p_n(m) = \sum_{\substack{x\in\Lambda_1\\ y\in\Lambda_2}} \frac{\nu\big(\alpha_y(1-\alpha_x)\mathbb{1}_{N_{\Lambda_1}=m}\big)}{(n-m)(|\Lambda_1|-m)}$$

and proceed as above.

In order to prove the estimate (3.57) one could use bounds on the chemical potentials λ_m, λ_{n-m}. We prefer however a different route and rewrite $p_n(m+1)$ as follows. For any ℓ we set $A_\ell = \Lambda_1 \cap \mathcal{A}_\ell$ and $B_\ell = \Lambda_2 \cap \mathcal{A}_\ell$. Let also $N_{A_\ell}(\alpha)$ be the number of particles in the plane A_ℓ, $V_{A_\ell}(\alpha) = 2L^2 - N_{A_\ell}(\alpha)$ the corresponding number of holes, and similarly for B_ℓ. Then

$$\begin{aligned}
p_n(m+1) &= \sum_{\ell=0}^{H-1}\sum_{\substack{x\in A_\ell\\ y\in B_\ell}} \nu\left(\frac{\alpha_x(1-\alpha_y)}{\sum_{h=0}^{H-1} N_{A_h}V_{B_h}}\mathbb{1}_{N_{\Lambda_1}=m+1}\right)\\
&= \sum_{\ell=0}^{H-1}\sum_{\substack{x\in A_\ell\\ y\in B_\ell}} \nu\left(\frac{(1-\alpha_x)\alpha_y}{\sum_{h=0}^{H-1}(N_{A_h}+\delta_{\ell,h})(V_{B_h}+\delta_{\ell,h})}\,|\,N_{\Lambda_1}=m\right)p_n(m)\\
&\leqslant \nu\left(\frac{\sum_{\ell=0}^{H-1} V_{A_\ell}N_{B_\ell}}{\sum_{h=0}^{H-1} N_{A_h}V_{B_h}}\,|\,N_{\Lambda_1}=m\right)p_n(m)\,. \qquad (3.59)
\end{aligned}$$

On the event $N_{\Lambda_1} = m, N_{\Lambda_2} = n-m$ we have

$$\frac{\sum_{\ell=0}^{H-1} V_{A_\ell}N_{B_\ell}}{\sum_{h=0}^{H-1} N_{A_h}V_{B_h}} = \frac{2L^2(n-m) - \sum_{\ell=0}^{H-1} N_{A_\ell}N_{B_\ell}}{2L^2m - \sum_{h=0}^{H-1} N_{A_h}N_{B_h}} \leqslant \frac{n-m}{2m-n} \leqslant \frac{1-\delta}{2\delta-1}$$

for $\delta n \leqslant m \leqslant n$. Therefore (3.57) follows from the estimate (3.59). □

Next we establish a Poincaré inequality for the marginal of ν on N_{Λ_1} with respect to the corresponding Metropolis-Glauber dynamics.

Proposition 3.11. *For all $q \in (0,1)$, there exists a constant $k < \infty$ such that for all integers L, H, for all $n \in \{0,1,\dots,4L^2H\}$ and for all functions $g:\mathbb{N}\to\mathbb{R}$*

$$\mathrm{Var}_\nu\big(g(N_{\Lambda_1})\big) \leqslant k\big(n\wedge L^2\big)\sum_m p_n(m)\wedge p_n(m+1)\,[g(m+1)-g(m)]^2\,. \qquad (3.60)$$

Proof. We follow closely the analogous result for translation invariant lattice gases proved in [7] (see Theorem 4.4 there). Assume without loss of generality that $m \geq \frac{n}{2}$ and write $p_n(m)$ as

$$p_n(m) = e^{-V_n(m)}\phi_n(m)$$

where

$$V_n(m) := 2\log(1/q)\,\big[\,m\,\lambda_m + (n-m)\lambda_{n-m}\,\big] - \log\Big(\frac{Z^m}{Z}\Big)$$

$$\phi_n(m) := \frac{\mu_{\Lambda_1}^{\lambda_m}(N_{\Lambda_1}=m)\mu_{\Lambda_2}^{\lambda_{n-m}}(N_{\Lambda_2}=n-m)}{\mu(N_\Lambda=n)}$$

Here the partition function $Z^{(m)}$ is given by

$$Z^{(m)} := \prod_{\substack{x\in\Lambda_1\\ y\in\Lambda_2}} \left(1+q^{2(\ell_x-\lambda_m)}\right)\left(1+q^{2(\ell_y-\lambda_{n-m})}\right)$$

and similarly for Z but without the chemical potentials λ_m, λ_{n-m}.
Were the factor $\phi_n(m)$ constant (better: of bounded variation uniformly in m, n) then the sought Poincaré inequality would follow at once, using e.g. Cheeger inequality or Hardy's inequality [20], from a convexity bound of the form (see (3.64) and (3.65) below)

$$\frac{d^2}{dm^2}V_n(m) \geq k'(\delta, q)\frac{1}{n\wedge L^2}$$

Unfortunately the ratio $\frac{\phi(m)}{\phi(m')}$ can be rather large, depending on n, if e.g. $m \approx n/2$ and $m' \approx n$. On the other hand Lemma 3.10 shows that the distribution $p_n(m)$ has at least exponential tails so that, as far as the Poincaré inequality is concerned, the tails should be irrelevant. That is indeed true and, according to section 4 of [7], the result follows if we can show that there exists $\delta < 1$ with $1-\delta \ll 1$ and a constant k such that

$$\sup_{m,m'\in[\frac{n}{2},\delta n]} \frac{\phi(m)}{\phi(m')} \leq k \tag{3.61}$$

$$\sup_{\delta n\leq m\leq n} \frac{p_n(m+1)}{p_n(m)} \leq \frac{1}{2} \tag{3.62}$$

$$\min_{m\in[\frac{n}{2},\delta n]} \left[\, V(m+1)-V(m)\,\right] \geq \frac{1}{k(n\wedge L^2)} \tag{3.63}$$

Inequality (3.61) follows at once from the fact, proved in the discussion of the equivalence of ensembles, that $\mu_{\Lambda_1}^{\lambda_m}\left(N_{\Lambda_1}=m\right)$ (and similarly for $\mu_{\Lambda_2}^{\lambda_{n-m}}(N_{\Lambda_2}=n-m)$) is comparable to the inverse of the standard deviation of the number of particles in Λ_1, together with (3.12).
Inequality (3.62) is nothing but (3.57) above. Finally (3.63) follows from the convexity of the "potential" $V_n(m)$. More precisely, since $V_n(m)$ is even w.r.t. $\frac{n}{2}$, all what we need is

$$\frac{d^2}{dm^2}V_n(m) = 2\log(1/q)\,\frac{d}{dm}\left[\,\lambda_m-\lambda_{n-m}\,\right] \tag{3.64}$$

together with

$$\frac{d}{dm}\lambda_m = \frac{1}{\log(1/q)}\frac{1}{\mathrm{Var}_{\mu_{\Lambda_1}^{\lambda_m}}(N_{\Lambda_1})} \geq k'(\delta,q)\frac{1}{n\wedge L^2} \qquad \forall m\in[\frac{n}{2},\,\delta n] \tag{3.65}$$

and analogously for λ_{n-m}. Above we have used once more (3.12) to control the variance of the number of particles in terms of its mean. ☐

3.5. **Moving particles.** In this paragraph we will show how to relate "long jump terms" of the form $\nu_{\Lambda,n}([\nabla_{xy}f]^2)$ with $x, y \in \Lambda$ to sum of nearest neighbor jumps along a path leading from x to y. In what follows the setting and the notation will be that of the preceding subsection, cf. (3.55).

We will analyze two different situations that we call, for convenience, the many particles case (MP) and the few particles case (FP). The definitions will depend on a parameter δ which will be forced to be sufficiently small when needed (see the proof of Theorem 4.1).

- *Many particles*:

$$H \geqslant 1\,,\ \delta L^2 \leqslant n \leqslant 2L^2 H\,. \tag{MP}$$

- *Few particles*:

$$H \geqslant 1\,,\ 1 \leqslant n \leqslant \delta L^2\,. \tag{FP}$$

In the MP case let A and B be two horizontal section of Λ_1 and Λ_2 at height ℓ_A and ℓ_B respectively, with $\ell_A \geqslant \ell_B$. In the FP case, the sets A and B are instead given by

$$A = \big\{x \in \Lambda_1 :\ \ell_x \leqslant h-1\big\}\,, \quad B = \big\{y \in \Lambda_2 :\ \ell_y = 0\big\}\,. \tag{3.66}$$

where $h \in \mathbb{N}$ will be suitably tuned later on. Below we use $\nu(\cdot\,|\,m)$ for $\nu(\cdot\,|\,N_{\Lambda_1} = m)$.

Proposition 3.12. *For any $q \in (0,1)$, any $n = 1,\dots \frac{|\Lambda|}{2}$ and any $m \in [\frac{n}{2}, n]$*

$$\sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)}\nu\big((\nabla_{xy}f)^2\alpha_y(1-\alpha_x)\ |\,m\big)$$
$$\leqslant C\Big\{L^2 \sum_{b\in\mathcal{O}_\Lambda} \nu\big((\nabla_b f)^2\ |\,m\big) + \sum_{b\in\mathcal{V}_\Lambda} \nu\big((\nabla_b f)^2\ |\,m\big)\Big\}\,, \tag{3.67}$$

where C is a suitable constant depending on q in the MP *case and on q,h in the* FP *case.*

The rest of this section is devoted to the proof of the above proposition. For each couple of sites $x \in A$, $y \in B$, define a third site $z = z(x,y)$ with $z_1 = y_1$, $z_2 = y_2$ and $\ell_z = \ell_x$. That is z is the unique element of $\mathcal{A}_{\ell_x} \cap \Sigma_y$. Since $T_{xy}\alpha = T_{yz}T_{xz}T_{yz}\alpha$ we decompose $\nabla_{xy}f$ in

$$\nabla_{xy}f(\alpha) = \nabla_{yz}f(T_{xz}T_{yz}\alpha) + \nabla_{xz}f(T_{yz}\alpha) + \nabla_{yz}f(\alpha)\,. \tag{3.68}$$

We then have two vertical moves corresponding to exchanges between y and z, and one horizontal move corresponding to the exchange between x and z. Thus

$$\sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)}\nu\big((\nabla_{xy}f)^2\alpha_y(1-\alpha_x)\ |\,m\big) \leqslant 3\,\{\mathbb{I}_{\mathcal{O}} + \mathbb{I}_{\mathcal{V}}\} \tag{3.69}$$

with

$$\mathbb{I}_{\mathcal{O}} = \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)}\nu\big((\nabla_{xz}f(T_{yz}\alpha))^2\alpha_y(1-\alpha_x)\,|\,m\big) \tag{3.70}$$

and

$$\mathbb{I}_{\mathcal{V}} = \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)}\nu\,\big[\big((\nabla_{yz}f(T_{xz}T_{yz}\alpha))^2 + (\nabla_{yz}f(\alpha))^2\big)\alpha_y(1-\alpha_x)\,|\,m\big] \tag{3.71}$$

We analyze these terms separately.

Vertical moves. If we have a particle at y and a hole at x then

$$\big(T_{xz}T_{yz}\alpha\big)_y = \alpha_z\,, \quad \big(T_{xz}T_{yz}\alpha\big)_x = \alpha_y = 1\,, \quad \big(T_{xz}T_{yz}\alpha\big)_z = \alpha_x = 0\,.$$

Computing $\nabla_{yz}f(T_{xz}T_{yz}\alpha)$ we may thus assume $\alpha_z = 1$ (it vanishes otherwise). Since we have a particle both at y and at z the change of variables $\alpha \to T_{xz}T_{yz}\alpha$ produces no extra factors and

$$\nu\big((\nabla_{yz}f(T_{xz}T_{yz}\alpha))^2\alpha_y(1-\alpha_x)\ |\,m\big) = \nu\big((\nabla_{yz}f(T_{xz}T_{yz}\alpha))^2\alpha_y(1-\alpha_x)\alpha_z\ |\,m\big)$$
$$= \nu\big((\nabla_{yz}f(\alpha))^2\alpha_y\alpha_x(1-\alpha_z)\ |\,m\big) \tag{3.72}$$

For the second term in (3.71) we have

$$\nu\big((\nabla_{yz}f(\alpha))^2\alpha_y(1-\alpha_x)\mid m\big)=\nu\big((\nabla_{yz}f(\alpha))^2\alpha_y(1-\alpha_x)(1-\alpha_z)\mid m\big)\,,$$

therefore (3.71) becomes

$$\mathbb{I}_{\mathcal{V}}=\sum_{\substack{x\in A\\ y\in B}}q^{2(\ell_x-\ell_y)}\nu\big((\nabla_{yz}f(\alpha))^2\alpha_y(1-\alpha_z)\mid m\big) \tag{3.73}$$

We need the following rather general result. Given any $y\in\Lambda$ and $z\in\Sigma_y$ we write γ_{zy} for the shortest path connecting z and y along the stick.

