BIRKHOFF NORMAL FORM FOR SOME QUASILINEAR HAMILTONIAN PDES

DARIO BAMBUSI
Dipartimento di Matematica, Via Saldini 50, 20 133 Milano, Italy

Consider a Hamiltonian PDE having an elliptic equilibrium at zero. Assuming a suitable condition on the frequencies we will construct a canonical transformation putting the system in Birkhoff normal form up to a small reminder. In the case of quasilinear systems the normal form will be used to describe the dynamics of smooth solutions of small amplitude. Applications to the water wave problem and to quasilinear wave equations will be given.

1. Introduction

During the last fifteen years perturbation theory of Hamiltonian partial differential equations has been quite extensively studied and remarkable results have been established. In particular the existence of quasiperiodic solutions has been proved through suitable extensions of KAM theory[18,20,17,11,6,8]. However very little is known on the behavior of the solutions lying outside KAM tori, solutions which correspond to the large majority of initial data. In the finite dimensional case they are described by Nekhoroshev's theorem, whose extension to PDEs is at present a completely open problem[a]. However there is a particular situation where the full power of Nekhoroshev theorem is not needed: the neighborhood of an elliptic equilibrium point, where (in the finite dimensional case) Birkhoff normal form theorem gives a quite precise description of the dynamics.

In the present paper we will present a result partially extending Birkhoff normal form theorem to PDEs. Assuming a suitable nonresonance condition we will show that for any positive integer M there exists a canonical transformation putting the Hamiltonian in Birkhoff normal form up to a reminder having a zero of order $M+1$ at the origin. In the case of quasilinear systems[14,15] the normal form will be used to show that smooth initial data with norm ϵ give rise to solutions remaining $\mathcal{O}(\epsilon^M)$ close to a torus up to times of order ϵ^{-1}.

The theory will be applied to the equations of the water wave problem[9] and to quasilinear wave equations in arbitrary space dimension. For the water wave problem we will obtain that the transformation putting the system in normal form is well defined, but we will not deduce any dynamical consequence. For the quasilinear wave equation we will also deduce that smooth solutions remain close to finite dimensional tori for long times. Some of the results presented here were proved in [2].

[a] See however the papers [1,5] where the dynamics close to finite dimensional small amplitude tori is studied.

2. The finite dimensional case

Consider a finite dimensional Hamiltonian system with a linearly stable equilibrium point. It is well known that there exist coordinates in which the Hamiltonian takes the form

$$H = H_0 + P \tag{1}$$

where

$$H_0(p,q) := \sum_{j=1}^{N} \omega_j \frac{p_j^2 + q_j^2}{2} \tag{2}$$

and P has a zero of order at least three at the origin. Assume that P is of class C^∞, then the following theorem holds.

Theorem 2.1. *[Birkhoff] For any positive integer $M \geq 2$ there exist a neighborhood of the origin $\mathcal{U}_M$ and a canonical transformation $\mathcal{T}_M : \mathcal{U}_M \to \mathbb{R}^{2N}$ which puts the system in Birkhoff normal form, namely such that*

$$H \circ T_M = H_0 + Z + R_M \tag{3}$$

where Z Poisson commutes with H_0, namely $\{H_0; Z\} \equiv 0$ and R_M is small, i.e.

$$\|X_{R_M}(p,q)\| \leq C_M \|(p,q)\|^{M+1} \tag{4}$$

where X_{R_M} is the Hamiltonian vector field of R_M

$$X_{R_M} = \left(-\frac{\partial R_M}{\partial q_j}, \frac{\partial R_M}{\partial p_j} \right)$$

Remark 2.1. In the case where the frequencies are nonresonant namely they fulfill

$$\omega \cdot k \neq 0 \ , \quad \forall k \in \mathbb{Z}^n \backslash \{0\} \ , \tag{5}$$

the function Z can be shown to depend on the actions $I_j := (p_j^2 + q_j^2)/2$ only.

