Semiclassical Analysis for Magnetic Scattering by Two Solenoidal Fields : Total Cross Sections

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1. Introduction

In quantum mechanics, vector potentials have a direct significance to particles moving in magnetic fields. This is known as the Aharonov–Bohm effect (A–B effect) ([3]). The aim of the present work is to study this quantum effect through the semiclassical analysis on total cross sections of magnetic scattering by two solenoidal (point–like) fields with total flux vanishing in two dimensions. Such a system seems to be important in practical aspects as well as in theoretical aspects, because it may be thought of a toroidal solenoid with zero cross section in three dimensions under the idealization that two parallel fields connect at infinity in their direction (direction perpendicular to the plane).

Let $\Lambda(x) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the vector potential defined by

$$\Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log |x|, \partial_1 \log |x|\right), \quad \partial_j = \partial/\partial x_j.$$

The potential has the point–like field

$$\nabla \times \Lambda = \left(\partial_1^2 + \partial_2^2\right) \log |x| = \Delta \log |x| = 2\pi\delta(x)$$

at the origin. The particle moving in two solenoidal fields with flux $\pm\alpha \in \boldsymbol{R}$ at center $e_\pm \in \boldsymbol{R}^2$ is governed by the Hamiltonian

$$H_h = (-ih\nabla - A)^2 = \sum_{j=1,2} \left(-ih\partial_j - a_j\right)^2, \quad 0 < h \ll 1, \tag{1.1}$$

where $A(x) = (a_1(x), a_2(x)) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ takes the form

$$A(x) = \alpha\Lambda(x - e_+) - \alpha\Lambda(x - e_-), \quad e_+ \neq e_-.$$

The potential $A(x)$ has a strong singularity at centers $e_\pm$ of two fields. For this reason, H_h formally defined by (1.1) is not necessarily essentially self–adjoint in $C_0^\infty(\boldsymbol{R}^2 \setminus \{e_-, e_+\})$. We have to impose boundary conditions at $e_\pm$ to define a self–adjoint realization in the space $L^2 = L^2(\boldsymbol{R}^2)$. We know that H_h admits the self–adjoint realization under the boundary conditions

$$\lim_{|x-e_\pm|\to 0} |u(x)| < \infty. \tag{1.2}$$

We denote by the same notation H_h this self–adjoint operator and by $f_h(\omega_- \to \omega)$ the amplitude for the scattering from incident direction $\omega_- \in S^1$ to final one ω at energy $E > 0$, where S^1 denotes the unit circle and E is fixed throughout. When the total flux vanishes, $A(x)$ falls off like $A(x) = O(|x|^{-2})$ at infinity, so that the forward amplitude $f_h(\omega_- \to \omega_-)$ is finite and the total scattering cross section

$$\sigma_h(\omega_-) = \int |f_h(\omega_- \to \omega)|^2 \, d\omega$$

is also well defined, where the integration with no domain attached is taken over the whole space. We often use this abbreviation. In the present work we derive the asymptotic formula with first three terms for $\sigma_h(\omega_-)$ in the semiclassical limit $h \to 0$. The system with two solenoidal fields has the trajectory oscillating between two centers $e_\pm$. Our motivation comes from analyzing how this trapping effect is reflected in the asymptotic formula through the A–B effect.

The asymptotic formula is described in terms of amplitudes for the scattering by a single solenoidal field. We begin by making a quick review on it. The system with single field is exactly solvable, and the explicit representation for amplitude has been calculated by [3] (see [1, 2, 5, 15] also). We now consider the Hamiltonian

$$H_{\alpha h} = (-ih\nabla - \alpha\Lambda)^2, \quad \Lambda = \Lambda(x), \tag{1.3}$$

which governs the particle moving in the point–like field $2\alpha\delta(x)$ at the origin. For the same reason as above, this operator is not essentially self–adjoint in $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$. In fact, it is known to be a symmetric operator with type $(2,2)$ of deficiency indices ([1, 5]). A self–adjoint extension $H_{\alpha h}$ is obtained by imposing the boundary condition

$$\lim_{|x|\to 0} |u(x)| < \infty. \tag{1.4}$$

If, in particular, α/h is not an integer, then (1.4) is equivalent to saying that $u(x) = 0$ at the origin. This means that the particle never touches the solenoid at the origin. We denote by the Gauss notation $[\alpha/h]$ the greatest integer not exceeding α/h, and we often identify the coordinates over S^1 with the azimuth angles from the positive x_1 axis. Then the amplitude $f_h = f_h(\omega_- \to \omega; \alpha)$ for the scattering from incident direction ω_- to final one ω, $\omega \neq \omega_-$, at energy E is known to take the form

$$f_h = (ic(E)/\pi)\, h^{1/2} \sin(\alpha/h)\pi \, \exp\left(i[\alpha/h](\omega - \omega_-)\right) F_0(\omega - \omega_-), \tag{1.5}$$

where $F_0(\theta)$ is defined by $F_0(\theta) = e^{i\theta}/(1 - e^{i\theta})$ for $\theta \neq 0$ and

$$c(E) = (2\pi)^{1/2} E^{-1/4} e^{-i\pi/4}.$$

For completeness, we shall establish this formula in Proposition 7.1, although it is an immediate consequence of the results obtained by the early works mentioned above. The amplitude $f_h = f_h(\omega_- \to \omega; \alpha)$ appears in the asymptotic form

$$\varphi \sim e^{i(\alpha/h)(\gamma(x;\omega_-)-\pi)} \exp(ih^{-1}E^{1/2}x \cdot \omega_-) + f_h \exp(ih^{-1}E^{1/2}|x|)|x|^{-1/2}$$

as $x = |x|\omega \to \infty$ of outgoing eigenfunction $\varphi(x)$ to equation $H_{\alpha h}\varphi = E\varphi$, where the notation $\cdot$ is the scalar product in $\boldsymbol{R}^2$ and

$$\gamma(x;\omega) = \gamma(\hat{x};\omega), \quad 0 \le \gamma < 2\pi, \quad \hat{x} = x/|x|, \tag{1.6}$$

denotes the azimuth angle from ω to $\hat{x}$. In particular, we write $\gamma(x)$ for the azimuth angle from the positive x_1 axis. The first term on the right side represents the wave incident from direction ω_- and the second one the wave scattered into final direction ω. The incident wave takes a form different from the usual plane wave $e^{ih^{-1}E^{1/2}x\cdot\omega_-}$. The reason is because the vector potential $\alpha\Lambda$ falls off like $\alpha\Lambda(x) = O(|x|^{-1})$ slowly at infinity. The modified factor $e^{i(\alpha/h)(\gamma(x;\omega_-)-\pi)}$ is required because of this long–range property. As is seen from (1.5),

$$f_h(\omega_- \to \omega; \alpha) \sim h^{1/2}, \quad \omega \ne \omega_-,$$

is as small as $h^{1/2}$ and the forward amplitude $f_h(\omega_- \to \omega_-; \alpha) = \infty$ is in general divergent. Hence the total cross section can not be defined for the scattering by a single field. If we denote by $f_h(\omega_- \to \omega; \alpha, p)$ the amplitude for the scattering by the field $2\pi\alpha\delta(x-p)$ with center at $p \in \boldsymbol{R}^2$, then it is calculated as

$$f_h(\omega_- \to \omega; \alpha, p) = \exp\left(-ih^{-1}E^{1/2}p \cdot (\omega - \omega_-)\right) f_h(\omega_- \to \omega; \alpha),$$

because $|x-p| = |x| - p\cdot\omega + O(|x|^{-1})$ as $|x| \to \infty$ along direction ω. With the notation above, we are now in a position to formulate the main theorem.

Theorem 1.1 *Let the notation be as above. Assume that the incident direction ω_- satisfies $\omega_- \ne \pm\hat{e}$, $\hat{e} = e/|e|$, with $e = e_+ - e_-$. Set*

$$F_\pm(\omega_-) = f_h(\omega_- \to \pm\hat{e}; \mp\alpha, e_\mp) f_h(\pm\hat{e} \to \omega_-; \pm\alpha, e_\pm)$$

and define $\tau_\pm$ by

$$\tau_\pm = \exp\Big(\pm i2(\alpha/h)\,(\pi - \gamma(\pm\hat{e};\omega_-))\Big).$$

Then the total cross section $\sigma_h(\omega_-)$ at energy E obeys the asymptotic formula

$$\begin{aligned}\sigma_h(\omega_-) &= 4\left(|e(\omega_-)| - 2\,(\alpha/h - [\alpha/h] - 1/2)\,E^{-1/2}h\right)\sin^2(\alpha/h)\pi \\ &- \ 2(2\pi)^{1/2}|e|^{-1/2}E^{-1/4}h^{1/2}\mathrm{Re}\left[e^{i\pi/4}\,(\tau_+F_-(\omega_-) + \tau_-F_+(\omega_-))\right] + o(h^{3/2})\end{aligned}$$

as $h \to 0$, where $e(\omega_-) = e - (e\cdot\omega_-)\,\omega_-$ is the projection of e onto the impact line perpendicular to ω_-.

We make a comment on how the A–B quantum effect is reflected in the asymptotic formula above. The total cross section is related to the forward scattering amplitude through relation

$$\sigma_h(\omega_-) = -2(2\pi)^{1/2}E^{-1/4}h^{1/2}\mathrm{Re}\left(e^{i\pi/4}f_h(\omega_- \to \omega_-)\right). \tag{1.7}$$

This relation is called the optical theorem ([4]). We establish it in the next section for completeness. When the total flux vanishes, a gauge transformation enables us to take a vector potential in such a way that its support is almost concentrated near the segment of line joining the two centers e_+ and e_-. The leading term indicates a contribution from particles passing over the support of such a vector potential. The second term with magnitude of order $O(h)$ is rather subtle and technical. It is not easy to explain its relation to the A–B effect intuitively. Roughly speaking, this term describes a contribution from the forward scattering trajectories of particles grazing each center $e_\pm$. On the other hand, the third term which consists of two terms with magnitude of order $O(h^{3/2})$ is clear. It is involved with two trajectories scattered from ω_- to itself. For example, $F_-(\omega_-)$ comes from the trajectory of particle which is scattered from $-\hat{e}$ to ω_- by the field $-2\pi\alpha\delta(x-e_-)$ after scattered from ω_- to $-\hat{e}$ by the field $2\pi\alpha\delta(x-e_+)$. A similar interpretation applies to the other term $F_+(\omega_-)$. We have shown in [10] how the vector potential $\alpha\Lambda(x-e_+)$ associated with field $2\pi\alpha\delta(x-e_+)$ influences the scattering by the other field $-2\pi\alpha\delta(x-e_-)$ through the A–B effect. In fact, the factor $\tau_\pm$ appears because of this quantum effect.

The proof of the theorem is done from section 4 through section 9 after the problem is reduced to the proof of three main lemmas in section 3. As stated above, the present work is closely related to the previous work [10] where we have established the semiclassical asymptotic formula of amplitudes with first three terms for the scattering by two solenoidal fields. A new difficulty which is added in the semiclassical analysis on total cross sections is to control a contribution from trajectories passing near individual solenoidal fields. To this end, we require a precise information on the asymptotic behavior at infinity of outgoing eigenfunctions of Hamiltonian with a single field. The main body of this work is occupied by the derivation of the second term. The idea developed here is general enough to cover the case of scattering by a finite number of solenoidal fields. The proof does not make an essential use of the special assumption that the system consists of two solenoidal fields only. There are a lot of physics articles dealing with the scattering by two or many solenoidal fields. We refer to [2, 11, 13, 16, 17] and to the references cited in the book [2].

We conclude the section by mentioning two problems related to the scattering by a finite number of fields. The scattering phase (spectral shift function) is also well defined for systems with total flux vanishing. One of problems is its asymptotic behavior in the semiclassical limit. To this end, we have to make an elaborate analysis on the trajectory oscillating between two centers. The other problem is concerned with the existence of resonances. As stated later (Remark 7.2), a system of single field never has any resonances under boundary condition (1.4). Then a natural question arises : Are there any resonances for systems of several solenoidal fields and how are they distributed ? We expect the results to depend heavily on the location of centers of fields.

2. Optical theorem

In this section we prove (1.7). We begin by introducing the standard notation used in scattering theory. For two given self–adjoint operators H and K acting on $L^2 = L^2(\boldsymbol{R}^2)$, we denote by $W_\pm(H, K)$ the wave operator

$$W_\pm(H, K) = s - \lim_{t\to\pm\infty} \exp(itH) \exp(-itK) : L^2 \to L^2$$

and by $S(H, K)$ the scattering operator

$$S(H, K) = W_+(H, K)^* W_-(H, K) : L^2 \to L^2.$$

Let

$$\varphi_0(x; \lambda, \omega) = \exp(i\lambda^{1/2} x \cdot \omega), \quad \lambda > 0, \quad \omega \in S^1,$$

be the generalized eigenfunction of the free Hamiltonian $H_0 = -\Delta$. We define the unitary mapping $F : L^2 \to L^2((0, \infty); d\lambda) \otimes L^2(S^1)$ by

$$(Fu)(\lambda, \omega) = 2^{-1/2}(2\pi)^{-1} \int \bar{\varphi}_0(x; \lambda, \omega) u(x)\, dx = 2^{-1/2} \hat{u}(\lambda^{1/2}\omega) \tag{2.1}$$

and F_h by

$$(F_h u)(\lambda, \omega) = 2^{-1/2}(2\pi h)^{-1} \int \bar{\varphi}_0(x/h; \lambda, \omega) u(x)\, dx, \tag{2.2}$$

where $\hat{u}(\xi)$ is the Fourier transform of u.

As stated in the previous section, H_h defined by (1.1) admits the self–adjoint realization with domain

$$\mathcal{D} = \{u \in L^2 : (-ih\nabla - A)^2 u \in L^2, \ \lim_{|x-e_\pm|\to 0} |u(x)| < \infty\},$$

where $(-ih\nabla - A)^2 u$ is understood in the distribution sense. The operator H_h can be shown to have the following spectral property ([8, section 7]) : H_h has no bound states and its spectrum is absolutely continuous. We can further show that the wave operator $W_\pm(H_h, H_{0h})$ exists and is asymptotically complete

$$\mathrm{Ran}\,(W_+(H_h, H_{0h})) = \mathrm{Ran}\,(W_-(H_h, H_{0h})) = L^2$$

for pair (H_h, H_{0h}) with $H_{0h} = -h^2\Delta$. Hence the scattering operator $S(H_h, H_{0h}) : L^2 \to L^2$ can be defined as a unitary operator. The mapping F_h defined by (2.2) decomposes $S(H_h, H_{0h})$ into the direct integral

$$S(H_h, H_{0h}) \simeq F_h S(H_h, H_{0h}) F_h^* \simeq \int_0^\infty \oplus\, S(\lambda; H_h, H_{0h})\, d\lambda,$$

where the fibre $S(\lambda; H_h, H_{0h}) : L^2(S^1) \to L^2(S^1)$ is called the scattering matrix at energy $\lambda > 0$ and it acts as

$$(S(\lambda; H_h, H_{0h})(F_h u)(\lambda, \cdot))(\omega) = (F_h S(H_h, H_{0h})u)(\lambda, \omega)$$

on $u \in L^2$. The operator $S(\lambda; H_h, H_{0h})$ is unitary and takes the form

$$S(\lambda; H_h, H_{0h}) = Id + T(\lambda; H_h, H_{0h}) \tag{2.3}$$

with $T(\lambda; H_h, H_{0h})$ of Hilbert–Schmidt class, where Id is the identity operator. If we denote by $S(\theta, \omega; H_h, H_{0h})$ the kernel of scattering matrix $S(E; H_h, H_{0h})$ at energy E, then the amplitude $f_h(\omega_- \to \omega)$ at energy E is defined by

$$f_h(\omega_- \to \omega) = c(E/h^2)\left(S(\omega, \omega_-; H_h, H_{0h}) - \delta(\omega - \omega_-)\right) \tag{2.4}$$

through the kernel $S(\theta, \omega; H_h, H_{0,h})$, where $c(E)$ is as in (1.5). If, in particular, $\omega \neq \omega_-$, then

$$f_h(\omega_- \to \omega) = c(E)h^{1/2} S(\omega, \omega_-; H_h, H_{0h}) \tag{2.5}$$

because of $c(E/h^2) = c(E)h^{1/2}$. We are now in a position to prove (1.7) in question.

Proposition 2.1 *We have*

$$\sigma_h(\omega_-) = -2(2\pi)^{1/2} E^{-1/4} h^{1/2} \mathrm{Re}\left(e^{i\pi/4} f_h(\omega_-)\right),$$

where $f_h(\omega_-) = f_h(\omega_- \to \omega_-)$ is the forward amplitude at energy E.

Proof. We denote by $T_h(\omega, \theta)$ the kernel of $T(E; H_h, H_{0h})$ defined in (2.3). Then we get

$$f_h(\omega_- \to \omega) = c(E)h^{1/2} T_h(\omega, \omega_-)$$

by (2.3) and (2.5). This remains true even for $\omega = \omega_-$. Since $S(E; H_h, H_{0h})$ is unitary, it follows that

$$T^*(E; H_h, H_{0h})T(E; H_h, H_{0h}) = -\left(T^*(E; H_h, H_{0h}) + T(E; H_h, H_{0h})\right).$$

We make use of this relation to calculate $\sigma_h(\omega_-)$ as

$$\begin{aligned}
\sigma_h &= \int |f_h(\omega_- \to \omega)|^2 \, d\omega = |c(E)|^2 h \int |T_h(\omega, \omega_-)|^2 \, d\omega \\
&= |c(E)|^2 h \int \overline{T}_h(\omega, \omega_-) T_h(\omega, \omega_-) \, d\omega = |c(E)|^2 h \int T_h^*(\omega_-, \omega) T_h(\omega, \omega_-) \, d\omega \\
&= -\overline{c}(E) c(E) h \left(\overline{T}_h(\omega_-) + T_h(\omega_-)\right),
\end{aligned}$$

where $T_h(\omega_-) = T_h(\omega_-, \omega_-)$. Hence we have

$$\sigma_h(\omega_-) = -c(E)h^{1/2} \overline{f}_h(\omega_-) - \overline{c}(E) h^{1/2} f_h(\omega_-).$$

Since $c(E) = (2\pi)^{1/2}E^{-1/4}e^{-i\pi/4}$, the desired relation follows at once. $\square$

3. Reduction to three main lemmas

It is convenient to discuss the problem as the scattering by solenoidal fields with two centers at large separation. In this section we put ourselves in such a framework and reduce the proof of Theorem 1.1 to those of three main lemmas (Lemmas 3.1, 3.2 and 3.3).

We write $H(B) = (-i\nabla - B)^2$ for the Schrödinger operator with vector potential $B(x) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. We recall that $\gamma(x;\omega)$ denotes the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$ ((1.6)). This function satisfies

$$\nabla\gamma(x;\omega) = \Lambda(x) = \left(-x_2/|x|^2, x_1/|x|^2\right). \tag{3.1}$$

We now define the two unitary operators

$$\left(U_1 f\right)(x) = h^{-1}f(h^{-1}x), \qquad \left(U_2 f\right)(x) = \exp(i\eta_0(x))f(x), \tag{3.2}$$

acting on L^2, where

$$\eta_0(x) = [\alpha/h]\gamma(x - d_+;\hat{e}) - [\alpha/h]\gamma(x - d_-;\hat{e}), \quad \hat{e} = e/|e|,$$

with $d_\pm = e_\pm/h$. As is easily seen, $\exp(i\eta_0(x))$ is well defined as a single valued function over $\boldsymbol{R}^2 \setminus \{d_+, d_-\}$. By definition, we have

$$U_1^*\left(h\nabla\right)U_1 f = \nabla f, \qquad U_1^* a(x) U_1 f = a(hx)f,$$

and $\eta_0(x)$ satisfies $\nabla\eta_0 = [\alpha/h]\Lambda(x - d_+) - [\alpha/h]\Lambda(x - d_-)$ by (3.1). If we further make use of relation $U_2^*\nabla U_2 f = \left(\nabla + i\nabla\eta_0\right)f$, then we have

$$K_d := \left(U_1U_2\right)^* H_h \left(U_1U_2\right) = \left(-i\nabla - B_d\right)^2 = H(B_d), \tag{3.3}$$

where

$$B_d(x) = \beta\Lambda(x - d_+) - \beta\Lambda(x - d_-), \quad \beta = \alpha/h - [\alpha/h]. \tag{3.4}$$

This operator is self–adjoint with domain

$$\mathcal{D}(K_d) = \{u \in L^2 : \left(-i\nabla - B_d\right)^2 u \in L^2, \ \lim_{|x-d_\pm|\to 0} |u(x)| < \infty\}$$

and enjoys the same spectral properties as the original operator H_h. We denote by $g_d(\omega_- \to \omega)$ the scattering amplitude at energy E for pair (K_d, H_0), $H_0 = -\Delta$ being the free Hamiltonian.