Proposition 3.13. *For any $q\in(0,1)$, for any $y\in\Lambda$ and $z\in\Sigma_y$*

$$q^{2[(\ell_z-\ell_y)\vee 0]}\nu\big((\nabla_{zy}f)^2\alpha_y(1-\alpha_z)\big)\leqslant 4q^2(1-q^2)^{-1}\sum_{b\in\gamma_{zy}}\nu\big((\nabla_b f)^2\big) \tag{3.74}$$

Proof. Assume first that $\ell_z\geqslant\ell_y$. Let $M=\ell_z-\ell_y$ and consider the sequence $y=x_0,x_1,\dots,x_M=z$, with $\ell_{x_i}=\ell_{x_{i-1}}+1$. Write $\alpha_i=\alpha_{x_i}$, $T_{i,i+1}$ for the exchange operator $T_{x_ix_{i+1}}$ and $\nabla_{i,i+1}$ for the corresponding gradient. We want to prove

$$q^{2M}\nu\big((\nabla_{0,M}f)^2\alpha_0(1-\alpha_M)\big)\leqslant 4q^2(1-q^2)^{-1}\sum_{i=1}^{M}\nu\big((\nabla_{i-1,i}f)^2\big) \tag{3.75}$$

We have a particle at x_0 and a hole at x_M. To compute $T_{0,M}$ we first bring the particle from x_0 to x_M and then bring the hole, which sits now at x_{M-1}, back to x_0. We write

$$T_{0,M}\alpha=T_{0,1}T_{1,2}\cdots T_{M-1,M}T_{M-2,M-1}\cdots T_{1,2}T_{0,1}\alpha\,. \tag{3.76}$$

To fit the picture described above, formula (3.76) should be read backwards. The first part of the transformation is described by operators

$$R_0\alpha=\alpha\,,\quad R_i\alpha=T_{i-1,i}\cdots T_{1,2}T_{0,1}\alpha\,,\quad i=1,2,\dots,M-1\,, \tag{3.77}$$

while the second part is given by operators

$$\begin{aligned}L_i\alpha&=T_{i,i+1}T_{i+1,i+2}\cdots T_{M-1,M}T_{M-2,M-1}\cdots T_{1,2}T_{0,1}\alpha\\ &=T_{i,i+1}T_{i+1,i+2}\cdots T_{M-1,M}R_{M-1}\alpha\,,\quad i=1,2,\dots,M-1\,.\end{aligned} \tag{3.78}$$

In this way a simple telescopic argument shows that

$$\begin{aligned}\nabla_{0,M}f(\alpha)&=f(T_{0,M}\alpha)-f(\alpha)=f(L_{M-1}\alpha)-f(\alpha)+\sum_{i=1}^{M-1}\nabla_{i-1,i}f(L_i\alpha)\\ &=\sum_{i=1}^{M}\nabla_{i-1,i}f(R_{i-1}\alpha)+\sum_{i=1}^{M-1}\nabla_{i-1,i}f(L_i\alpha)\end{aligned} \tag{3.79}$$

Let us study these two contributions separately. We start with the R_i's. Observe that

$$\big(R_i\alpha\big)_j=\begin{cases}\alpha_0 & j=i\\ \alpha_{j+1} & 0\leqslant j\leqslant i-1\\ \alpha_j & i+1\leqslant j\leqslant M\end{cases}\qquad i=1,\dots,M-1\,. \tag{3.80}$$

The change of variable $\alpha\to R_i\alpha$ produces then a factor

$$r_i(\alpha)=\frac{\nu(\alpha)}{\nu(R_i\alpha)}=q^{2\sum_{j=0}^{i-1}\ell_{x_j}(\alpha_j-\alpha_{j+1})}q^{2\ell_{x_i}(\alpha_i-\alpha_0)}\,. \tag{3.81}$$

Writing $\ell_{x_j} = \ell_y + j$ we have

$$\sum_{j=0}^{i-1} \ell_{x_j} \left(\alpha_j - \alpha_{j+1}\right) + \ell_{x_i} \left(\alpha_i - \alpha_0\right)$$
$$= \sum_{j=0}^{i-1} j \left(\alpha_j - \alpha_{j+1}\right) + i \left(\alpha_i - \alpha_0\right) = N_{[1,i]}(\alpha) - i\alpha_0 \,, \tag{3.82}$$

where we have used the identity

$$\sum_{j=0}^{i-1} j \left(\alpha_j - \alpha_{j+1}\right) = \alpha_1 - \alpha_2 + 2\alpha_2 - 2\alpha_3 + \cdots + (i-1)\alpha_{i-1} - (i-1)\alpha_i$$
$$= (\alpha_1 + \cdots + \alpha_{i-1}) - (i-1)\alpha_i = N_{[1,i]}(\alpha) - i\alpha_i \,, \tag{3.83}$$

and $N_{[1,i]}$ stands for the number of particles between in $\{x_1, \dots, x_i\}$. Therefore we may estimate (3.81) simply with $r_i(\alpha) \leqslant q^{-2i}$. In particular,

$$\nu\left(\left(\nabla_{i-1,i} f(R_{i-1}\alpha)\right)^2\right) \leqslant q^{-2(i-1)} \nu\left(\left(\nabla_{i-1,i} f(\alpha)\right)^2\right) \,. \tag{3.84}$$

On the other hand by Schwarz' inequality

$$\left\{\sum_{i=1}^{M} \nabla_{i-1,i} f(R_{i-1}\alpha)\right\}^2 \leqslant \sum_{j=1}^{M} q^{-2j} \sum_{i=1}^{M} q^{2i} \left(\nabla_{i-1,i} f(R_{i-1}\alpha)\right)^2 . \tag{3.85}$$

From (3.84) and (3.85) we arrive at

$$q^{2M} \nu\left(\left\{\sum_{i=1}^{M} \nabla_{i-1,i} f(R_{i-1}\alpha)\right\}^2\right) \leqslant \sum_{j=1}^{M} q^{2(M-j)} \sum_{i=1}^{M} q^2 \nu\left(\left(\nabla_{i-1,i} f\right)^2\right)$$
$$\leqslant q^2 (1-q^2)^{-1} \sum_{i=1}^{M} \nu\left(\left(\nabla_{i-1,i} f\right)^2\right) \,. \tag{3.86}$$

We turn to estimate the contribution of terms with L_i in (3.79). Notice that

$$\left(L_i \alpha\right)_j = \begin{cases} \alpha_M & j = i \\ \alpha_0 & j = M \\ \alpha_{j+1} & 0 \leqslant j \leqslant i-1 \\ \alpha_j & i+1 \leqslant j \leqslant M-1 \end{cases} \qquad i = 1, \dots, M-1 \,. \tag{3.87}$$

The change of variable $\alpha \to L_i\alpha$ gives a factor

$$l_i(\alpha) = \frac{\nu(\alpha)}{\nu(L_i\alpha)} = q^{2\sum_{j=0}^{i-1} \ell_{x_j} (\alpha_j - \alpha_{j+1})} q^{2\ell_{x_i}(\alpha_i - \alpha_M) + 2\ell_{x_M}(\alpha_M - \alpha_0)} \,. \tag{3.88}$$

As in (3.83) and (3.82) we can write

$$l_i(\alpha) = q^{2N_{[1,i]}(\alpha)} q^{2(M-i)\alpha_M} q^{-2M\alpha_0} \leqslant q^{-2M} q^{2N_{[1,i]}(\alpha)} \,. \tag{3.89}$$

In particular,

$$\nu\left(q^{-2N_{[1,i]}(\alpha)} \left(\nabla_{i-1,i} f(L_i\alpha)\right)^2\right) \leqslant q^{-2M} \nu\left(\left(\nabla_{i-1,i} f(\alpha)\right)^2\right) \,. \tag{3.90}$$

Since we are assuming $\alpha_M = 0$, we also have $(L_i\alpha)_i = \alpha_M = 0$. Thus in order to compute $\nabla_{i-1,i} f(L_i\alpha)$ we may assume $(L_i\alpha)_{i-1} = \alpha_i = 1$ and write directly $\alpha_i \nabla_{i-1,i} f(L_i\alpha)$. Using again Schwarz' inequality

$$\Big\{\sum_{i=1}^{M-1} \alpha_i \nabla_{i-1,i} f(L_i\alpha)\Big\}^2 \leqslant \sum_{j=1}^{M-1} \alpha_j q^{2N_{[1,j]}(\alpha)} \sum_{i=1}^{M-1} q^{-2N_{[1,i]}(\alpha)} \big(\nabla_{i-1,i} f(L_i\alpha)\big)^2 . \tag{3.91}$$

But

$$\sum_{j=1}^{M-1} \alpha_j q^{2N_{[1,j]}(\alpha)} = \sum_{j=1}^{N_{[1,M-1]}(\alpha)} q^{2j} \leqslant q^2(1-q^2)^{-1} .$$

Now we can estimate as in (3.86), using (3.90):

$$\begin{aligned} q^{2M}\, \nu\left(\Big\{\sum_{i=1}^{M-1} \nabla_{i-1,i} f(L_i\alpha)\Big\}^2 \alpha_0(1-\alpha_M)\right) \\ \leqslant q^2(1-q^2)^{-1} \sum_{i=1}^{M-1} \nu\left(\big(\nabla_{i-1,i} f\big)^2\right) . \end{aligned} \tag{3.92}$$

The estimates of (3.86) and (3.92) together with (3.79) imply the claim (3.75).

It is not difficult to adapt the above argument to the case $\ell_z < \ell_y$. In this case, writing $M = \ell_y - \ell_z$, (3.75) has to be replaced by

$$\nu\big((\nabla_{0,M} f)^2 \alpha_M (1-\alpha_0)\big) \leqslant 4q^2(1-q^2)^{-1} \sum_{i=1}^{M} \nu\big((\nabla_{i-1,i} f)^2\big) . \tag{3.93}$$

Now $r_i(\alpha) \leqslant q^{2N_{[1,i]}(\alpha)}$ and $l_i(\alpha) \leqslant q^{2(M-i)}$, so (3.93) follows using the estimate (3.92) for R_i-terms and (3.86) for L_i-terms. This ends the proof of Proposition 3.13. □

We can now go back to (3.73) and continue the proof of Proposition 3.12. Suppose first we are in case (MP), i.e. the sets A and B are the planar sections at level ℓ_A and ℓ_B respectively. Then, summing over $x \in A$ in (3.73),

$$\mathbb{I}_{\mathcal{V}} = 2L^2 q^{2(\ell_A - \ell_B)} \sum_{y \in B} \nu\big((\nabla_{yz} f(\alpha))^2 \alpha_y (1-\alpha_z) \mid m\big) , \tag{3.94}$$

where z is the unique element of $\Sigma_y \cap \mathcal{A}_{\ell_A}$. Since $\ell_A \geqslant \ell_B$ using Proposition 3.13 (with ν replaced by $\nu(\cdot \mid m)$) we easily obtain

$$\mathbb{I}_{\mathcal{V}} \leqslant 8q^2(1-q^2)^{-1} L^2 \sum_{b \in \mathcal{V}_\Lambda} \nu\big((\nabla_b f)^2 \mid m\big) . \tag{3.95}$$

Suppose now we are in the case (FP). Here A is a sub-cylinder with height h while B is the planar section at height 0, cf. (3.66). Then from (3.73), using Proposition 3.13 (with

ν replaced by $\nu(\cdot\,|\,m)$), we see that

$$\begin{aligned}\mathbb{I}_{\mathcal{V}} &= \sum_{x\in A}\sum_{y\in B} q^{2\ell_x}\nu\big((\nabla_{yz}f(\alpha))^2\alpha_y(1-\alpha_z)\ |\,m\big)\\ &\leqslant \sum_{x\in A} 4q^2(1-q^2)^{-1}\sum_{b\in\mathcal{V}_\Lambda}\nu\big((\nabla_b f)^2\ |\,m\big)\\ &= 8q^2(1-q^2)^{-1}hL^2\sum_{b\in\mathcal{V}_\Lambda}\nu\big((\nabla_b f)^2\ |\,m\big)\,. \end{aligned}\tag{3.96}$$

Horizontal moves. We go back to (3.70). Observe that if there is a particle at y and a hole at z the change of variable $\alpha\to T_{yz}\alpha$ produces the factor $q^{2(\ell_y-\ell_z)}$, thus canceling $q^{2(\ell_x-\ell_y)}$ in (3.70). We can estimate

$$\mathbb{I}_{\mathcal{O}} \leqslant \sum_{\substack{x\in A\\ y\in B}} \nu\big((\nabla_{xz}f)^2\ |\,m\big)\tag{3.97}$$

Consider the case (MP) first. Now both x and z lie on the plane $\mathcal{A}_{\ell_A}$. We fix a choice of paths on this plane as follows. For each couple $x,z\in\mathcal{A}_{\ell_A}$ we take the path γ_{xz} obtained by connecting x to z first along the direction e_1 and then along the direction e_2. As in the case of vertical moves we use a telescopic sum to write $\nabla_{xz}f$, thus obtaining two sums over all bonds in the path γ_{xz}, cf. (3.79). Since here we only have horizontal exchanges there are no factors when we change variables and we simply use Schwarz' inequality to obtain

$$\begin{aligned}\nu\big((\nabla_{xz}f)^2\ |\,m\big) &\leqslant 2|\gamma_{xz}|\sum_{b\in\gamma_{xz}}\nu\big((\nabla_b f)^2\ |\,m\big)\\ &\leqslant 8L\sum_{b\in\gamma_{xz}}\nu\big((\nabla_b f)^2\ |\,m\big)\,,\end{aligned}\tag{3.98}$$

where we used $|\gamma_{xz}|\leqslant 4L$. Moreover, for any bond b in the plane

$$\sum_{\substack{x\in A\\ y\in B}} 1_{\{b\in\gamma_{xz}\}}\leqslant 4L^3\,.\tag{3.99}$$

When we sum in (3.97) we obtain

$$\mathbb{I}_{\mathcal{O}}\leqslant 32L^4\sum_{b\in\mathcal{O}_\Lambda}\nu\big((\nabla_b f)^2\ |\,m\big)\,.\tag{3.100}$$

Consider now the case (FP). Here A is the sub-cylinder at height h and B is the planar section at height 0, see (3.66). The same estimate (3.100) applies since when summing over $x\in A$ we are now summing over all layers up to level $h-1$ and the r.h.s. in (3.100) contains all bonds in such planes.

