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Dissipative Schrödinger-Poisson systems

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Abstract

The paper is devoted to the dissipative Schödinger-Poisson system. We prove that the system always admits a solution and that all solutions of a given Schrödinger-Poisson system are included in a uniform ball whose radius depends only on the data of the system.

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dissipative Schrödinger-type operators, dissipative Schrödinger-Poisson systems, carrier and current densities, density matrices, a priori estimates.

Contents

1 Introduction

Schrödinger-Poisson systems are of great interest in semiconductor physics. In the following we consider a Schrödinger-Poisson system on a subinterval $\Omega = (a, b)$ of the real axis $\mathbb{R}$. A system of this type was considered in [2], [5], [12]. By φ we denote the electrostatic potential on Ω which is determined by Poisson's equation

$$-\frac{d}{dx}\epsilon(x)\frac{d}{dx}\varphi(x) = q\left(C(x) + u^+(x) - u^-(x)\right), \quad x \in [a, b], \tag{1.1}$$

where u^+ and u^- are the densities of holes and electrons, q is the magnitude of the elementary charge, $C(\cdot)$ is the doping profile of the semiconductor devices and $\epsilon = \epsilon(x) > 0$ denotes the dielectric permittivity. We regard the following mixed boundary conditions for the Poisson equation (1.1)

$$\begin{aligned} \varphi(x) &= \varphi_\Gamma(x) & \text{if } x \in \Gamma, \\ -\epsilon(x)\tfrac{d}{dx}\varphi(x) &= k(x)(\varphi(x) - \varphi_\Gamma(x)) & \text{if } x \in \partial\Omega \setminus \Gamma, \end{aligned} \tag{1.2}$$

where $\Gamma \subseteq \partial\Omega = \{a, b\}$. The function φ_Γ, defined on the closure of Ω, represents the boundary values given on Γ and the inhomogeneous boundary conditions of third kind on $\partial\Omega \setminus \Gamma$. The function $k \geq 0$ is defined on $\partial\Omega$. Such kind of boundary condition occurs in semiconducter device modeling, see [3].

The densities $u^\pm$ in (1.1) are determined by Schrödinger-type operators

$$H^\pm(V) = -\frac{1}{2}\frac{d}{dx}\frac{1}{m_\pm(x)}\frac{d}{dx} + V(x), \tag{1.3}$$

which act on the Hilbert space $L^2[a, b]$ ($m_\pm$ is the position dependent effective mass of holes and electrons and $\hbar \equiv 1$) and density matrices $\varrho_\pm$ which describe the collective behaviour of holes and electrons. In the following we investigate stationary Schrödinger-Poisson systems. This case happens if $\varrho_\pm$ are steady states.

Since the formalism of quantum mechanics is well developed only for self-adjoint Schrödinger-type operators, usually self-adjoint boundary conditions are chosen, cf. [5]. Self-adjoint boundary conditions imply that the system under consideration is closed. In particular, this means that no carrier exchange with the environment is possible. However, from the semiconductor physics point of view this consequence is unacceptable since a net current flow through the boundary is natural. Thus one has to devise boundary conditions which allow those flows.

A simple proposal to replace the self-adjoint boundary conditions by non-self-adjoint ones was made in [5]. The treatment of the resulting non-selfadjoint operators $H^\pm(V)$ leads to several complications. In particular, the important notion of carrier density has to be redefined.

The situation can be improved if we choose dissipative boundary conditions, cf. [7]. This enables us to use the dilation theory for dissipative operators as the technical tool to overcome difficulties arising from the non-selfadjointness. From the physical point of view the minimal self-adjoint dilations $K^{\pm}(V)$ play the role of the Hamiltonians of a larger, closed systems which contains the original system described by $H^{\pm}(V)$. Using this fact one defines steady states, carrier and current densities, cf. [9]. On the basis of these notions a so-called dissipative Schrödinger-Poisson system is set up, see [9]. Our goal is to show that this dissipative system has a solution. We note that dissipative Schrödinger-Poisson systems were also considered in [4] and [11] in a non-stationary setup which is quite different from the present stationary one.

The paper is organized as follows. In Section 2 we rigorously define Schrödinger-type operators, cf. [7], briefly introduce their dilations and generalized eigenfunction expansions, cf. [8], and recall the definition of the carrier density given in [9]. Following [5] we introduce the (nonlinear) carrier density operator assigning to a Schrödinger potential the corresponding carrier density. In Section 3 we investigate the convergence properties of Schrödinger-type operators, their dilations and eigenfunction expansions as well as of the carrier density operators with respect to the potentials in $L^\infty([a,b])$. In Section 4 we rigorously define dissipative Schrödinger-Poisson systems and show that such systems always have a solution.

2 Notions and definitions

2.1 Notations and general assumptions

In this paper we use the following notations: the Schrödinger Poisson system will be regarded on a one dimensional interval which will always be denoted by $(a,b) := \Omega$. By L^1 we denote the space of (equivalence classes of) real-valued Lebesgue integrable functions on the interval $[a,b]$. The space of real Lebesgue measurable and essentially bounded functions on $[a,b]$ will be denoted by L^∞ in the sequel.

In order to avoid confusion we denote the space of complex valued, square integrable functions on the interval $[a,b]$ by $\mathfrak{H}$. Furthermore, we denote by $W^{1,2}$ the usual complex Sobolev space $W^{1,2}[a,b]$ and by $C[a,b]$ the space of complex valued, continuous functions on $[a,b]$.

If $\mathcal{H}$ is any Hilbert space then $\mathfrak{L}_1(\mathcal{H})$ denotes the space of nuclear operators on $\mathcal{H}$ and $\mathfrak{L}_2(\mathcal{H})$ denotes the space of Hilbert Schmidt operators, each with its canonic norm. For Banach spaces X and Y, we denote by $\mathcal{B}(X;Y)$ the space of all linear, continous operators from X into Y. If $X = Y$ we write $\mathcal{B}(X)$. Finally, $\mathbb{N}$ is used as the symbol for the set of natural numbers.

In order to avoid reiterations of the same conditions the following conventions are made throughout section 2 and 3:

Assumptions 2.1

(A_1) The functions $m_\pm$, which are called the 'effective masses', are positive and obey $m_\pm$, $\frac{1}{m_\pm} \in L^\infty$. We set $\|m_\pm\| := \|m_\pm\|_{L^\infty}$.

(A_2) The boundary coefficients $\kappa_a^\pm, \kappa_b^\pm$ are from the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im\mathrm{m}(z) > 0\}$.

The distinction between '+' and '−' is not relevant for the investigations in the sections 2 and 3, therefore we will abbreviate the notations to m, κ_a and κ_b.