Proposition 3.1 *We have $f_h(\omega_- \to \omega) = h^{1/2}g_d(\omega_- \to \omega)$.*

Proof. We define $W_\pm(H,K;J)$ by

$$W_\pm(H,K;J) = s - \lim_{t\to\pm\infty} \exp(itH)J\exp(-itK) \tag{3.5}$$

for self–adjoint operators H, K and a bounded operator J on L^2. The function $\eta_0(x)$ in (3.2) falls off at infinity, so that

$$W_+(H_0,H_0;U_2) = W_-(H_0,H_0;U_2^*) = Id.$$

Since H_0 satisfies $\exp(-itH_0) = U_1^*\exp(-itH_{0h})U_1$ for $H_{0h} = -h^2\Delta$, we have

$$W_\pm(H_h,H_{0h}) = U_1U_2W_\pm(K_d,H_0)U_1^*$$

by (3.3), and hence

$$S(H_h,H_{0h}) = U_1S(K_d,H_0)U_1^*. \tag{3.6}$$

Let F and F_h be defined by (2.1) and (2.2) respectively. A simple computation shows that

$$F = F_hU_1. \tag{3.7}$$

This enables us to decompose $S(K_d,H_0)$ into the direct integral

$$S(K_d,H_0) \simeq FS(K_d,H_0)F^* \simeq \int_0^\infty \oplus\, S(\lambda;K_d,H_0)\,d\lambda$$

and we have

$$g_d(\omega_-\to\omega) = c(E)\left(S(\omega,\omega_-;K_d,H_0) - \delta(\omega-\omega_-)\right)$$

by (2.4) with $h=1$, where $S(\omega,\theta;K_d,H_0)$ is the kernel of $S(E;K_d,H_0)$. It follows from (3.6) and (3.7) that $S(\omega,\theta;H_h,H_{0h})$ equals $S(\omega,\theta;K_d,H_0)$. Since $c(E/h^2) = c(E)h^{1/2}$, we obtain the desired relation. $\square$

This proposition, together with Proposition 2.1, implies that

$$\sigma_h(\omega_-) = -2(2\pi)^{1/2}E^{-1/4}h\mathrm{Re}\left(e^{i\pi/4}g_d(\omega_-)\right), \tag{3.8}$$

where $g_d(\omega_-) = g_d(\omega_-\to\omega_-)$. The problem is now reduced to analyzing the asymptotic behavior of $\sigma_h(\omega_-)$ as the distance

$$|d| = |d_+ - d_-| = |e_+ - e_-|/h = |e|/h \to \infty$$

between centers d_+ and d_- of two solenoidal fields obtained from potential $B_d(x)$ goes to infinity.

We continue a reduction. We set $\omega_1 = (1,0)$ and $\omega_2 = (0,1)$. For brevity, we fix the incident direction ω_- as $\omega_- = \omega_2 = (0,1)$ throughout the discussion in the sequel. We introduce a smooth cut–off function $\chi \in C_0^\infty[0,\infty)$ such that $0\le\chi\le 1$ and

$$\chi(s) = 1 \quad \text{on } [0,1], \qquad \chi(s) = 0 \quad \text{on } [2,\infty). \tag{3.9}$$

Then we define

$$\eta(x) = (1 - \chi_M(x))\,\eta_1(x), \quad \eta_1 = \beta\gamma(x - d_+; \hat{e}) - \beta\gamma(x - d_-; \hat{e}), \tag{3.10}$$

where $\chi_M(x) = \chi(2h|x|/M)$ for $M \gg 1$, and β, $0 \le \beta < 1$, is as in (3.4). Even if β is not necessarily an integer, we can take $M \gg 1$ so large that $\exp(i\eta(x))$ becomes a single valued function. If we further set

$$\tilde{B}_d(x) = \chi_M(x) B_d(x) + (\nabla\chi_M(x))\,\eta_1,$$

then $\tilde{B}_d$ has compact support in $\{|x| < M/h\}$, and $B_d(x)$ admits the decomposition $B_d = \tilde{B}_d + \nabla\eta$. This is shown by use of $\nabla\eta_1 = B_d(x)$ for $|x| \gg 1$. In fact, B_d is calculated as

$$B_d = \chi_M B_d + (1 - \chi_M)\nabla\eta_1 = \chi_M B_d + \nabla\eta + (\nabla\chi_M)\,\eta_1 = \tilde{B}_d(x) + \nabla\eta.$$

Thus we have

$$K_d = H(B_d) = H(\tilde{B}_d + \nabla\eta) = e^{i\eta} H(\tilde{B}_d) e^{-i\eta} = e^{i\eta} \tilde{K}_d e^{-i\eta} \tag{3.11}$$

by the gauge transformation. The operator $\tilde{K}_d = H(\tilde{B}_d)$ has the same forward scattering amplitude as K_d. It is known by the principle of limiting absorption that

$$R(E \pm i0; K_d) = \lim_{\varepsilon \downarrow 0} (K_d - E \mp i\varepsilon)^{-1} : L^2_s \to L^2_{-s}$$

is bounded for $s > 1/2$, where L^2_s denotes the weighted L^2 space $L^2(\boldsymbol{R}^2; \langle x\rangle^{2s}\,dx)$ with $\langle x\rangle = (1 + |x|^2)^{1/2}$. We use the auxiliary operator $\tilde{K}_d$ to represent $\sigma_h(\omega_-)$ in terms of $R(E + i0; K_d)$.

We define the following functions :

$$v(x_1) = \chi(|x_1|/M), \qquad w(x_2) = \chi(|x_2|/M) \tag{3.12}$$

$$V(x_1) = v(hx_1) = \chi(h|x_1|/M), \quad W(x_2) = w(hx_2) = \chi(h|x_2|/M) \tag{3.13}$$

and $U(x) = V(x_1)W(x_2)$. By definition, $U = 1$ on $\operatorname{supp}\chi_M$, χ_M being as in (3.10). Let φ be the outgoing eigenfunction to equation $\tilde{K}_d\varphi = E\,\varphi$. Since $\tilde{B}_d$ has support in $\{|x| < M/h\}$,

$$\tilde{K}_d = H(\tilde{B}_d) = H_0 \tag{3.14}$$

over $\{|x| > M/h\}$ and hence the relation remains true on supp $(1 - U)$. If we put $\varphi = (1 - U)\varphi_- + \psi$ with $\varphi_- = \exp(iE^{1/2}x \cdot \omega_-)$, then ψ solves

$$\left(\tilde{K}_d - E\right)\psi = (H_0 U - U H_0)\,\varphi_- = [H_0, U]\varphi_-$$

by (3.14). Hence φ is represented as

$$\varphi = (1 - U)\,\varphi_- + R(E + i0; \tilde{K}_d)[H_0, U]\varphi_-. \tag{3.15}$$

Similarly we have

$$(1-U)\varphi = \varphi_- - R(E+i0;H_0)[H_0,U]\varphi, \tag{3.16}$$

because $[\tilde{K}_d, U] = [H_0, U]$ by (3.14). As is well known, $R(E+i0;H_0)$ has the integral kernel

$$G_0(x,y;E) = (i/4)H_0^{(1)}(E^{1/2}|x-y|)$$

and the Hankel function $H_0^{(1)}(z)$ of first kind and order zero behaves like

$$H_0^{(1)}(z) = (2/\pi)^{1/2}\exp(i(z-\pi/4))z^{-1/2}\left(1+O(|z|^{-1})\right), \quad |z|\to\infty.$$

This yields

$$G_0(x,y;E) = c_0(E)\exp(iE^{1/2}|x-y|)|x-y|^{-1/2}\left(1+O(|x-y|^{-1})\right) \tag{3.17}$$

as $|x-y|\to\infty$, where $c_0(E) = (8\pi)^{-1/2}E^{-1/4}e^{i\pi/4} = ic(E)/4\pi$, $c(E)$ being as in (1.5). We denote by $\langle\ ,\ \rangle$ the L^2 scalar product. Then it follows from (3.16) that φ behaves like

$$\varphi = \varphi_- - (ic(E)/4\pi)\,\langle[H_0,U]\varphi,\varphi_-\rangle\, e^{iE^{1/2}|x|}|x|^{-1/2} + o(|x|^{-1/2})$$

as $|x|\to\infty$ along direction ω_-. We insert (3.15) into φ in the scalar product on the right side to obtain that

$$g_d(\omega_-) = (ic(E)/4\pi)\left\langle\left((1-U)+R(E+i0;\tilde{K}_d)[H_0,U]\right)\varphi_-, [H_0,U]\varphi_-\right\rangle.$$

This enables us to represent $\sigma_h(\omega_-)$ in terms of the spectral resolution

$$\Pi(K_d) = (2\pi i)^{-1}\left(R(E+i0;K_d) - R(E-i0;K_d)\right) \tag{3.18}$$

at energy E of self–adjoint operator K_d.

Proposition 3.2 *Let* $\Pi(K_d)$ *be as above. Then*

$$\sigma_h(\omega_-) = \pi E^{-1/2}h\left\langle\Pi(K_d)e^{i\eta}[H_0,U]\varphi_-, e^{i\eta}[H_0,U]\varphi_-\right\rangle,$$

where $\varphi_-(x) = \exp(iE^{1/2}x\cdot\omega_-)$ *and* $\eta(x)$ *is defined by (3.10).*

Proof. We start from (3.8). We have $\Pi(\tilde{K}_d) = e^{-i\eta}\Pi(K_d)e^{i\eta}$ by (3.11), and the coefficient $\pi E^{-1/2}h$ arises from the product

$$-\left(2(2\pi)^{1/2}E^{-1/4}h\right)\times(ic(E)/4\pi)\times(2\pi i)\times e^{i\pi/4}\times 2^{-1} = \pi E^{-1/2}h.$$

Thus it suffices to show that $\mathrm{Im}\,\langle(1-U)\varphi_-, [H_0,U]\varphi_-\rangle = 0$. Since $U\in C_0^\infty(\boldsymbol{R}^2)$ is a real function with compact support and since $H_0\varphi_- = E\,\varphi_-$, it is easy to see that $\langle\varphi_-, [H_0,U]\varphi_-\rangle = 0$. We further have

$$\mathrm{Im}\,\langle U\varphi_-, [H_0,U]\varphi_-\rangle = \mathrm{Im}\,\langle U\varphi_-, -UH_0\varphi_-\rangle = E\,\mathrm{Im}\,\langle U\varphi_-, -U\varphi_-\rangle = 0.$$

This yields the desired representation. $\square$

The function $U(x)$ takes the form $U = V(x_1)W(x_2)$. Hence the commutator $[H_0, U]$ acts as

$$[H_0, U]\varphi_- = -2iE^{1/2}V\,(\partial_2 W)\,\varphi_- - (\Delta U)\,\varphi_-$$

on $\varphi_- = \exp(iE^{1/2}x_2)$. We insert it into the representation in Proposition 3.2 to get the scalar product

$$\Phi = 4\pi E^{1/2}h\left\langle \Pi(K_d)e^{i\eta}V\,(\partial_2 W)\,\varphi_-, e^{i\eta}V\,(\partial_2 W)\,\varphi_-\right\rangle. \tag{3.19}$$

We confine ourselves to analyzing the behavior as $h \to 0$ of this scalar product throughout the argument in the sequel. In fact, it makes only a contribution to the first three terms of the asymptotic formula in Theorem 1.1. In section 9, we make only a comment on the scalar products associated with the other pairs. These terms are shown to obey $o(h^{3/2})$.

We now write $e_\pm = (e_{\pm 1}, e_{\pm 2})$. By assumption, $\omega_- \neq \pm\hat{e}$ for $\omega_- = \omega_2 = (0, 1)$, and hence $e_{-1} \neq e_{+1}$. We may assume that $e_{-1} < e_{+1}$. Let v and w be as in (3.12) and let $v_\pm = \chi(M|x_1 - e_{\pm 1}|)$ for $M \gg 1$. Then $v\,(\partial_2 w)$ has support in $\{x : |x_1| \le 2M,\ M \le |x_2| \le 2M\}$. We introduce a smooth partition of unity normalized by

$$v_{-\infty} + v_- + v_0 + v_+ + v_{+\infty} = 1 \tag{3.20}$$

over the interval $[-2M, 2M]$, where the functions v_0 and $v_{\pm\infty}$ are nonnegative and have the following properties :

$$\begin{aligned} \operatorname{supp} v_{-\infty} &\subset (-3M, e_{-1} - 1/M), & v_{-\infty} &= 1 \text{ on } [-2M, e_{-1} - 2/M] \\ \operatorname{supp} v_0 &\subset (e_{-1} + 1/M, e_{+1} - 1/M), & v_0 &= 1 \text{ on } [e_{-1} + 2/M, e_{+1} - 2/M] \\ \operatorname{supp} v_{+\infty} &\subset (e_{+1} + 1/M, 3M), & v_{+\infty} &= 1 \text{ on } [e_{+1} + 2/M, 2M]. \end{aligned}$$

We further introduce nonnegative smooth functions $w^\pm$ such that

$$\begin{aligned} \operatorname{supp} w^+ &\subset (M/2, 3M), & w^+ &= 1 \text{ on } [M, 2M], \\ \operatorname{supp} w^- &\subset (-3M, -M/2), & w^- &= 1 \text{ on } [-2M, -M]. \end{aligned}$$

If we set $W^\pm(x_2) = w^\pm(hx_2)$ and

$$V_{\pm\infty}(x_1) = v_{\pm\infty}(hx_1), \quad V_\pm(x_1) = v_\pm(hx_1), \quad V_0(x_1) = v_0(hx_1),$$

then it follows from (3.20) that Φ defined by (3.19) is decomposed into the sum

$$\Phi = \Phi^+_{-\infty} + \Phi^+_- + \Phi^+_0 + \Phi^+_+ + \Phi^+_{+\infty} + \Phi^-_{-\infty} + \Phi^-_- + \Phi^-_0 + \Phi^-_+ + \Phi^-_{+\infty}, \tag{3.21}$$

where

$$\Phi^\pm_{+\infty} = 4\pi E^{1/2}h\left\langle \Pi(K_d)e^{i\eta}V_{+\infty}W^\pm\,(\partial_2 W)\,\varphi_-, e^{i\eta}V\,(\partial_2 W)\,\varphi_-\right\rangle,$$

$$\Phi_0^{\pm} = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_0 W^{\pm} \left(\partial_2 W\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle,$$
$$\Phi_+^{\pm} = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^{\pm} \left(\partial_2 W\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle,$$
$$\Phi_-^{\pm} = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_- W^{\pm} \left(\partial_2 W\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle,$$
$$\Phi_{-\infty}^{\pm} = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_{-\infty} W^{\pm} \left(\partial_2 W\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle.$$

The remaining sections are devoted to analyzing the behavior as $h \to 0$ of each term on the right sides of (3.21). We prove the following three main lemmas, which yield the first three terms of the asymptotic formula in Theorem 1.1.

Lemma 3.1 *We have $\Phi_{-\infty}^{\pm} = O(h^N)$ and $\Phi_{+\infty}^{\pm} = O(h^N)$ for any $N \gg 1$.*

Lemma 3.2 *We have $\Phi_0^{\pm} = \left(1 - e^{\pm i2\beta\pi}\right) \int v_0(x_1)\, dx_1 + O(h^N)$.*

Lemma 3.3 *Let $F_{\pm}(\omega_-)$ and $\tau_{\pm}$ be as in Theorem 1.1, and let k_0 and k_1 be the real numbers defined by*

$$k_0 = 2(\beta - 1/2) E^{-1/2} \sin \beta\pi, \qquad k_1 = (2\pi)^{1/2} |e|^{-1/2} E^{-1/4}$$

with $\beta = \alpha/h - [\alpha/h]$. Then we have

$$\Phi_+^- = \left(1 - e^{-i2\beta\pi}\right) \int_{-\infty}^{e_{+1}} v_+(x_1)\, dx_1 - ik_0 e^{-i\beta\pi} h - k_1 e^{i\pi/4} \tau_+ F_-(\omega_-) h^{1/2} + o(h^{3/2})$$
$$\Phi_-^+ = \left(1 - e^{i2\beta\pi}\right) \int_{e_{-1}}^{\infty} v_-(x_1)\, dx_1 + ik_0 e^{i\beta\pi} h - k_1 \overline{e^{i\pi/4} \tau_+ F_-(\omega_-)} h^{1/2} + o(h^{3/2})$$
$$\Phi_-^- = \left(1 - e^{-i2\beta\pi}\right) \int_{e_{-1}}^{\infty} v_-(x_1)\, dx_1 - ik_0 e^{-i\beta\pi} h - k_1 e^{i\pi/4} \tau_- F_+(\omega_-) h^{1/2} + o(h^{3/2})$$
$$\Phi_+^+ = \left(1 - e^{i2\beta\pi}\right) \int_{-\infty}^{e_{+1}} v_+(x_1)\, dx_1 + ik_0 e^{i\beta\pi} h - k_1 \overline{e^{i\pi/4} \tau_- F_+(\omega_-)} h^{1/2} + o(h^{3/2}).$$

We make repeated use of stationary phase method to prove these lemmas. We end the section by formulating this method under the notation with the same meaning as ascribed in [6, Theorem 7.7.5]. In particular, f' and f'' denote the gradient ∇f and the Hessian $\operatorname{Hess} f$ of a real function $f(x)$ respectively.

Proposition 3.3 *Let $K \subset \boldsymbol{R}^n$ be a compact set and let X be an open neighborhood of K. Assume that $f \in C^{3k+1}(X)$ is a real function and $a \in C_0^{2k}(X)$ for some positive integer k. Suppose that there exists a unique $x_0 \in K$ such that*

$$f'(x_0) = 0, \quad \det f''(x_0) \neq 0, \quad f'(x_0) \neq 0 \ \text{in } K \setminus \{x_0\}.$$

Then

$$\left|\int \exp(i\lambda f(x))a(x)\,dx - \exp(i\lambda f(x_0))\Big(\det\,(\lambda f''(x_0)/2\pi i)\,\Big)^{-1/2}\sum_{0\le j<k}\lambda^{-j}L_j u\right|$$
$$\le\ C\,\lambda^{-k}\sum_{|\alpha|\le 2k}\sup\,|D^\alpha a|$$

for $\lambda \gg 1$. *Here* C *is bounded when* f *remains in a bounded set in* $C^{3k+1}(X)$ *and* $|x-x_0|/|f'(x)|$ *has a uniform bound, while*

$$L_j a = \sum_{\nu-\mu=j}\ \sum_{2\nu\ge 3\mu} i^{-j}2^{-\nu}\langle f''(x_0)^{-1}D, D\rangle^\nu\,(g_0^\mu a)\,(x_0)/\mu!\,\nu!$$

is a differential operator of order $2j$ *acting on* $a(x)$ *at* x_0, *and* $g_0(x)$ *is defined by*

$$g_0(x) = f(x) - f(x_0) - \langle f''(x_0)(x-x_0), (x-x_0)\rangle/2.$$

We often use the proposition when f is a quadratic function. In this case, $g_0(x)=0$ vanishes identically (g_0^0 is interpreted as $g_0^0=1$), and L_j acts as

$$L_j a = (2i)^{-j}\langle f''(x_0)^{-1}D, D\rangle^j a(x_0)/j! = \Big(L^j a\Big)\,(x_0)/j!$$

with $L=(2i)^{-1}\langle f''(x_0)^{-1}D, D\rangle$.