In the nonresonant case one can immediately deduce some dynamical consequences:

Corollary 2.1. *Assume that condition (5) holds, then for any M_1, M_2 with $M := 2M_1 + M_2 \geq 2$ there exists ϵ_M with the following property. Consider a solution $z(t)$ corresponding to an initial datum $z_0 \equiv (p^0, q^0)$ fulfilling*

$$\epsilon := \|z_0\| < \epsilon_M \ .$$

Then there exists a smooth torus $\mathbb{T}_{z_0}$ such that

$$d(z(t), \mathbb{T}_{z_0}) \leq C_M \epsilon^{M_1} \quad \textit{for} \quad |t| \leq (C_M \epsilon)^{-M_2} \tag{6}$$

Moreover, up to the same time one has

$$\|z(t)\| \leq 2\epsilon \ , \qquad |I_j(t) - I_j(0)| \leq C\epsilon^3 \tag{7}$$

The main difficulty in order to extend theorem 2.1 to the infinite dimensional case is due to the appearance of small denominators. In the finite dimensional case one has that, having fixed M,

$$[0<|k|\le M+2\ ,\quad \omega\cdot k\neq 0]\ \text{implies}\ |\omega\cdot k|\ge\gamma\ , \tag{8}$$

a condition needed in the proof of Birkhoff's theorem. In the infinite dimensional case ($N=\infty$) the set considered in (8) is infinite and in all interesting cases one has

$$\inf_{\substack{0<|k|\le M+2\\ \omega\cdot k\neq 0}}|\omega\cdot k|=0\ .$$

This makes impossible a straightforward generalization of Birkhoff's theorem to PDEs.

3. An abstract result for the infinite dimensional case

Consider a (formal) Hamiltonian system of the form (1) with

$$H_0(p,q):=\sum_{j\ge1}\omega_j\frac{p_j^2+q_j^2}{2} \tag{9}$$

To define precisely the phase space consider the Banach space ℓ^2_s of the sequences $\{x_j\}_{j\ge1}$ such that

$$\|x\|_s^2:=\sum_{j\ge1}j^{2s}x_j^2<\infty\ ,$$

and denote $\mathcal{P}_s:=\ell^2_s\times\ell^2_s$. If $z\equiv(p,q)\in\mathcal{P}_s$ is a phase point, we will denote by $\|z\|_s^2:=\|p\|_s^2+\|q\|_s^2$ (the square of) its norm and by $B_s(R)$ the open ball of radius R in $\mathcal{P}_s$.

For any positive (large) N denote by $\omega^{(N)}:=(\omega_1,...,\omega_N)$ the truncation of length N of the frequency vector. We assume that

H1) For any positive (large) r there exist $\alpha=\alpha(r)$ and $\gamma=\gamma(r)>0$ such that for any N and $\forall k\in\mathbb{Z}^N$, with $0<|k|\le r+2$ one has

$$\text{either}\quad \omega^{(N)}\cdot k=0\quad\text{or}\quad|\omega^{(N)}\cdot k|\ge\frac{\gamma}{N^\alpha}\ ,$$

where $|k|:=|k_1|+...+|k_N|$.

H2) There exists s_0, and, for any positive r there exists d_r with the following properties: for any $s\ge s_0$ there exists an open neighborhood of the origin $\mathcal{U}_{s+d_r}\subset\mathcal{P}_{s+d_r}$ such that the Hamiltonian vector field of H is defined on $\mathcal{U}_{s+d_r}$ and fulfills $X_H\in C^{r+2}(\mathcal{U}_{s+d_r},\mathcal{P}_s)$.

Definition 3.1. Let N be a positive integer, a functional Z will be said to be in N-normal form if it is independent of all the variables $\{p_j,q_j\}_{j>N}$ and

$$\{Z,H_0\}\equiv0\ . \tag{10}$$

For fixed N we will denote

$$H_0^{(N)}(p,q)=\sum_{j=1}^N\omega_j\frac{p_j^2+q_j^2}{2} \tag{11}$$

Theorem 3.1. *Assume H1-H2 and fix a positive M. Then there exist constants s', s_*, $R_*>0$, a function $N(\epsilon)$, and an analytic canonical transformation $\mathcal{T}:B_{s_*}(R_*)\to\mathcal{P}_{s_*}$ that puts the Hamiltonian in normal form up to order M. Precisely such that*