Collecting all the estimates in (3.95), (3.96) and (3.100) and plugging into (3.69) we have obtained the desired bound (3.67). This completes the proof of Proposition 3.12.

4. Recursive Proof of Theorem 2.7

We begin by describing the main ideas behind the recursive proof of Theorem 2.7. Let $\mathrm{Var}_{\Lambda,n}(f)$ denote the variance of a function f w.r.t. $\nu_{\Lambda,n}$ and let

$$W(\Lambda)=\max_n \sup_f \frac{\mathrm{Var}_{\Lambda,n}(f)}{\mathcal{E}_{\Lambda,n}(f,f)}\,, \tag{4.1}$$

where the supremum is taken over all non-constant $f:\Omega_\Lambda\to\mathbb{R}$. When $\Lambda=\Sigma_{L,H}$ we write $W(L)$ for $\sup_H W(\Sigma_{L,H})$. The lower bound in Theorem 2.7 follows if we can prove that for any $q\in(0,1)$ there exists $k<\infty$ such that

$$W(L)\leqslant kL^2 \tag{4.2}$$

for any L of the form $L=2^j$, $j\in\mathbb{N}$. In turn (4.2) follows at once if we can prove that for any $q\in(0,1)$ there exist $k<\infty$ and $L_0>0$ such that

$$W(2L)\le 3W(L)+kL^2 \qquad L\ge L_0 \tag{4.3}$$

$$W(2L)\le kW(L)+k \qquad L\le L_0 \tag{4.4}$$

$$W(1)\leqslant k\,. \tag{4.5}$$

4.1. **Transport theorem and proof of the recursive inequalities.** The starting point to prove the recursive inequalities is the formula for conditional variance that we now describe.

Consider the cylinder $\Sigma_{2L,H}$ and divide it into two parts (cf. (3.55))

$$\Lambda=\Sigma_{2L,H}=\Lambda_1\cup\Lambda_2\,,\quad \Lambda_1=\Sigma_{R_{L,2L},H}\,,\ \Lambda_2=(L,0,0)+\Sigma_{R_{L,2L},H}\,.$$

Fix $n\in\{1,\dots,2L^2H=|\Lambda|/2\}$ and let $\nu_{\Lambda,n}$ denote as usual the canonical measure on $\Lambda=\Sigma_{2L,H}$ with total particle number n. Conditioning on the number of particles in Λ_1 decomposes the variance as follows:

$$\mathrm{Var}_{\Lambda,n}(f)=\nu_{\Lambda,n}\left(\mathrm{Var}_{\Lambda,n}(f\,|N_{\Lambda_1})\right)+\mathrm{Var}_{\Lambda,n}\left(\nu_{\Lambda,n}(f\,|N_{\Lambda_1})\right)\,. \tag{4.6}$$

Moreover, the above conditioning breaks $\nu_{\Lambda,n}$ into the product $\nu_{\Lambda_1,N_{\Lambda_1}}\otimes\nu_{\Lambda_2,n-N_{\Lambda_1}}$, and therefore

$$\nu_{\Lambda,n}\left(\mathrm{Var}_{\Lambda,n}(f\,|N_{\Lambda_1})\right)\leqslant \nu_{\Lambda,n}\Big(\mathrm{Var}_{\Lambda_1,N_{\Lambda_1}}(f)+\mathrm{Var}_{\Lambda_2,n-N_{\Lambda_1}}(f)\Big)\,. \tag{4.7}$$

The first term in (4.6) is then estimated above using (4.1):

$$\begin{aligned}\nu_{\Lambda,n}\left(\mathrm{Var}_{\Lambda,n}(f\,|N_{\Lambda_1})\right)&\leqslant W(\Sigma_{R_{L,2L},H})\,\nu_{\Lambda,n}\Big[\mathcal{E}_{\Lambda_1,N_{\Lambda_1}}(f,f)+\mathcal{E}_{\Lambda_2,n-N_{\Lambda_1}}(f,f)\Big]\\&\leqslant W(\Sigma_{R_{L,2L},H})\,\mathcal{E}_{\Lambda,n}(f,f)\end{aligned} \tag{4.8}$$

The analysis of the second term in (4.6) is more delicate and is directly related to transport of particles. In a sense it represents the core of the proof. As we will see we will provide two different bounds on the transport term: the first one is rather subtle but it is valid only for large enough L. The second one, valid for any value of L, is much more rough and therefore it will be used only for those values of L for which the first bound is not known to hold. For simplicity, in what follows, we will always refer to these two situations as the "large" or "small" L case.

Recall now the definition (2.30) and (2.31) of the horizontal and vertical part of the Dirichlet form. Then we have

Theorem 4.1.

(i) `Large` L.
For any $\epsilon > 0$, $q \in (0,1)$, *there exists a finite constants* $C_\epsilon = C(\epsilon, q)$, $k = k(q)$ *and* $L_0 = L_0(\epsilon, q)$ *such that for any* $L > L_0$, $H \geqslant 1$ *and for any* $n = 1, 2, \dots, |\Lambda| - 1$

$$\mathrm{Var}_{\Lambda,n}\left(\nu_{\Lambda,n}(f\,|N_{\Lambda_1})\right) \leqslant$$
$$k\left\{L^2\mathcal{E}_n^{\mathcal{O}_\Lambda}(f,f) + \mathcal{E}_n^{\mathcal{V}_\Lambda}(f,f)\right\} + C_\epsilon\,\mathcal{E}_n(f,f) + \epsilon\,\mathrm{Var}_{\Lambda,n}(f)\,,$$

(ii) `Small` L.
For any $q \in (0,1)$ *and for any* $L \geq 1$ *there exists a finite constant* $C = C(L,q)$ *such that, for any* $H \geqslant 1$ *and any* $n = 1, 2, \dots, |\Lambda| - 1$

$$\mathrm{Var}_{\Lambda,n}\left(\nu_{\Lambda,n}(f\,|N_{\Lambda_1})\right) \leqslant C\Big\{\,\mathcal{E}_n(f,f) + \nu_{\Lambda,n}\big(\,\mathrm{Var}_{\Lambda,n}\left(f\,|\,N_{\Lambda_1}\right)\big)\,\Big\}$$

Once Theorem 4.1 is proven, we use (4.6) and (4.8) to obtain

$$\mathrm{Var}_{\Lambda,n}(f) \leqslant (1-\epsilon)^{-1}\left\{W(\Sigma_{R_{L,2L},H}) + kL^2\right\}\mathcal{E}_{\Lambda,n}(f,f) \qquad \texttt{for } L \geq L_0(\epsilon,q) \tag{4.9}$$

$$\mathrm{Var}_{\Lambda,n}(f) \leqslant \left\{C_0\,W(\Sigma_{R_{L,2L},H}) + C_0\,\right\}\mathcal{E}_{\Lambda,n}(f,f) \qquad \texttt{for } L \leq L_0(\epsilon,q) \tag{4.10}$$

where $C_0 = C(L_0, q)$. In the large L case (4.9) proves in particular that

$$W(\Sigma_{R_{2L,2L},H}) \leqslant (1-\epsilon)^{-1}\left\{W(\Sigma_{R_{L,2L},H}) + kL^2\right\}. \tag{4.11}$$

We repeat now the decomposition (4.6) for $\Lambda = \Sigma_{R_{L,2L},H}$, writing the latter cylinder as $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 = \Sigma_{L,H}$ and $\Lambda_2 = (0,L,0) + \Sigma_{L,H}$. Applying the same reasoning as above we arrive at

$$W(\Sigma_{R_{L,2L},H}) \leqslant (1-\epsilon)^{-1}\left\{W(\Sigma_{L,H}) + kL^2\right\}, \qquad \forall\, H \geqslant 1 \text{ and } L \geqslant L_0(\epsilon,q) \tag{4.12}$$

From (4.11) and (4.12) we finally obtain

$$W(2L) \leqslant (1-\epsilon)^{-2}\left\{W(L) + kL^2\right\} \qquad \forall\, L \geq L_0(\epsilon,q) \tag{4.13}$$

which proves (4.3) due to the arbitrariness of ϵ. Equation (4.4) is proved similarly starting from (4.10). The bound (4.5) is given in the next subsection.

4.2. **Spectral gap in the one dimensional case.** In this final paragraph we prove that $W(1) < \infty$. In other words we show that the spectral gap for the one dimensional asymmetric simple exclusion process with generator (2.28) in the interval $\Sigma_H := \{x = (0,0,l) : 0 \leq l \leq H-1\}$ is bounded away from zero uniformly in the number of particles and in the length of the interval H. Such a result has already been proved for the one dimensional XXZ model in [22] but we decided to present a "probabilistic" proof for completeness. Here is our formal statement. Below we write $\nu := \nu_{\Sigma_H,n}$, $\mathcal{E} := \mathcal{E}_{\Sigma_H,n}$

Theorem 4.2. *For any* $q \in (0,1)$ *there exists a constant* k *such that for any* $H \geqslant 1$, *any* $n \leqslant H$ *and any function* f

$$\mathrm{Var}_\nu(f) \leq k\,\mathcal{E}(f,f)\,.$$

In particular, $W(1) \leqslant k$.

Proof. Let $\gamma(n,H)$ denote the inverse spectral gap for the process in Σ_H with n particles and let $\gamma(n) = \sup_H \gamma(n,H)$. Notice that, by the particle–hole duality, $\gamma(n,H) = \gamma(H-n,H)$ and therefore we will always assume, without loss of generality, that $n \leqslant \frac{H}{2}$. If $n = 1$ then it is well known, by e.g. Hardy's [20] or Cheeger inequality [11], that $\gamma(1) < \infty$. Our idea is to perform a sort of induction on the number of particles. For this purpose, for each configuration α with n particles we denote by $\xi := \xi(\alpha)$ the position of the last particle, namely $\xi = \max\{x \in \Sigma_H : \alpha_x = 1\}$, and we set $\rho(x) = \nu(\xi = x)$ the probability that $\xi = x$. It is not difficult to see that the distribution of ξ has an exponential falloff so that, in particular, it satisfies a Poincareé inequality with constant depending only on q. More precisely we have the following

Lemma 4.3. *For any $q \in (0,1)$ there exists k such that for any $f(\alpha) := F(\xi(\alpha))$*

$$\mathrm{Var}_\nu(f) \le k \sum_{x \ge n-1} \big(\rho(x) \wedge \rho(x+1)\big)[F(x+1) - F(x)]^2 \qquad \forall H \geqslant n\,. \tag{4.14}$$

Proof. Using Cheeger inequality it is enough to prove that there exists $x_0 \ge n-1$ and $\beta < 1$ depending only on q such that $\frac{\rho(x+1)}{\rho(x)} \le \beta$ for any $x \ge x_0$. A simple change of variables (see (4.21) below) shows that

$$\frac{\rho(x+1)}{q\rho(x)} \nu\big((1-\alpha_x)\,|\, \xi = x+1\big) = 1 \tag{4.15}$$

In order to complete the proof it is enough to prove that $\nu\big(\alpha_x \,|\, \xi = x+1\big)$ tends to zero for large x uniformly in $n \le \frac{H}{2}$. For any $x \ge n$ we have

$$\begin{aligned}\nu\Big(\alpha_x \,|\, \xi = x+1\Big) \;&\le\; \frac{\mu(\alpha_x = 1\,;\, \alpha_{x+1} = 1\,;\, \alpha_y = 0\ \forall y > x+1)}{\mu(N_{[0,n-2]} = n-1\,;\, \alpha_{x+1} = 1\,;\, \alpha_y = 0\ \forall y \geqslant n-1, y \ne x+1)} \\ &\le \frac{\mu(\alpha_x)}{\big[\prod_{x=0}^{n-2} \mu(\alpha_y)\big]\ \big[\prod_{\substack{y\ne x:\\ y\,\geqslant\, n-1}} \mu(1-\alpha_y)\big]} \le k\, q^{2(x-n)}\end{aligned} \tag{4.16}$$

for some constant $k = k(q)$. Above we have used the explicit product structure of the measure $\mu := \mu^{\lambda(n)}$ together with the fact proved in (3.1) that $|\lambda(n) - n| \le k'$. ☐

We are now in a position to prove the theorem. We write

$$\mathrm{Var}_\nu(f) = \nu\big(\mathrm{Var}_\nu(f\,|\,\xi)\big) + \mathrm{Var}_\nu\big(\nu(f\,|\,\xi)\big) \tag{4.17}$$