4. Proof of Lemmas 3.1, 3.2 and 3.3

The first half of the present section is devoted to proving Lemmas 3.1 and 3.2 and the second one to reducing the proof of Lemma 3.3 to those of three lemmas (Lemmas 4.3, 4.4 and 4.5). We fix ε and δ $(0<\varepsilon,\ \delta\ll 1)$ small enough throughout the discussion in the sequel. Let $\kappa_\pm(s)\in C^\infty(\boldsymbol{R})$ be a real periodic function with period 2π such that $\kappa_\pm(s)=\pm\beta s$ on the interval $[\varepsilon, 2\pi-\varepsilon]$. Then we define the real function $j_\pm(x;\omega)\in C^\infty(\boldsymbol{R}^2)$ by

$$j_\pm(x;\omega) = (1-\chi(h^\varepsilon|x|/\delta))\,\kappa_\pm(\gamma(x;\omega)) \tag{4.1}$$

for $\omega\in S^1$, where $\chi\in C_0^\infty(\boldsymbol{R}^2)$ is a smooth cut–off function with property (3.9). For $g\in C_0^\infty$ with compact support, we also use the notation $\tilde g\in C_0^\infty$ to denote a function such that $\tilde g$ has a little wider support than g and satisfies $\tilde g = g$ on $\operatorname{supp} g$.

We prove Lemma 3.1 only for

$$\Phi_{+\infty}^+ = 4\pi E^{1/2}h\left\langle \Pi(K_d)e^{i\eta}V_{+\infty}W^+\,(\partial_2 W)\,\varphi_-, e^{i\eta}V\,(\partial_2 W)\,\varphi_-\right\rangle,$$

where η is defined by (3.10). A similar argument applies to the other terms $\Phi_{+\infty}^-$ and $\Phi_{-\infty}^\pm$. The proof relies on the lemma below. The proof of this lemma is done after completing the proof of Lemmas 3.1 and 3.2.

Lemma 4.1 *Let $j_\pm(x;\omega)$ be as in (4.1). Define the real smooth function $\zeta(x)$ by*

$$\zeta(x) = j_+(x - d_+; -\hat{e}) + j_-(x - d_-; -\hat{e})$$

and the operator L_0 by $L_0 = \exp(i\zeta)H_0\exp(-i\zeta) = H(\nabla\zeta)$. Then there exists $\tilde{v}_{+\infty}(x_1)$ such that

$$\begin{aligned}&\left\langle \Pi(K_d)e^{i\eta}V_{+\infty}W^+\left(\partial_2 W\right)\varphi_-, e^{i\eta}V\left(\partial_2 W\right)\varphi_-\right\rangle = \\ &\qquad \left\langle \Pi(L_0)e^{i\eta}V_{+\infty}W^+\left(\partial_2 W\right)\varphi_-, e^{i\eta}\tilde{V}_{+\infty}\left(W^+ + W^-\right)\left(\partial_2 W\right)\varphi_-\right\rangle + O(h^N)\end{aligned}$$

for any $N \gg 1$, where $\tilde{V}_{+\infty}(x_1) = \tilde{v}_{+\infty}(hx_1)$.

Proof of Lemma 3.1. We set

$$\Sigma_\pm(\omega) = \{x : |x - d_\pm| > h^{-\varepsilon},\ \ |\gamma(x - d_\pm; -\omega) - \pi| > \varepsilon\} \tag{4.2}$$

for $\omega \in S^1$. Then $\nabla\zeta = B_d(x)$ over $\Sigma_-(-\hat{e}) \cap \Sigma_+(-\hat{e})$ by (3.1) and (4.1). Since $e_{-1} < e_{+1}$, $\Sigma_+(-\hat{e}) \subset \Sigma_-(-\hat{e})$ and hence $K_d = L_0$ over $\Sigma_+(-\hat{e})$. The particle starting from $\operatorname{supp} V_{+\infty}W^+$ at velocity $2E^{1/2}\omega_-$ moves like a free particle without passing over the outside of $\Sigma_+(-\hat{e})$ for all t, $-\infty < t < \infty$. Thus $R(E \pm i0; K_d)$ is approximated by $R(E \pm i0; L_0)$. This is the motivation to introduce the auxiliary operator L_0 as in Lemma 4.1. This lemma yields the decomposition $\Phi^+_{+\infty} = \Psi_+ + \Psi_- + O(h^N)$, where

$$\begin{aligned}\Psi_+ &= 4\pi E^{1/2}h\left\langle \Pi(L_0)e^{i\eta}V_{+\infty}W^+\left(\partial_2 W\right)\varphi_-, e^{i\eta}\tilde{V}_{+\infty}W^+\left(\partial_2 W\right)\varphi_-\right\rangle, \\ \Psi_- &= 4\pi E^{1/2}h\left\langle \Pi(L_0)e^{i\eta}V_{+\infty}W^+\left(\partial_2 W\right)\varphi_-, e^{i\eta}\tilde{V}_{+\infty}W^-\left(\partial_2 W\right)\varphi_-\right\rangle.\end{aligned}$$

We write Ψ_+ in the integral form to analyze its behavior as $h \to 0$. By definition, $\Pi(L_0) = e^{i\zeta}\Pi(H_0)e^{-i\zeta}$ and also $\Pi(H_0)$ has the kernel

$$\Pi(x, y; H_0) = \left(8\pi^2\right)^{-1}\int \varphi_0(x;\omega)\overline{\varphi}_0(y;\omega)\,d\omega$$

with $\varphi_0(x;\omega) = \exp(iE^{1/2}x\cdot\omega)$. Since $\gamma(x - d_\pm; \hat{e}) - \gamma(x - d_\pm; -\hat{e}) = -\pi$ for $x \in \operatorname{supp} V_{+\infty}W^+$, $\eta - \zeta = 0$ there. Similarly $\eta - \zeta$ vanishes over $\operatorname{supp} \tilde{V}_{+\infty}W^+$. If we make a change of variables $hy \to y$, then $V_{+\infty}(y_1)$ and $\left(\partial_2 W\right)(y_2)$ undergo the changes

$$V_{+\infty}(y_1) = v_{+\infty}(hy_1) \to v_{+\infty}(y_1), \quad \left(\partial_2 W\right) = \left(\partial_2 w\right)(hy_2) \to h\left(\partial_2 w\right)(y_2).$$

If we work in the coordinate system $\omega = (-\sin\theta, \cos\theta) \in S^1$, then we also have

$$|\nabla\left(y\cdot(\omega_- - \omega)\right)| = |\omega_- - \omega| \ge c\,h^{1-\varepsilon}, \quad c > 0,$$

for $|\theta| > 2\,h^{1-\varepsilon}$. Thus we make a change of variable $h^{-(1-\varepsilon)}\theta \to \theta$ and make repeated use of partial integration to see that Ψ_+ takes the integral form

$$\Psi_+ = (2\pi)^{-1}E^{1/2}h^{-\varepsilon}\int\int\left[\int\int\exp(ih^{-\varepsilon}f)a\,d\theta\,dy_1\right]g\,dy_2\,dx + O(h^N),$$

where $f = E^{1/2}\,(y_1 - x_1)\,\theta$ and

$$\begin{aligned} a &= \chi(\theta) v_{+\infty}(y_1) \exp\left(iE^{1/2} r(y/h, x/h, h^{1-\varepsilon}\theta)\right) \\ r &= (y_1 - x_1)\,(\sin\theta - \theta) + (y_2 - x_2)(1 - \cos\theta) \\ g &= w^+(y_2)\,(\partial_2 w)\,(y_2) \tilde{v}_{+\infty}(x_1) w^+(x_2)\,(\partial_2 w)\,(x_2). \end{aligned}$$

We apply the stationary phase method to the integral in the bracket. The function a is a smooth function with derivatives bounded uniformly in h and satisfies

$$a = v_{+\infty}(x_1), \qquad L_j a = \left(-iE^{-1/2} D_\theta D_{y_1}\right)^j a = 0, \quad j \geq 1,$$

at the stationary point $(\theta, y_1) = (0, x_1)$. Since

$$\int w^{\pm}(y_2)\,(\partial_2 w)\,(y_2)\,dy_2 = \mp 1 \tag{4.3}$$

and since $v_{+\infty}(x_1)\tilde{v}_{+\infty}(x_1) = v_{+\infty}(x_1)$, we get $\Psi_+ = \int v_{+\infty}(x_1)\,dx_1 + O(h^N)$. A similar argument applies to Ψ_-. Since $\eta - \zeta = 0$ on $\operatorname{supp} \tilde{V}_{+\infty} W^-$, it follows from (4.3) that $\Psi_- = -\int v_{+\infty}(x_1)\,dx_1 + O(h^N)$ and the proof is completed. $\square$

We move to the proof of Lemma 3.2. We consider only the scalar product

$$\Phi_0^+ = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_0 W^+\,(\partial_2 W)\,\varphi_-, e^{i\eta} V\,(\partial_2 W)\,\varphi_- \right\rangle.$$

The proof is based on the lemma below. We proceed with the argument, accepting it as proved.

Lemma 4.2 *Set $\zeta = j_+(x - d_+; \hat{e}) + j_-(x - d_-; -\hat{e})$ and define $L_0 = H(\nabla\zeta)$. Then there exists $\tilde{v}_0(x_1)$ such that*

$$\begin{aligned} &\left\langle \Pi(K_d) e^{i\eta} V_0 W^+\,(\partial_2 W)\,\varphi_-, e^{i\eta} V\,(\partial_2 W)\,\varphi_- \right\rangle = \\ &\qquad \left\langle \Pi(L_0) e^{i\eta} V_0 W^+\,(\partial_2 W)\,\varphi_-, e^{i\eta} \tilde{V}_0 \left(W^+ + W^-\right)(\partial_2 W)\,\varphi_- \right\rangle + O(h^N), \end{aligned}$$

where $\tilde{V}_0(x_1) = \tilde{v}_0(hx_1)$.

Proof of Lemma 3.2. The proof is based on the same idea as in the proof of Lemma 3.1. The lemma above enables us to decompose Φ_0^+ into $\Phi_0^+ = \Psi_+ + \Psi_- + O(h^N)$, where

$$\begin{aligned} \Psi_+ &= 4\pi E^{1/2} h \left\langle \Pi(L_0) e^{i\eta} V_0 W^+\,(\partial_2 W)\,\varphi_-, e^{i\eta} \tilde{V}_0 W^+\,(\partial_2 W)\,\varphi_- \right\rangle, \\ \Psi_- &= 4\pi E^{1/2} h \left\langle \Pi(L_0) e^{i\eta} V_0 W^+\,(\partial_2 W)\,\varphi_-, e^{i\eta} \tilde{V}_0 W^-\,(\partial_2 W)\,\varphi_- \right\rangle. \end{aligned}$$

If $x \in \operatorname{supp} V_0 W^+$, then $\gamma(x - d_-; \hat{e}) - \gamma(x - d_-; -\hat{e}) = -\pi$ and hence $\eta - \zeta = \beta\pi$ over $\operatorname{supp} V_0 W^+$. On the other hand, $\eta - \zeta = -\beta\pi$ over $\operatorname{supp} V_0 W^-$. Thus we obtain

$$\Psi_+ = \int v_0(x_1)\, dx_1 + O(h^N), \qquad \Psi_- = -e^{i2\beta\pi} \int v_0(x_1)\, dx_1 + O(h^N)$$

by use of the stationary phase method. $\square$

The proof of Lemmas 4.1 and 4.2 relies on the resolvent estimate uniform in h for $R(E + i0; K_d)$.

Proposition 4.1 *Denote by $\| \ \|$ the operator norm of bounded operators on L^2.*

(1) *Let X_ρ be the multiplication by $\langle x \rangle^\rho$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then there exists $\nu \gg 1$ large enough such that*

$$\|X_{-\nu} R(E \pm i0; K_d) X_{-\nu}\| = O(h^{-\nu}).$$

(2) *Let $s_\pm$ be the characteristic function of $S_\pm = \{|x - d_\pm| < h^{-\varepsilon}\}$. Then there exists $c > 0$ independent of ε such that*

$$\|s_\pm R(E + i0; K_d) s_\pm\| = O(h^{-c\varepsilon}), \quad \|s_+ R(E \pm i0; K_d) s_-\| = O(h^{1/2 - c\varepsilon}).$$

The proposition has been already obtained as [9, Theorem 4.1], where the vector potential of $K_d = H(B_d)$ has a flux independent of h. We need to add a comment. The proposition is still true in the case that $B_d(x)$ has the flux $\beta = \alpha/h - [\alpha/h]$, $0 \le \beta < 1$, bounded uniformly in h. We further remark that the proposition remains true even in the case of a finite number of solenoidal fields.

Proof of Lemma 4.1. Throughout the proof, we denote by $O_p(h^\rho)$ bounded operators the norm of which obeys $O(h^\rho)$. We define $\psi(x)$ by

$$\psi = 1 - \chi(2h^\varepsilon |x - d_+|) - \chi(2h^\varepsilon |x - d_-|)$$

and $p(\xi)$ by

$$p(\xi) = \chi(|\xi - E^{1/2}\omega_-|/\delta), \quad 0 < \delta \ll 1. \tag{4.4}$$

We write $p(D_x)$ for the pseudodifferential operator with symbol $p(\xi)$. We further define

$$Q = \psi e^{i\zeta} R(E + i0; H_0) \tilde{p}(D_x) e^{-i\zeta} U_{+\infty}^- p(D_x),$$

where $U_{+\infty}^- = e^{i\eta} V_{+\infty} W^- \, (\partial_2 W)$. Since $L_0 = e^{i\zeta} H_0 e^{-i\zeta}$, Q satisfies

$$(K_d - E)Q = (L_0 - E)Q + Q_0 = U_{+\infty}^- p(D_x) + Q_2 + Q_1 + Q_0,$$

where $Q_0 = (K_d - L_0)Q$ and

$$Q_1 = e^{i\zeta}[H_0, \psi] \tilde{p}(D_x) e^{-i\zeta} U_{+\infty}^- p(D_x), \quad Q_2 = e^{i\zeta} \left(Id - \tilde{p}(D_x) \right) e^{-i\zeta} U_{+\infty}^- p(D_x).$$

Hence we have

$$R(E+i0;K_d)U_{+\infty}^{-}p(D_x)=Q-R(E+i0;K_d)\left(Q_0+Q_1+Q_2\right).$$

We now assert that

$$X_\nu Q_0 X_N = O_p(h^N) \tag{4.5}$$

for any $N \gg 1$, where $\nu > 0$ is as in Proposition 4.1. To see it, we first note that $K_d = L_0$ over $\Sigma_+(-\hat{e})$ as stated previously, where $\Sigma_\pm(\omega)$ is defined by (4.2). Next we write $R(E+i0;H_0)$ in the integral form

$$R(E+i0;H_0)=i\int_0^\infty e^{-itE}\exp(-itH_0)\,dt.$$

If x is in the outside of $\Sigma_+(-\hat{e})$ and if $y\in\operatorname{supp}V_{+\infty}W^-$ and $\xi\in\operatorname{supp}\tilde{p}$, then

$$\left|\nabla_\xi\left(x\cdot\xi-(y\cdot\xi+t|\xi|^2)\right)\right|=|x-(y+2t\xi)|>c(|t|+|x|+|y|+1/h)$$

for some $c>0$. Hence (4.5) is obtained by repeated use of integration by parts, and this, together with Proposition 4.1, implies that

$$V\left(\partial_2 W\right)R(E+i0;K_d)Q_0X_N=O_p(h^N). \tag{4.6}$$

A similar result is shown to be true for Q_1, if we take account of the fact that the particle starting from $\operatorname{supp}V_{+\infty}W^-$ at velocity $2E^{1/2}\omega_-$ never passes over S_+ and S_- in which the coefficients of $[H_0,\psi]$ have support. The symbol $p(\xi)$ vanishes on supp $(1-\tilde{p})$, so that $Q_2 = X_{-N}O_p(h^N)X_{-N}$ by pseudodifferential calculus. This proves (4.6) with Q_0 replaced by Q_2. Since $R(E+i0;L_0)$ equals $e^{i\zeta}R(E+i0;H_0)e^{-i\zeta}$, we have

$$V\left(\partial_2W\right)Q=V\left(\partial_2W\right)R(E+i0;L_0)U_{+\infty}^{-}p(D_x)+X_{-N}O_p(h^N)X_{-N}.$$

We can also obtain by use of a similar argument that

$$\left(\partial_2W\right)\left(V-\tilde{V}_{+\infty}\right)R(E+i0;L_0)U_{+\infty}^{-}p(D_x)=X_{-N}O_p(h^N)X_{-N}$$

and hence it follows that

$$\left(\partial_2W\right)\left(VR(E+i0;K_d)-\tilde{V}_{+\infty}R(E+i0;L_0)\right)U_{+\infty}^{-}p(D_x)=X_{-N}O_p(h^N)X_{-N}.$$

This remains true between $R(E-i0;K_d)$ and $R(E-i0;L_0)$. Since $p(D_x)\varphi_-=\varphi_-$, the desired relation is obtained and the proof is complete. □

Proof of Lemma 4.2. The lemma is verified in almost the same way as Lemma 4.1. We note that the particle starting from $\operatorname{supp}V_0W^+$ at velocity $2E^{1/2}\omega_-$ moves like a free particle for all t without touching the centers d_+ and d_-. This enables us to repeat the same argument as in the proof of Lemma 4.1, and we get the desired relation. We skip the details. □

The proof of Lemma 3.3 is done by reducing it to proofs of three lemmas. For brevity, we analyze the behavior of only the term

$$\Phi_+^- = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle$$

in some details and we only comment on necessary modifications for the other terms in section 9. We now assume that $z(x_1) \in C_0^\infty(\boldsymbol{R})$ is a real function the support of which does not intersect with $\operatorname{supp} v_+$. Then it follows that

$$\left\langle \Pi(K_d) e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} Z W^- \left(\partial_2 W\right) \varphi_- \right\rangle = O(h^N) \tag{4.7}$$

for $Z(x_1) = z(hx_1)$. In fact, the particle starting from $\operatorname{supp} ZW^-$ at velocity $2E^{1/2}\omega_-$ moves like a free particle for $t > 0$ and never passes over $\operatorname{supp} V_+ W^-$, so that we have

$$\left\langle e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, R(E + i0; K_d) e^{i\eta} Z W^- \left(\partial_2 W\right) \varphi_- \right\rangle = O(h^N)$$

in the same way as in the proof of Lemma 4.1. We can also easily see that

$$\left\langle e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, R(E - i0; K_d) e^{i\eta} Z W^- \left(\partial_2 W\right) \varphi_- \right\rangle = O(h^N).$$

Thus (4.7) is verified. If $z(x_1)$ vanishes on $\operatorname{supp} v_\pm$, then we follow the same reasoning as above to obtain that

$$\left\langle \Pi(K_d) e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} Z W^+ \left(\partial_2 W\right) \varphi_- \right\rangle = O(h^N).$$

Thus Φ_+^- admits the decomposition $\Phi_+^- = \Gamma_+ + \Gamma + \Gamma_- + O(h^N)$, where

$$\Gamma_+ = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} \tilde{V}_+ W^- \left(\partial_2 W\right) \varphi_- \right\rangle,$$

$$\Gamma = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} \tilde{V}_+ W^+ \left(\partial_2 W\right) \varphi_- \right\rangle, \tag{4.8}$$

$$\Gamma_- = 4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} \tilde{V}_- W^+ \left(\partial_2 W\right) \varphi_- \right\rangle. \tag{4.9}$$

The asymptotic formula of Φ_+^- in Lemma 3.3 is obtained from the following three lemmas.