2.2 Schrödinger-type operators

Following the proposal of [5] we consider the non-selfadjoint Schrödinger-type operator $H(V)$ on the Hilbert space $\mathfrak{H}$ defined by

$$\mathrm{dom}(H(V)) = \left\{ g \in W^{1,2} : \begin{array}{c} \frac{1}{m(x)} g'(x) \in W^{1,2}, \\ \frac{1}{2m(a)} g'(a) = -\kappa_a g(a), \\ \frac{1}{2m(b)} g'(b) = \kappa_b g(b) \end{array} \right\} \tag{2.1}$$

and

$$(H(V)g)(x) = (l(g))(x), \quad g \in \mathrm{dom}(H(V)), \quad g \in \mathrm{dom}(H(V)), \tag{2.2}$$

where

$$(l(g))(x) := -\frac{1}{2}\frac{d}{dx}\frac{1}{m(x)}\frac{d}{dx}g(x) + V(x)g(x), \tag{2.3}$$

cf. [7, 8]. Furthermore, it is always assumed that the occuring Schrödinger potentials are from L^∞. The operator H is maximal dissipative and completely non-selfadjoint, see [7]. The spectrum of $H(V)$ consists of isolated eigenvalues in the lower half-plane with the only accumulation point at infinity. Since the operator $H(V)$ is completely non-selfadjoint there do not exist real eigenvalues.

Moreover, $H(V)$ can be entirely characterized by its characteristic function $\Theta_{H(V)}(z)$, with $z \in \varrho(H(V)) \cap \varrho(H(V)^*)$, cf. [1]. In our case the definition of the characteristic function relies on the boundary operators $T(V)(z) : \mathfrak{H} \longrightarrow \mathbb{C}^2$, $z \in \varrho(H(V))$, and $T_*(V)(z) : \mathfrak{H} \longrightarrow \mathbb{C}^2$, $z \in \varrho(H(V)^*)$. Let us introduce the unclosed operator $\alpha : \mathfrak{H} \longrightarrow \mathbb{C}^2$,

$$\alpha f = \begin{pmatrix} \alpha_b f(b) \\ -\alpha_a f(a) \end{pmatrix}, \qquad f \in \mathrm{dom}(\alpha) = C[a,b], \tag{2.4}$$

where we have set

$$\kappa_a = q_a + \frac{i}{2}\alpha_a^2 \quad \text{and} \quad \kappa_b = q_b + \frac{i}{2}\alpha_b^2, \quad q_a, q_b \in \mathbb{R}, \quad \alpha_a, \alpha_b > 0. \tag{2.5}$$

The boundary operators are then defined by

$$T(V)(z)f := \alpha(H(V) - z)^{-1} f \tag{2.6}$$

and

$$T_*(V)(z)f := \alpha(H(V)^* - z)^{-1}f, \tag{2.7}$$

$f \in \mathfrak{H}$. The characteristic function $\Theta_{H(V)}(\cdot)$ of the maximal dissipative operator $H(V)$ is a two-by-two matrix-valued function which satisfies the relation

$$\Theta_{H(V)}(z)T(V)(z)f = T_*(V)(z)f, \quad z \in \varrho(H(V)) \cap \varrho(H(V)^*), \tag{2.8}$$

$f \in \mathfrak{H}$. It is a holomorphic function on $\varrho(H(V)) \cap \varrho(H(V)^*)$ and contractive on $\mathbb{C}_- \cup \mathbb{R}$, i.e. it satisfies

$$\|\Theta_{H(V)}(z)\| \le 1 \quad \text{for} \quad z \in \mathbb{C}_- \cup \mathbb{R}. \tag{2.9}$$

The characteristic function is given by

$$\Theta_H(z) = I_{\mathbb{C}^2} - i\alpha T(V)(\overline{z})^*, \tag{2.10}$$

cf. [8].

The operator $H(V)$ admits a description in terms of quadratic forms. To this end we introduce the sesquilinear form $\mathfrak{h}_0[\cdot,\cdot]$,

$$\mathfrak{h}_0[g,f] := \int_a^b dx \,\left\{\frac{1}{2m(x)}g'(x)\overline{f'(x)} + g(x)\overline{f(x)}\right\}, \tag{2.11}$$

$f, g \in \mathrm{dom}(\mathfrak{h}_0) = W^{1,2}$, see [7]. The form $\mathfrak{h}_0$ is symmetric and non-negative. Since $\mathfrak{h}_0$ is closed there is a self-adjoint operator H_0 such that the representation

$$\mathfrak{h}_0[g,f] = (H_0 g, f), \quad g \in \mathrm{dom}(H_0), \quad f \in \mathrm{dom}(\mathfrak{h}_0), \tag{2.12}$$

holds. This operator H_0 can be explicitly described as follows:

$$\mathrm{dom}(H_0) := \left\{ g \in W^{1,2} : \begin{array}{l} \frac{1}{2m(x)}g'(x) \in W^{1,2} \\ \frac{1}{2m(b)}g'(b) = \frac{1}{m(a)}g'(a) = 0 \end{array} \right\}, \tag{2.13}$$

and

$$(H_0 g)(x) = -\frac{d}{dx}\frac{1}{2m(x)}\frac{d}{dx}g(x) + g(x), \quad g \in \mathrm{dom}(H_0). \tag{2.14}$$

Obviously, the operator H_0 is non-negative. In order to obtain further properties of the operators $H(V)$ we introduce certain quadratic forms in terms of which $H(V)$ can be understood as a (form) perturbation of H_0. We start with the boundary form $\mathfrak{t}_{\partial\Omega}[\cdot,\cdot]$ defined by

$$\mathfrak{t}_{\partial\Omega}[g,f] := -\kappa_a g(a)\overline{f(a)} - \kappa_b g(b)\overline{f(b)}, \tag{2.15}$$

$f, g \in \mathrm{dom}(\mathfrak{t}_{\partial\Omega}) = W^{1,2}$. Next we define the potential form $\mathfrak{t}_V[\cdot,\cdot]$,

$$\mathfrak{t}_V[g,f] := \int_a^b dx \; V(x)g(x)\overline{f(x)}, \tag{2.16}$$

$f, g \in \mathrm{dom}(\mathfrak{t}_V) = W^{1,2}$ and the form sum

$$\mathfrak{t}_{\partial\Omega,V} := \mathfrak{t}_{\partial\Omega} + \mathfrak{t}_V. \tag{2.17}$$