The first term in the r.h.s. of (4.17) coincides with

$$\sum_{x\,\geqslant\, n-1} \rho(x)\, \mathrm{Var}_{\nu_{[0,x-1],n-1} \otimes \nu_{[x,H-1],0}}(f)$$

and therefore it can be bounded from above, using the definition of $\gamma(n,x)$, by

$$\sum_{x \ge n-1} \rho(x)[\gamma(n-1,x) \wedge \gamma(x-n+1,x)]\mathcal{E}_{[0,x-1],n-1}(f,f) \tag{4.18}$$

because of the holes–particles duality. Here and below $\gamma(0,x) = 0$ for all x.
Let us examine the second term. Here we apply Lemma 4.3 to write

$$\mathrm{Var}_\nu\big(\nu(f\,|\,\xi)\big) \le k \sum_{x\,\geqslant\, n-1} \rho(x) \wedge \rho(x+1)\big[F(x+1) - F(x)\big]^2 \tag{4.19}$$

where $F(x) = \nu(f \,|\, \xi = x)$. In order to compute the "gradient" of $F(x)$ we write

$$
\begin{aligned}
F(x) &= \sum_{\alpha;\,\xi(\alpha)=x} \frac{\nu(\alpha)}{\rho(x)} f(\alpha)\alpha_x \\
&= \frac{\rho(x+1)}{q\rho(x)} \sum_{\alpha;\,\xi(\alpha)=x+1} \frac{\nu(\alpha)}{\rho(x+1)} f(\alpha^{x,x+1})(1-\alpha_x) \\
&= \frac{\rho(x+1)}{q\rho(x)} \Big[\nu\Big([\nabla_{x,x+1} f](1-\alpha_x) \,|\, \xi = x+1 \Big) + \nu\Big(f, (1-\alpha_x) \,|\, \xi = x+1 \Big) \Big] + \\
&\quad + \frac{\rho(x+1)}{q\rho(x)} \nu(f \,|\, \xi = x+1) \nu\big((1-\alpha_x) \,|\, \xi = x+1 \big)
\end{aligned}
\tag{4.20}
$$

Setting $f = 1$ gives

$$
\frac{\rho(x+1)}{q\rho(x)} \nu\big((1-\alpha_x) \,|\, \xi = x+1 \big) = 1 \tag{4.21}
$$

Therefore the last term in the r.h.s. of (4.20) is equal to $F(x+1)$.
Thanks to (4.16) $\left(\frac{\rho(x+1)}{q\rho(x)} \right) \le k$ uniformly in n, H. In conclusion

$$
\begin{aligned}
&\big[F(x+1) - F(x) \big]^2 \le \\
&k' \nu\Big((\nabla_{x,x+1} f)^2 \,|\, \xi = x+1 \Big) + k' \varepsilon(x) \nu\Big(f, f \,|\, \xi = x+1 \Big)
\end{aligned}
\tag{4.22}
$$

where

$$
\varepsilon(x) := \nu\Big((1-\alpha_x), (1-\alpha_x) \,|\, \xi = x+1 \Big) \le k\, q^{2(x-n)} \tag{4.23}
$$

because of (4.16). Thus the r.h.s. of (4.17) is bounded from above by

$$
\sup_{n-1 \,\leqslant\, x \,\leqslant\, H-1} \Big[\big\{ \big[(\gamma(n-1, x) \wedge \gamma(x-n+1, x) \big] (1 + k\, q^{2(x-n)}) \big\} \vee k' \Big] \mathcal{E}(f, f) \tag{4.24}
$$

In other words we have proved the recursive inequality

$$
\begin{aligned}
\gamma(n) &\leqslant \sup_{x \,\geqslant\, n} \Big[\big\{ \big(\gamma(n-1) \wedge \gamma(x-n) \big) (1 + k\, q^{2(x-n)}) \big\} \vee k' \Big] \\
&\leqslant \sup_{n \,\leqslant\, x \,\leqslant\, 2n-1} \Big[\big(\gamma(x-n) \vee k'' \big) (1 + k\, q^{2(x-n)}) \Big].
\end{aligned}
\tag{4.25}
$$

It is quite simple now to conclude that $\gamma(n)$ is uniformly bounded. Indeed if $\tilde{\gamma}(m) := (\gamma(m) \vee k'')$ then (4.25) tells us that

$$
\tilde{\gamma}(m) \leqslant \sup_{1 \,\leqslant\, \ell \,\leqslant\, m-1} \Big[\tilde{\gamma}(\ell)(1 + kq^{2\ell}) \Big]. \tag{4.26}
$$

We then have a sequence $\ell_1 < \ell_2 < \cdots < \ell_s$, $s \leqslant m-1$ such that

$$
\tilde{\gamma}(m) \leqslant \tilde{\gamma}(1) \prod_{i=1}^{s} (1 + kq^{2\ell_i})
$$

which is finite since $\tilde{\gamma}(1) < \infty$ and $\sum_i q^{2\ell_i} < \infty$. Thus $\tilde{\gamma}(m)$ is uniformly bounded and so is $\gamma(n)$. ◻

5. Proof of Theorem 4.1

The setting in this section is as in (3.55). For notation convenience in what follows we will drop the subscripts Λ, n. We also use $\nu(\cdot\,|\,m)$ for $\nu(\cdot\,|\,N_{\Lambda_1} = m)$. If we apply Proposition 3.11 to the function $g(N_{\Lambda_1}) = \nu\big(f\,|N_{\Lambda_1}\big)$ we get

$$\mathrm{Var}_\nu\,\big(g(N_{\Lambda_1})\big) \leqslant k\big(n \wedge L^2\big) \sum_m p_n(m) \wedge p_n(m+1)\,[g(m+1) - g(m)]^2 \tag{5.1}$$

where $p_n(m) = \nu\big(N_{\Lambda_1} = m\big)$.
Therefore we need to study the gradient $g(m+1) - g(m)$. For this purpose the main idea (very roughly) is the following.
Pick a configuration α such that $N_{\Lambda_1}(\alpha) = m+1$ and $N_\Lambda(\alpha) = n$, choose two sites $x \in \Lambda_1$, $y \in \Lambda_2$ such that $\alpha(x) = 1$, $\alpha(y) = 0$ and consider the exchanged configuration $\eta = \alpha^{xy}$. Clearly $N_{\Lambda_1}(\eta) = m$ and $N_\Lambda(\eta) = n$. Using this kind of change of variables it is not difficult to write an expression for the gradient $g(m+1) - g(m)$ in terms of suitable spatial averages of $\nabla_{xy} f$ plus a covariance term $\nu(f, F_{xy})$, where the latter originates from the action of the change of variables on the probability measures.
One possibility to concretely implement this program is to write

$$\nu\big(f\,|\,m+1\big) = \frac{1}{(m+1)(n-m-1)} \sum_{x\in\Lambda_1, y\in\Lambda_2} \nu\big(f\,\alpha_x(1-\alpha_y)\,|\,m+1\big)$$

and to make the change of variables described above for each pair (x, y). This idea works just fine in the context of translation invariant lattice gases [7], but has some drawback in our context due to the nature of the typical configurations of the measure $\nu\big(\cdot\,|\,m+1\big)$. As already shown, the $m+1$ particles in Λ_1 tend to fill the cylinder Λ_1 up to a well specified height and the same for Λ_2. Without loss of generality we can assume $m \geqslant n/2$ so that the resulting surface in Λ_1 will stay higher than the surface in Λ_2. Thus, if we don't want to transform a typical configuration of $\nu\big(\,\cdot\,|\,m+1\big)$ into an atypical one for $\nu\big(\,\cdot\,|\,m\big)$ via the exchange T_{xy}, we should only try to exchange the holes that sit on the surface in Λ_2 with the particles on the surface in Λ_1. In other words the above (deterministic) sum

$$\sum_{x\in\Lambda_1, y\in\Lambda_2} \alpha_x(1-\alpha_y)$$

should be replaced by a random variable

$$\sum_{x\in A,\, y\in B} \alpha_x(1-\alpha_y)$$

where A, B denote the two surfaces. Of course, for certain rare configurations, the surfaces either do not exist or their density of particles is far from its typical value. We are forced therefore to split according to some criterium the contribution to the gradient $g(m+1)-g(m)$ coming from typical and rare configurations and apply the above reasoning only to the typical cases. The contribution coming from the rare configurations should be estimated via moderate deviation bounds for the measure $\nu\big(\,\cdot\,|\,m+1\big)$.
We will now make precise what we just said. In the rest of this section we will always assume $m \geq \frac{n}{2}$.
For any event $G \subset \Omega_\Lambda$ we write

$$\begin{gathered}\big[\nu\big(f\,|\,m+1\big) - \nu\big(f\,|\,m\big)\big]\nu(G\,|\,m+1) = \\ \big[\nu\big(f\,\mathbb{1}_G\,|\,m+1\big) - \nu\big(f\,|\,m\big)\nu\big(G\,|\,m+1\big)\big] - \nu\big(f, \mathbb{1}_G\,|\,m+1\big)\,.\end{gathered} \tag{5.2}$$

We then estimate

$$\Big[\nu\big(f\mid m+1\big)-\nu\big(f\mid m\big)\Big]^2 \leqslant 2\frac{\nu\big(G^c\mid m+1\big)}{\nu\big(G\mid m+1\big)}\operatorname{Var}_\nu\big(f\,|\,m+1\big)$$
$$+\,2\frac{1}{\nu\big(G\mid m+1\big)^2}\left[\nu\big(f\,\mathbb{1}_G\mid m+1\big)-\nu\big(f\mid m\big)\nu\big(\mathbb{1}_G\mid m+1\big)\right]^2 \tag{5.3}$$

5.1. **The typical events.** We will provide different definitions of the typical event G according to whether L is "large" or "small" and whether we have "many" MP or "few" FP particles (see beginning of §3.5).

L large.
We start with the MP case.
Take $\lambda,\lambda'\in\mathbb{R}$ such that

$$\mu_{\Lambda_1}^{\lambda}(N_{\Lambda_1})=m\,,\quad \mu_{\Lambda_2}^{\lambda'}(N_{\Lambda_2})=n-m\,. \tag{5.4}$$

Set

$$\ell_A=[\lambda\vee 0]\wedge(H-1)\,,\quad \ell_B=[\lambda'\vee 0]\wedge(H-1) \tag{5.5}$$

and define A,B as the planar sections of Λ_1 and Λ_2 at height ℓ_A and ℓ_B respectively. More precisely

$$A=\Lambda_1\cap\mathcal{A}_{\ell_A}\qquad B=\Lambda_2\cap\mathcal{A}_{\ell_B} \tag{5.6}$$

Define the number of particles in A and the number of holes in B

$$N_A(\alpha)=\sum_{x\in A}\alpha_x\,,\quad V_B(\alpha)=\sum_{y\in B}(1-\alpha_y)\,, \tag{5.7}$$

and define $\bar N_A=\nu(N_A\,|\,m)$, $\bar V_B=\nu(V_B\,|\,m)$.

Definition 5.1. *The event G in the* MP, L large *case.*
We set $G=G_A\cap G_B$, where

$$G_A=\{|N_A(\alpha)-\bar N_A|\leqslant(n\wedge L^2)^{\frac12+\gamma}\}\,,\quad G_B=\{|V_B(\alpha)-\bar V_B|\leqslant L^{1+2\gamma}\}\,. \tag{5.8}$$

Here γ is a small positive number, say $\gamma=.001$.

Note that in any case when L is large G implies

$$N_A\geqslant\frac14(n\wedge L^2)\,,\quad V_B\geqslant L^2\,. \tag{5.9}$$

We turn to the case (FP). Here we do not fix two planar sections but rather confine most of the particles in a cylinder with finite height. Let $h\in\mathbb{N}$, $1\leqslant h\leqslant H$ and define

$$A=\big\{x\in\Lambda_1:\ \ell_x\leqslant h-1\big\}\,,\quad B=\big\{y\in\Lambda_2:\ \ell_y=0\big\}\,. \tag{5.10}$$

Definition 5.2. *The event G in the* FP, L large *case.*
We set $G=G_A\cap G_B$, with

$$G_A=\{N_A(\alpha)\geqslant m/2\}\,,\quad G_B=\big\{|V_B(\alpha)-\bar V_B|\leqslant L^{1+2\gamma}\big\}\,. \tag{5.11}$$

Note that here too when L is large G implies

$$N_A\geqslant\frac14 n\,,\quad V_B\geqslant L^2\,. \tag{5.12}$$

Finally we analyze the case of L small. The construction of the sets A and B is done exactly as before in the two cases MP and FP but the definition of G changes.

Definition 5.3. *The event G in the L `small` case.*
We set $G = G_A \cap G_B$, where

$$G_A = \{N_A(\alpha) \geq 1\} \ , \quad G_B = \{V_B(\alpha) \geq 1\} \ . \tag{5.13}$$

and A and B are as in (5.6) or as in (5.10) depending on whether we are in the `MP` *or the* `FP` *case.*

5.2. **Bounds on the probability of the typical events.** In what follows we will provide some simple estimates on the probability of the event G^c in the various cases of few/many particles and large/small L. We begin by stating our bounds to be used when L is large.