Lemma 4.3 *We have* $\Gamma_+ = \int v_+(x_1)\, dx_1 + O(h^N)$.

Lemma 4.4 *Let k_0 be as in Lemma 3.3. Then we have*

$$\Gamma = -e^{-i2\beta\pi} \int_{-\infty}^{e_{+1}} v_+(x_1)\, dx_1 - \int_{e_{+1}}^{\infty} v_+(x_1)\, dx_1 - ik_0 e^{-i\beta\pi} h + o(h^{3/2}).$$

Lemma 4.5 *Let k_1 be as in Lemma 3.3. Then we have*

$$\Gamma_- = -k_1 e^{i\pi/4}\tau_+ F_-(\omega_-)h^{1/2} + o(h^{3/2}).$$

Among these three lemmas, Lemma 4.3 can be verified in the same way as in the proof of Lemmas 3.1. We skip its proof. Next we analyze the behavior of Γ. This term describes a contribution from the trajectroies of particles starting from $\operatorname{supp} V_+W^-$ and arriving at $\operatorname{supp} V_+W^+$. Thus the behavior is closely related to the trajectories passing near center $d_+ = e_+/h$. We define the self–adjoint operator

$$K_\pm = H(B_\pm) = (-i\nabla - B_\pm)^2\,, \quad B_\pm(x) = \pm\beta\Lambda(x-d_\pm), \tag{4.10}$$

with domain

$$\mathcal{D}_\pm = \{u \in L^2 : (-i\nabla - B_\pm)^2 u \in L^2,\ \lim_{|x-d_\pm|\to 0} |u(x)| < \infty\}.$$

Let $j_\pm(x;\omega)$ be as in (4.1). Then we introduce the auxiliary operator

$$L_\pm = \exp(i\zeta_\mp)K_\pm \exp(-i\zeta_\mp) = H(B_\pm + \nabla\zeta_\mp), \tag{4.11}$$

where $\zeta_\pm = j_\pm(x-d_\pm;\pm\hat{e})$. The operator $L_\pm$ is also self–adjoint with the same domain as $K_\pm$. We prove the following lemma.

Lemma 4.6 *Let the notation be as above. Then $\Gamma = \Gamma_1 + o(h^{3/2})$, where Γ_1 is defined by*

$$\Gamma_1 = 4\pi E^{1/2} e^{-i2\beta\pi} h \left\langle \Pi(K_+)e^{i\zeta_+}V_+W^-\left(\partial_2 W\right)\varphi_-, e^{i\zeta_+}\tilde{V}_+W^+\left(\partial_2 W\right)\varphi_-\right\rangle$$

with $\zeta_+ = j_+(x-d_+;\hat{e})$ as in (4.11).

Lemma 4.6 is verified through a series of lemmas. In the statements and proofs of these lemmas, we use the following notation :

$$\begin{aligned} T_- &= \psi_- L_+ - K_d\psi_- = [\psi_-, L_+] + (L_+ - K_d)\psi_- = T_{-0} + T_{-1} \\ T_+ &= \psi_+ L_0 - L_+\psi_+ = [\psi_+, L_0] + (L_0 - L_+)\psi_+ = T_{+0} + T_{+1}, \end{aligned}$$

where $\psi_\pm = 1 - \chi(2h^\varepsilon|x-d_\pm|)$ and $L_0 = \exp(-i\zeta_+)H_0\exp(i\zeta_+) = H(\nabla\zeta_+)$.

Lemma 4.7 *We have*

$$\begin{aligned} \tilde{V}_+W^+R(E+i0;K_d)V_+W^- &= \tilde{V}_+W^+R(E+i0;L_+)V_+W^- \\ &+\ \tilde{V}_+W^+\left(R(E+i0;K_d)T_{-0}R(E+i0;L_+) + O_p(h^N)\right)V_+W^-. \end{aligned}$$

Proof. We give only a sketch for the proof. The detailed argument can be found in the proof of [10, Lemma 4.3]. Since $\psi_- = 1$ on both $\operatorname{supp} \tilde{V}_+ W^+$ and $\operatorname{supp} V_+ W^-$, we get

$$\begin{aligned} \tilde{V}_+ W^+ R(E+i0; K_d) V_+ W^- &= \tilde{V}_+ W^+ R(E+i0; L_+) V_+ W^- \\ &+ \tilde{V}_+ W^+ R(E+i0; K_d)\,(T_{-0}+T_{-1})\, R(E+i0; L_+) V_+ W^-. \end{aligned}$$

Hence we have only to show that

$$\tilde{V}_+ W^+ R(E+i0; K_d) T_{-1} R(E+i0; L_+) V_+ W^- = O_p(h^N). \tag{4.12}$$

We recall the notation $\Sigma_\pm(\omega)$ ((4.2)) and define $\Omega_\pm(\omega)$ as

$$\Omega_\pm(\omega) = \{x : |x-d_\pm| > h^{-\varepsilon}, \ \ |\gamma(x-d_\pm; -\omega) - \pi| < \varepsilon\}. \tag{4.13}$$

Then $\nabla\zeta_- = B_-(x)$ on $\Sigma(-\hat{e})$, so that

$$K_d = H(B_+ + B_-) = H(B_+ + \nabla\zeta_-) = \exp(i\zeta_-) H(B_+) \exp(-i\zeta_-) = L_+$$

there. Thus the coefficients of T_{-1} have support in $\Omega_-(-\hat{e})$. The relation (4.12) is intuitively obvious. In fact, the particle starting from $\operatorname{supp} V_+ W^-$ never passes over $\operatorname{supp} \tilde{V}_+ W^+$ after scattered into $-\hat{e}$ by the field $2\beta\pi\delta(x-d_+)$, because the scattered particle moves like a free particle over $\Omega_-(-\hat{e})$. The detailed proof is done by constructing approximations to $R(E+i0; K_d)$ and $R(E+i0; L_+)$ as in the proof of Lemma 4.1. Then Proposition 4.1 is used to control error terms arising from such approximations. $\square$

If we set $\tilde{\psi}_- = 1 - \chi(2^2 h^\varepsilon |x-d_-|)$, then $\tilde{\psi}_- \psi_- = \psi_-$ and hence $T_{-0}\tilde{\psi}_- = T_{-0}$. Then the same argument as above shows that

$$\begin{aligned} T_{-0} R(E-i0; K_d) \tilde{V}_+ W^+ &= T_{-0} R(E-i0; L_+) \tilde{V}_+ W^+ \\ &+ T_{-0} \Big(R(E-i0; K_d) \tilde{T}_{-0} R(E-i0; L_+) + O_p(h^N) \Big) \tilde{V}_+ W^+, \end{aligned} \tag{4.14}$$

where $\tilde{T}_{-0}$ is defined as $\tilde{T}_{-0} = [\tilde{\psi}_-, L_+]$.

Lemma 4.8 *We have*

$$\begin{aligned} & s_- R(E+i0; L_+) V_+ W^- p(D_x) \\ &\quad = s_- R(E+i0; L_+) T_{+0} R(E+i0; L_0) V_+ W^- p(D_x) + O_p(h^N) X_{-N} \\ & s_- R(E-i0; L_+) \tilde{V}_+ W^+ p(D_x) \\ &\quad = s_- R(E-i0; L_+) T_{+0} R(E-i0; L_0) \tilde{V}_+ W^+ p(D_x) + O_p(h^N) X_{-N}, \end{aligned}$$

where $s_\pm$ is the characteristic function of $S_\pm = \{|x-d_\pm| < h^{-\varepsilon}\}$ and $p(\xi)$ is defined by (4.4).

Proof. We give only a sketch for the proof of this lemma also. To prove the first relation, it suffices to show that

$$s_- R(E+i0;L_+)T_{+1}R(E+i0;L_0)V_+W^- p(D_x) = O_p(h^N)X_{-N}.$$

The coefficients of T_{+1} have support in $\Omega_+(\hat{e})$. The free particle passing there has $2E^{1/2}\omega_-$ as a velocity and it does not touch the center d_+. This means that the particle never passes over S_-. Thus the lemma is proved by constructing an approximation to $R(E+i0;L_+)$ as in the proof of Lemma 4.1. $\square$

Lemma 4.9 *We have*

$$\begin{aligned} & s_+ R(E+i0;H_0)e^{i\zeta_+}V_+W^-\left(\partial_2 W\right)\varphi_- \\ & \qquad = \ s_+\left((i/2)e^{i\beta\gamma(-\omega_-;\hat{e})}E^{-1/2}\varphi_- + O(h^{1/2-\varepsilon})\right) \\ & s_+ R(E-i0;H_0)e^{i\zeta_+}\tilde{V}_+W^+\left(\partial_2 W\right)\varphi_- \\ & \qquad = \ s_+\left((i/2)e^{i\beta\gamma(-\omega_-;\hat{e})}E^{-1/2}\varphi_- + O(h^{1/2-\varepsilon})\right). \end{aligned}$$

Proof. We prove only the first relation in some details. For brevity, we take the center d_+ as the origin. As is seen from (3.17), the kernel $G_0(x,y;E)$ of $R(E+i0;H_0)$ behaves like

$$G_0(x,y;E) = c_0(E)\exp(iE^{1/2}|x-y|)|x-y|^{-1/2}\left(1+O(|x-y|^{-1})\right)$$

as $|x-y| \to \infty$, where $c_0(E) = (8\pi)^{-1/2}E^{-1/4}e^{i\pi/4}$. Assume that $x \in S_+$ and $y \in \operatorname{supp} V_+W^-$. Then $|x| = |x-d_+| = O(h^{-\varepsilon})$ and $h^{-1}/c \le |y| \le c\,h^{-1}$ for some $c > 1$. Hence $|y-x| = |y| - \hat{y}\cdot x + O(h^{1-\varepsilon})$. This implies that

$$\exp(iE^{1/2}|x-y|) = \exp(-iE^{1/2}\hat{y}\cdot x)\exp(iE^{1/2}|y|)\left(1+O(h^{1-\varepsilon})\right)$$

and that $|y-x|^{-1/2} = |y|^{-1/2}\left(1+O(h^{1-\varepsilon})\right)$. Thus $s_+R(E+i0;H_0)V_+W^-$ has the kernel which behaves like

$$s_+\left(c_0(E)\exp(-iE^{1/2}\hat{y}\cdot x)\exp(iE^{1/2}|y|)|y|^{-1/2} + O(h^{3/2-\varepsilon})\right)V_+W^-.$$

We analyze the behavior of integral

$$I = \int \exp(-iE^{1/2}\hat{y}\cdot x)\exp(iE^{1/2}|y|)|y|^{-1/2}V_+W^-\left(\partial_2 W\right)\varphi_-\,dy.$$

To do this, we work in the coordinate system $y = (|y|\sin\nu, -|y|\cos\nu)$ with $-\pi \le \nu < \pi$. Then $dy = |y|\,d|y|\,d\nu$ and

$$\varphi_- = \exp(iE^{1/2}y\cdot\omega_-) = \exp\left(-iE^{1/2}|y|\cos\nu\right).$$

We have also $e^{i\zeta_+} = e^{i\beta(\nu+3\pi/2-\gamma(\hat{e}))}$ on $\operatorname{supp} V_+W^-$ and, in particular, $e^{i\zeta_+}$ equals $e^{i\beta\gamma(-\omega_-;\hat{e})}$ at $\nu = 0$. Thus I takes the form

$$I = \int_0^\infty \left[\int \exp(i|y|f)\, a\, d\nu \right] d|y|,$$

where $f = E^{1/2}(1 - \cos\nu)$ and

$$a = \exp(-iE^{1/2}\hat{y}\cdot x)e^{i\beta(\nu+3\pi/2-\gamma(\hat{e}))}|y|^{1/2}V_+W^-\,(\partial_2 W)\,.$$

We apply the stationary phase method to the integral in the bracket. Since $V_+ = 1$ at $y = d_+$ (origin), a takes the value

$$a = \exp(iE^{1/2}x\cdot\omega_-)e^{i\beta\gamma(-\omega_-;\hat{e})}|y|^{1/2}W^-(-|y|)\,(\partial_2 W)\,(-|y|)$$

at the stationary point $\nu = 0$. If we take account of the relation

$$\int W^-(-|y|)\,(\partial_2 W)\,(-|y|)\,d|y| = \int W^-(y_2)\,(\partial_2 W)\,(y_2)\,dy_2 = 1$$

and of the bound $\partial_2 W = O(h)$, then it follows that

$$I = (2\pi)^{1/2}E^{-1/4}e^{i\pi/4}e^{i\beta\gamma(-\omega_-;\hat{e})}\varphi_- + O(h).$$

Since $\int O(h^{3/2-\varepsilon})V_+W^-\,(\partial_2 W)\,dy = O(h^{1/2-\varepsilon})$, the first relation is obtained. To derive the second relation, we work in the coordinates $y = (-|y|\sin\nu, |y|\cos\nu)$. The kernel of $R(E - i0; H_0)$ behaves like

$$\overline{c}_0(E)\exp(-i\sqrt{E}|x-y|)|x-y|^{-1/2}\left(1 + O(|x-y|^{-1})\right)$$

and also $\int W^+(y_2)\,(\partial_2 W)\,(y_2)\,dy_2 = -1$. These facts imply the second relation under natural modification. $\square$

Proof of Lemma 4.6. The scalar product Γ in question is defined by (4.8). It can be easily seen that $\tilde{V}_+W^+R(E - i0; K_d)V_+W^-p(D_x) = O_p(h^N)X_{-N}$. In fact, the particle starting from $\operatorname{supp} V_+W^-$ at velocity $2E^{1/2}\omega_-$ moves like a free particle for $t < 0$ without passing over $\operatorname{supp} \tilde{V}_+W^+$. Thus we have by Lemma 4.7 and (4.14) that

$$\begin{aligned}\Gamma = {} & 4\pi E^{1/2}h\left\langle \Pi(L_+)e^{i\eta}V_+W^-\,(\partial_2 W)\,\varphi_-, e^{i\eta}\tilde{V}_+W^+\,(\partial_2 W)\,\varphi_-\right\rangle \\ & + 4\pi E^{1/2}h\left\langle Q_2e^{i\eta}V_+W^-\,(\partial_2 W)\,\varphi_-, e^{i\eta}\tilde{V}_+W^+\,(\partial_2 W)\,\varphi_-\right\rangle \\ & + 4\pi E^{1/2}h\left\langle Q_3e^{i\eta}V_+W^-\,(\partial_2 W)\,\varphi_-, e^{i\eta}\tilde{V}_+W^+\,(\partial_2 W)\,\varphi_-\right\rangle + O(h^N),\end{aligned}$$

where $Q_2 = R(E + i0; L_+)T_{-0}R(E + i0; L_+)$ and

$$Q_3 = -R(E + i0; L_+)\tilde{T}_{-0}R(E + i0; K_d)T_{-0}R(E + i0; L_+).$$

The first term on the right side just coincides with Γ_1. In fact, we have $\Pi(L_+) = e^{i\zeta_-}\Pi(K_+)e^{-i\zeta_-}$ by (4.11), and a simple computation yields

$$\eta - \zeta_- = \beta\gamma(x - d_+; \hat{e}) \mp \beta\pi = \zeta_+(x) \mp \beta\pi$$

on $\operatorname{supp} V_+W^\mp$. Thus the first term equals Γ_1. We denote by Γ_2 and Γ_3 the second and third terms. We assert that these terms obey $o(h^{3/2})$. Proposition 4.1 remains true for $R(E \pm i0; L_+)$. The coefficients of $T_{\pm 0}$ has support in $S_\pm$. Hence we have by elliptic estimate that $\|T_{-0}R(E \pm i0; L_+)T_{+0}\| = O(h^{1/2-c\varepsilon})$ for some $c > 0$. Since $\|s_-R(E + i0; K_d)s_-\| = O(h^{c\varepsilon})$ by Proposition 4.1 again, it follows from Lemmas 4.8 and 4.9 that $\Gamma_2 = o(h^{3/2})$ and $\Gamma_3 = o(h^{3/2})$. Thus the proof is complete. □

5. Eigenfunction of Hamiltonian with single solenoidal field

The scalar product Γ_1 in Lemma 4.6 contains the spectral resolution $\Pi(K_+)$ associated with self–adjoint operator $K_+ = H(B_+)$ with single field $2\beta\pi\delta(x - d_+)$, and the kernel of $\Pi(K_+)$ is described in terms of the eigenfunction. Thus we require an information on it to analyze the behavior of Γ_1. In this section we study the asymptotic behavior at infinity of the outgoing eigenfunction of Hamiltonian

$$K_\beta = H(\beta\Lambda) = (-i\nabla - \beta\Lambda)^2, \quad \Lambda = \Lambda(x).$$

The operator has the field $2\pi\beta\delta(x)$ with flux $\beta = \alpha/h - [\alpha/h]$, $0 \le \beta < 1$, and it is self–adjoint under boundary condition (1.4) at the origin. We denote by $\varphi(x; \omega) = \varphi(x; E, \omega)$ the outgoing eigenfunction with incident direction ω to equation

$$K_\beta\varphi = E\,\varphi. \tag{5.1}$$

The flux β depends on h, and hence so does $\varphi(x; \omega)$. However the estimates on $\varphi(x; \omega)$ hold true uniformly in h.

The operator K_β is rotationally invariant. We work in the polar coordinate system (r, θ) to calculate the eigenfunction. Let U be the unitary operator defined by

$$(Uu)(r, \theta) = r^{1/2}u(r\theta) : L^2 \to L^2((0, \infty); dr) \otimes L^2(S^1).$$

Then K_β admits the partial wave expansion

$$K_\beta \simeq UK_\beta U^* = \sum_{l \in Z} \oplus\ (D_l \otimes Id)\,,$$

where $D_l = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}$ with $\nu = |l - \beta|$ is self–adjoint in $L^2((0, \infty); dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2}|u(r)| < \infty$. As is well known, $e^{z(t/-1/t)/2}$ is the generating function with Bessel coefficients. Hence $\varphi_0(x; \omega) = \exp(iE^{1/2}x \cdot \omega)$ is expanded as

$$\varphi_0(x; \omega) = \sum_{l=-\infty}^{\infty} \exp(i|l|\pi/2)\exp(il\gamma(x; \omega))J_{|l|}(E^{1/2}|x|) \tag{5.2}$$

in terms of Bessel functions $J_p(r)$ ([18, p.15]), and $J_p(r)$ satisfies the asymptotic formula

$$J_p(r) = (2/\pi)^{1/2} r^{-1/2} \cos(r - (2p+1)\pi/4)\,(1 + s_N(r)) + O(r^{-N}), \quad r \to \infty,$$

where s_N obeys $(d/dr)^k s_N(r) = O(r^{-1-k})$. If we set

$$e_l(r) = \exp(-i|l|\pi/2) J_{|l|}(r) - \exp(-i\nu\pi/2) J_\nu(r)$$

for $\nu = |l - \beta|$, then it follows that

$$e_l(r) = \exp(ir)\left(C_l r^{-1/2} + O(r^{-3/2})\right) + \exp(-ir) O(r^{-3/2})$$

for some constant $C_l \neq 0$, and it fulfills the outgoing radiation condition $e_l' - ie_l = O(r^{-3/2})$. Since the relation

$$\exp(il\gamma(x;-\omega)) = \exp(i|l|\pi + il\gamma(x;\omega)) \tag{5.3}$$

holds true between $\gamma(x;\omega)$ and $\gamma(x;-\omega)$, (5.2) enables us to calculate the eigenfunction $\varphi(x;\omega)$ to (5.1) as

$$\varphi(x;\omega) = \sum_{l=-\infty}^{\infty} \exp(-i\nu\pi/2) \exp(il\gamma(x;-\omega)) J_\nu(E^{1/2}|x|), \quad \nu = |l - \beta|. \tag{5.4}$$

As is easily seen, the series converges locally uniformly in $\boldsymbol{R}^2$.