As usual, we will denote the corresponding quadratic forms by the same symbols, but only one argument occuring. In the following we will supply relative form estimates for $\mathfrak{t}_{\partial\Omega}$, $\mathfrak{t}_V$ and $\mathfrak{t}_{\partial\Omega,V}$ with respect to $\mathfrak{h}_0$. Let

$$\mathfrak{g}_1 := \sup_{0 \neq \psi \in W^{1,2}} \frac{\|\psi\|_{L^\infty}}{\|\psi\|_{W^{1,2}}^{1/2}\,\|\psi\|_{\mathfrak{H}}^{1/2}} = \sup_{0 \neq \psi \in W^{1,2}} \frac{\|\psi\|_{C[a,b]}}{\|\psi\|_{W^{1,2}}^{1/2}\,\|\psi\|_{\mathfrak{H}}^{1/2}} \tag{2.18}$$

be the Gagliardo-Nirenberg constant and

$$\tilde{m} := \max\{1, \|m\|\}. \tag{2.19}$$

Applying the Gagliardo-Nirenberg inequality we can estimate the form $\mathfrak{t}_{\partial\Omega}$ as follows:

$$\begin{aligned}
|\mathfrak{t}_{\partial\Omega}[g,f]| &\leq (|\kappa_a| + |\kappa_b|) \cdot \|f\|_{C[a,b]} \cdot \|g\|_{C[a,b]} \qquad (2.20)\\
&\leq \Big((|\kappa_a| + |\kappa_b|)\mathfrak{g}_1^2 \|f\|_{W^{1,2}}\,\|f\|_{\mathfrak{H}}\Big)^{1/2} \Big((|\kappa_a| + |\kappa_b|)\mathfrak{g}_1^2 \|g\|_{W^{1,2}}\,\|g\|_{\mathfrak{H}}\Big)^{1/2}\\
&\leq \Big((|\kappa_a| + |\kappa_b|)\mathfrak{g}_1^2 \tilde{m}^{1/2} \mathfrak{h}_0[f]^{1/2} \|f\|_{\mathfrak{H}}\Big)^{1/2} \Big((|\kappa_a| + |\kappa_b|)\mathfrak{g}_1^2 \tilde{m}^{1/2} \mathfrak{h}_0[g]^{1/2} \|g\|_{\mathfrak{H}}\Big)^{1/2}\\
&\leq \left(\delta\mathfrak{h}_0[g] + \frac{(|\kappa_a| + |\kappa_b|)^2 \mathfrak{g}_1^4 \tilde{m}}{4\delta}\|g\|^2\right)^{1/2} \left(\delta\mathfrak{h}_0[f] + \frac{(|\kappa_a| + |\kappa_b|)^2 \mathfrak{g}_1^4 \tilde{m}}{4\delta}\|f\|^2\right)^{1/2}.
\end{aligned}$$

Setting

$$c := \frac{\big(|\kappa_a| + |\kappa_b|\big)^2 \mathfrak{g}_1^4 \tilde{m}}{4} \tag{2.21}$$

and summing up with the obvious inequality

$$\mathfrak{t}_V[g,f]| \leq \|V\|_{L^\infty} \cdot \|f\|_{\mathfrak{H}} \cdot \|g\|_{\mathfrak{H}}$$

one gets the following estimate for the form $\mathfrak{t}_{\partial\Omega,V}$:

$$\begin{aligned}
&|\mathfrak{t}_{\partial\Omega,V}[g,f]| \qquad (2.22)\\
&\quad\leq \left(\delta\mathfrak{h}_0[g] + \frac{c}{\delta}\|g\|_{\mathfrak{H}}^2\right)^{1/2} \left(\delta\mathfrak{h}_0[f] + \frac{c}{\delta}\|f\|_{\mathfrak{H}}^2\right)^{1/2} + \|V\|_{L^\infty}\|f\|_{\mathfrak{H}}\|g\|_{\mathfrak{H}}\\
&\quad\leq \left(\delta\mathfrak{h}_0[g] + \big(\frac{c}{\delta} + \|V\|_{L^\infty}\big)\|g\|_{\mathfrak{H}}^2\right)^{1/2} \left(\delta\mathfrak{h}_0[f] + \big(\frac{c}{\delta} + \|V\|_{L^\infty}\big)\|f\|_{\mathfrak{H}}^2\right)^{1/2}
\end{aligned}$$

$f, g \in \mathrm{dom}(\mathfrak{t}_{\partial\Omega,V})$. By the last inequality it turns out that the quadratic form $\mathfrak{t}_{\partial\Omega,V}$ is infinitesimally small with respect to $\mathfrak{h}_0$. Hence, the quadratic form corresponding to the sesquilinear form $\mathfrak{h}_V[g,f]$ is given by

$$\mathfrak{h}_V[g,f] := \mathfrak{h}_0[g,f] + \mathfrak{t}_{\partial\Omega,V}[g,f] - (g,f) \;=\; \mathfrak{h}_0[g,f] + \mathfrak{t}_{\partial\Omega,V-1}[g,f], \tag{2.23}$$

$f, g \in \mathrm{dom}(\mathfrak{h}_V) = W^{1,2}$, is closed and sectorial. Consequently, there is a (unique) maximal sectorial operator $H(V)$ such that the representation $\mathfrak{h}_V[g,f] = (H(V)g, f)$

is valid for $g \in \mathrm{dom}(H(V))$ and $f \in \mathrm{dom}(\mathfrak{h}_V)$. The so defined operator $H(V)$ coincides with that one given by (2.1), (2.2) and (2.3).

Next we intend for $\mu \geq 0$ and $V \in L^\infty$ to define an operator $B_\mu(V)$ which will allow us a certain factorization of the resolvent of $H(V)$. For this, let us introduce the sesquilinear form

$$\mathfrak{b}_\mu(V)[f,g] := \mathfrak{t}_{\partial\Omega,V^{-1}}[(H_0+\mu)^{-1/2}f,(H_0+\mu)^{-1/2}g], \quad \mu \geq 0, \tag{2.24}$$

$f,g \in \mathrm{dom}(\mathfrak{b}_\mu(V)) = \mathfrak{H}$. The form $\mathfrak{b}_\mu(V)$ defines a bounded operator $B_\mu(V)$ on $\mathfrak{H}$. For all what follows the norm of the operator $B_\mu(V)$ is of fundamental interest:

Lemma 2.2 *Assume $V \in L^\infty$ and $\delta \in]0,1[$. If $\mu \geq 0$, then*

$$\|B_\mu(V)\|_{\mathcal{B}(\mathfrak{H})} \leq \delta + \left(\frac{c}{\delta} + 1 + \|V\|_{L^\infty}\right) \cdot \frac{1}{1+\mu}. \tag{2.25}$$

In particular, if

$$\mu \geq 4c + 2 + 2\|V\|_{L^\infty}, \tag{2.26}$$

then $\|B_\mu(V)\|_{\mathcal{B}(\mathfrak{H})} < 1$ and

$$\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})} \leq \frac{2(\mu+1)}{1+\mu-4c-2-2\|V\|_{L^\infty}}. \tag{2.27}$$

Proof. (2.25) follows from (2.22). Setting $\delta = 1/2$ we get from (2.25) and (2.26) that $\|B_\mu(V)\|_{\mathcal{B}(\mathfrak{H})} < 1$. The last assertion follows from (2.25), (2.26) and the representation of the resolvent by Neumann's series. □

Lemma 2.3 *Assume $V \in L^\infty$. If $\mu \geq 4c + 2 + 2\|V\|_{L^\infty}$, then the representation*

$$(H(V)+\mu)^{-1} = (H_0+\mu)^{-1/2}(I+B_\mu(V))^{-1}(H_0+\mu)^{-1/2} \tag{2.28}$$

holds.