Lemma 5.4. *Assume* (`MP`). *For any $q \in (0,1)$ there exist $k < \infty$ such that*

$$\nu\big(G^c \,|\, m\big) \leqslant k \exp(-k^{-1} L^{4\gamma}) \, . \tag{5.14}$$

Proof. Observe that

$$\nu\big(G^c \,|\, m\big) \leqslant \nu_{\Lambda_1,m}\big[G^c_A\big] + \nu_{\Lambda_2,n-m}\big[G^c_B\big] \, .$$

We first prove

$$\nu_{\Lambda_1,m}\big[G^c_A\big] \leqslant k \exp(-k^{-1} L^{4\gamma}) \, . \tag{5.15}$$

We write $A = A_1 \cup A_2$ with $A_1 = \Sigma_{L,H} \cap \mathcal{A}_{\ell_A}$ and $A_2 = \{(0,L,0) + \Sigma_{L,H}\} \cap \mathcal{A}_{\ell_A}$. Letting

$$G_i = \big\{|N_{A_i}(\alpha) - \bar{N}_A/2| \leqslant \frac{1}{2}(n \wedge L^2)^{1/2+\gamma}\big\}, \quad i = 1, 2 \, ,$$

we see that

$$\nu_{\Lambda_1,m}\big[G^c_A\big] \leqslant \nu_{\Lambda_1,m}\big[G^c_1\big] + \nu_{\Lambda_1,m}\big[(G_2)^c\big] \tag{5.16}$$

By Proposition 3.8 we can estimate (5.16) with the help of the grand canonical distribution μ^λ where λ is given by (5.4). Thus

$$\nu_{\Lambda_1,m}\big[G^c_A\big] \leqslant k\mu^\lambda\big[G^c_1\big] + k\mu^\lambda\big[(G_2)^c\big] = 2k\mu^\lambda\big[G^c_1\big] \, . \tag{5.17}$$

Note that α_x, $x \in A_1$ are i.i.d. random variables under μ^λ, with mean value $\rho_x = \mu^\lambda(\alpha_x)$. Let us consider the case $n \leqslant L^2$ in detail. For the case $n \geqslant L^2$ simply replace n by L^2 in the lines below. We have, for any $t \geqslant 0$

$$\mu^\lambda\big[G^c_1\big] \leqslant \exp\left(-t\, n^{1/2+\gamma}\right) \left[\exp|A_1|\varphi(t) + \exp|A_1|\varphi(-t)\right] \, , \tag{5.18}$$

where

$$\varphi(t) = \log \mu^\lambda_{\Lambda_1}\big(\exp t\,(\alpha_x - \rho_x)\big) \, . \tag{5.19}$$

Then $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(t) = \mathrm{Var}^{\lambda_t}(\alpha_x)$, where $\lambda_t = \lambda + t/(-2\log q)$. Now, for any $|t| \leqslant 1$,

$$\mathrm{Var}_{\mu^{\lambda_t}}(\alpha_x) = \mu^{\lambda_t}\left[(\alpha_x - \mu^{\lambda_t}(\alpha_x))^2\right] \leqslant \mu^{\lambda_t}\left[(\alpha_x - \mu^{\lambda}(\alpha_x))^2\right] \leqslant \mathrm{e}^2 \mathrm{Var}_{\mu^\lambda}(\alpha_x) \, . \tag{5.20}$$

Then

$$|\varphi(t)| \leqslant 5\mathrm{Var}_{\mu^\lambda}(\alpha_x) t^2 \, , \quad |t| \leqslant 1 \, . \tag{5.21}$$

Using Lemma 3.2 and Lemma 3.1 we have

$$|A_1|\mathrm{Var}_{\mu^\lambda}(\alpha_x) = \mathrm{Var}_{\mu^\lambda}(N_{A_1}) \leqslant km \leqslant kn \, .$$

Therefore by (5.18), choosing $t = O(n^{-1/2+\gamma})$ we obtain

$$\mu^\lambda\big[G^c_1\big] \leqslant k \exp(-k^{-1} n^{2\gamma}) \leqslant k' \exp(-k'^{-1} L^{4\gamma}) \tag{5.22}$$

The estimate for V_B is obtained in a similar fashion. $\square$

We turn to analyze the case of few particles. Again our estimate will be meaningful only if L is large enough.

Lemma 5.5. *Assume* (FP)*. For any $q \in (0,1)$ there exist $k < \infty$, $h_0 < \infty$ such that for all $m \in [\frac{n}{2}, n]$ and $h \geqslant h_0$ we have*

$$\nu\big(G^c \,|\, m\big) \leqslant k\big(n^{-1}q^{2h} + \exp\left(-k^{-1}L^{4\gamma}\right)\big) . \tag{5.23}$$

Proof. Repeating the argument leading to (5.18) and (5.21), choosing $t = O(L^{-1+2\gamma})$, we easily obtain

$$\nu_{\Lambda_2, n-m}\big[G_B^c\big] \leqslant k \exp\left(-tL^{1+2\gamma}\right) \exp\left(knt^2\right) \leqslant k' \exp\left(-k'^{-1}L^{4\gamma}\right) . \tag{5.24}$$

Let $\bar{A} = \Lambda_1 \setminus A$ and write

$$\nu_{\Lambda_1, m}\big[G_A^c\big] \leqslant \nu_{\Lambda_1, m}\big[N_{\bar{A}} \geqslant m/2\big] .$$

Dividing $\bar{A}$ in two parts $\bar{A} = A_1 \cup A_2$ with $A_1 = \Sigma_{L,H} \cap \bar{A}$ and $A_2 = \{(0, L, 0) + \Sigma_{L,H}\} \cap \bar{A}$, we may estimate

$$\nu_{\Lambda_1, m}\big[N_{\bar{A}} \geqslant m/2\big] \leqslant \nu_{\Lambda_1, m}\big[N_{A_1} \geqslant m/4\big] + \nu_{\Lambda_1, m}\big[N_{A_2} \geqslant m/4\big] .$$

Then by Proposition 3.8 it is sufficient to estimate $\mu^\lambda_{\Lambda_1}[N_{A_1} \geqslant m/4]$, where λ is given by (5.4). Since $m \leqslant n \leqslant \delta L^2$ we have $\lambda \leqslant 0$ from Lemma 3.1. Therefore

$$\begin{aligned}
\mu^\lambda_{\Lambda_1}(N_{A_1}) &= L^2 \sum_{j=h}^{H-1} \frac{q^{2(j-\lambda)}}{1 + q^{2(j-\lambda)}} \leqslant 2q^{2h} L^2 \sum_{j=0}^{H-1} \frac{q^{2(j-\lambda)}}{1 + q^{2(j-\lambda)}} \\
&= 2q^{2h} \mu^\lambda_{\Lambda_1}(N_{\Lambda_1}) = 2q^{2h} m .
\end{aligned}$$

We then estimate

$$\mu^\lambda_{\Lambda_1}\big[N_{A_1} \geqslant m/4\big] \leqslant \mu^\lambda_{\Lambda_1}\big[|N_{A_1} - \mu^\lambda_{\Lambda_1}(N_{A_1})| \geqslant cm\big] ,$$

with $c > 0$, if $h \geqslant h_0(q)$ for some $h_0(q) < \infty$. Then

$$\begin{aligned}
\mu^\lambda_{\Lambda_1}[N_{A_1} \geqslant m/4] &\leqslant \frac{1}{c^2 m^2} \mathrm{Var}_{\mu^\lambda_{\Lambda_1}}(N_{A_1}) = \frac{L^2}{c^2 m^2} \sum_{j=h}^{H} \frac{q^{2(j-\lambda)}}{(1 + q^{2(j-\lambda)})^2} \\
&\leqslant \frac{(1-q^2)^{-1} q^{2(h-\lambda)} L^2}{c^2 m^2} \leqslant \frac{4q^{2h}}{c^2 m} ,
\end{aligned} \tag{5.25}$$

where in the last bound we use

$$q^{-2\lambda} \leqslant \frac{4m}{L^2}(1 - q^2) ,$$

which follows from Lemma 3.1. Since $m \geqslant n/2$ (5.25) and (5.24) yield (5.23). $\square$

Finally we analyze the case L small. Below the event G will be that appearing in (5.13)

Lemma 5.6.

$$\nu(G \,|\, m) \geqslant c(q, L) > 0 , \tag{5.26}$$

with a constant $c(q, L)$ independent of the height H of the cylinder.

Proof. Since

$$\nu(G_A \,|\, m) = \frac{\mu^\lambda_{\Lambda_1}(G_A \cap \{N_{\Lambda_1} = m\})}{\mu^\lambda_{\Lambda_1}(\{N_{\Lambda_1} = m\})}\,, \tag{5.27}$$

the claim easily follows from a slight modification of the argument given at the end of Theorem 3.4 (packing all particles at the bottom of the cylinder). The same can be done for the event G_B. □

5.3. **Bounding the gradient** $\left[\nu\big(f\mathbb{1}_G \,|\, m+1\big) - \nu\big(f \,|\, m\big)\nu\big(G \,|\, m+1\big)\right]^2$**.** From Lemmas 5.4, 5.5 and 5.6 we see that, for any $\epsilon \in (0,1)$ the first term in the r.h.s. of (5.3) can be bounded from above by

`(i)` L `large`.

$$2\frac{\nu\big(G^c \,|\, m+1\big)}{\nu\big(G \,|\, m+1\big)}\operatorname{Var}_\nu\big(f \,|\, m+1\big) \le \frac{\epsilon}{n \wedge L^2}\operatorname{Var}_\nu\big(f \,|\, m+1\big) \tag{5.28}$$

provided that L and the constant h in Lemma 5.5 are large enough depending on ϵ.

`(ii)` L `small`.

$$2\frac{\nu\big(G^c \,|\, m+1\big)}{\nu\big(G \,|\, m+1\big)}\operatorname{Var}_\nu\big(f \,|\, m+1\big) \le C\operatorname{Var}_\nu\big(f \,|\, m+1\big) \tag{5.29}$$

where $C = C(L)$ is some finite constant independent of m.

We now turn our attention to the second term appearing in the r.h.s. of (5.3). As before, the factor $2\nu\big(G \,|\, m+1\big)^{-2}$ can be bounded from below by either $\frac{2}{(1-\epsilon)^2}$ or by $C'(L)$ for a suitable constant $C'(L)$ according to whether L is large enough (depending on ϵ) or it is small (i.e. smaller than some L_0).
We thus concentrate on the computation of the relevant term

$$\left[\nu\big(f\mathbb{1}_G \,|\, m+1\big) - \nu\big(f \,|\, m\big)\nu\big(G \,|\, m+1\big)\right]^2 .$$

The following calculation holds irrespectively of which definition of G is adopted.
Defining

$$\phi_{xy}(\alpha) = \frac{\alpha_x(1-\alpha_y)}{N_A(\alpha)V_B(\alpha)}\,, \quad x \in A,\, y \in B\,,$$

we may write

$$\nu\big(f\mathbb{1}_G \,|\, m+1\big) = \sum_{\substack{x\in A\\ y\in B}} \nu\big(f\mathbb{1}_G\phi_{xy} \,|\, m+1\big)\,. \tag{5.30}$$

With the change of variables $\alpha \to T_{xy}\alpha$, (5.30) becomes

$$\begin{aligned}\nu\big(f\mathbb{1}_G \,|\, m+1\big) &= \frac{p_n(m)}{p_n(m+1)}\sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)}\nu\big(T_{xy}\left[f\mathbb{1}_G\phi_{xy}\right] \,|\, m\big)\\ &= \sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)}\nu\big(T_{xy}fF_{xy} \,|\, m\big)\,,\end{aligned} \tag{5.31}$$

where

$$\sigma_m = \frac{p_n(m)}{p_n(m+1)}\,, \quad F_{xy}(\alpha) = \mathbb{1}_G(\alpha^{xy})\phi_{xy}(\alpha^{xy})\,. \tag{5.32}$$

Subtracting and adding f inside averages gives

$$\nu\big(f\mathbb{I}_G \mid m+1\big) = \sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \left\{\nu\big(\nabla_{xy} f F_{xy} \mid m\big) + \nu\big(f F_{xy} \mid m\big)\right\} . \tag{5.33}$$

When $f = \mathbf{1}$ we see that

$$\nu\big(G \mid m+1\big) = \sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \nu\big(F_{xy} \mid m\big) . \tag{5.34}$$

Therefore, by subtracting $\nu\big(f \mid m\big)\nu\big(G \mid m+1\big)$ the last term in (5.33) becomes a covariance

$$\begin{aligned}&\nu\big(f\mathbb{I}_G \mid m+1\big) - \nu\big(f \mid m\big)\nu\big(G \mid m+1\big)\\ &\quad= \sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \left\{\nu\big([\nabla_{xy} f]\, F_{xy} \mid m\big) + \nu\big(f, F_{xy} \mid m\big)\right\} .\end{aligned} \tag{5.35}$$

We then estimate the square of the l.h.s. of (5.35) by

$$\left[\nu\big(f\mathbb{I}_G \mid m+1\big) - \nu\big(f \mid m\big)\nu\big(G \mid m+1\big)\right]^2 \leqslant \mathbb{I}_1 + \mathbb{I}_2 , \tag{5.36}$$

with

$$\mathbb{I}_1 = 2\Big\{\sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \nu\big(\nabla_{xy} f F_{xy} \mid m\big)\Big\}^2 , \tag{5.37}$$

and

$$\mathbb{I}_2 = 2\Big\{\sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \nu\big(f, F_{xy} \mid m\big)\Big\}^2 . \tag{5.38}$$