We decompose $\varphi(x;\omega)$ into the sum of incident and scattering waves. We define $\sigma = \sigma(x;\omega)$ by

$$\sigma = \sigma(x;\omega) = \gamma(x;\omega) - \pi, \qquad -\pi \leq \sigma < \pi. \tag{5.5}$$

If we take account of (5.3), then it follows from (5.4) that

$$\varphi(x;\omega) = \sum_{l=-\infty}^{\infty} (-i)^\nu \exp(il\sigma) J_\nu(E^{1/2}|x|).$$

We now use the integral representation formula

$$J_\nu(r) = e^{i\nu\pi/2} I_\nu(-ir) = (i^\nu/\pi)\left(\int_0^\pi e^{-ir\cos t} \cos\nu t\, dt - \sin\nu\pi \int_0^\infty e^{-\nu t + ir\cosh t}\, dt\right)$$

for Bessel function $J_\nu(r)$ ([18, p.181]) to obtain

$$\varphi = (1/\pi) \sum_{l=-\infty}^{\infty} \left(\int_0^\pi e^{-iE^{1/2}|x|\cos t} \cos\nu t\, dt - \sin\nu\pi \int_0^\infty e^{-\nu t + iE^{1/2}|x|\cosh t}\, dt\right) e^{il\sigma}.$$

We take

$$\varphi_{\mathrm{in}}(x;\omega) = \exp(i\beta(\sigma(x;\omega))\varphi_0(x;\omega) = \exp(i\beta\sigma - iE^{1/2}|x|\cos\sigma) \tag{5.6}$$

as the incident wave. Then $\varphi_{\rm in}(x;\omega)$ is expanded into the Fourier series

$$\varphi_{\rm in} = (1/\pi) \sum_{l=-\infty}^{\infty} \left(\int_0^\pi e^{-iE^{1/2}|x|\cos t} \cos\, \nu t\, dt \right) e^{il\sigma}, \quad \nu = |l-\beta|,$$

so that the scattering wave $\varphi_{\rm sc}(x;\omega)$ takes the form

$$\varphi_{\rm sc} = (1/\pi) \sum_{l=-\infty}^{\infty} \left(-\sin \nu\pi \int_0^\infty e^{-\nu t + iE^{1/2}|x|\cosh t}\, dt \right) e^{il\sigma}.$$

If $\hat{x} = x/|x| \neq \omega$, then $|\sigma| = |\sigma(x;\omega)| < \pi$ and $e^{\pm i\sigma} \neq -1$. Since $0 \le \beta < 1$, we have

$$\sum_{l=-\infty}^{\infty} \sin \nu\pi\, e^{il\sigma} e^{-\nu t} = \sin \beta\pi \left(\frac{e^{\beta t}}{1 + e^{-i\sigma} e^t} + \frac{e^{-\beta t}}{1 + e^{-i\sigma} e^{-t}} \right)$$

by a simple computation. This yields

$$\varphi_{\rm sc}(x;\omega) = (-\sin \beta\pi/\pi) \int \left(\frac{e^{-\beta t}}{e^{-t} + e^{i\sigma}} \right) e^{iE^{1/2}|x|\cosh t}\, dt\, e^{i\sigma}, \quad x/|x| \neq \omega. \tag{5.7}$$

This integral representation is known as Takabayashi's formula ([12]) (see [2, p. 164] also).

Remark 5.1. Let $K_\pm$ be defined by (4.10). It is easy to see that the eigenfunction of K_+ is given by $\varphi(x - d_+;\omega)$. However we require a comment on the construction of eigenfunction of K_-. To define it, we consider the auxiliary operator $K_{1-\beta} = H\left((1-\beta)\Lambda\right)$, $0 \le 1-\beta < 1$, and denote by $\tilde{\varphi}(x;\omega)$ the eigenfunction which is defined by (5.4) with β replaced by $1-\beta$. Since

$$K_{-\beta} = H(-\beta\Lambda) = e^{-i\gamma(x;\omega)} H((1-\beta)\Lambda) e^{i\gamma(x;\omega)} = e^{-i\gamma(x;\omega)} K_{1-\beta} e^{i\gamma(x;\omega)},$$

the eigenfunction is given by $e^{-i\gamma(x-d_-;\omega)}\tilde{\varphi}(x - d_-;\omega)$.

Proposition 5.1 *The scattering wave $\varphi_{\rm sc}(x;\omega)$ has the following properties :*

(1) *If $1/c < |x| < c$ and $|\hat{x} - \omega| > 1/c$ for some $c > 1$, then $\varphi_{\rm sc}(x/h;\omega)$ admits the asymptotic expansion*

$$\varphi_{\rm sc}(x/h;\omega) = h^{1/2} \left(\sum_{j=0}^{N-1} a_j(x,\omega) h^j \right) e^{iE^{1/2}|x|/h} + O(h^{N+1/2}),$$

where $a_j(x,\omega)$ is smooth in $\omega \in S^1$ and in x as above.

(2) *If $|x| > c$ for some $c > 0$, then $\varphi_{\rm sc}(x/h;\omega)$ is uniformly bounded.*

Proof. (1) This is shown by use of the stationary phase method. Since $\sigma = \sigma(x;\omega)$ defined by (5.5) is homogeneous of degree zero as a function of x, we have

$$\varphi_{\mathrm{sc}}(x/h;\omega) = (-\sin\beta\pi/\pi)\left(\int \exp(ih^{-1}E^{1/2}|x|\cosh t)\frac{e^{-\beta t}}{e^{-t}+e^{i\sigma}}\,dt\right)e^{i\sigma}.$$

As is easily seen, $t = 0$ is a unique stationary point. If x fulfills the assumption, then the integrand is smooth and bounded uniformly in x and ω in a neighborhood of $t = 0$. In particular, $|e^{i\sigma}+1|$ is away from zero. Thus we can get the desired asymptotic expansion.

(2) We consider the integral

$$I = \int \left(e^{-\beta t}/(e^{-t}+e^{i\sigma})\right) e^{i|y|\cosh t}\,dt$$

with $y = E^{1/2}x/h$. If $|\hat{y}-\omega| > |y|^{-1/2}$, then

$$I = \int_{|y|^{-1/2}<|t|<1} \left(e^{-t}+e^{i\sigma}\right)^{-1} e^{i|y|\cosh t}\,dt + O(1), \qquad |y|\to\infty.$$

Since $|\partial_t \left(e^{-t}+e^{i\sigma}\right)^{-1}| = O(t^{-2})$ for $|y|^{-1/2} < |t| < 1$, we see by partial integration that the integral on the right side is bounded. If, on the other hand, $0 < |\hat{y}-\omega| < |y|^{-1/2}$, then I behaves like

$$I = -\int_{|t|<1} (t+i\zeta)^{-1} e^{i|y|\cosh t}\,dt + O(1), \qquad |y|\to\infty,$$

with $\zeta = i(e^{i\sigma}+1)$. We set $\sigma = -\pi+\delta$ with $\delta > 0$. By assumption, $\delta = O(|y|^{-1/2})$ and hence $|\zeta| = O(|y|^{-1/2})$. If we take account of the fact that the integral

$$\int_{|y|^{-1/2}<|t|<1} t^{-1}\, e^{i|y|\cosh t}\,dt = 0$$

vanishes, then it follows that

$$\int_{|y|^{-1/2}<|t|<1} \left((t+i\zeta)^{-1} - t^{-1}\right) e^{i|y|\cosh t}\,dt = O(1).$$

This implies that

$$I = \int_{|t|<|y|^{-1/2}} (t+i\zeta)^{-1}\,dt\, e^{i|y|} + O(1) = O(1).$$

The case $\sigma = \pi - \delta$ with $\delta > 0$ is dealt with similarly. Thus (2) is verified. $\square$

Proposition 5.2 *Let $\varphi(x;\omega)$ be as above and let $\tau=\tau(x,\omega)$ be defined by*

$$\tau(x,\omega)=\gamma(x;-\omega)-\gamma(\omega;-\omega)=\gamma(x;-\omega)-\pi.$$

Assume that

$$1/c<|x|<c, \qquad |\tau(x,\omega)|<c\,h^{1-\varepsilon}$$

for some $c>1$. Then there exists a smooth function $b(|x|,\tau;h)$ such that :

(1) $\varphi(x/h;\omega)=b(|x|,\tau;h)\exp(ih^{-1}E^{1/2}x\cdot\omega)+O(h^{1-4\varepsilon})$

(2) *$b(|x|,\tau;h)$ is expanded as $b(|x|,\tau;h)=\sum_{j=0}^{n}b_j(|x|;h)\tau^j$ for some $n\gg 1$ large enough, and b_j satisfies $(d/d\,|x|)^m\,b_j=O(h^{-(1/2+\varepsilon)j-2m\varepsilon})$.*

We postpone the proof of the proposition until the end of the paper (section 10).

6. Proof of Lemma 4.4 : calculation of first and second terms

In this section we prove Lemma 4.4 by analyzing the behavior of Γ_1 defined in Lemma 4.6. As a result, the first and second terms of the asymptotic formula in Theorem 1.1 are calculated.

For brevity, we fix the center d_+ as the origin, and we denote by $\varphi(x;\omega)=\varphi(x;E,\omega)$ the outgoing eigenfunction at energy E of operator $K_+=K_\beta=H(B_+)$ with $B_+(x)=\beta\Lambda(x)$. The spectral resolution $\Pi(K_+)$ at energy E has the kernel

$$\Pi(x,y)=\left(8\pi^2\right)^{-1}\int\varphi(x;\omega)\overline{\varphi}(y;\omega)\,d\omega.$$

Hence we have

$$\Gamma_1=(2\pi)^{-1}E^{1/2}e^{-i2\beta\pi}h\int I_-(\omega)I_+(\omega)\,d\omega,$$

where

$$I_-=\left\langle e^{i\beta\gamma(\cdot;\hat e)}V_+W^-\left(\partial_2W\right)\varphi_-,\varphi(\cdot;\omega)\right\rangle,\ \ I_+=\left\langle\varphi(\cdot;\omega),e^{i\beta\gamma(\cdot;\hat e)}\tilde V_+W^+\left(\partial_2W\right)\varphi_-\right\rangle.$$

We now make change of variables $hx\to x$ and $hy\to y$ to define

$$\begin{aligned}X(\omega)&=\left\langle\varphi(\cdot/h;\omega),e^{i\beta\gamma(\cdot;\hat e)}\tilde v_+w^+\left(\partial_2w\right)\varphi_-(\cdot/h)\right\rangle\\ Y(\omega)&=\left\langle e^{i\beta\gamma(\cdot;\hat e)}v_+w^-\left(\partial_2w\right)\varphi_-(\cdot/h),\varphi(\cdot/h;\omega)\right\rangle,\end{aligned}$$

where $v_+,\ \tilde v_+,\ w^\pm$ and w are as in sections 3 and 4. We use the variables $x=(x_1,x_2)$ and $y=(y_1,y_2)$ to represent the scalar products $X(\omega)$ and $Y(\omega)$ in the integral forms respectively. Since $\partial_2W=h\left(\partial_2w\right)$, we obtain

$$\Gamma_1=(2\pi)^{-1}E^{1/2}e^{-i2\beta\pi}h^{-1}\int Y(\omega)X(\omega)\,d\omega.$$

Lemma 6.1 *Let $X(\omega)$ and $Y(\omega)$ be as above, and define*

$$Y_{\rm sc}(\omega) = \left\langle e^{i\beta\gamma(\cdot;\hat{e})} v_+ w^- \,(\partial_2 w)\, \varphi_-(\cdot/h), \varphi_{\rm sc}(\cdot/h;\omega) \right\rangle.$$

Then we have the following statements :

(1) *If $h^{1-\varepsilon} < |\omega - \omega_-| < \delta \ll 1$, then $X(\omega) = O(1)$ and $Y(\omega) = O(h^N)$.*

(2) *If $|\omega - \omega_-| > \delta$, then $X(\omega) = O(h)$ and $Y(\omega) = O(h^{2-5\varepsilon})$.*

(3) *If $|\omega - \omega_-| < h^{1-\varepsilon}$, then $X(\omega) = O(1)$ and $Y_{\rm sc}(\omega) = O(h^N)$.*

Proof. (1) The fact that $X(\omega) = O(1)$ is an immediate consequence of Proposition 5.1 (2). To prove the bound on $Y(\omega)$, we use Proposition 5.1 (1). We represent $Y(\omega)$ in the integral form. The function $\sigma(y/h;\omega) = \sigma(y;\omega)$ defined by (5.5) is smooth over $\operatorname{supp} v_+ w^-$, and

$$\varphi(y/h;\omega) = \varphi_{\rm in}(y/h;\omega) + \varphi_{\rm sc}(y/h;\omega) = e^{i\beta\sigma(y;\omega)}\varphi_0(y/h;\omega) + \varphi_{\rm sc}(y/h;\omega)$$

admits the asymptotic expansion as in Proposition 5.1. Since

$$|\nabla\,(y\cdot(\omega-\omega_-))| = |\omega-\omega_-| > h^{1-\varepsilon}, \quad |\nabla(|y| - y\cdot\omega)| = |\hat{y} - \omega| > c > 0$$

over $\operatorname{supp} v_+ w^-$, the desired bound is obtained by repeated use of integration by parts.

(2) We write $X(\omega)$ as $X(\omega) = X_{\rm in}(\omega) + X_{\rm sc}(\omega)$ according to the decomposition $\varphi = \varphi_{\rm in} + \varphi_{\rm sc}$. Then it is easy to see that $X_{\rm in}(\omega) = O(h)$, and also it follows from Proposition 5.1 (1) that $X_{\rm sc}(\omega) = O(h)$. In fact, it is a simple application of stationary phase method. To prove the bound on $Y(\omega)$, Proposition 5.2 plays an important role. We use the polar coordinate system $y = (|y|\cos\theta, |y|\sin\theta)$ to represent $Y(\omega)$ in the integral form and we divide the integral into two parts

$$Y(\omega) = \int\!\!\int ((1-\chi_h(|\theta-\omega|)) + \chi_h(|\theta-\omega|))\cdots\, d|y|\, d\theta,$$

where $\chi_h(|\theta|) = \chi(h^{-(1-\varepsilon)}|\theta|)$. To evaluate the first integral on the right side, we decompose $\varphi = \varphi_{\rm in} + \varphi_{\rm sc}$ and use the integral representation (5.7) for $\varphi_{\rm sc}$. If $y \in \operatorname{supp} v_+ w^-$, then $|\nabla\,(y\cdot(\omega-\omega_-))| > \delta > 0$ and

$$|(\partial/\partial|y|)\,(|y|\cosh t - y\cdot\omega_-)| > 1 - \cos(\theta - \omega_-) > c$$

for some constant $c > 0$. Hence a repeated use of integration by parts shows that the first integral obeys $O(h^N)$. We use Proposition 5.2 to evaluate the second integral. We have

$$|(\partial/\partial|y|)\,(y\cdot\omega - y\cdot\omega_-)| = |\cos(\theta-\omega) - \cos(\theta-\omega_-)| > c > 0$$

when $|\hat{y}-\omega| < 2h^{1-\varepsilon}$, and also it follows that $\int \chi_h(|\theta-\omega|)\,d\theta = O(h^{1-\varepsilon})$. Thus the second integral is verified to obey $O(h^{2-5\varepsilon})$.

(3) This is verified in the same way as (1). We skip the detail. $\square$

We take the coordinates over S^1 in such a way that

$$\omega = (-\sin\theta, \cos\theta), \qquad -\pi \le \theta < \pi. \tag{6.1}$$

Then $\omega_- = \omega_2 = (0,1)$ corresponds to $\theta = 0$ in this coordinate system. By Lemma 6.1, we have

$$\Gamma_1 = (2\pi)^{-1}E^{1/2}e^{-i2\beta\pi}h^{-1}\int \chi_h(\theta)Y_{\mathrm{in}}(\omega)X(\omega)\,d\omega + o(h^{3/2}),$$

where $\chi_h(\theta) = \chi(h^{-(1-\varepsilon)}|\theta|)$ again and

$$Y_{\mathrm{in}}(\omega) = \left\langle e^{i\beta\gamma(\cdot;\hat{e})}v_+w^-\,(\partial_2 w)\,\varphi_-(\cdot/h), \varphi_{\mathrm{in}}(\cdot/h;\omega)\right\rangle.$$

We decompose $\tilde{v}_+(x_1)$ into the sum

$$\tilde{v}_+ = \left(1-\chi(h^{-(1-\varepsilon)}|x_1|/M)\right)\tilde{v}_+(x_1) + z_0(x_1) = z_-(x_1) + z_0(x_1) + z_+(x_1)$$

for $M \gg 1$ large enough, where $z_0 = \chi(h^{-(1-\varepsilon)}|x_1|/M)\tilde{v}_+(x_1)$ and $z_\pm$ has the following properties :

$$\begin{aligned} \operatorname{supp} z_+ &\subset (2Mh^{1-\varepsilon}, 2/M), \qquad & z_+ = 1 \ \text{ on } \ [Mh^{1-\varepsilon}, 1/M], \\ \operatorname{supp} z_- &\subset (-2/M, -2Mh^{1-\varepsilon}), \qquad & z_- = 1 \ \text{ on } \ [-1/M, -Mh^{1-\varepsilon}]. \end{aligned}$$

Then $X(\omega)$ admits the decomposition

$$X(\omega) = X_{\mathrm{in}-}(\omega) + X_{\mathrm{sc}-}(\omega) + X_0(\omega) + X_{\mathrm{in}+}(\omega) + X_{\mathrm{sc}+}(\omega),$$

where

$$\begin{aligned} X_{\mathrm{in}\pm}(\omega) &= \left\langle \varphi_{\mathrm{in}}(\cdot/h;\omega), e^{i\beta\gamma(\cdot;\hat{e})}z_\pm w^+\,(\partial_2 w)\,\varphi_-(\cdot/h)\right\rangle \\ X_{\mathrm{sc}\pm}(\omega) &= \left\langle \varphi_{\mathrm{sc}}(\cdot/h;\omega), e^{i\beta\gamma(\cdot;\hat{e})}z_\pm w^+\,(\partial_2 w)\,\varphi_-(\cdot/h)\right\rangle \\ X_0(\omega) &= \left\langle \varphi(\cdot/h;\omega), e^{i\beta\gamma(\cdot;\hat{e})}z_0 w^+\,(\partial_2 w)\,\varphi_-(\cdot/h)\right\rangle. \end{aligned}$$

We also have

$$\Gamma_1 = \Gamma_{\mathrm{in}-} + \Gamma_{\mathrm{sc}-} + \Gamma_0 + \Gamma_{\mathrm{sc}+} + \Gamma_{\mathrm{in}+} + o(h^{3/2}),$$

where

$$\begin{aligned} \Gamma_{\mathrm{in}\pm} &= (2\pi)^{-1}E^{1/2}e^{-i2\beta\pi}h^{-1}\int \chi_h(\theta)Y_{\mathrm{in}}(\omega)X_{\mathrm{in}\pm}(\omega)\,d\omega \\ \Gamma_{\mathrm{sc}\pm} &= (2\pi)^{-1}E^{1/2}e^{-i2\beta\pi}h^{-1}\int \chi_h(\theta)Y_{\mathrm{in}}(\omega)X_{\mathrm{sc}\pm}(\omega)\,d\omega \\ \Gamma_0 &= (2\pi)^{-1}E^{1/2}e^{-i2\beta\pi}h^{-1}\int \chi_h(\theta)Y_{\mathrm{in}}(\omega)X_0(\omega)\,d\omega. \end{aligned}$$

We complete the proof of Lemma 4.4 in question, accepting a series of the lemmas below as proved.