Proof. One has for any $f,g \in W^{1,2}$

$$\mathfrak{h}_V[g,f] + \mu(g,f) = \left(\sqrt{H_0+\mu}g, \sqrt{H_0+\mu}f\right) + \mathfrak{t}_{\partial\Omega,V^{-1}}[g,f] \tag{2.29}$$

which yields

$$\mathfrak{h}_V[g,f] + \mu(g,f) = \left((I+B_\mu(V))\sqrt{H_0+\mu}g, \sqrt{H_0+\lambda}f\right) \tag{2.30}$$

for $\mu > 0$. From the previous lemma we get $\|B_\mu(V)\| < 1$. Hence, the inverse operator of $I + B_\mu(V)$ exists and is bounded. Therefore, the definition

$$R_\mu(V) := (H_0+\mu)^{-1/2}(I+B_\mu(V))^{-1}(H_0+\mu)^{-1/2} \tag{2.31}$$

makes sense. Since $R_\mu(V)g \in W^{1,2}([a,b])$ we find

$$\begin{aligned}
&\mathfrak{h}_V[R_\mu(V)g, f] + \lambda(R_\mu(V)g, f) \\
&\quad = \left(\sqrt{H_0+\mu}R_\mu(V)g, \sqrt{H_0+\mu}f\right) + \mathfrak{t}_{\partial\Omega,V^{-1}}[R_\mu(V)g, f],
\end{aligned} \tag{2.32}$$

$g \in \mathfrak{H}$ and $f \in W^{1,2}$. Consequently, we obtain

$$\begin{aligned}
&\mathfrak{h}_V[R_\mu(V)g, f] + \mu(R_\mu(V)g, f) \\
&\quad = \left((I + B_\mu(V))^{-1}(H_0+\mu)^{-1/2}g, (H_0+\mu)^{1/2}f\right) \\
&\qquad + \left(B_\mu(V)(I + B_\mu(V))^{-1}(H_0+\mu)^{-1/2}g, (H_0+\mu)^{1/2}f\right),
\end{aligned} \tag{2.33}$$

which shows that

$$\mathfrak{h}_V[R_\mu(V)g, f] + \mu(R_\mu(V)g, f) = (g, f), \tag{2.34}$$

$g \in \mathfrak{H}$, $f \in W^{1,2}$. However, the relation (2.34) implies that $R_\mu(V)g \in \mathrm{dom}(H(V))$ and $(H(V)+\mu)R_\mu(V)g = g$ for any $g \in \mathfrak{H}$. Similarly, one proves that $R_\mu(V)(H(V)+\mu)g = g$ for any $g \in \mathrm{dom}(H(V))$. Both relations imply that $(H(V)+\mu)^{-1} = R_\mu(V)$. □

2.3 Dilations

Since $H(V)$ is a maximal dissipative operator there is a larger Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ and a self-adjoint operator $K(V)$ on $\mathfrak{K}$ such that

$$P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-z)^{-1}|\mathfrak{H} = (H(V)-z)^{-1}, \quad \Im\mathrm{m}(z) > 0, \tag{2.35}$$

see [1]. The operator $K(V)$ is called a self-adjoint dilation of the maximal dissipative operator $H(V)$. Obviously, from the condition (2.35) one gets

$$P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-z)^{-1}|\mathfrak{H} = (H(V)^*-z)^{-1}, \quad \Im\mathrm{m}(z) < 0. \tag{2.36}$$

If the condition

$$\bigvee_{z\in\mathbb{C}\setminus\mathbb{R}} (K(V)-z)^{-1}\mathfrak{H} = \mathfrak{K} \tag{2.37}$$

is satisfied, then $K(V)$ is called a minimal self-adjoint dilation of $H(V)$. Minimal self-adjoint dilations of maximal dissipative operators are determined up to a certain isomorphism, in particular, all minimal self-adjoint dilations are unitarily equivalent.

In the present case the minimal self-adjoint dilation of the maximal dissipative operator $H(V)$ can be constructed in an explicit manner, cf. [8]. The dilation space $\mathfrak{K}$ is given by

$$\mathfrak{K} = \mathcal{D}_- \oplus \mathfrak{H} \oplus \mathcal{D}_+, \tag{2.38}$$

where $\mathcal{D}_\pm := L^2(\mathbb{R}_\pm, \mathbb{C}^2)$. Introducing the domain $\hat{\Omega} := \mathbb{R}_- \times [a,b] \times \mathbb{R}_+$, we may write $\mathfrak{K} = L^2(\hat{\Omega}, dx)$. In accordance with (2.38) one writes

$$\vec{f} := f_- \oplus f \oplus f_+ \in \mathfrak{K}. \tag{2.39}$$

The resolvent of K is given by

$$\begin{aligned}
&(K(V)-z)^{-1}(f_- \oplus f \oplus f_+) \qquad (2.40)\\
&= i\int_{-\infty}^{x} dy\, e^{i(x-y)z} f_-(y) \;\oplus\; (H(V)-z)^{-1}f + iT_*(V)(\overline{z})^* \int_{-\infty}^{0} dy\, e^{-iyz} f_-(y)\\
&\oplus\, i\int_{0}^{x} dy\, e^{i(x-y)z} f_+(y) + ie^{izx}T(V)(z)f + i\Theta_{H(V)}(\overline{z})^* \int_{-\infty}^{0} dy\, e^{i(x-y)z} f_-(y)
\end{aligned}$$

for $\Im\mathrm{m}(z) > 0$ and

$$\begin{aligned}
&(K(V)-z)^{-1}(f_- \oplus f \oplus f_+) \qquad (2.41)\\
&= -i\int_{x}^{0} dy\, e^{i(x-y)z} f_-(y) - ie^{izx}T_*(V)(z)f - i\Theta_{H(V)}(z)\int_{0}^{\infty} dy\, e^{i(x-y)z} f_+(y)\\
&\oplus\, (H(V)^* - z)^{-1}f - iT(V)(\overline{z})^* \int_{0}^{\infty} dy\, e^{-iyz} f_+(y) \;\oplus\; -i\int_{x}^{\infty} dy\, e^{i(x-y)z} f_+(y)
\end{aligned}$$

for $\Im\mathrm{m}(z) < 0$. The self-adjoint operator K is absolutely continuous and its spectrum coincides with the real axis, i.e. $\sigma(K) = \mathbb{R}$. The multiplicity of its spectrum is two. For more details the reader is referred to [8].