`Estimate of` $\mathbb{I}_1$

Using (5.34), the non-negativity of F_{xy} and the Schwarz' inequality we obtain

$$\begin{aligned}\mathbb{I}_1 &\leqslant 2\sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \left\{\frac{\nu\left(\nabla_{xy} f F_{xy} \mid m\right)}{\nu\big(F_{xy} \mid m\big)}\right\}^2 \nu\big(F_{xy} \mid m\big)\\ &\leqslant 2\sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \nu\big((\nabla_{xy} f)^2 F_{xy} \mid m\big)\end{aligned} \tag{5.39}$$

Next we observe that, by the definition of the event G, using (5.9) and (5.12) we have

$$F_{xy}(\alpha) = \mathbb{I}_G(\alpha^{xy}) \frac{\alpha_y(1-\alpha_x)}{N_A(\alpha^{xy})V_B(\alpha^{xy})} \leqslant \begin{cases} 4(n\wedge L^2)^{-1}L^{-2}\alpha_y(1-\alpha_x) & (\texttt{MP})\ L \text{ large}\\ 4n^{-1}L^{-2}\alpha_y(1-\alpha_x) & (\texttt{FP})\ L \text{ large}\\ \alpha_y(1-\alpha_x) & L \text{ small}\end{cases} \tag{5.40}$$

Therefore in both cases (MP), (FP)

$$\mathbb{I}_1 \leqslant 4\sigma_m L^{-2}(n\wedge L^2)^{-1} \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \nu\big((\nabla_{xy} f)^2 \alpha_y(1-\alpha_x) \mid m\big) \tag{5.41}$$

if L is large, while

$$\mathbb{I}_1 \leqslant \sigma_m \sum_{\substack{x\in A\\ y\in B}} q^{2(\ell_x-\ell_y)} \nu\big((\nabla_{xy} f)^2 \alpha_y(1-\alpha_x) \mid m\big) \tag{5.42}$$

if L is small. We can finally apply Proposition 3.12 to obtain

$$\mathbb{I}_1 \leqslant \sigma_m \begin{cases} \frac{C}{n\wedge L^2}\Big\{L^2 \sum_{b\in\mathcal{O}_\Lambda} \nu\big((\nabla_b f)^2 \mid m\big) + \sum_{b\in\mathcal{V}_\Lambda} \nu\big((\nabla_b f)^2 \mid m\big)\Big\} & \text{if } L \text{ is large} \\ \Big\{L^2 \sum_{b\in\mathcal{O}_\Lambda} \nu\big((\nabla_b f)^2 \mid m\big) + \sum_{b\in\mathcal{V}_\Lambda} \nu\big((\nabla_b f)^2 \mid m\big)\Big\} & \text{if } L \text{ is small} \end{cases} \tag{5.43}$$

where C is a suitable constant depending on q and h (h is the constant in Lemma 5.5).

`Estimate of` $\mathbb{I}_2$

Recall the definition of $\mathbb{I}_2$ given in (5.38). It is quite clear from (5.40), Lemma 3.10 and the Schwartz inequality that

$$\mathbb{I}_2 \le k\, L^8 \mathrm{Var}_\nu(f \mid m) \tag{5.44}$$

where $k = k(q)$. Such a bound will turn out useful when L is "small". The case L large is more involved and requires a more subtle analysis. We start with the case (`MP`).

Lemma 5.7. *For every $\epsilon > 0$ and $q \in (0,1)$ there exist finite constants C_ϵ and L_0 such that for any $L \ge L_0$, H, n satisfying* (`MP`) *the following estimate holds*

$$\mathbb{I}_2 \leqslant (n\wedge L^2)^{-1} \left\{ C_\epsilon\, \mathcal{E}_\nu(f,f \mid m) + \epsilon\ \mathrm{Var}_\nu(f \mid m) \right\} . \tag{5.45}$$

Proof. From (5.38) and Lemma 3.10 we have a first estimate

$$\mathbb{I}_2 \leqslant k \Big\{ \sum_{\substack{x\in A\\ y\in B}} \nu\big(f, F_{xy} \mid m\big) \Big\}^2 . \tag{5.46}$$

Observe that

$$\sum_{\substack{x\in A\\ y\in B}} F_{xy} = \frac{V_A N_B \mathbb{1}_{\widetilde G}}{(N_A+1)(V_B+1)} \tag{5.47}$$

where $\widetilde G = \widetilde G_A \cap \widetilde G_B$ with

$$\widetilde G_A = \{ |N_A(\alpha) + 1 - \bar N_A| \leqslant (n\wedge L^2)^{1/2+\gamma} \}$$

and

$$\widetilde G_B = \{ |V_B(\alpha) + 1 - \bar V_B| \leqslant L^{1+2\gamma} \} .$$

As in Lemma 5.4 we have the bounds

$$\nu(\widetilde G_A^c \mid m) \leqslant k \exp\left(-k^{-1} L^{4\gamma}\right), \quad \nu(\widetilde G_B^c \mid m) \leqslant k \exp\left(-k^{-1} L^{4\gamma}\right). \tag{5.48}$$

Writing

$$F_A = \frac{V_A}{N_A+1} \mathbb{1}_{\widetilde G_A}, \quad F_B = \frac{N_B}{V_B+1} \mathbb{1}_{\widetilde G_B},$$

(5.46) says that

$$\mathbb{I}_2 \leqslant k\, \nu\big(f, F_A F_B \big| m\big)^2 \tag{5.49}$$

We write $\nu(\cdot\,|\,m)=\nu_1\otimes\nu_2$ where $\nu_1=\nu_{\Lambda_1,m}$ and $\nu_2=\nu_{\Lambda_2,n-m}$ and use the decomposition

$$\nu\big(f,F_AF_B\,|\,m\big)=\nu_2(F_B)\nu\big(f,F_A\,|\,m\big)+\nu\big(F_A\nu_2\big(f,F_B\big)\,|\,m\big)\,. \tag{5.50}$$

We start by estimating $\nu\big(f,F_A\,|\,m\big)^2$. Defining

$$\rho_A=\frac{N_A}{|A|},\quad \bar\rho_A=\frac{\bar N_A}{|A|}$$

we may write

$$F_A=\frac{1\!\!1_{\widetilde G_A}}{\rho_A}-1\!\!1_{\widetilde G_A}+1\!\!1_{\widetilde G_A}\left[1-\frac{1}{\rho_A}\right]\frac{1}{N_A+1}\,. \tag{5.51}$$

For the second term in the right side of (5.51) one can use (5.48). For the third term, recalling (5.9), one has an upper bound of order kL^{-2}. Therefore apart from the first term the rest contributes at most $kL^{-4}\,\mathrm{Var}_\nu(f\,|\,m)$ to the upper bound on $\nu\big(f,F_A\,|\,m\big)^2$. The first term in (5.51) is handled as follows. We expand

$$\frac{1\!\!1_{\widetilde G_A}}{\rho_A}=\frac{1\!\!1_{\widetilde G_A}}{\bar\rho_A}\left(2-\frac{\rho_A}{\bar\rho_A}+\mathcal{R}_A\right) \tag{5.52}$$

where, on $\widetilde G_A$,

$$|\mathcal{R}_A|\leqslant k\left(\frac{\rho_A}{\bar\rho_A}-1\right)^2\leqslant k(n\wedge L^2)^{-1+2\gamma}\,. \tag{5.53}$$

In view of (5.53) and using again (5.48) to depress the term proportional to $1\!\!1_{\widetilde G_A}$ we have obtained

$$\nu\big(f,F_A\,|\,m\big)^2\leqslant k(\bar\rho_A^2\bar N_A^2)^{-1}\,\nu\big(f,N_A\,|\,m\big)^2+kL^{-4+8\gamma}\,\mathrm{Var}_\nu(f\,|\,m)\,. \tag{5.54}$$

An application of Proposition 3.9 together with the bound $(\bar\rho_A^2\bar N_A^2)^{-1}\leqslant kL^4(n\wedge L^2)^{-4}$ yields the estimate

$$\nu\big(f,F_A\,|\,m\big)^2\leqslant L^4(n\wedge L^2)^{-3}\left\{C\,\mathcal{E}_\nu(f,f\,|\,m)+\epsilon\,\mathrm{Var}_\nu(f\,|\,m)\right\}\,. \tag{5.55}$$

Using $F_B\leqslant(n\wedge L^2)L^{-2}$, the first term in (5.50) can be finally estimated by

$$\nu_2(F_B)^2\nu\big(f,F_A\,|\,m\big)^2\leqslant(n\wedge L^2)^{-1}\left\{C\,\mathcal{E}_\nu(f,f\,|\,m)+\epsilon\,\mathrm{Var}_\nu(f\,|\,m)\right\}\,. \tag{5.56}$$

We turn to the second term in (5.50). Repeating the arguments given above and using $|V_B-\bar V_B|\leqslant L^{1+2\gamma}$ we obtain the upper bound

$$\nu\big(\nu_2\big(f,F_B\big)^2\,|\,m\big)\leqslant kL^{-4}\nu\big(\nu_2\big(f,N_B\big)^2\,|\,m\big)+kL^{-4+8\gamma}\,\mathrm{Var}_\nu(f\,|\,m)\,. \tag{5.57}$$

By Proposition 3.9

$$\nu\big(\nu_2\big(f,F_B\big)^2\,|\,m\big)\leqslant(n\wedge L^2)L^{-4}\left\{C\,\mathcal{E}_\nu(f,f\,|\,m)+\epsilon\,\mathrm{Var}_\nu(f\,|\,m)\right\}\,. \tag{5.58}$$

Recalling that $F_A\leqslant L^2(n\wedge L^2)^{-1}$ we can estimate the square of the second term in (5.50) by

$$\nu(F_A^2\,|\,m)\nu\big(\nu_2\big(f,F_B\big)^2\,|\,m\big)\leqslant(n\wedge L^2)^{-1}\left\{C\,\mathcal{E}_\nu(f,f\,|\,m)+\epsilon\,\mathrm{Var}_\nu(f\,|\,m)\right\}\,. \tag{5.59}$$

$\square$

We now turn to estimate $\mathbb{I}_2$ in the case (FP). Recall that here A is the cylinder with height h, see (5.10).

Lemma 5.8. *For every $\epsilon > 0$, $q \in (0,1)$, there exists $\delta_0(\epsilon, q) > 0$ and finite constants C_ϵ, L_ϵ and h_ϵ such that for any $L \geq L_\epsilon$, H, n satisfying* (FP) *with $\delta \leqslant \delta_0$, any $m \geq \frac{n}{2}$ and $h \geqslant h_\epsilon$*

$$\mathbb{I}_2 \leqslant n^{-1} \left\{ C_\epsilon \, \mathcal{E}_\nu(f, f \,|\, m) + \epsilon \operatorname{Var}_\nu(f \,|\, m) \right\} . \tag{5.60}$$

Proof. Define

$$F_A = \frac{1\!\!1_{\widetilde{G}_A}}{N_A + 1} \sum_{j=0}^{h-1} q^{2j} V_j \,, \quad F_B = \frac{N_B}{V_B + 1} 1\!\!1_{\widetilde{G}_B} \,,$$

with

$$V_j = \sum_{x \in A_j} (1 - \alpha_x) \,, \quad A_j = \{x \in A : \ell_x = j\} \,,$$

and $\widetilde{G}_A = \{N_A + 1 \geqslant m/2\}$, $\widetilde{G}_B = \{|V_B + 1 - \bar{V}_B| \leqslant L^{1+2\gamma}\}$. Then as in (5.46) we have

$$\mathbb{I}_2 \leqslant k \, \nu\big(f, F_A F_B \,|\, m\big)^2 . \tag{5.61}$$

and we decompose as in (5.50). Let us first estimate

$$\begin{aligned} \nu(f, F_A \,|\, m)^2 &\leqslant \operatorname{Var}_\nu(f \,|\, m) \operatorname{Var}_\nu(F_A \,|\, m) \\ &= \operatorname{Var}_\nu(f \,|\, m) \left[\nu\big(\operatorname{Var}_\nu(F_A \,|\, N_A) \,|\, m\big) + \operatorname{Var}_\nu\big(\nu(F_A \,|\, N_A) \,|\, m\big) \right] . \end{aligned} \tag{5.62}$$

Observe that

$$\begin{aligned} \operatorname{Var}_\nu(F_A \,|\, N_A) &= \frac{1\!\!1_{\widetilde{G}_A}}{(N_A + 1)^2} \operatorname{Var}_\nu\Big(\sum_j q^{2j} V_j \,|\, N_A \Big) \\ &\leqslant k n^{-2} \sum_j q^{2j} \operatorname{Var}_\nu\big(V_j \,|\, N_A\big) \leqslant k n^{-1} \,, \end{aligned} \tag{5.63}$$

where we used the fact that $N_A + 1 \geqslant m/2 \geqslant n/4$ and that $\operatorname{Var}_\nu(V_j \,|\, N_A) = \operatorname{Var}_\nu(N_{A_j} \,|\, N_A) \leqslant kn$. The latter estimate can be derived as usual from Proposition 3.8 and Lemma 3.2. For the second term in (5.62) we claim that

$$\operatorname{Var}_\nu\big(\nu(F_A \,|\, N_A) \,|\, m\big) \leqslant k L^4 n^{-3} q^{2h} . \tag{5.64}$$