Lemma 6.2 *We have*

$$\Gamma_{\rm in+} = -\int z_+(x_1)v_+(x_1)\,dx_1 + O(h^N)$$
$$\Gamma_{\rm in-} = -e^{-i2\beta\pi}\int z_-(x_1)v_+(x_1)\,dx_1 + O(h^N).$$

Lemma 6.3 *Let* $\gamma_- = \gamma(\omega_-;\hat{e})$. *Then we have*

$$\Gamma_{\rm sc\pm} = e^{-i\beta(\pi-\gamma_-)}\int e^{-i\beta\gamma(x;\hat{e})}\varphi_{\rm sc}(x/h;\omega_-)z_\pm w^+\,(\partial_2 w)\,\overline{\varphi}_-(x/h)\,dx + O(h^N).$$

Lemma 6.4 *We have*

$$\Gamma_0 = e^{-i\beta(\pi-\gamma_-)}\int e^{-i\beta\gamma(x;\hat{e})}\varphi(x/h;\omega_-)z_0 w^+\,(\partial_2 w)\,\overline{\varphi}_-(x/h)\,dx + o(h^{3/2}).$$

Lemma 6.5 *Define*

$$\Gamma_{\rm sc} = e^{-i\beta(\pi-\gamma_-)}\int e^{-i\beta\gamma(x;\hat{e})}\varphi_{\rm sc}(x/h;\omega_-)\tilde{v}_+ w^+\,(\partial_2 w)\,\overline{\varphi}_-(x/h)\,dx.$$

Then we have $\Gamma_{\rm sc} = -ik_0e^{-i\beta\pi}h + o(h^{3/2})$, k_0 *being as in Lemma 3.3.*

Proof of Lemma 4.4. By Lemmas 6.3, 6.4 and 6.5, we have

$$\Gamma_{\rm sc+} + \Gamma_0 + \Gamma_{\rm sc-} = \Gamma_{\rm sc} + \Gamma_{\rm in} + o(h^{3/2}) = -ik_0e^{-i\beta\pi}h + \Gamma_{\rm in} + o(h^{3/2}),$$

where

$$\Gamma_{\rm in} = e^{-i\beta(\pi-\gamma_-)}\int e^{-i\beta\gamma(x;\hat{e})}\varphi_{\rm in}(x/h;\omega_-)z_0 w^+\,(\partial_2 w)\,\overline{\varphi}_-(x/h)\,dx.$$

We calculate $\Gamma_{\rm in}$. If $x_1 > 0$, then

$$\sigma(x;\omega_-) = \gamma(x;\omega_-) - \pi = \gamma(x;\hat{e}) - \gamma_- + \pi$$

and if $x_1 < 0$, then $\sigma(x;\omega_-) = \gamma(x;\hat{e}) - \gamma_- - \pi$. Hence we have

$$\Gamma_{\rm in} = -e^{-i2\beta\pi}\int_{-\infty}^0 z_0(x_1)\,dx_1 - \int_0^\infty z_0(x_1)\,dx_1.$$

Since $z_\pm(x_1)v_+(x_1) + z_0(x_1) = v_+(x_1)$ for $\pm x_1 > 0$, the lemma follows from Lemma 6.2. $\square$

It remains to prove Lemmas 6.2 $\sim$ 6.5. These lemmas are proved by making repeated use of the stationary phase method.

Proof of Lemma 6.2. We consider only the behavior of $\Gamma_{\mathrm{in}+}$ in some details. Recall that ω is represented as (6.1), and θ ranges over $[-2h^{1-\varepsilon}, 2h^{1-\varepsilon}]$. If $y \in \operatorname{supp} v_+ w^-$, then

$$\sigma(y;\omega) = \gamma(y;\omega) - \pi = \gamma(y;\hat{e}) - \theta - \gamma_- - \pi$$

with $\gamma_- = \gamma(\omega_-;\hat{e})$, and hence $\gamma(y;\hat{e}) - \sigma(y;\omega) = \theta + \gamma_- + \pi$. If, on the other hand, we write

$$x = (x_1, x_2) = (-|x|\sin\nu, |x|\cos\nu), \quad -\pi \le \nu < \pi, \tag{6.2}$$

then there exists $c > 0$ such that $\nu < -cMh^{1-\varepsilon}$ for $x \in \operatorname{supp} z_+ w^+$, and we have

$$\gamma(x;\omega) = 2\pi - (\theta + \gamma_- - \gamma(x;\hat{e})).$$

Hence $\sigma(x;\omega) - \gamma(x;\hat{e}) = -\theta - \gamma_- + \pi$. We make change of variable $h^{-(1-\varepsilon)}\theta \to \theta$ to represent $\Gamma_{\mathrm{in}+}$ in the integral form

$$\Gamma_{\mathrm{in}+} = (2\pi)^{-1} E^{1/2} h^{-\varepsilon} \times \int\int \left[\int\int \exp(ih^{-\varepsilon} f) a \, d\theta \, dy_1\right] g \, dy_2 dx,$$

where $f = E^{1/2}(y_1 - x_1)\theta$ and

$$\begin{aligned} a &= \chi(|\theta|) v_+(y_1) \exp(ir(y/h, x/h, h^{1-\varepsilon}\theta)) \\ r &= E^{1/2}\left((y_1 - x_1)(\sin\theta - \theta) + (y_2 - x_2)(1 - \cos\theta)\right) \\ g &= w^-(y_2)\,(\partial_2 w)\,(y_2) z_+(x_1) w^+(x_2)\,(\partial_2 w)\,(x_2). \end{aligned}$$

We apply the stationary phase method to the integral in the bracket to obtain

$$\Gamma_{\mathrm{in}+} = \int\int w^-\,(\partial_2 w)\, v_+ z_+ w^+\,(\partial_2 w)\; dy_2 dx + O(h^N)$$

in the same way as in the proof of Lemma 3.1 in section 4. This, together with (4.3), yields the desired relation. The asymptotic formula for $\Gamma_{\mathrm{in}-}$ can be derived in a similar way. If we have only to note that $\sigma(x;\omega) - \gamma(x;\hat{e}) = -\theta - \gamma_- - \pi$ on $\operatorname{supp} z_- w^+$, the factor $e^{-i2\beta\pi}$ is left without cancellation. $\square$

Proof of Lemma 6.3. We give a proof for $\Gamma_{\mathrm{sc}+}$ only. The same argument applies to $\Gamma_{\mathrm{sc}-}$ also. We set $\omega_h = (-\sin h^{1-\varepsilon}\theta, \cos h^{1-\varepsilon}\theta)$ after making change of variable $h^{-(1-\varepsilon)}\theta \to \theta$. Then we have

$$\Gamma_{\mathrm{sc}+} = (2\pi)^{-1} E^{1/2} e^{-i\beta(\pi - \gamma_-)} h^{-\varepsilon} \int\int \left[\int\int \exp(ih^{-\varepsilon} f) a \, d\theta \, dy_1\right] g \, dy_2 dx,$$

where $f = E^{1/2} y_1 \theta$ and

$$\begin{aligned} a &= \exp(i\beta h^{1-\varepsilon}\theta)\chi(|\theta|) v_+(y_1)\varphi_{\mathrm{sc}}(x/h;\omega_h)\exp(ir(y/h, h^{1-\varepsilon}\theta)) \\ r &= E^{1/2}\left(y_1(\sin\theta - \theta) + y_2(1 - \cos\theta)\right) \\ g &= w^-(y_2)\,(\partial_2 w)\,(y_2)\left(e^{-i\beta\gamma(x;\hat{e})} z_+(x_1) w^+(x_2)\,(\partial_2 w)\,(x_2)\right)\overline{\varphi}_-(x/h). \end{aligned}$$

We apply the stationary phase method to the integral in the bracket. We take a look at the denominator $e^{-t} + e^{i\sigma}$ in the integral representation (5.7) for $\varphi_{\rm sc}$. If $|t| \ll 1$ and $x \in \operatorname{supp} z_+ w^+$, then

$$\left| e^{-t} + e^{i\sigma(x/h;\omega_h)} \right| \sim |t + i\gamma(x;\omega_h)| \ge c\, h^{1-\varepsilon}, \quad c > 0,$$

and hence we have

$$|(\partial/\partial\theta)^m \varphi_{\rm sc}(x/h;\omega_h)| \le C_m h^{-(1-\varepsilon)}.$$

This enables us to get the desired relation. $\square$

Proof of Lemma 6.4. The proof of this lemma is based on Proposition 5.2. According to the proposition,

$$\varphi(x/h;\omega) = b(|x|,\tau;h)\exp(ih^{-1}E^{1/2}x\cdot\omega) + O(h^{1-4\varepsilon})$$

for x and ω as in the lemma, where $\tau = \tau(x/h, h^{1-\varepsilon}\omega) = \tau(x,\omega) = \nu - \theta$ in the coordinates (6.1) and (6.2). We insert this relation into the term Γ_0 in question and represent it in the integral form. Then we have

$$\Gamma_0 = (2\pi)^{-1}E^{1/2}e^{-i\beta(\pi-\gamma_-)}h^{-\varepsilon}\int\int\left[\int\int \exp(ih^{-\varepsilon}f)a\,d\theta\,dy_1\right] g\,dy_2 dx + o(h^{3/2}),$$

where $f = E^{1/2}(y_1 - x_1)\theta$ and

$$\begin{aligned} a &= \exp(i\beta h^{1-\varepsilon}\theta)\chi(|\theta|)v_+(y_1)b(|x|,\tau;h)\exp(ir(y/h,x/h,h^{1-\varepsilon}\theta)) \\ r &= E^{1/2}\left((y_1-x_1)(\sin\theta-\theta)+(y_2-x_2)(1-\cos\theta)\right) \\ g &= w^-(y_2)\,(\partial_2 w)\,(y_2)\left(e^{-i\beta\gamma(x;\hat e)}z_0(x_1)w^+(x_2)\,(\partial_2 w)\,(x_2)\right). \end{aligned}$$

The remainder estimate $o(h^{3/2})$ above comes from the estimate

$$O(h^{-\varepsilon})O(h^{1-4\varepsilon}) \times \int z_0(x_1)\,dx_1 = O(h^{2-6\varepsilon}) = o(h^{3/2}).$$

If we take account of the property of $b(|x|,\tau;h)$ in Proposition 5.2, then we obtain

$$\Gamma_0 = e^{-i\beta(\pi-\gamma_-)}\int e^{-i\beta\gamma(x;\hat e)}b(|x|,\tau_0;h)z_0(x_1)w^+(x_2)\,(\partial_2 w)\,(x_2)\,dx + o(h^{3/2})$$

by the stationary phase method, where $\tau_0 = \tau(x,\omega_-) = \nu$. By Proposition 5.2 again, we have

$$\varphi(x/h;\omega_-) = b(|x|,\tau_0;h)\varphi_-(x/h) + O(h^{1-4\varepsilon})$$

for $x \in \operatorname{supp} z_+ w^+$, and hence $b(|x|,\tau_0;h) = \varphi(x/h;\omega_-)\overline{\varphi}_-(x/h) + O(h^{1-4\varepsilon})$. This proves the lemma. $\square$

Proof of Lemma 6.5. We work in the coordinate system (6.2). Then we have

$$\sigma = \sigma(x;\omega_-) = \gamma(x;\omega_-) - \pi = \begin{cases} \nu - \pi, & \nu > 0 \\ \nu + \pi, & \nu < 0 \end{cases} \tag{6.3}$$

and hence $e^{i\sigma} = -e^{i\nu}$. We also have $\gamma(x;\hat{e}) = \nu + \gamma_-$ with $\gamma_- = \gamma(\omega_-;\hat{e})$. If we insert the representation (5.7) for $\varphi_{\rm sc}(x/h;\omega_-)$, then $\Gamma_{\rm sc}$ takes the form

$$\Gamma_{\rm sc} = (\sin\beta\pi/\pi)e^{-i\beta\pi}\int_0^\infty \left[\int\int \exp(ih^{-1}f)g\,d\nu\,dt\right]|x|\,d|x|,$$

where $f = E^{1/2}|x|(t^2+\nu^2)/2$ and

$$\begin{aligned} g &= e^{i\nu}e^{-\beta(t+i\nu)}\left(e^{-t}-e^{i\nu}\right)^{-1}\tilde{v}_+w^+\,(\partial_2 w)\exp(ih^{-1}r(x,t,\nu)) \\ r &= E^{1/2}|x|\left((\cosh t - 1 - t^2/2) - (\cos\nu - 1 + \nu^2/2)\right). \end{aligned}$$

We denote by $I_{\rm sc}(|x|)$ the integral in the bracket and represent it in the polar coordinates $(t,\nu) = (\rho\cos\mu, \rho\sin\mu)$. If we write

$$\left(e^{-t}-e^{i\nu}\right)^{-1} = (t-i\nu)^{-1}\left((t-i\nu)/(e^{-t}-e^{i\nu})\right) = \rho^{-1}e^{i\mu}\left((t-i\nu)/(e^{-t}-e^{i\nu})\right),$$

then $I_{\rm sc}(|x|)$ takes the form

$$I_{\rm sc} = \int_0^{2\pi}\left[\int_0^\infty \exp(ih^{-1}E^{1/2}|x|\rho^2/2)a\,d\rho\right]d\mu + o(h^{3/2})$$

uniformly in $|x|$ ($x \in \operatorname{supp}\tilde{v}_+w^+$), where $a = \chi_h(\rho)e^{i\mu}b$ with $\chi_h = \chi(h^{-(1-\varepsilon)/2}\rho)$, and

$$b = e^{i\nu}e^{-\beta(t+i\nu)}\left((t-i\nu)/(e^{-t}-e^{i\nu})\right)\tilde{v}_+w^+\,(\partial_2 w)\exp(ih^{-1}r).$$

We note that $(t-i\nu)/(e^{-t}-e^{i\nu})$ is smooth as a function of ρ in a neighborhood of $\rho = 0$ and that it has the limit $(t-i\nu)/(e^{-t}-e^{i\nu}) \to -e^{-i2\mu}$ as $\rho \to 0$. Hence b is of the form $b = e^{-i2\mu} \times$ (smooth function) and is expanded as

$$b = e^{-i2\mu}\left(b_0 + \left(b_+e^{i\mu}+b_-e^{-i\mu}\right)\rho + \left(\sum_{k=0}^{2} b_{2k}\cos^k\mu\sin^{2-k}\mu\right)\rho^2 + O(\rho^3)\right)$$

in a neighborhood of $\rho = 0$ by the Taylor series, where all the coefficients depend on $|x|$ only. If we apply the stationary phase method to the integral in ρ, then the stationary point $\rho = 0$ is the endpoint of the integral. Thus we have to take a more careful look at the contribution from such a point. The function $h^{-1}r$ behaves like

$$h^{-1}r = E^{1/2}|x|h^{-1}\left(\cos^4\mu - \sin^4\mu\right)\rho^4/4! + O(h^{2-3\varepsilon})$$

for $|\rho| < 2h^{(1-\varepsilon)/2}$. All the integrals below

$$\int e^{-i\mu}\,d\mu,\ \int b_-e^{-i2\mu}\,d\mu,\ \int e^{-i\mu}\cos^k\mu\sin^{2-k}\mu\,d\mu,\ \int e^{-i\mu}\cos^4\mu\,d\mu,\ \int e^{-i\mu}\sin^4\mu\,d\mu$$

over the interval $(0,2\pi)$ vanish. On the other hand, the integral

$$\int_0^\infty O(\rho^3)\chi_h(\rho)\,d\rho = O(h^{2-2\varepsilon}) = o(h^{3/2})$$

can be dealt with as a remainder term. Thus the integral associated with $b_+e^{i\mu}$ only makes a contribution to the leading term. We now calculate the term $b_+(|x|)$. We can expand $\tilde{v}_+,\ w^+$ and $\partial_2 w$ as follows : $\tilde{v}_+(x_1) = \tilde{v}_+(-|x|\sin\nu) = 1 + O(\rho^2)$,

$$w^+(x_2) = w^+(|x|\cos\nu) = w^+(|x|) + O(\rho^2), \quad (\partial_2 w)(x_2) = (\partial_2 w)(|x|) + O(\rho^2).$$

We further obtain that

$$\begin{aligned} e^{i\nu} &= 1 + i\rho\sin\mu + O(\rho^2) = 1 + \left(e^{i\mu} - e^{-i\mu}\right)\rho/2 + O(\rho^2) \\ e^{-\beta(t+i\nu)} &= 1 - \beta(t+i\nu) + O(\rho^2) = 1 - \beta e^{i\mu}\rho + O(\rho^2) \end{aligned}$$

and that

$$(t-i\nu)(e^{-t}-e^{i\nu})^{-1} = e^{-i\mu}\left(-e^{i\mu} + \rho/2 + O(\rho^2)\right)^{-1} = -e^{-i2\mu}\left(1 + e^{-i\mu}\rho/2 + O(\rho^2)\right).$$

Thus we combine these results to get

$$b_+(|x|) = (\beta - 1/2)w^+(|x|)(\partial_2 w)(|x|).$$

Hence the leading term comes from the integral

$$\begin{aligned} & 2\sin\beta\pi e^{-i\beta\pi}\int_0^\infty \left[\int_0^\infty \exp(ih^{-1}E^{1/2}|x|\rho^2/2)\chi_h(\rho)\rho\,d\rho\right] b_+(|x|)|x|\,d|x| \\ &= 2i(\beta - 1/2)\sin\beta\pi E^{-1/2}e^{-i\beta\pi}h\int_0^\infty w^+(\partial_2 w)\,d|x| + o(h^{3/2}) \\ &= -ik_0 e^{-i\beta\pi}h + o(h^{3/2}). \end{aligned}$$

This completes the proof of the lemma. $\square$

7. Scattering by single solenoidal field

The calculation of the third term of the asymptotic formula relies on the representation for scattering amplitudes by single solenoidal field in terms of resolvents. We here make a quick review on it without detailed proof.

We use the notation $W_\pm(H,K)$ and $S(H,K)$ with the same meanings ascribed in section 2. We consider the self–adjoint operator $H_{\alpha h}$ defined by (1.3) with boundary condition (1.4). Let U_1 be as in (3.2) and let $\tilde{U}_2$ be defined by

$$\left(\tilde{U}_2 f\right)(x) = \exp(i[\alpha/h]\gamma(x)) \times f(x),$$

where $\gamma(x) = \gamma(\hat{x})$ denotes the azimuth angle from the x_1 axis. Then we obtain

$$K_\beta := \left(U_1\tilde{U}_2\right)^* H_{\alpha h}\left(U_1\tilde{U}_2\right) = H(\beta\Lambda) = (-i\nabla - \beta\Lambda)^2$$

with $\beta = \alpha/h - [\alpha/h]$. This operator is also self–adjoint under (1.4). According to notation (3.5), we set

$$\tilde{W}_+ = W_+(H_0, H_0; \tilde{U}_2), \qquad \tilde{W}_- = W_-(H_0, H_0; \tilde{U}_2^*), \tag{7.1}$$

Then we get the relation

$$S(H_{\alpha h}, H_{0h}) = U_1 \tilde{W}_+ S(K_\beta, H_0) \tilde{W}_- U_1^*. \tag{7.2}$$

The existence of $\tilde{W}_\pm$ follows as a consequence of the well known lemma below (see [14, Theorem IX. 31] for example).