$$H\circ\mathcal{T}=H_0^{(N)}+Z+\mathcal{R} \tag{12}$$

where Z is in $N(\epsilon)$-normal form. Moreover, for any $s>s_*$ there exists R_s such that $\mathcal{T}$ restricts to an analytic map from $B_s(R_s)$ to $\mathcal{P}_s$ and, for any $z\in B_{s+s'}(R_{s+s'})$ the following estimate holds

$$\left\|X_{\mathcal{R}}(z)\right\|_s\le C_s\|z\|_{s+s'}^{M+1} \tag{13}$$

The proof is obtained in three steps: First make a Galerkin cutoff of the original system, second put in normal form the finite dimensional cutoffed system, third choose the cutoff so that the error due to the Galerkin approximation is of the same order of magnitude as the size of the reminder of the Birkhoff normal form.

Concerning theorem 3.1 note in particular that the reminder is small in the norm of $\mathcal{P}_s$ when it is evaluated at a point of $\mathcal{P}_{s+s'}$. Here s' is a large M dependent number. So, the situation is that the reminder is small provided it is considered as a 'differential operator' of large order. For this reason it is impossible to deduce from (12) any bound on the norm of the solution. Furthermore it is non trivial to deduce some dynamical consequences. Nevertheless, adding some assumptions one can compare the solution of the normalized system to the solution of the original one. To this end we will restrict to quasilinear systems and assume that the Lyapunov exponents are of order ϵ in a ball of radius ϵ.

Denote by $\mathcal{B}(\mathcal{P}_{s+d},\mathcal{P}_s)$ the space of bounded linear operators from $\mathcal{P}_{s+d}$ to $\mathcal{P}_s$. First of all we assume that the system has the form

$$\dot z=A(z)z+g(z)\equiv X_H(z)\ . \tag{14}$$

H2') There exists d, and, for any $s\ge s_0$ a positive R_{s+d}, such that the map

$$B_{s+d}(R_{s+d})\ni z\mapsto A(z)\in\mathcal{B}(\mathcal{P}_{s+d},\mathcal{P}_s)$$

is of class C^∞. Moreover g is smooth, i.e. $g\in C^\infty(B_{s+d}(R_{s+d}),\mathcal{P}_{s+d})$.

For any ϵ small enough consider the set of the functions $\zeta\in C^0([0,T],\mathcal{P}_{s+d})\cap C^1([0,T],\mathcal{P}_s)$ fulfilling

$$\sup_{t\in[0,T]}\|\zeta(t)\|_{s+d}+\sup_{t\in[0,T]}\left\|\dot\zeta(t)\right\|_s\le\epsilon \tag{15}$$

and the linear time dependent equation

$$\dot z=A(\zeta(t))z \tag{16}$$

H3) There exists $\nu\ge1$ such that the evolution operator $U(t,s)$ associated to equation (16) exists and fulfills the estimate

$$\sup_{0\le t\le\tau\le T}\|U(t,\tau)\|_{\ell^2_{s+d}\to\ell^2_{s+d}}\le Me^{\beta\epsilon^\nu T}\ , \tag{17}$$

with some constants M,β independent of ζ,T,ϵ.

H4) g has a zero of order at least $\nu+1$ at the origin.

Due to the form of the Hamiltonian, one has $A(z) = A_0 + B(z)$, with $A_0 = X_{H_0}$ and $B(.)$ an operator valued map vanishing at zero. In assumption H3 we are thinking of the case where B has a zero of order ν at the origin. In the semilinear case one has $B \equiv 0$, and therefore H3 is automatic.

Remark 3.1. Using Kato's theory[14] one can prove that under the assumptions H2',H3,H4 the dynamics of the system is locally well posed in any space $\mathcal{P}_s$ with $s \geq s_0 + d$.

We fix now the notations concerning the objects we are going to compare. Given an initial datum z_0 we consider the corresponding solution $z(t)$ of the equations of motion (14) of the original system. Then we consider the solution $z_N(t)$ of the finite dimensional normalized system

$$\dot{z} = X_{H_0^{(N)}}(z) + X_Z(z)$$

with initial datum $\Pi_N \mathcal{T}^{-1}(z_0)$, where Π_N is the projector on the first N modes. Finally we define the approximating solution $z_a(t) := \mathcal{T}(z_N(t))$, which is the solution of the normalized system in the original coordinates.