2.4 Eigenfunction expansions

The generalized eigenfunctions $\vec{\psi}^-(V)(\cdot,\lambda,\tau)$, $\lambda \in \mathbb{R}$, $\tau = a, b$, of $K(V)$ are given by

$$\begin{aligned}
\vec{\psi}^-(V)(x,\lambda,\tau) &:= \psi_-^-(V)(x,\lambda,\tau) \oplus \psi^-(V)(x,\lambda,\tau) \oplus \psi_+^-(V)(x,\lambda,\tau) \qquad (2.42)\\
&= \frac{1}{\sqrt{2\pi}} e^{ix\lambda} e_\tau \oplus \frac{1}{\sqrt{2\pi}}\big((T_*(V)(\lambda))^* e_\tau\big)(x) \oplus \frac{1}{\sqrt{2\pi}} e^{ix\lambda}\Theta_{H(V)}(\lambda)^* e_\tau
\end{aligned}$$

where

$$e_b := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_a := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (2.43)$$

The eigenfunctions are orthogonal, i.e.

$$\left(\vec{\psi}^-(V)(\cdot,\lambda,\tau), \vec{\psi}^-(V)(\cdot,\lambda',\tau')\right)_{L^2(\hat{\Omega})} = \delta(\lambda-\lambda')\delta_{\tau\tau'}, \qquad (2.44)$$

$\lambda, \lambda' \in \mathbb{R}$, $\tau, \tau' = a, b$ in the sense of distributions, cf. [8], and their linear span (modulo the scalar, continuous, compactly supported functions) is dense in $\mathfrak{K}$. We note that the generalized eigenfunctions $\vec{\psi}^-(V)(\cdot,\lambda,\tau)$ are usually called the incoming eigenfunctions. Using the incoming eigenfunctions one defines a transformation $\Phi_-(V) : \mathfrak{K} \longrightarrow \hat{\mathfrak{K}} = L^2(\mathbb{R}, \mathbb{C}^2)$

$$(\Phi_-(V)\vec{g}))(\lambda) =: \hat{g}(\lambda) = \begin{pmatrix} \hat{g}^b(\lambda) \\ \hat{g}^a(\lambda) \end{pmatrix}, \qquad (2.45)$$

where

$$\hat{g}^\tau(\lambda) := \int_{\hat{\Omega}} dx \, \left(\vec{g}(x), \vec{\psi^-}(V)(x,\lambda,\tau)\right), \quad \tau = a, b. \tag{2.46}$$

$\Phi_-(V)$ is unitary and called the incoming Fourier transformation. The inverse incoming Fourier transformation $\Phi_-(V)^{-1}$ is given by

$$(\Phi_-(V)^{-1}\hat{g})(x) = \int_{\mathbb{R}} d\lambda \sum_{\tau=a,b} \vec{\psi^-}(V)(x,\lambda,\tau)\hat{g}^\tau(\lambda), \quad \hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2). \tag{2.47}$$

We note that

$$\Phi_-(V)K(V)\Phi_-(V)^{-1} = M, \tag{2.48}$$

where M is the multiplication operator by the independent variable λ on $\hat{\mathfrak{K}}$, i.e.

$$\begin{aligned} \mathrm{dom}(M) &:= \left\{\hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) : \lambda\hat{g}(\lambda) \in L^2(\mathbb{R}, \mathbb{C}^2)\right\}, \\ (M\hat{g})(\lambda) &:= \lambda\hat{g}(\lambda), \quad \hat{g} \in \mathrm{dom}(M). \end{aligned} \tag{2.49}$$

The representation (2.49) induced by $\Phi_-(V)$ is called the incoming spectral representation of K.

Finally, we note that each bounded self-adjoint operator G on $\mathfrak{K}$, which commutes with K, corresponds to a measurable family $\{G(\lambda)\}_{\lambda\in\mathbb{R}}$ of two-by-two matrices which is uniformly bounded, i.e. $G(\cdot) \in L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{C}^2))$, such that the multiplication operator $\hat{G}$ on $L^2(\mathbb{R}, \mathbb{C}^2)$ defined by

$$\begin{aligned} \mathrm{dom}(\hat{G}) &:= \left\{\hat{g} \in L^2(\mathbb{R}, \mathbb{C}^2) : G(\lambda)\hat{g}(\lambda) \in L^2(\mathbb{R}, \mathbb{C}^2)\right\}, \\ (\hat{G}\hat{g})(\lambda) &:= G(\lambda)\hat{g}(\lambda), \quad \hat{g} \in \mathrm{dom}(\hat{G}) \end{aligned} \tag{2.50}$$

is unitarily equivalent to G, i.e.

$$\Phi_-(V)G\Phi_-(V)^{-1} = \hat{G}. \tag{2.51}$$

The representation (2.50) is called in incoming spectral representation of G.

2.5 Carrier densities

In the following we call an operator $\varrho : \mathfrak{K} \longrightarrow \mathfrak{K}$ a density matrix if ϱ is a bounded, non-negative, self-adjoint operator. The operator ϱ is called a steady state, if ϱ commutes with $K(V)$, see [9]. Thus any steady state ϱ is unitarily equivalent to a multiplication operator $\hat{\varrho}$ on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$ induced by a measurable function $\varrho(\cdot) \in L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{C}^2))$. In the following we assume that the function $\varrho(\cdot)$ is fixed, i.e. the function $\varrho(\cdot)$ does not depend on the potential V. This leads to a steady state of the form

$$\varrho(V) = \Phi_-(V)^{-1}\hat{\varrho}\Phi_-(V), \tag{2.52}$$

which depends on V.

In order to define the carrier density $u_{\hat{\varrho}}(V)(\cdot)$ one has (in accordance with [9]) to introduce the carrier density observable

$$D(V)(x,\lambda) := \tag{2.53}$$
$$\begin{pmatrix} |\psi^-(V)(x,\lambda,b)|^2 & \psi^-(V)(x,\lambda,a)\overline{\psi^-(V)(x,\lambda,b)} \\ \psi^-(V)(x,\lambda,b)\overline{\psi^-(V)(x,\lambda,a)} & |\psi^-(V)(x,\lambda,a)|^2 \end{pmatrix}.$$

With respect to the carrier density observable $D(V)(x,\lambda)$ one defines the carrier density $u_{\hat{\varrho}}(V)(x,\lambda)$ at $x \in [a,b]$ and at energy $\lambda \in \mathbb{R}$ by

$$u_{\hat{\varrho}}(V)(x,\lambda) := \operatorname{tr}(\varrho(\lambda)D(V)(x,\lambda)) \geq 0. \tag{2.54}$$

The carrier density $u_{\varrho(\hat{V})}(\cdot)$ is given by

$$u_{\hat{\varrho}}(V)(x) = \int_{\mathbb{R}} d\lambda\, u_{\hat{\varrho}}(V)(x,\lambda). \tag{2.55}$$

If the function $\varrho(\cdot)$ satisfies the condition

$$C_{\hat{\varrho}} := \sup_{\lambda\in\mathbb{R}} \sqrt{\lambda^2+1}\|\varrho(\lambda)\|_{\mathcal{B}(\mathbb{C}^2)} < \infty, \tag{2.56}$$

then the definition (2.55) makes sense for a.e. $x \in [a,b]$. Moreover, in this case $u_{\hat{\varrho}}(V)(\cdot)$ is a positive and integrable function. Furthermore, $P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-i)^{-1} \in \mathfrak{L}_1(\mathfrak{K})$ and the estimate

$$\|u_{\hat{\varrho}}(V)\|_{L^1} = \operatorname{tr}(\varrho(V)P^{\mathfrak{K}}_{\mathfrak{H}}) \leq C_{\hat{\varrho}}\|(K(V)-i)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\|_{\mathfrak{L}_1(\mathfrak{K})} \tag{2.57}$$

is valid, cf. [9]. Let us introduce the operator

$$(M(h)\vec{f})(x) := 0 \oplus h(x)f(x) \oplus 0, \quad \vec{f} \in \operatorname{dom}(M(h)) = \mathfrak{K}, \tag{2.58}$$

for functions $h \in L^\infty$. If the condition (2.56) is satisfied, then

$$\int_a^b dx\, u_{\hat{\varrho}}(V)(x)h(x) = \operatorname{tr}(\varrho(V)M(h)) \tag{2.59}$$

for any $h \in L^\infty$, cf. [9].