Lemma 7.1 *The free solution* $\exp(-itH_0)f$ *with initial state* $f \in L^2$ *behaves like*

$$(\exp(-itH_0)f)(x) = (2it)^{-1} \exp(i|x|^2/4t)\widehat{f}(x/2t) + o(1), \quad |t| \to \infty,$$

on L^2*, where* $\hat{f}$ *denotes the Fourier transform of* f*.*

This lemma also implies that $\tilde{W}_+$ defined by (7.1) acts as the multiplication

$$F\tilde{W}_+F^* = \exp(i[\alpha/h]\gamma(\omega))\times = \exp(i[\alpha/h]\omega)\times$$

on $L^2((0,\infty); d\lambda) \otimes L^2(S^1)$, where the position vector ω on S^1 is identified with the azimuth angle $\gamma(\omega)$. Since $\gamma(-\omega) = \omega + \pi$ or $\omega - \pi$ according as $0 \le \omega < \pi$ or $\pi \le \omega < 2\pi$, we have

$$F\tilde{W}_-F^* = \exp(-i[\alpha/h]\gamma(-\omega))\times = (-1)^{[\alpha/h]} \exp(-i[\alpha/h]\omega)\times$$

in a similar way. We denote by $S(\theta, \omega; H_{\alpha h}, H_{0h})$ and $S(\theta, \omega; K_\beta, H_0)$ the kernels of scattering matrices $S(E; H_{\alpha h}, H_{0h})$ and $S(E; K_\beta, H_0)$ at energy E respectively. Then it follows from (7.2) that

$$S(\theta, \omega; H_{\alpha h}, H_{0h}) = (-1)^{[\alpha/h]} e^{i[\alpha/h](\theta-\omega)} S(\theta, \omega; K_\beta, H_0), \quad \theta \ne \omega.$$

If $0 \le \beta < 1$, then the kernel for scattering by single field has been known to be explicitly calculated as

$$S(\theta, \omega; K_\beta, H_0) = (i/\pi) \sin \beta\pi F_0(\theta - \omega)$$

by the early works [2, 3, 15], as already mentioned in Introduction, where $F_0(\theta)$ is as in (1.5). We recall (see (2.4)) that the amplitude $f_h(\omega \to \theta; \alpha)$ for the scattering from incident direction ω to final one θ at energy E is defined by

$$f_h(\omega \to \theta; \alpha) = c(E/h^2) S(\theta, \omega; H_{\alpha h}, H_{0h})$$

for pair $(H_{\alpha h}, H_{0h})$, where $c(E)$ is as in (1.5). Similarly the amplitude $g_\beta(\omega \to \theta)$ for pair (K_β, H_0) is defined by

$$g_\beta(\omega \to \theta) = c(E) S(\theta, \omega; K_\beta, H_0).$$

Thus we obtain the following relation between $f_h(\theta \to \omega; \alpha)$ and $g_\beta(\theta \to \omega)$, and hence (1.5) follows immediately.

Proposition 7.1 *Let the notation be as above. Then we have*

$$f_h(\omega \to \theta; \alpha) = (-1)^{[\alpha/h]} e^{i[\alpha/h](\theta-\omega)} h^{1/2} g_\beta(\omega \to \theta).$$

Remark 7.1. We recall that $f_h(\omega \to \theta; \pm\alpha, e_\pm)$ denotes the amplitude for the scattering by field $\pm 2\pi\alpha\delta(x - e_\pm)$. Let $K_\pm = H(B_\pm)$ be defined by (4.10) and let $g_\pm(\theta \to \omega)$ denote the scattering amplitude for pair $(K_\pm, H_0)$. Then the same argument as used to prove Proposition 7.1 enables us to get

$$f_h(\omega \to \theta; \alpha, e_+) = (-1)^{[\alpha/h]} e^{i[\alpha/h](\theta-\omega)} h^{1/2} g_+(\omega \to \theta).$$

If we write $-\beta$ as $-\beta = -\alpha/h - (-[\alpha/h])$, then we also have

$$f_h(\omega \to \theta; -\alpha, e_-) = (-1)^{[\alpha/h]} e^{-i[\alpha/h](\theta-\omega)} h^{1/2} g_-(\omega \to \theta).$$

Remark 7.2. As is easily seen, the scattering matrix $S(\lambda; H_{\alpha h}, H_{0h})$ is independent of energy λ as a function with values in bounded operators from $L^2(S^1)$ into itself for the self–adjoint operator $H_{\alpha h}$ defined by (1.3) with (1.4). This means that a system of single field does not have any resonances under boundary condition (1.4).

We represent $g_\beta(\omega \to \theta)$ in terms of $R(E + i0; K_\beta)$. The proposition below has been verified as [10, Lemma 3.2]. The derivation is based on the analysis in the phase space and is due to the idea developed in [7, 19].

Proposition 7.2 *Let $j_\pm(x;\omega)$ be defined by (4.1). Set $\psi(x) = 1 - \chi(2h^\varepsilon|x|)$, where $\chi \in C_0^\infty[0,\infty)$ is a cut–off function with property (3.9). Then we have*

$$g_\beta(\omega \to \theta) = (ic(E)/4\pi) \left\langle (R(E + i0; K_\beta) Q_1 \varphi_0(\omega), Q_2 \varphi_0(\theta) \right\rangle + O(h^N), \quad \omega \neq \theta,$$

for any $N \gg 1$, where $\varphi_0(\omega) = \exp(iE^{1/2} x \cdot \omega)$ and

$$Q_1 = \exp(ij_+(x;\omega))[\psi, H_0], \qquad Q_2 = \exp(ij_+(x;-\theta))[\psi, H_0].$$

We apply the above proposition to $g_\pm = g_\pm(\omega \to \theta)$ to obtain

$$g_\pm = (ic(E)/4\pi) \left\langle R(E + i0; K_\pm) Q_{\pm 1} \varphi_0(\omega), Q_{\pm 2} \varphi_0(\theta) \right\rangle + O(h^N) \tag{7.3}$$

where

$$Q_{\pm 1} = \exp(ij_\pm(x - d_\pm;\omega))[\psi_\pm, H_0], \quad Q_{\pm 2} = \exp(ij_\pm(x - d_\pm;-\theta))[\psi_\pm, H_0]$$

with $\psi_\pm = 1 - \chi(2h^\varepsilon|x - d_\pm|)$.

8. Proof of Lemma 4.5 : calculation of third term

In this section we prove Lemma 4.5. As a result, the third term of the asymptotic formula is calculated and also the proof of Lemma 3.3 is completed for Φ_+^-. The calculation is based on an idea similar to that used to prove [10, Theorem 1.1] (see the argument in [10, sections 4 and 5]). We give only a rather sketchy proof of the lemma.

Proof of Lemma 4.5. Let Γ_- be defined by (4.9). The aim is to establish the asymptotic formula

$$\Gamma_- = -k_1 e^{i\pi/4}\tau_+ F_-(\omega_-)h^{1/2} + o(h^{3/2}),$$

where $k_1 = (2\pi)^{1/2}|e|^{-1/2}E^{-1/4}$ and

$$\tau_+ = \exp(i2(\alpha/h)(\pi - \gamma(\hat{e};\omega_-)) = \exp(i2(\alpha/h)(\gamma_- - \pi)), \quad \gamma_- = \gamma(\omega_-;\hat{e}).$$

The particle starting from $\operatorname{supp} V_+W^-$ at velocity $2E^{1/2}\omega_-$ moves like a free particle for $t < 0$, and it never passes over $\operatorname{supp} \tilde{V}_-W^+$. Hence we see that

$$\Gamma_- = -2iE^{1/2}h \left\langle R(E+i0;K_d)e^{i\eta}V_+W^-(\partial_2 W)\varphi_-, e^{i\eta}\tilde{V}_-W^+(\partial_2 W)\varphi_-\right\rangle + O(h^N).$$

We proceed with the argument, accepting a series of the lemmas below as proved.

Lemma 8.1 *Let $\tilde{\zeta}(x)$ be defined by*

$$\tilde{\zeta}(x) = j_+(x - d_+;\omega_-) + j_-(x - d_-;-\omega_-), \tag{8.1}$$

where $j_\pm$ is defined by (4.1). Then we have

$$\begin{aligned}&\tilde{V}_-W^+R(E+i0;K_d)e^{i\eta}U_+^-p(D_x) = \\ &\qquad e^{-i\beta\pi}\tilde{V}_-W^+R(E+i0;K_d)e^{i\tilde{\zeta}}[\psi_+, H_0]R(E+i0;H_0)U_+^-p(D_x) + O_p(h^N)X_{-N}\end{aligned}$$

with $U_+^- = V_+W^-(\partial_2 W)$, and

$$\begin{aligned}&V_+W^-R(E-i0;K_d)e^{i\eta}U_-^+p(D_x) = \\ &\qquad e^{i\beta\pi}V_+W^-R(E-i0;K_d)e^{i\tilde{\zeta}}[\psi_-, H_0]R(E-i0;H_0)U_-^+p(D_x) + O_p(h^N)X_{-N}\end{aligned}$$

with $U_-^+ = \tilde{V}_-W^+(\partial_2 W)$, where $\psi_\pm(x) = 1 - \chi(2h^\varepsilon|x - d_\pm|)$ and $p(\xi)$ is defined by (4.4).

Lemma 8.2 *We have*

$$\begin{aligned}s_+R(E+i0;H_0)V_+W^-(\partial_2 W)\varphi_- &= s_+\left((i/2)E^{-1/2}\varphi_- + O_2(h^{1/2-\varepsilon})\right)\\ s_-R(E-i0;H_0)\tilde{V}_-W^+(\partial_2 W)\varphi_- &= s_-\left((i/2)E^{-1/2}\varphi_- + O_2(h^{1/2-\varepsilon})\right).\end{aligned}$$

where $s_\pm$ is the characteristic function of $S_\pm = \{|x - d_\pm| < h^{-\varepsilon}\}$.

By Proposition 4.1, we combine these two lemmas to obtain that

$$\Gamma_- = -(i/2)E^{-1/2}e^{-i2\beta\pi}h\left\langle R(E+i0;K_d)e^{i\tilde{\zeta}}[\psi_+,H_0]\varphi_-, e^{i\tilde{\zeta}}[\psi_-,H_0]\varphi_-\right\rangle + o(h^{3/2}).$$

Lemma 8.3 *Let $L_\pm$ and $\zeta_\pm$ be as in (4.11) and let $\psi_\pm$ be as above. Then*

$$s_-R(E+i0;K_d)s_+ = s_-R(E+i0;L_-)YR(E+i0;L_+)s_+ + O_p(h^{3/2-c\varepsilon})$$

for $c>0$ independent of ε, where

$$Y = \exp\left(i(\zeta_-+\zeta_+)\right)[\psi_-,H_0]R(E+i0;H_0)[H_0,\psi_+]\exp\left(-i(\zeta_-+\zeta_+)\right).$$

Lemma 8.4 *We have*

$$s_-R(E+i0;H_0)s_+ = (ic(E)/4\pi)|e|^{-1/2}h^{1/2}Y_0 + O_p(h^{3/2-c\varepsilon})$$

for $c>0$ independent of ε, where Y_0 acts as

$$(Y_0u)(x) = \langle u, \varphi_0(-\hat{e})\rangle\,\varphi_0(-\hat{e}) = \int u(y)\overline{\varphi}_0(y;E,-\hat{e})\,dy\,\varphi_0(x;E,-\hat{e})$$

on u, where $\varphi_0(\omega) = \varphi_0(x;E,\omega) = \exp(iE^{1/2}x\cdot\omega)$.

Since $R(E+i0;L_\pm) = e^{i\zeta_\mp}R(E+i0;K_\pm)e^{-i\zeta_\mp}$, we make use of Lemmas 8.3 and 8.4 to calculate $s_-R(E+i0;K_d)s_+$ as

$$\begin{aligned} s_-R(E+i0;K_d)s_+ = {} & (ic(E)/4\pi)|e|^{-1/2}h^{1/2}s_-\exp(i\zeta_+)R(E+i0;K_-) \\ & \times\ \exp(i\zeta_-)[\psi_-,H_0]Y_0[H_0,\psi_+]\exp(-i\zeta_+) \\ & \times\ R(E+i0;K_+)\exp(-i\zeta_-)s_+ + O_p(h^{3/2-c\varepsilon}). \end{aligned}$$

This implies that

$$\begin{aligned} \Gamma_- = {} & -(i/2)|e|^{-1/2}E^{-1/2}e^{-i2\beta\pi}(ic(E)/4\pi)h^{3/2}\times \\ & \left\langle R(E+i0;K_-)\exp(i\zeta_-)[\psi_-,H_0]\varphi_0(-\hat{e}), \exp(i(\tilde{\zeta}-\zeta_+)[\psi_-,H_0]\varphi_-\right\rangle \times \\ & \left\langle R(E+i0;K_+)\exp(i(\tilde{\zeta}-\zeta_-))[\psi_+,H_0]\varphi_-, \exp(i\zeta_+)[\psi_+,H_0]\varphi_0(-\hat{e})\right\rangle \\ & +\ o(h^{3/2}). \end{aligned}$$

We further calculate $\tilde{\zeta}-\zeta_\mp = \pm\beta\left(\gamma(x-d_\pm;\pm\omega_-)+\gamma_-\right)$ over $S_\pm$, where $\tilde{\zeta}$ is defined by (8.1). Thus it follows from (7.3) that Γ_- in question behaves like

$$-(2\pi)E^{-1/2}|e|^{-1/2}(1/c(E))e^{-i2\beta(\pi-\gamma_-)}h^{3/2}g_-(-\hat{e}\to\omega_-)g_+(\omega_-\to-\hat{e}) + o(h^{3/2}),$$

which, together with Remark 7.1, implies that Γ_- equals

$$-(2\pi)E^{-1/2}|e|^{-1/2}(1/c(E))h^{1/2}e^{-i2\beta(\pi-\gamma_-)}e^{i[\alpha/h](\omega_--(-\hat{e}))}e^{-i[\alpha/h]((-\hat{e})-\omega_-)}F_-(\omega_-)$$

with error term $o(h^{3/2})$, where the position vectors ω_- and $(-\hat{e})$ in the exponent denote the azimuth angles $\gamma(\omega_-)$ and $\gamma(-\hat{e})$ from the positive x_1 axis. Since $c(E) = (2\pi)^{1/2}E^{-1/4}e^{-i\pi/4}$, the constant factor is calculated as

$$(2\pi)E^{-1/2}|e|^{-1/2}(1/c(E)) = (2\pi)^{1/2}|e|^{-1/2}E^{-1/4}e^{i\pi/4} = k_1 e^{i\pi/4}$$

and also a direct computation yields

$$e^{-i2\beta(\pi-\gamma_-)}e^{i[\alpha/h](\omega_- -(-\hat{e}))}e^{-i[\alpha/h]((-\hat{e})-\omega_-)} = \exp(i2(\alpha/h)(\gamma_- - \pi)) = \tau_+.$$

Thus the proof of the lemma is complete. $\square$

Proof of Lemma 8.1. The lemma is physically clear. We prove it in a rather intuitive way. The rigorous proof is done by constructing approximations to resolvent $R(E \pm i0; K_d)$ along scattering trajectories and by making use of Proposition 4.1 to evaluate error terms arising from approximations (see the proof of Lemma 4.7 or the proof of [10, Lemma 4.3] for details).

We prove the first relation only. A similar argument applies to the second one. Let $\tilde{\zeta}$ be as in the lemma. We introduce the auxiliary operator $L_0 = e^{i\tilde{\zeta}}H_0e^{-i\tilde{\zeta}} = H(\nabla\tilde{\zeta})$. If we set $\psi = 1 - \chi(2h^\varepsilon|x-d_-|) - \chi(2h^\varepsilon|x-d_+|)$, then we have

$$\begin{aligned}\tilde{V}_-W^+R(E+i0;K_d)e^{i\eta}V_+W^- &= \tilde{V}_-W^+R(E+i0;L_0)e^{i\eta}V_+W^- \\ &+ \tilde{V}_-W^+R(E+i0;K_d)\,(T_0+T_1)\,R(E+i0;L_0)e^{i\eta}V_+W^-\end{aligned}$$

as in the proof of Lemma 4.7, where $T_0 + T_1 = [\psi, L_0] + (L_0 - K_d)\psi$. The first operator on the right side fulfills

$$\tilde{V}_-W^+R(E+i0;L_0)e^{i\eta}V_+W^-\,(\partial_2 W)\,p(D_x) = O_p(h^N)X_{-N}. \tag{8.2}$$

In fact, the free particle starting from $\operatorname{supp} V_+W^-$ at velocity $2E^{1/2}\omega_-$ never passes over $\operatorname{supp}\tilde{V}_-W^+$, which implies (8.2). Let $\psi_\pm$ be as in the lemma. Then

$$T_0 = T_{+0} + T_{-0} = [\psi_+, L_0] + [\psi_-, L_0] = e^{i\tilde{\zeta}}\left([\psi_+, H_0] + [\psi_-, H_0]\right)e^{-i\tilde{\zeta}}.$$

The coefficients of $[\psi_-, H_0]$ have support in S_-. Hence the same reason as above implies that

$$\tilde{V}_-W^+R(E+i0;K_d)T_{-0}R(E+i0;L_0)e^{i\eta}V_+W^-\,(\partial_2 W)\,p(D_x) = O_p(h^N)X_{-N}.$$

The function $\eta(x)$ is defined by (3.10). We can calculate $\eta - \tilde{\zeta}$ as

$$\beta(\gamma(x-d_+;\hat{e}) - \gamma(x-d_+;\omega_-)) - \beta(\gamma(x-d_-;\hat{e}) - \gamma(x-d_-;-\omega_-)) = -\beta\pi$$

on $\operatorname{supp} V_+W^-$ and hence we have

$$T_{+0}R(E+i0;L_0)e^{i\eta}V_+W^- = e^{-i\beta\pi}e^{i\tilde{\zeta}}[\psi_+, H_0]R(E+i0;H_0)V_+W^-,$$

which gives rise to the leading operator of the relation in question. The coefficients of T_1 have support in $\Omega_+(\omega_-) \cup \Omega_-(-\omega_-)$, where $\Omega_\pm(\omega)$ is defined by (4.13). Since $\Omega_+(\omega_-) \cap \Omega_-(-\omega_-) = \emptyset$, T_1 admits the decomposition $T_1 = T_{+1} + T_{-1}$, where $T_{\pm 1} = (L_0 - K_d)\psi_\pm$ has coefficients with support in $\Omega_\pm(\pm\omega_-)$. We assert that

$$\tilde{V}_- W^+ R(E + i0; K_d) T_{\pm 1} R(E + i0; L_0) e^{i\eta} V_+ W^- \left(\partial_2 W\right) p(D_x) = O_p(h^N) X_{-N}.$$

These relations also follow for the same reason as used to derive (8.2). For example, consider the free particle passing over $\Omega_+(\omega_-)$ at velocity $2E^{1/2}\omega_-$. Such a particle continues to move like a free particle without touching the centers $d_\pm$, and hence it never passes over $\operatorname{supp} \tilde{V}_- W^+$. This implies the "+" case. If we note that the free particle starting from $\operatorname{supp} V_+ W^-$ at velocity $2E^{1/2}\omega_-$ never passes over $\Omega_-(-\omega_-)$, a similar intuitive argument applies to the "-" case. Thus the first relation is established. □

Proof of Lemma 8.2. The lemma is verified in almost the same way as in the proof of Lemma 4.9. We skip the proof. □

Proof of Lemma 8.3. This lemma has been verified as [10, Proposition 4.2]. □

Proof of Lemma 8.4. The kernel of $R(E + i0; H_0)$ obeys (3.17). If $x \in S_-$ and $y \in S_+$, then $|x - y| = (x - y) \cdot \hat{e} + O(h^{1-\varepsilon})$ and hence we have

$$\exp(iE^{1/2}|x - y|) = \exp(-iE^{1/2} x \cdot \hat{e}) \exp(iE^{1/2} y \cdot \hat{e}) \left(1 + O(h^{1-\varepsilon})\right).$$

If we have only to note that the constant $c_0(E)$ in (3.17) just coincides with $ic(E)/4\pi$, the lemma is obtained. □

9. Completion of Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 by making three comments on necessary modifications.