Set

$$\varphi_j(N_A) = \frac{\nu(V_j \,|\, N_A)}{N_A + 1} \,,$$

so that

$$\operatorname{Var}_\nu\big(\nu(F_A \,|\, N_A) \,|\, m\big) \leqslant k \sum_j q^{2j} \operatorname{Var}_\nu(\varphi_j(N_A) 1\!\!1_{\widetilde{G}_A} \,|\, m) . \tag{5.65}$$

We have

$$\begin{aligned} \operatorname{Var}_\nu(\varphi_j(N_A) 1\!\!1_{\widetilde{G}_A} \,|\, m) &= \sum_{\ell, \ell'} \nu(N_A = \ell \,|\, m) \nu(N_A = \ell' \,|\, m) \times \\ &\times \big(\varphi_j(\ell) - \varphi_j(\ell')\big)^2 \left[1\!\!1_{\widetilde{G}_A}(\ell) 1\!\!1_{\widetilde{G}_A}(\ell') + 2 1\!\!1_{\widetilde{G}_A}(\ell) 1\!\!1_{\widetilde{G}^c_A}(\ell') \right] . \end{aligned} \tag{5.66}$$

Using $N_A \in [n/4, n]$, $V_j \leqslant kL^2$ and $|\nu(V_j \,|\, N_A = \ell) - \nu(V_j \,|\, N_A = \ell')| \leqslant n$, (5.66) implies

$$\operatorname{Var}_\nu(\varphi_j(N_A) \,|\, m) \leqslant k + kL^4 n^{-4} \operatorname{Var}_\nu(N_A \,|\, m) + kL^4 n^{-2} \nu(\widetilde{G}^c_A \,|\, m) . \tag{5.67}$$

From the equivalence of ensembles and Lemma 3.2 we have

$$\operatorname{Var}_\nu(N_A \,|\, m) \leqslant kmq^{2h} \leqslant knq^{2h} .$$

Moreover, by Lemma 5.5 we know that $\nu(\widetilde{G}_A^c\,|\,m) \leqslant kn^{-1}q^{2h}$. Thus (5.67) combined with (5.65) yields the claim (5.64). Going back to (5.62) and recalling that $F_B \leqslant knL^{-2}$ we have the estimate

$$\nu_2(F_B)^2\nu\big(f, F_A\,|\,m\big)^2 \leqslant k\big(q^{2h}n^{-1} + nL^{-4}\big)\mathrm{Var}_\nu(f\,|\,m)\,. \tag{5.68}$$

Recall that $n \leqslant \delta L^2$ so that $nL^{-4} \leqslant \delta n^{-1}$ and we have to choose δ small depending on ϵ. We now estimate the term $\nu\big(\nu_2\big(f, F_B\big)^2\,|\,m\big)$ in (5.50). As in (5.58) we have

$$\nu\big(\nu_2\big(f, F_B)^2\,|\,m\big) \leqslant nL^{-4}\left\{C_\epsilon\,\mathcal{E}_\nu(f, f\,|\,m) + \epsilon\,\mathrm{Var}_\nu(f\,|\,m)\right\}\,. \tag{5.69}$$

At this point the bound $F_A \leqslant kn^{-1}L^2$ gives

$$\nu(F_A^2\,|\,m)\nu\big(\nu_2\big(f, F_B)^2\,|\,m\big) \leqslant n^{-1}\left\{C_\epsilon\,\mathcal{E}_\nu(f, f\,|\,m) + \epsilon\,\mathrm{Var}_\nu(f\,|\,m)\right\}\,. \tag{5.70}$$

Choosing h sufficiently large in (5.68) and combining with (5.70) the proof of (5.60) is complete. ☐

5.4. **The proof of the theorem completed.** `(i)` L `large.`
From the estimate of Proposition 3.11applied to $g(N_{\Lambda_1}) := \nu(f\,|\,N_{\Lambda_1})$, (5.3), the bound (5.28) and (5.36), we see that

$$\begin{aligned}\mathrm{Var}_\nu\,\big(\nu(f\,|\,N_{\Lambda_1})\big) &\leqslant \\ \epsilon\,\mathrm{Var}_\nu\,\big(f\big) + k(n\wedge L^2)&\sum_m p_n(m)\wedge p_n(m+1)\left[\mathbb{I}_1 + \mathbb{I}_2\right]\,,\end{aligned} \tag{5.71}$$

provided that L is large enough depending on q, ϵ. Thanks to (5.43)

$$\begin{aligned}(n\wedge L^2)&\sum_m p_n(m)\wedge p_n(m+1)\,\mathbb{I}_1 \\ &\leqslant k\sum_m p_n(m)\Big\{L^2\sum_{b\in\mathcal{O}_\Lambda}\nu\big((\nabla_b f)^2\,|m\big) + \sum_{b\in\mathcal{V}_\Lambda}\nu\big((\nabla_b f)^2\,|m\big)\Big\} \\ &= k\left\{L^2\mathcal{E}_\nu^{\mathcal{O}_\Lambda}(f,f) + \mathcal{E}_\nu^{\mathcal{V}_\Lambda}(f,f)\right\}\end{aligned} \tag{5.72}$$

On the other hand the estimates on $\mathbb{I}_2$ given in Lemma 5.7 and 5.8 yield

$$(n\wedge L^2)\sum_m p_n(m)\wedge p_n(m+1)\,\mathbb{I}_2 \leqslant C_\epsilon\,\mathcal{E}_\nu(f,f) + \epsilon\mathrm{Var}_\nu(f)\,. \tag{5.73}$$

for any $\epsilon > 0$ and a suitable constant C_ϵ independent of L. In conclusion, for any $\epsilon > 0$ and $q \in (0,1)$ we can choose $L_0 = L_0(\epsilon, q)$ such that, by combining together (5.72) and (5.73), the r.h.s. of (5.71) is bounded from above by

$$k\left\{L^2\mathcal{E}_\nu^{\mathcal{O}_\Lambda}(f,f) + \mathcal{E}_\nu^{\mathcal{V}_\Lambda}(f,f)\right\} + C_\epsilon\,\mathcal{E}_\nu(f,f) + \epsilon\mathrm{Var}_\nu(f) \tag{5.74}$$

for any $L \geq L_0$.

`(ii)` L `small.`
Using Proposition 3.11 together with (5.29), and (5.36), we see that

$$\begin{aligned}&\mathrm{Var}_\nu\,\big(\nu\big(f\,|N_{\Lambda_1}\big)\big) \leqslant \\ C\Big\{&\sum_m p_n(m)\wedge p_n(m+1)\left[\mathbb{I}_1 + \mathbb{I}_2\right] + \mathrm{Var}_\nu(f)\,\Big\}\end{aligned} \tag{5.75}$$

for a suitable constant $C = C(L, q)$.
It is enough to use at this point (5.43) together with the rough estimate (5.44) to get the sought bound. ☐

6. Proof of the upper bound in Theorem 2.7 and of Theorem 2.4

In this final section we prove the upper bound on the spectral gap of the generator $\mathcal{L}_{\Sigma_{H,L},n}$ and the bound on the spectral projection.

6.1. **Proof of (2.34).** Consider the cylinder $\Lambda := \Sigma_{L,H}$ which has the square Q_L (containing L^2 sites) as basis. A generic point of Q_L will be denoted by z and N_z stands for the number of particles in the stick going through z,

$$N_z(\alpha) = \sum_{x\in\Sigma_{z,H}} \alpha_x \,.$$

Given a smooth function $\varphi : [0,1]^2 \to \mathbb{R}$, we define $f_\varphi : \Omega_\Lambda \to \mathbb{R}$ by

$$f_\varphi(\alpha) = \sum_{z\in Q_L} \varphi_L(z) N_z(\alpha) \,, \tag{6.1}$$

where φ_L denotes the rescaled profile

$$\varphi_L(z) = \varphi(z/L) \,, \quad z \in Q_L \,. \tag{6.2}$$

We will use the notation

$$e(\varphi) = \int_{[0,1]^2} \|\nabla\varphi(u)\|^2 \,\mathrm{d}u \,, \qquad \|\nabla\varphi(u)\|^2 := (\partial_{u_1}\varphi(u))^2 + (\partial_{u_2}\varphi(u))^2 \,.$$

The upper bound in Theorem 2.7 is obtained as follows.

Proposition 6.1. *For every $q \in (0,1)$, there exists $k = k(q) < \infty$ such that the following holds. For any smooth function $\varphi : [0,1]^2 \to \mathbb{R}$ satisfying $\int \varphi(u)\,\mathrm{d}u = 0$ and $\int \varphi(u)^2\,\mathrm{d}u = 1$ there exists L_0 such that for any $L \geqslant L_0$, $H \geqslant 1$ and $n = 1, \dots, HL^2 - 1$ one has*

$$\mathcal{E}_\nu(f_\varphi, f_\varphi) \leqslant k\, e(\varphi)\, L^{-2}\, \mathrm{Var}_\nu(f_\varphi) \,, \tag{6.3}$$

Proof. Observing that N_z, $z \in Q_L$ are identically distributed under ν we easily see that

$$\nu(N_z, N_{z'}) = -\frac{\sigma_\nu^2}{L^2 - 1} \,, \quad z \neq z' \,, \tag{6.4}$$

where $\sigma_\nu^2 := \nu(N_z, N_z)$ is the variance of the number of particles in a single stick. Thus

$$\begin{aligned} \mathrm{Var}_\nu(f_\varphi) &= \sigma_\nu^2 \sum_{z\in Q_L} \varphi_L(z)^2 - \frac{\sigma_\nu^2}{L^2-1} \sum_{\substack{z,z'\in Q_L:\\ z\neq z'}} \varphi_L(z)\varphi_L(z') \\ &\geqslant \sigma_\nu^2 \sum_{z\in Q_L} \varphi_L(z)^2 - \frac{\sigma_\nu^2}{L^2-1} \left(\sum_{z\in Q_L} \varphi_L(z) \right)^2 \,. \end{aligned} \tag{6.5}$$

Since $\int \varphi(u)\,\mathrm{d}u = 0$ and $\int \varphi(u)^2\,\mathrm{d}u = 1$, from Riemann integration we conclude that there exists a finite L_0 such that for any $L > L_0$

$$\mathrm{Var}_\nu(f_\varphi) \geqslant \frac{\sigma_\nu^2}{2} L^2 \,. \tag{6.6}$$

Let us now estimate the Dirichlet form. In view of (6.6) all we have to prove is

$$\mathcal{E}_\nu(f_\varphi, f_\varphi) \leqslant k\, e(\varphi)\, \sigma_\nu^2 \,. \tag{6.7}$$

Consider a bond $(x,y) = b \in \mathcal{O}_\Lambda$. Clearly $\nabla_{xy} f_\varphi = 0$ if x, y belong to the same stick since an exchange between x and y does not change the number of particles in any stick. In particular, only horizontal bonds $b \in \mathcal{O}_\Lambda$ contribute to $\mathcal{E}_\nu(f,f)$. Take $z, z' \in Q_L$ such that $x \in \Sigma_{z,H}$, $y \in \Sigma_{z',H}$ and $b = (x,y) \in \mathcal{O}_\Lambda$. One has

$$\nabla_{xy} f_\varphi(\alpha) = (\alpha_y - \alpha_x)(\varphi_L(z) - \varphi_L(z')) \tag{6.8}$$

and since $\|z - z'\|_1 = 1$,

$$|\varphi_L(z) - \varphi_L(z')| \leqslant 2L^{-1}\|\nabla\varphi(\tilde z/L)\| + O(L^{-2})\,.$$

From (6.8) we obtain,

$$\mathcal{E}_\nu(f_\varphi, f_\varphi) \leqslant 2L^{-2} \sum_{z\in Q_L} \big(\|\nabla\varphi(\tilde z/L)\|^2 + O(L^{-2})\big)\, \mathcal{C}(\nu, z)\,, \tag{6.9}$$

with

$$\mathcal{C}(\nu, z) := \sum_{x\in\Sigma_{z,H}} \sum_{\substack{y\notin\Sigma_{z,H}:\\ \|x-y\|_1=1}} \nu\big((\alpha_x - \alpha_y)^2\big)\,.$$

Since

$$L^{-2} \sum_{z\in Q_L} \big(\|\nabla\varphi(\tilde z/L)\|^2 + O(L^{-2})\big) \to e(\varphi)\,, \qquad L\to\infty\,,$$

the claim (6.7) is proven once we show that there exists $k < \infty$ such that for any $z \in Q_L$

$$\mathcal{C}(\nu, z) \leqslant k\sigma_\nu^2\,. \tag{6.10}$$

We start the proof of (6.10) by estimating with the help of Proposition 3.8:

$$\nu\big((\alpha_x - \alpha_y)^2\big) \leqslant k\mu^\lambda\big((\alpha_x - \alpha_y)^2\big)\,, \tag{6.11}$$

with μ^λ the grand canonical measure corresponding to n particles. We observe that, since x and y are at the same height

$$\mu^\lambda\big((\alpha_x - \alpha_y)^2\big) = \mu^\lambda(\alpha_x)(1 - \mu^\lambda(\alpha_x)) + \mu^\lambda(\alpha_y)(1 - \mu^\lambda(\alpha_y)) = 2\mathrm{Var}_{\mu^\lambda}(\alpha_x)$$

For every $x \in \Sigma_z$ there are at most 4 horizontal neighbours $y \notin \Sigma_z$ so that $\mathcal{C}(\nu, z) \leqslant 8\,\sigma^2(\lambda)$, with $\sigma^2(\lambda) := \mathrm{Var}_{\mu^\lambda}(N_z)$. The rest of the proof is now concerned with the estimate

$$\sigma^2(\lambda) \leqslant k\sigma_\nu^2(n) \tag{6.12}$$

with a constant k only depending on q. Once (6.12) is established we obtain (6.10) and the proposition follows.