2.6 Carrier density operator

By the previous considerations it seems to be useful to introduce the so-called carrier density operator $\mathcal{N}_{\hat{\varrho}}(\cdot) : L^\infty \longrightarrow L^1$ which is defined by

$$\mathcal{N}_{\hat{\varrho}}(V) := u_{\hat{\varrho}}(V), \quad V \in \operatorname{dom}(\mathcal{N}_{\hat{\varrho}}) = L^\infty, \tag{2.60}$$

where $u_{\hat{\varrho}}(V)$ is the carrier density defined by (2.55). The operator is, of course, nonlinear.

Proposition 2.4 *Suppose $V \in L^\infty$. If the density matrix ϱ satisfies the condition (2.56), then*

$$\begin{aligned}\|\mathcal{N}_{\hat\varrho}(V)\|_{L^1} \le C_{\hat\varrho} \cdot \Big[& 8 + 4\sqrt{2}\,(b-a)\sqrt{\|m\|}\cdot\sqrt{8c+5+4\|V\|_{L^\infty}} \\ & +8\mathfrak{g}_1\big(\alpha_a^2+\alpha_b^2\big)^{1/2}\|H_0^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H};W^{1,2})}(8c+5+4\|V\|_{L^\infty})^{1/4}\Big],\end{aligned} \tag{2.61}$$

where c is defined by (2.21).

Proof. In view of (2.57) it suffices to estimate $\|(K(V)-i)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\|_{\mathfrak{L}_1(\mathfrak{K})}$. Using (2.40), we obtain the equation

$$\big(K(V)-i\big)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\vec{f} = \big(0,(H(V)-i)^{-1}f, e^{-\bullet}\,iT(V)(i)f\big)$$

where $\vec{f} = f_- \oplus f \oplus f_+$. Thus, one can estimate

$$\|\big(K(V)-i\big)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\|_{\mathfrak{L}_1(\mathfrak{K})} \le \|\big(H(V)-i\big)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} + \|ie^{-\bullet}\,T(V)(i)\|_{\mathfrak{L}_1(\mathfrak{H};\mathcal{D}_+)}. \tag{2.62}$$

We estimate the first addend on the right hand side. Let μ be a sufficiently large positive number (to be specified later). We write

$$(H(V)-i)^{-1} = (H(V)+\mu)^{-1}\Big(1+(\mu+i)(H(V)-i)^{-1}\Big). \tag{2.63}$$

Since $H(V)$ is a maximal dissipative operator one has $\|(H(V)-i)^{-1}\|_{\mathcal{B}(\mathfrak{H})} \le 1$. Thus, (2.63) implies

$$\|(H(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \le (2+\mu)\|(H(V)+\mu)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})}. \tag{2.64}$$

Applying the factorization formula (2.28) one gets

$$\|(H(V)+\mu)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \le \|(H_0+\mu)^{-1/2}\|^2_{\mathfrak{L}_2(\mathfrak{H})}\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})}.$$

The first factor of the right hand side is calculated to $\sum_{l=0}^{\infty}\frac{1}{\zeta_l+1+\mu}$, where the numbers ζ_l are, of course, the eigenvalues of the operator H_0-1. Let $\hat H_0$ be the self-adjoint operator defined by (2.13) and (2.14) where m is specified to $m(x)\equiv 1$. Obviously, one has $\frac{1}{\|m\|}(\hat H_0 - 1) \le H_0 - 1$. The eigenvalues of $\hat H_0 - 1$ are given by $\frac{\pi^2 l^2}{2(b-a)^2}$, $l = 0,1,\ldots$. Thus the minimax principle implies $\frac{1}{\|m\|}\frac{\pi^2 l^2}{2(b-a)^2} \le \zeta_l$, $l=0,1,\ldots$. Hence we obtain

$$\|(H_0+\mu)^{-1/2}\|^2_{\mathfrak{L}_2(\mathfrak{H})} \le \sum_{l=0}^{\infty}\frac{1}{\frac{1}{\|m\|}\frac{\pi^2}{2(b-a)^2}l^2+1+\mu}.$$

For any $l \ge 1$ we have

$$\frac{1}{\frac{1}{\|m\|}\frac{4\pi^2}{(b-a)^2}l^2+1+\mu} \le \int_{l-1}^{l}\frac{ds}{\frac{1}{\|m\|}\frac{4\pi^2}{(b-a)^2}s^2+1+\mu}.$$

Thus, we get

$$\|(H_0+\mu)^{-1/2}\|^2_{\mathfrak{L}_2(\mathfrak{H})} \le \frac{1}{1+\mu} + \sum_{l=1}^{\infty}\int_{l-1}^{l} \frac{ds}{\frac{1}{\|m\|}\frac{\pi^2}{2(b-a)^2}s^2+1+\mu},$$

which yields

$$\|(H_0+\mu)^{-1/2}\|^2_{\mathfrak{L}_2(\mathfrak{H})} \le \frac{1}{1+\mu} + \sqrt{\|m\|}\,\frac{b-a}{\sqrt{2}}\cdot\frac{1}{\sqrt{1+\mu}}.$$

This altogether gives

$$\|(H(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \le \left[\frac{2+\mu}{1+\mu} + \sqrt{\|m\|}\,\frac{b-a}{\sqrt{2}}\cdot\frac{2+\mu}{\sqrt{1+\mu}}\right]\cdot\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})}$$

and, consequently

$$\|(H(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \le \left[2+\sqrt{2}\sqrt{\|m\|}\,(b-a)\cdot\sqrt{1+\mu}\right]\cdot\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})}.$$

Setting $\mu = 2(4c+2+2\|V\|_{L^\infty})$ and taking into account (2.27) one gets

$$\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})} \le 4,$$

what finally implies

$$\|\big(H(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \le 8+4\sqrt{2}(b-a)\sqrt{\|m\|}\cdot\sqrt{8c+5+4\|V\|_{L^\infty}}. \tag{2.65}$$

Now we are going to estimate the second term of the right hand side of (2.62). Since $\|e^{-\bullet}\otimes I_{\mathbb{C}^2}\|_{\mathfrak{L}_1(\mathbb{C}^2;\mathcal{D}_+)} = \sqrt{2}$, we get using equation (2.6)

$$\|ie^{-\bullet}T(V)(i)\|_{\mathfrak{L}_1(\mathfrak{H};\mathcal{D}_+)} \le \sqrt{2}(\alpha_a^2+\alpha_b^2)^{1/2}\|(H(V)-i)^{-1}\|_{\mathcal{B}(\mathfrak{H};C[a,b])}. \tag{2.66}$$