Completion of Proof of Theorem 1.1. (1) We have to evaluate the scalar products

$$4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} \left(\Delta U\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle$$
$$4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} \left(\Delta U\right) \varphi_-, e^{i\eta} \left(\Delta U\right) \varphi_- \right\rangle,$$

as mentioned in section 3, where $\Delta U = \Delta\left(VW\right) = \left(\partial_1^2 V\right) W + V\left(\partial_2^2 W\right)$. The function $\left(\partial_1^2 V\right) W$ has support in

$$\{x = (x_1, x_2) : M/h < |x_1| < 2M/h, \; |x_2| < 2M/h\}.$$

The particle starting from this region at velocity $2E^{1/2}\omega_-$ moves like a free particle for all t, $-\infty < t < \infty$, without touching the centers $d_\pm$. Hence we can show in the same way as in the proof of Lemma 3.1 that

$$4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} \left(\partial_1^2 V\right) W \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle = o(h^{3/2}).$$

Since the integral $\int w^{\pm}\left(\partial_2^2 w\right)\, dx_2 = 0$ vanishes, we can also show that

$$4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V \left(\partial_2^2 W\right) \varphi_-, e^{i\eta} V \left(\partial_2 W\right) \varphi_- \right\rangle = o(h^{3/2}).$$

A similar argument applies to the other scalar product. Thus the first three terms of the asymptotic formula are shown to be obtained only from Φ defined by (3.19).

(2) We consider the scalar products

$$\begin{aligned} &4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^+ \left(\partial_2 W\right) \varphi_-, e^{i\eta} \tilde{V}_+ W^- \left(\partial_2 W\right) \varphi_- \right\rangle, \\ &4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} V_+ W^+ \left(\partial_2 W\right) \varphi_-, e^{i\eta} \tilde{V}_- W^- \left(\partial_2 W\right) \varphi_- \right\rangle, \end{aligned}$$

which are obtained from Φ_+^+ by decomposition (see (4.8) and (4.9)). If we regard the first product as the conjugate of

$$4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} \tilde{V}_+ W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} V_+ W^+ \left(\partial_2 W\right) \varphi_- \right\rangle,$$

then the second term $ik_0 e^{i\beta\pi} h$ is obtained, and if we regard the second product as the conjugate of

$$4\pi E^{1/2} h \left\langle \Pi(K_d) e^{i\eta} \tilde{V}_- W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta} V_+ W^+ \left(\partial_2 W\right) \varphi_- \right\rangle,$$

then the third term $-k_1 \overline{e^{i\pi/4} \tau_- F_+(\omega_-)} h^{1/2}$ is obtained (see Lemma 3.3).

(3) Finally we explain how to obtain the second term from Φ_-^- in Lemma 3.3. For brevity, we take the center d_- as the origin and consider the scalar product

$$4\pi E^{1/2} h \left\langle \Pi(K_-) e^{i\eta_-} V_- W^- \left(\partial_2 W\right) \varphi_-, e^{i\eta_-} \tilde{V}_- W^+ \left(\partial_2 W\right) \varphi_- \right\rangle$$

with $\eta_- = -\beta\gamma(x; \hat{e})$. This product corresponds to Γ_1 in Lemma 4.6 and it is derived by use of the relation $\eta - \zeta_+ = -\beta\gamma(x - d_-; \hat{e}) = \eta_-$ on $\operatorname{supp} V_- W^{\mp}$. As stated in Remark 5.1, the outgoing eigenfunction $\varphi(x; \omega)$ of K_- is defined by

$$\varphi(x; \omega) = e^{-i\gamma(x;\omega)} \tilde{\varphi}(x; \omega),$$

where $\tilde{\varphi}$ is the eigenfunction of $K_{1-\beta} = H((1-\beta)\Lambda)$ associated with energy E. The eigenfunction φ admits the decomposition $\varphi = \varphi_{\mathrm{in}} + \varphi_{\mathrm{sc}}$, where

$$\varphi_{\mathrm{in}}(x; \omega) = e^{-i\gamma(x;\omega)} e^{i(1-\beta)\sigma(x;\omega)} \varphi_0(x; \omega), \quad \sigma(x; \omega) = \gamma(x; \omega) - \pi,$$

and $\varphi_{\mathrm{sc}}(x; \omega) = e^{-i\gamma(x;\omega)} \tilde{\varphi}_{\mathrm{sc}}(x; \omega)$. If $y \in \operatorname{supp} v_- w^-$, then

$$e^{i\eta_-(y/h;\hat{e})} \overline{\varphi}_{\mathrm{in}}(y/h; \omega_-) = e^{i((1-\beta)\pi - \beta\gamma_-)} \overline{\varphi}_-(y/h)$$

with $\gamma_- = \gamma(\omega_-; \hat{e})$. Thus the second term comes from the integral

$$I = e^{i((1-\beta)\pi - \beta\gamma_-)} \int e^{i\beta\gamma(x;\hat{e})} \varphi_{\mathrm{sc}}(x/h; \omega_-) \tilde{v}_- w^+ \left(\partial_2 w\right) \overline{\varphi}_-(x/h)\, dx.$$

We rewrite it as

$$\begin{aligned} I &= e^{i((1-\beta)\pi-\beta\gamma_-)}e^{i\gamma_-}\int e^{-i(1-\beta)\gamma(x;\hat e)}\tilde\varphi_{\mathrm{sc}}(x/h;\omega_-)\tilde v_- w^+\,(\partial_2 w)\,\overline{\varphi}_-(x/h)\,dx \\ &= e^{-i2\beta\pi}\left[e^{-i(1-\beta)(\pi-\gamma_-)}\int e^{-i(1-\beta)\gamma(x;\hat e)}\tilde\varphi_{\mathrm{sc}}(x/h;\omega_-)\tilde v_- w^+\,(\partial_2 w)\,\overline{\varphi}_-(x/h)\,dx\right] \end{aligned}$$

and we apply Lemma 6.5 with β replaced by $1-\beta$ to the integral in the bracket. Then I has the leading term

$$e^{-i2\beta\pi}\left(-2i((1-\beta)-1/2)\sin(1-\beta)\pi e^{-i(1-\beta)\pi}\right)h = -ik_0e^{-i\beta\pi}h.$$

This is just the second term obtained from Φ_-^-.

10. Appendix : Proof of Proposition 5.2. We end the paper by proving Proposition 5.2 which remains unproved.

Proof of Proposition 5.2. We begin by recalling the notation: the eigenfunction $\varphi(x;\omega)$ under consideration admits the decomposition $\varphi = \varphi_{\mathrm{in}}+\varphi_{\mathrm{sc}}$, where

$$\begin{aligned} \varphi_{\mathrm{in}}(x;\omega) &= \exp(i\beta\sigma)\varphi_0(x;\omega) = \exp(i\beta\sigma)\exp(iE^{1/2}x\cdot\omega) \\ \varphi_{\mathrm{sc}}(x;\omega) &= -\frac{\sin\beta\pi}{\pi}\int \exp(ih^{-1}E^{1/2}|x|\cosh t)\left(\frac{e^{-\beta t}}{e^{-t}+e^{i\sigma}}\right)dt\,e^{i\sigma} \end{aligned}$$

with $\sigma=\sigma(x;\omega)=\gamma(x;\omega)-\pi$ (see (5.6) and (5.7)). For brevity, we prove the proposition for the case $\omega=\omega_-=(0,1)$, and we write $\varphi(x)$ for $\varphi(x;\omega_-)$ (and similarly for φ_{in} and φ_{sc}). We work in the coordinate system (6.2) :

$$x=(x_1,x_2)=(-|x|\sin\nu,|x|\cos\nu),\qquad \nu=\gamma(x)-\pi/2.$$

Then

$$\tau(x,\omega_-)=\gamma(x:-\omega_-)-\pi=\gamma(x)-\pi/2=\nu$$

and it follows by assumption that

$$|\nu|\le c\,h^{1-\varepsilon},\qquad |x_1|\le c\,h^{1-\varepsilon},\qquad 1/c<x_2<c \tag{10.1}$$

for some constant $c>1$. We also recall the relation (6.3) : $\sigma=\sigma(x;\omega_-)=\nu-\pi$ or $\sigma=\nu+\pi$ according as $\nu>0$ or $\nu<0$, so that $e^{i\sigma}=-e^{i\nu}$. By (10.1), $e^{i\beta\nu}=1+O(h^{1-\varepsilon})$, and hence we have

$$\varphi_{\mathrm{in}}(x/h)=\begin{cases} e^{-i\beta\pi}\exp(ih^{-1}E^{1/2}x\cdot\omega_-)+O(h^{1-\varepsilon}), & \nu>0,\\ e^{i\beta\pi}\exp(ih^{-1}E^{1/2}x\cdot\omega_-)+O(h^{1-\varepsilon}), & \nu<0.\end{cases} \tag{10.2}$$

We analyze the behavior of

$$\varphi_{\mathrm{sc}}(x/h)=\frac{\sin\beta\pi}{\pi}\int\exp(ih^{-1}E^{1/2}|x|\cosh t)\left(\frac{e^{-\beta t}}{e^{-t}-e^{i\nu}}\right)dt\,e^{i\nu}.$$

To do this, we introduce the auxiliary function

$$\tilde{\varphi}_{\rm sc}(x/h) = \frac{\sin\beta\pi}{\pi}\int \exp(ih^{-1}E^{1/2}|x|\cosh t)\chi_h(t)\left(\frac{e^{-\beta t}}{e^{-t}-e^{i\nu}}\right) dt\, e^{i\nu},$$

where $\chi_h = \chi(h^{-(1/2-\varepsilon)}|t|)$ and $\chi \in C_0^\infty[0,\infty)$ have the property as in (3.9). The function $\chi_h(t)$ vanishes for $|t| > 2h^{1/2-\varepsilon}$. Hence we see by repeating partial integration that $\varphi_{\rm sc}(x/h) = \tilde{\varphi}_{\rm sc}(x/h) + O(h^N)$ for any $N \gg 1$. If $|t| < 2h^{1/2-\varepsilon}$, then

$$\exp(ih^{-1}E^{1/2}|x|\cosh t) = \exp(ih^{-1}E^{1/2}|x|)\exp(ih^{-1}E^{1/2}|x|t^2/2)\left(1+h^{-1}O(t^4)\right)$$

as $t \to 0$. Since $(e^{-t}-e^{i\nu})^{-1} = O(|t|^{-1})$ and $e^{-\beta t} = 1-\beta t + O(t^2)$, $\tilde{\varphi}_{\rm sc}(x/h)$ obeys

$$\tilde{\varphi}_{\rm sc} = \frac{\sin\beta\pi}{\pi}\int \exp(ih^{-1}E^{1/2}|x|t^2/2)\chi_h(t)\left(\frac{1-\beta t}{e^{-t}-e^{i\nu}}\right) dt\, e^{iE^{1/2}|x|/h}e^{i\nu} + O(h^{1-4\varepsilon}).$$

We set $s = t + i\nu$ and expand $(e^{-t}-e^{i\nu})^{-1}$ as

$$\left(e^{-t}-e^{i\nu}\right)^{-1} = e^{-i\nu}s^{-1}\left(s/(e^{-s}-1)\right) = e^{-i\nu}s^{-1}\left(-1-s/2+O(|s|^2)\right).$$

Then it follows that

$$\begin{aligned}(1-\beta t)/\left(e^{-t}-e^{i\nu}\right) &= (1-\beta(s-i\nu))\,e^{-i\nu}s^{-1}\left(-1-s/2+O(|s|^2)\right)\\ &= e^{-i\nu}\left(-\left(1+i\beta\nu\right)/s + ((\beta-1/2)-i\beta\nu/2)+O(|s|)\right).\end{aligned}$$

By (10.1), $s = t + O(h^{1-\varepsilon})$, and hence $\int \chi_h(t)O(|s|)\,dt = O(h^{1-2\varepsilon})$. Thus $\varphi_{\rm sc}$ admits the decomposition

$$\varphi_{\rm sc}(x/h) = -\left(\sin\beta\pi/\pi\right)\left(\varphi_1(x/h)-\tilde{\varphi}_0(x/h)\right)e^{iE^{1/2}|x|/h} + O(h^{1-4\varepsilon}), \tag{10.3}$$

where

$$\begin{aligned}\tilde{\varphi}_0(x/h) &= \int \exp(ih^{-1}E^{1/2}|x|t^2/2)\chi_h(t)\left((\beta-1/2)-i\beta\nu/2\right)\, dt\\ \varphi_1(x/h) &= \int \exp(ih^{-1}E^{1/2}|x|t^2/2)\chi_h(t)\left(1+i\beta\nu\right)/s\, dt.\end{aligned}$$

The stationary phase method shows that $\tilde{\varphi}_0(x/h) = \alpha_0|x|^{-1/2}h^{1/2} + O(h)$ for some constant $\alpha_0 \neq 0$. Its explicit form does not matter.

We proceed to analyzing the behavior of $\varphi_1(x/h)$. We divide the integral into the sum

$$\varphi_2 + \varphi_3 = \left(\int_{|t|>h^{1/2+\varepsilon}} + \int_{|t|<h^{1/2+\varepsilon}}\right)\exp(ih^{-1}E^{1/2}|x|t^2/2)\chi_h(t)(1+i\beta\nu)/s\, dt.$$

If $|t| > h^{1/2+\varepsilon}$, then $|\nu/t| = O(h^{1/2-2\varepsilon})$ by (10.1). Hence we can take $n \gg 1$ so large that

$$s^{-1} = (t+i\nu)^{-1} = t^{-1}\left(1+i\nu/t\right)^{-1} = t^{-1}\sum_{k=0}^{n}(-i\nu/t)^k + O(h). \tag{10.4}$$

If we note that

$$\int_{|t|>h^{1/2+\varepsilon}} \exp(ih^{-1}\sqrt{E}|x|t^2/2)\chi_h(t)t^{-1}\,dt = 0,$$

then we insert this expansion into the integral to obtain that

$$\varphi_2(x/h) = \sum_{k=1}^{n}\alpha_k(|x|;h)\nu^k + O(h),$$

where α_k is given by

$$\alpha_k(|x|;h) = (-i)^k\int_{|t|>h^{1/2+\varepsilon}}\chi_h(t)t^{-k-1}\exp(ih^{-1}\sqrt{E}|x|t^2/2)\,dt$$

and it satisfies $(d/d|x|)^m\,\alpha_k = O(h^{-(1/2+\varepsilon)k-2m\varepsilon})$.

We look at the behavior of $\varphi_3(x/h)$. If $|t| < h^{1/2+\varepsilon}$, then $\chi_h(t) = 1$ and $t^2/h = O(h^{2\varepsilon})$. Hence we can expand $\exp(ih^{-1}E^{1/2}|x|t^2/2)$ as

$$\exp(ih^{-1}E^{1/2}|x|t^2/2) = \sum_{k=0}^{n}\left(ih^{-1}E^{1/2}|x|t^2/2\right)^k/k! + O(h)|t|$$

for some $n \gg 1$. We insert $t = s - i\nu$ into this expansion. Then

$$\exp(ih^{-1}E^{1/2}|x|t^2/2) = \sum_{k=0}^{n}\left(\beta_k(|x|)h^{-k}\nu^{2k} + \sum_{j=0}^{2k-1}\beta_{kj}(|x|)h^{-k}s^{2k-j}\nu^j\right) + O(h)|t|,$$

where $\beta_0(|x|) = 1$. We further rewrite it in the form

$$\begin{aligned}\exp(ih^{-1}\sqrt{E}|x|t^2/2) &= \sum_{k=0}^{n}\beta_k(|x|)h^{-k}\nu^{2k}\\ &+ \sum_{l=0}^{n-1}\left(\sum_{k=l+1}^{n}\gamma_{kl}(|x|)h^{-k}s^{2(k-l)}\right)\nu^{2l}\\ &+ \sum_{l=0}^{n-1}\left(\sum_{k=l+1}^{n}\tilde{\gamma}_{kl}(|x|)h^{-k}s^{2(k-l)-1}\right)\nu^{2l+1} + O(h)|t|.\end{aligned}$$

We insert this expansion into

$$\varphi_3(x/h) = \int_{|t|<h^{1/2+\varepsilon}}\exp(ih^{-1}E^{1/2}|x|t^2/2)(1+i\beta\nu)/s\,dt$$

and evaluate the integrals as follows :

$$\int_{|t|<h^{1/2+\varepsilon}} s^{2k-2l-1}\,dt = \sum_{j=0}^{2k-2l-1} O(h^{k-(1/2+\varepsilon)(2l+j)})\nu^j, \quad l+1\le k\le n$$

$$\int_{|t|<h^{1/2+\varepsilon}} s^{2k-2l-2}\,dt = \sum_{j=0}^{2k-2l-2} O(h^{k-(1/2+\varepsilon)(2l+j+1)})\nu^j, \quad l+1\le k\le n$$

for $0\le l\le n-1$, and

$$\left(h^{-k}\nu^{2k}\right)\int_{|t|<h^{1/2+\varepsilon}} s^{-1}\,dt = O(h^{1-2\varepsilon}), \quad 1\le k\le n.$$

Thus $\varphi_3(x/h)$ takes the form

$$\varphi_3(x/h) = \int_{|t|<h^{1/2+\varepsilon}} s^{-1}\,dt + \sum_{j=0}^{n}\delta_j(|x|;h)\nu^j + O(h^{1-2\varepsilon}),$$

where δ_j is a polynomial of $|x|$ and it obeys the bound $\delta_j = O(h^{-(1/2+\varepsilon)j})$. We write the leading integral as

$$\int_{|t|<h^{1/2+\varepsilon}} s^{-1}\,dt = \left\{\int_{|t|<h^{\varepsilon}} - \int_{h^{1/2+\varepsilon}<|t|<h^{\varepsilon}}\right\} s^{-1}\,dt.$$

Then the first integral on the right side equals

$$\int_{|t|<h^{\varepsilon}} (t+i\nu)^{-1}\;dt = \begin{cases} -i\pi + O(h^{1-2\varepsilon}), & \nu>0 \\ i\pi + O(h^{1-2\varepsilon}), & \nu<0 \end{cases}$$

and the second one is shown to be expanded as

$$\int_{h^{1/2+\varepsilon}<|t|<h^{\varepsilon}} s^{-1}\,dt = \sum_{j=1}^{n} O(h^{-(1/2+\varepsilon)j})\nu^j + O(h^{1-\varepsilon}).$$

To see this, we expand $s^{-1} = (t+i\nu)^{-1}$ as in (10.4). If we have only to note that the integral $\int_{h^{1/2+\varepsilon}<|t|<h^{\varepsilon}} t^{-1}\,dt = 0$, then it is verified.

We now combine all the obtained results. Since

$$(|x| - x\cdot\omega_-) = |x|\,(1-\cos\nu) = O(h^{2-2\varepsilon})$$

by (10.1), we have

$$\exp(ih^{-1}E^{1/2}|x|) = \exp(ih^{-1}E^{1/2}x\cdot\omega_-) + O(h^{1-2\varepsilon}).$$

Hence it follows from (10.3) that $\varphi_{\mathrm{sc}}(x/h)$ is expanded as

$$\varphi_{\mathrm{sc}}(x/h) \sim \begin{cases} i\sin\beta\pi\exp(ih^{-1}E^{1/2}x\cdot\omega_-), & \nu>0 \\ -i\sin\beta\pi\exp(ih^{-1}E^{1/2}x\cdot\omega_-), & \nu<0 \end{cases}$$

with remainder term having the form

$$\left(\sum_{j=0}^{n} b_j(|x|;h)\tau^j\right)\exp(ih^{-1}E^{1/2}x\cdot\omega_-)+O(h^{1-4\varepsilon}),$$

where $b_j(|x|;h)$ satisfies the bound in the proposition. This, together with (10.2), completes the proof. □

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