Theorem 3.2. *Assume H1,H2',H3,H4 and fix $M \geq 2$, then there exists $s' >> 1$, and, for any s large enough, there exists ϵ_s with the following property. If the initial datum is smooth and small enough, precisely if*

$$\epsilon := \|z_0\|_{s+s'} \leq \epsilon_s \tag{18}$$

then

$$d_s(z(t), z_a(t)) \leq C\epsilon^M \quad \text{for} \quad |t| \leq \frac{1}{C\epsilon^\nu} \tag{19}$$

where $d_s(.,.)$ is the distance in the norm of $\mathcal{P}_s$.

Remark 3.2. The above theorem is essentially an averaging theorem, indeed the time of validity of its description of the dynamics is of the same order of magnitude as the inverse of the size of the perturbation.

Remark 3.3. When the frequencies are nonresonant the solution of the normalized system lie on an invariant torus, and so the same is true for $z_a(t)$. From this one can conclude that there exists a finite dimensional smooth torus $\mathbb{T}_{z_0}$ with the property that, up to the considered times ones has

$$d_s(\mathbb{T}_{z_0}, z(t)) \leq C\epsilon^M \ .$$

4. Applications

4.1. *The water wave problem*

The water wave problem consists in describing the motion of the free surface of a fluid subjected to the gravitational force. Here we will consider the case of a fluid lying in a two dimensional domain of infinite depth. We will study space periodic solutions. In terms of the velocity potential $\varphi(x,y)$ and of the profile of the surface $\eta(x)$, the equations of motion are given by

$$\Delta\varphi = 0\,, \quad 0 \le x \le 2\pi\,,\ -\infty < y < \eta(x) \tag{20}$$

$$\varphi \to 0\,, \qquad y \to -\infty \tag{21}$$

$$\eta_t = \varphi_y - \eta_x \varphi_x\,, \quad \varphi_t = -g\eta - (\nabla\varphi)^2\,, \quad y = \eta(x) \tag{22}$$

$$\eta(x,y) = \eta(x+2\pi,y)\,, \quad \varphi(x,y) = \varphi(x+2\pi,y) \tag{23}$$

Zakharov[22] pointed out that this is a Hamiltonian system. The corresponding Hamiltonian function is the energy of the fluid, and conjugated variables are given by the wave profile $\eta(x)$ and the velocity potential $\varphi(x,\eta(x))$ at the free surface.

It was shown by Zakharov that close to the equilibrium solution in which the fluid is at rest and the surface is horizontal the Hamiltonian has the form (1) with $\omega_k = \sqrt{g|k|}$, $k \in \mathbb{Z}\backslash\{0\}$. So it is easy to verify that the frequencies fulfill the assumption H1. The regularity of the nonlinear part P can be proved by classical theory of elliptic equations. So assumption H2 holds. It follows that according to theorem 2.1 there exists a canonical transformation putting the Hamiltonian in normal form up to any finite order.

We point out that in this problem nothing is known on the Lyapunov exponents of the system, and therefore we are not able to deduce any (rigorous) conclusion on the dynamics of the system.

The first terms of the function Z were computed explicitly in a series of paper[13,9,10,12], with the surprising outcome that up to order four $H_0 + Z$ is integrable. The dynamics of this integrable system has also been studied in detail. However as far as we know the rigorous existence of the normalizing transformation was, up to now, established only for the transformation putting the system in third order normal form[9].

4.2. *A quasilinear wave equation*

Consider the n dimensional parallelepiped $\mathcal{R}$ with sides of length L_i, namely

$$\mathcal{R} := \{x \equiv (x_1, ..., x_n) \in \mathbb{R}^n \ :\ 0 < x_i < L_i\}$$

and the formal Hamiltonian system defined by the Hamiltonian

$$H(u,v) = \int_{\mathcal{R}} \left[\frac{v^2}{2} + \frac{\|\nabla u\|^2}{2} + \frac{mu^2}{2} + W(u,\nabla u) \right] d^n x\ , \tag{24}$$

where u is a function on $\mathcal{R}$ vanishing at its boundary, $v = \dot{u}$ is the momentum conjugated to u, and W is a smooth function. The corresponding Hamilton equations have the form

$$u_{tt} - \Delta u + mu - b_{ij}(u,\nabla u)\partial_i\partial_j u + g(u,\nabla u) = 0 \tag{25}$$