It remains to estimate $\|(H(V)-i)^{-1}\|_{\mathcal{B}(\mathfrak{H};C[a,b])}$. Taking into account (2.63) one obtains (analogous to (2.64))

$$\|(H(V)-i)^{-1}\|_{\mathcal{B}(\mathfrak{H};C[a,b])} \le (2+\mu)\|(H(V)+\mu)^{-1}\|_{\mathcal{B}(\mathfrak{H};C[a,b])}. \tag{2.67}$$

As in the previous part of the proof we put $\mu = 2(4c+2+2\|V\|_{L^\infty})$ and afterwards substitute $(H(V)+\mu)^{-1}$ via the factorization formula (2.28). This leads to the following estimate:

$$\begin{aligned}\|(H(V)-i)^{-1}\|_{\mathcal{B}(\mathfrak{H};C[a,b])} &\le (\mu+2)\cdot \\ &\cdot\|(H_0+\mu)^{-1/2}\|_{\mathcal{B}(\mathfrak{H};C[a,b])}\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})}\|(H_0+\mu)^{-1/2}\|_{\mathcal{B}(\mathfrak{H})}.\end{aligned} \tag{2.68}$$

By $\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})} \le 4$ and $\|(H_0+\mu)^{-1/2}\|_{\mathcal{B}(\mathfrak{H})} \le \frac{1}{\sqrt{1+\mu}}$ one gets

$$\|(H(V)-i)^{-1}\|_{\mathcal{B}(\mathfrak{H};C[a,b])} \le 4\,\frac{2+\mu}{\sqrt{1+\mu}}\cdot\|(H_0+\mu)^{-1/2}\|_{\mathcal{B}(\mathfrak{H};C[a,b])}. \tag{2.69}$$

We estimate the last factor in this last inequality by the Gagliardo-Nirenberg inequality. For any $\psi \in \mathfrak{H}$ one has

$$\|(H_0+\mu)^{-1/2}\psi\|_{C[a,b]} \le \mathfrak{g}_1 \cdot \|(H_0+\mu)^{-1/2}\psi\|^{1/2}_{W^{1,2}} \cdot \|(H_0+\mu)^{-1/2}\psi\|^{1/2}_{\mathfrak{H}}$$
$$\le \mathfrak{g}_1 \cdot \|H_0^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}\|(H_0+\mu)^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H})} \cdot \|\psi\|_{\mathfrak{H}} \le \mathfrak{g}_1 \frac{\|H_0^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}}{(1+\mu)^{1/4}} \cdot \|\psi\|_{\mathfrak{H}}.$$

Clearly, this yields

$$\|(H_0+\mu)^{-1/2}\|_{\mathcal{B}(\mathfrak{H},C[a,b])} \le \mathfrak{g}_1 \frac{\|H_0^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}}{(1+\mu)^{1/4}}.$$

Together with (2.69) this gives

$$\|(H(V)-i)^{-1}\|_{\mathfrak{B}(\mathfrak{H},C[a,b])} \le 8\mathfrak{g}_1\|H_0^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}(1+\mu)^{1/4}.$$

Inserting the chosen $\mu = 2(4c+2+2\|V\|_{L^\infty})$, we obtain for the second addend in the right hand side of (2.62):

$$\|ie^{-\bullet}T(V)(i)\|_{\mathfrak{L}_1(\mathfrak{H},\mathcal{D}_+)}$$
$$\le 8\sqrt{2}\mathfrak{g}_1(\alpha_a^2+\alpha_b^2)^{1/2}\|H_0^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}(8c+5+4\|V\|_{L^\infty})^{1/4}.$$

Fitting together (2.57), (2.62), (2.65) and this last estimate we prove (2.61). □

Remark 2.5 One can even prove that the carrier density operator takes its values not only in L^1 but in (real) L^2. Additionally, one can prove estimates similar to (2.61) but more involved. We do not need these things in this paper, however, this fact becomes essential in the moment when one wants to include recombination effects of electrons and holes in the model.

3 Convergence

3.1 Schrödinger-type operators

First we want to prove the continuity properties of the Schrödinger operator $H(V)$, boundary operator $T(V)$ and characteristic function $\Theta_{H(V)}$:

Proposition 3.1 *Assume $V \in L^\infty$, $V_n \in L^\infty$, $n \in \mathbb{N}$. Let $V_n \xrightarrow{L^\infty} V$ as $n \to \infty$.*

(i) If $\mathcal{C} \subset \varrho(H(V))$ is a compact subset, then for sufficiently large $n \in \mathbb{N}$, one has $\mathcal{C} \subset \varrho(H(V_n))$,

$$\lim_{n\to\infty}\sup_{z\in\mathcal{C}}\left\|(H(V_n)-z)^{-1}-(H(V)-z)^{-1}\right\|_{\mathfrak{L}_1(\mathfrak{H})} = 0 \tag{3.1}$$

and

$$\lim_{n\to\infty}\sup_{z\in\mathcal{C}}\|T(V_n)(z)-T(V)(z)\|_{\mathfrak{L}_1(\mathfrak{H},\mathbb{C}^2)} = 0. \tag{3.2}$$

(ii) If $\mathcal{C} \subset \varrho(H(V)^*)$ is a compact subset, then for sufficiently large $n \in \mathbb{N}$ one has $\mathcal{C} \subset \varrho(H(V_n)^*)$,

$$\lim_{n\to\infty} \sup_{z\in\mathcal{C}} \left\|(H(V_n)^* - z)^{-1} - (H(V)^* - z)^{-1}\right\|_{\mathfrak{L}_1(\mathfrak{H})} = 0 \tag{3.3}$$

and

$$\lim_{n\to\infty} \sup_{z\in\mathcal{C}} \|T_*(V_n)(z) - T_*(V)(z)\|_{\mathfrak{L}_1(\mathfrak{H},\mathbb{C}^2)} = 0. \tag{3.4}$$

(iii) If $\mathcal{C} \subset \overline{\mathbb{C}_-}$ is a compact set, then

$$\lim_{n\to\infty} \sup_{z\in\mathcal{C}} \|\Theta_{H(V_n)}(z) - \Theta_{H(V)}(z)\|_{\mathfrak{L}_1(\mathbb{C}^2)} = 0. \tag{3.5}$$

Similarly, if $\mathcal{C} \subset \overline{\mathbb{C}_+}$ is a compact set, then

$$\lim_{n\to\infty} \sup_{z\in\mathcal{C}} \|\Theta_{H(V_n)^*}(z) - \Theta_{H(V)^*}(z)\|_{\mathfrak{L}_1(\mathbb{C}^2)} = 0. \tag{3.6}$$