Below we restrict to the case $n \leqslant HL^2/2$, which is no loss of generality in view of particle-hole duality. From (6.4) we have

$$\sigma_\nu^2 = \frac{L^2 - 1}{2L^2}\nu\big((N_z - N_{z'})^2\big)\,, \quad z \neq z'\,. \tag{6.13}$$

For any integer $m \geqslant -1$ consider the event $E_{z,m}$ that the stick $\Sigma_{z,H}$ is filled with particles up to level m and is empty above level m. More precisely if $x_0 = z, x_1, \dots, x_{H-1}$ are the sites of $\Sigma_{z,H}$ with $\ell_{x_i} = i$ we define

$$E_{z,m} = \{\alpha_{x_0} = \dots = \alpha_{x_m} = 1\,,\ \alpha_{m+1} = \dots = \alpha_{x_{H-1}} = 0\}\,, \quad E_{z,-1} = \{N_z = 0\}\,.$$

For any integer $0 \leqslant m \leqslant H - 1$ we have the bound

$$\nu\big((N_z - N_{z'})^2\big) \geqslant \nu\big(E_{z,m} \cap E_{z',m-1}\big)\big)\,. \tag{6.14}$$

The right hand side above should be maximal around $m = [n/L^2]$. Indeed, simple computations as in Lemma 3.1 show that there exists $\delta = \delta(q) > 0$ such that uniformly in the height H one has

$$\mu^\lambda(E_{z,[n/L^2]}) \geqslant \delta q^{-2(\lambda\wedge 0)}\,, \quad \mu^\lambda(E_{z,[n/L^2]-1}) \geqslant \delta\,. \tag{6.15}$$

Therefore using Theorem 3.4 we have

$$\begin{aligned}\nu\big(E_{z,[n/L^2]} \cap E_{z',[n/L^2]-1}\big) &\geqslant \mu^\lambda(E_{z,[n/L^2]})\mu^\lambda(E_{z,[n/L^2]-1}) - kL^{-2} \\ &\geqslant \delta^2 q^{-2(\lambda\wedge 0)} - kL^{-2} \geqslant k^{-1}\delta^2\sigma^2(\lambda) - kL^{-2}\,,\end{aligned} \tag{6.16}$$

with the last inequality coming from Lemma 3.2. But we know (Remark 3.12) that $\sigma^2(\lambda) \geqslant k^{-1}(1 \wedge \frac{n}{L^2})$, thus (6.13), (6.14) and (6.16) imply that there exist finite constants L_0, N_0 and k only depending on q such that (6.12) holds whenever $L \geqslant L_0$ and $n \geqslant N_0$.

It remains to treat the case $n < N_0$. It will suffice to show

$$\sigma_\nu^2 \geqslant \frac{n}{kL^2}\,. \tag{6.17}$$

We write

$$\begin{aligned}\nu\big((N_z - N_{z'})^2\big) &\geqslant \nu\big(N_z = 1, N_{z'} = 0\big) \\ &\geqslant \nu\big(N_z = 1, N_{z'} = 0 \,|\, N_w \leqslant 1\,,\ \forall w \in Q_L\big)\nu\big(N_w \leqslant 1\,,\ \forall w \in Q_L\big) \\ &\geqslant \nu\big(N_z = 1, N_{z'} = 0 \,|\, N_w \leqslant 1\,,\ \forall w \in Q_L\big)\nu\Big(\sum_{w\in Q_L} \alpha_w = n\Big)\end{aligned}$$

But

$$\begin{aligned}\nu\Big(\sum_{w\in Q_L} \alpha_w = n\Big) &= \frac{\mu^\lambda(\sum_{w\in Q_L}\alpha_w = n\,,\ N_{\Lambda\setminus Q_L} = 0)}{\mu^\lambda(N_\Lambda = n)} \\ &\geqslant \mu^\lambda\Big(\sum_{w\in Q_L}\alpha_w = n\,,\ N_{\Lambda\setminus Q_L} = 0\Big)\,,\end{aligned}$$

and the latter is bounded away from 0 uniformly as in the proof of Theorem 3.4 (see (3.24)). On the other hand

$$\nu\big(N_z = 1, N_{z'} = 0 \,|\, N_w \leqslant 1\,,\ \forall w \in Q_L\big) = \frac{\binom{L^2-2}{n-1}}{\binom{L^2}{n}} = \frac{n(L^2-n)}{L^2(L^2-1)} \geqslant \frac{n}{2L^2}\,,$$

as soon as $L^2 \geqslant 2N_0$. This yields the desired bound (6.17). □

Remark 6.2. The above proposition allows to produce low-lying excitations which are localized in a sub-cylinder $\Sigma_{R,H} \subset \Sigma_{L,H}$ with $R \leqslant L$, much in the spirit of [3]. Indeed, one can always choose a function φ supported on $[0, R/L]$ with $\int \varphi^2 = 1$ and $e(\varphi) = O(R^{-2}L^2)$ and the resulting states f_φ have energy $O(R^{-2})$.

6.2. **Proof of Theorem 2.4.** For simplicity we prove the result for the generator $\mathcal{L}_{\Lambda,n}$ instead of $\mathcal{G}_{\Lambda,n}$, but the argument applies essentially without modifications to the original setting of Theorem 2.4.

We follow quite closely the proof of an analogous result for translation invariant lattice gases (see Theorem 2.4 in [7]). The main idea is to establish the following inequality

$$\nu(g,f)^2 \leqslant k_\epsilon\big\{\,\ell^\epsilon\,\mathcal{E}_\nu(g,g) + \ell^{-2}\nu(g,g)\,\big\} \tag{6.18}$$

for any ϵ and any ℓ, with the constant k_ϵ uniform in ℓ, Λ. Once we have (6.18) we obtain Theorem 2.4 by choosing $g := E_s f$ and optimizing over ℓ. Indeed, with this choice we have $\nu(fE_sf) = \nu(f,g) = \nu(g,g)$ since f (and therefore E_sf) has zero mean. Moreover $\mathcal{E}_\nu(E_sf, E_sf) \leqslant s\,\nu(fE_sf)$, so that (6.18) implies

$$\nu(fE_sf) \leqslant k_\epsilon\big\{s\,\ell^\epsilon + \ell^{-2}\big\}\,,$$

and the claim follows.
In order to prove (6.18) we need the following technical lemma. In what follows f is as in Theorem 2.4.

Lemma 6.3. *There exists a constant k depending on f such that*

$$\mathrm{Var}_\nu\big(\,\nu(f\,|\,N_{\Sigma_{\ell,H}})\,\big) \leqslant \frac{k}{\ell^2} \qquad \forall\,\ell \leqslant \frac{L}{2} \tag{6.19}$$

Proof. Without loss of generality we can assume that ℓ is so large that the support of f is contained in $\Sigma_{\ell,H}$. For notation convenience we set $N_\ell := N_{\Sigma_{\ell,H}}$ and $\mu_\ell^\lambda := \mu_{\Sigma_{\ell,H}}^\lambda$. Using the result on the equivalence of ensembles, see Theorem 3.4, we can safely replace $\nu(f\,|\,N_\ell = m)\,\big)$ with its grand-canonical average $F(m) := \mu_\ell^{\lambda(m,\ell)}(f)$, where $\lambda(m,\ell)$ is such that $\mu_\ell^{\lambda(m,\ell)}(N_\ell) = m$. Moreover, thanks to Proposition 3.8, we can bound the canonical variance w.r.t. ν by the grand canonical one w.r.t. $\mu := \mu_\Lambda^{\lambda(n)}$ with self explanatory notation. In conclusion

$$\mathrm{Var}_\nu\big(\,\nu(f\,|\,N_\ell)\,\big) \le k\,\mu\big(F,F\big) + C\frac{1}{\ell^2} \tag{6.20}$$

for some $k = k(q)$ and $C = C(f,q)$.
Since the measure μ is a product measure over the sites of Λ, it is immediate to check (see also (3.45))

$$\mu\big(F,F\big) \le \Big\{\sum_{x\in\Sigma_{\ell,H}} \sigma_x^2\Big\}\,\mu\big(2[F(N_\ell+1) - F(N_\ell)]^2 + 2[F(N_\ell) - F(N_\ell-1)]^2\big) \tag{6.21}$$

where $\sigma_x^2 := \mu(\alpha_x, \alpha_x)$.
We now bound the gradient $[F(m+1) - F(m)]$. Let $\lambda_s := s\lambda(m+1,\ell) + (1-s)\lambda(m,\ell)$ and let $F_s := \mu_\ell^{\lambda_s}(f)$. Then, setting $a(q) = \log 1/q$ we have

$$F(m+1) - F(m) = \int_0^1 \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}F_s = a(q)\int_0^1 \mathrm{d}s\,\mu_\ell^{\lambda_s}(N_\ell, f)[\lambda(m+1,\ell) - \lambda(m,\ell)] \tag{6.22}$$

In turn

$$\lambda(m+1,\ell) - \lambda(m,\ell) = \int_m^{m+1} \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t,\ell) = \int_m^{m+1} \mathrm{d}t\,\big[a(q)\mu_\ell^{\lambda(t,\ell)}(N_\ell, N_\ell)\big]^{-1} \tag{6.23}$$

It is easy to check at this point, thanks to the results of §3.1, that $|\lambda(m+1,\ell) - \lambda(m,\ell)| \leqslant k(m\wedge\ell^2)^{-1}$ for some $k = k(q)$. Since $|\mu_\ell^{\lambda_s}(N_\ell, f)| \le C_f$ we get that the r.h.s. of (6.22) is bounded from above by $C_f k(m\wedge\ell^2)^{-1}$.
In conclusion, the r.h.s. of (6.21) is bounded from above by

$$K_f\ell^2\big(\frac{n}{L^2}\wedge 1\big)\mu\Big(\big(N_\ell\wedge\ell^2\big)^{-2}\Big) \tag{6.24}$$

for some constant K_f depending on f. Standard large deviations for the product measure μ imply that the r.h.s. of (6.24) is bounded from above by $K_f'\ell^{-2}$. $\square$

We can now complete the proof of the theorem following step by step the proof of Theorem 2.4 in [7]. We first establish (6.18) for $\epsilon = 2$ and then show how to improve it to all values of $\epsilon > 0$.
The main ingredients are the lower bound on the spectral gap given in Theorem 2.2 together with the formula

$$\nu(g,f) = \nu\big(\,\nu(g,f\,|\,\mathcal{F})\,\big) + \nu\big(\,g, \nu(f\,|\,\mathcal{F})\,\big)$$

valid for any σ–algebra $\mathcal{F}$. If we take $\mathcal{F}$ as the σ–algebra generated by N_ℓ, we get, after one Schwartz inequality,

$$\begin{aligned}\nu(g,f)^2 &\leqslant 2\nu\big(\,\nu(g,f\,|\,N_\ell)^2\,\big) + 2\nu\big(\,g,\nu(f\,|\,N_\ell)\,\big)^2 \\ &\leqslant C_f\Big[\,\ell^2\mathcal{E}_\nu(g,g) + \frac{1}{\ell^2}\,\mathrm{Var}_\nu(g)\,\Big]\end{aligned} \tag{6.25}$$

where we used Lemma 6.3 and the Poincaré inequality $\mathrm{Var}_\nu(g\,|\,N_\ell) \leqslant k\,\ell^2\,\mathcal{E}_\nu(g,g\,|\,N_\ell)$, which follows from Theorem 2.2.
Now we assume inductively that we have been able to prove (6.25) with ℓ^2 replaced by ℓ^ϵ and C_f replaced by some constant $C_{f,\epsilon}$ for all $\ell \leq \frac{L}{2}$. Then the term $\nu(g,f\,|\,N_\ell)^2$ in the r.h.s. of the first line of (6.25) can be bounded from above by

$$\begin{aligned}\nu(g,f\,|\,N_\ell)^2 &\leqslant C_{f,\epsilon}\Big[\,\ell_1^\epsilon\mathcal{E}_\nu(g,g\,|\,N_\ell) + \frac{1}{\ell_1^2}\,\mathrm{Var}_\nu(g\,|\,N_\ell)\,\Big] \\ &\leqslant C'_{f,\epsilon}\Big[\,\ell_1^\epsilon + \frac{\ell^2}{\ell_1^2}\,\Big]\mathcal{E}_\nu(g,g\,|\,N_\ell)\end{aligned}$$

for any $\ell_1 \leqslant \frac{\ell}{2}$. If we optimize over ℓ_1 for a given ℓ we get

$$\nu(g,f\,|\,N_\ell)^2 \leqslant C''_{f,\epsilon}\,\ell^{\frac{2\epsilon}{2+\epsilon}}\,\mathcal{E}_\nu(g,g\,|\,N_\ell) \tag{6.26}$$

In other words we have been able to replace the assumed ℓ^ϵ factor in front of the Dirichlet form of g by $\ell^{\frac{2\epsilon}{2+\epsilon}}$. The price is an increase of the constant $C_{f,\epsilon}$. Since the discrete map $x \to \frac{2x}{2+x}$, $x_0 = 2$ has as unique fixed point the origin, (6.18) follows. $\square$

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