(with summation convention over the indexes i,j and suitable functions b_{ij} and g) which should be supplemented with Dirichlet boundary conditions. In order to fit the abstract scheme we introduce the phase spaces $\mathcal{F}_s$ composed of the functions $(u,v) \in H^{\tilde{s}+1} \oplus H^{\tilde{s}}$ with $\tilde{s} := ns/2$, fulfilling the compatibility conditions

$$(-\Delta)^j u\big|_{\partial\Omega} = 0\ , \quad 0 \le j \le \left[\frac{\tilde{s}}{2}\right]\ ; \qquad (-\Delta)^j v\big|_{\partial\Omega} = 0\ , \quad 0 \le j \le \left[\frac{\tilde{s}+1}{2}\right] - 1\ . \tag{26}$$

Passing to Fourier coefficient it is easy to see that $\mathcal{F}_s$ is isomorphic to $\mathcal{P}_s$.

In order to satisfy the nonresonance condition H1 remark that the frequencies depend parametrically on the sides L_i of the domain and on the mass $m > 0$:

$$\omega_{(j_1,..,j_n)} = \sqrt{\left(\frac{2\pi}{L_1}j_1\right)^2 + ... + \left(\frac{2\pi}{L_n}j_n\right)^2 + m} \tag{27}$$

Theorem 4.1. *[See[2]] For any M there exists a set $\mathcal{N} \subset \mathbb{R}_+^{n+1}$ with full measure, such that, if $(L_1, ..., L_n, m) \in \mathcal{N}$ then the frequencies are nonresonant and fulfill the condition H1.*

Assume that the potential W is even in each of its arguments and that it has a zero of order $\nu + 2$ at the origin, then H2',H3,H4 hold and therefore theorem 3.2 applies. Consider an initial datum of the form

$$u(x,0) = \epsilon u_0(x) \ , \quad \dot{u}(x,0) = \epsilon v_0(x) \ . \tag{28}$$

Theorem 4.2. *[See[2]] Fix M, assume that the parameters belong to $\mathcal{N}$ and that the potential W is even and smooth; then there exists s' with the following property. If*

$$\|u_0\|_{s+s'} \ , \quad \|v_0\|_{s+s'} \leq 1$$

then, for small enough ϵ there exists a finite dimensional torus $\mathbb{T}_{z_0,\epsilon}$ such that one has

$$d_s(z(t), \mathbb{T}_{z_0,\epsilon}) \leq C_M \epsilon^{M/2} \quad for \quad |t| \leq (C_M \epsilon)^{-\nu} \ . \tag{29}$$

5. Concluding remarks

The main limitation of theorem 3.2 rests in the time of validity of its prediction. Indeed, in the finite dimensional case and also in some infinite dimensional cases[3,4] one has that the solution remains very close to a 'quasi–invariant' torus up to times $\mathcal{O}(\epsilon^{-M_1})$ for any M_1. So it is natural to ask whether a similar result holds also in the case considered by theorem 3.2.

To discuss this point consider the particular case of equation (25) given by

$$u_{tt} - (1 + u_x^\nu) u_{xx} + mu = 0 \tag{30}$$

with Dirichlet boundary conditions on $[0, \pi]$. According to the previous theory one has that for almost all values of the mass m the frequencies fulfill H1. Theorem 4.2 applies provided ν is even.

Now, it has been proved by Klainerman and Majda[16] that in the case $m = 0$ the solution of (30) with initial data (28) develops a singularity in the second derivative at a time of order $\epsilon^{-\nu}$. In the case $m = 0$ the nonresonance condition of our theorem is violated, so Klainerman and Majda's result is not a counterexample to theorem 3.2, but we think that it poses serious doubts on the validity of our description for longer times scales.

Finally we recall that up to now very little was known concerning equations in more than one space dimensions[6,8,19,4] and on the dynamics of quasilinear equations. We also mention the papers[7,21] that are strongly related to the present one.

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