Proof. (i) By Theorem IV 1.16 in [10] we get that $H(V_n) - z$ is boundedly invertible if

$$\|V - V_n\|_{L^\infty} < \left(\|(H(V) - z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}\right)^{-1} \tag{3.7}$$

(what is true for $n \in \mathbb{N}$ large enough). Furthermore, we get in this case

$$\|(H(V_n) - z)^{-1}\|_{\mathcal{B}(\mathfrak{H})} \le \frac{\|(H(V) - z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}}{1 - \|V_n - V\|_{L^\infty}\|(H(V) - z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}}. \tag{3.8}$$

We write for positive, sufficiently large μ

$$\begin{aligned}(H(V_n) - z)^{-1} - (H(V) - z)^{-1} &= (H(V) - z)^{-1}(V - V_n)(H(V_n) - z)^{-1}\\ &=(H(V) + \mu)^{-1}[1 + (\mu + z)(H(V) - z)^{-1}](V - V_n)(H(V_n) - z)^{-1}\\ &=(H_0 + \mu)^{-1/2}(1 + B_\mu(V))^{-1}(H_0 + \mu)^{-1/2}\cdot\\ &\quad\cdot [1 + (\mu + z)(H(V) - z)^{-1}](V - V_n)(H(V_n) - z)^{-1}.\end{aligned}$$

This gives

$$\begin{aligned}\|(H(V_n) - z)^{-1} - (H(V) - z)^{-1}\|_{\mathcal{L}_1(\mathfrak{H})} &\le \|H_0^{-1/2}\|^2_{\mathcal{L}_2(\mathfrak{H})}\|(1 + B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})}\cdot\\ &\cdot \left(1 + |\mu + z| \cdot \|(H(V) - z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}\right) \cdot \|V - V_n\|_{L^\infty}\|(H(V_n) - z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}.\end{aligned} \tag{3.9}$$

Hence we get by (3.8) and (3.9)

$$\lim_{n\to\infty} \|(H(V_n) - z)^{-1} - (H(V) - z)^{-1}\|_{\mathcal{L}_1(\mathfrak{H})} = 0,$$

for every $z \in \varrho(H(V))$. Since $\mathcal{C}$ is compact, this implies (3.1).

As in the proof of Proposition 2.4 (cf. (2.66)) one sees that

$$\|T(V_n)(z) - T(V)(z)\|_{\mathfrak{L}_1(\mathfrak{H};\mathbb{C}^2)} \le 2(\alpha_a^2 + \alpha_b^2)^{1/2}\cdot\left\|(H(V_n) - z)^{-1} - (H(V) - z)^{-1}\right\|_{\mathcal{B}(\mathfrak{H};C[a,b])}.$$

The latter factor is estimated, completely analogous to the preceding considerations, by

$$\|H_0^{-1/2}\|_{\mathcal{B}(\mathfrak{H};C[a,b])} \cdot \|H_0^{-1/2}\|_{\mathcal{B}(\mathfrak{H})}\|(1+B_\mu(V))^{-1}\|_{\mathcal{B}(\mathfrak{H})}\cdot$$
$$\cdot\left(1+|\mu+z|\cdot\|(H(V)-z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}\right)\cdot\|V-V_n\|_{L^\infty}\|(H(V_n)-z)^{-1}\|_{\mathcal{B}(\mathfrak{H})}.$$

This proves (3.4).
(ii) We note that (3.3) is a consequence of (3.1). The assertion (3.4) can be proven similarly to (3.2).

(iii) Clearly, it suffices to prove the convergence properties only with respect to the strong operator topology of $\mathcal{B}(\mathbb{C}^2)$. Secondly, for every $z \in \overline{\mathbb{C}_-} \cap \varrho(H(V))$ the mapping $\mathfrak{H} \ni f \mapsto T(V)(z)f \in \mathbb{C}^2$ is a surjection. If $U, V \in L^\infty$ then one gets from (2.8)

$$\begin{aligned}&\left[\Theta_{H(U)}(z) - \Theta_{H(V)}(z)\right]T(V)(z)f\\ &\qquad = (T_*(U)(z) - T_*(V)(z))f + \Theta_{H(U)}(z)\left[T(V)(z)f - T(U)(z)f\right].\end{aligned}$$

Taking into account the contractivity of $\Theta_{H(U)}(z)$ in case of $z \in \overline{\mathbb{C}_-}$ (see (2.9)), this leads to the estimate

$$\begin{aligned}&\|\left[\Theta_{H(U)}(z) - \Theta_{H(V)}(z)\right]T(V)(z)f\|_{\mathbb{C}^2}\\ &\qquad \le \|(T_*(U)(z) - T_*(V)(z))f\|_{\mathbb{C}^2} + \|T(U)(z)f - T(V)(z)f\|_{\mathbb{C}^2}.\end{aligned}$$

Thus, $V_n \xrightarrow{L^\infty} V$ together with (3.2) and (3.4) implies

$$\lim_{n\mapsto\infty}\|\Theta_{H(V_n)}(z) - \Theta_{H(V)}(z)\|_{\mathcal{B}(\mathbb{C}^2)} = 0 \tag{3.10}$$

for each single $z \in \overline{\mathbb{C}_-} \cap \varrho(H(V))$. Since $\Theta_{H(V)}(z)$ and $\Theta_{H(V_n)}(z)$ admit holomorphic extensions to whole $\overline{\mathbb{C}_-}$ equation (3.10) extents to all $z \in \overline{\mathbb{C}_-}$. Uniformity over a compact subset of $z's$ is derived by the continuity of the map $\overline{\mathbb{C}} \ni z \mapsto \Theta_{H(V)}(z)$ and a simple compactness argument. Since $\Theta_{H(V)}(\overline{z})^* = \Theta_{H(V)^*}(z)$ for $z \in \varrho(H(V))$ we obtain (3.6) from (3.5). □

3.2 Dilations

Proposition 3.2 *Assume $V \in L^\infty$, $V_n \in L^\infty$, $n = 1, 2, \ldots$. If $V_n \xrightarrow{L^\infty} V$ as $n \to \infty$, then*

$$\lim_{n\to\infty}\sup_{z\in\mathcal{C}}\|(K(V_n)-z)^{-1} - (K(V)-z)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})} = 0 \tag{3.11}$$

for any compact set $\mathcal{C} \subseteq \mathbb{C}\setminus\mathbb{R}$.

Proof. To prove (3.11) it is enough to verify it for a single point $z \in \mathbb{C}_+$. By (2.40) we get that

$$\begin{aligned}&(K(V_n)-z)^{-1}\vec{f} - (K(V)-z)^{-1}\vec{f} = \\ &\quad 0 \oplus \left((H(V_n)-z)^{-1} - (H(V)-z)^{-1}\right)f + i\left(T_*(V_n)(\overline{z})^* - (T_*(V)(\overline{z})^*\right)\xi\\ &\quad \oplus ie^{ixz}\left(T(V_n)(z) - T(V)(z)\right)f + ie^{ixz}\left(\Theta_{H(V_n)}(\overline{z})^* - \Theta_{H(V)}(\overline{z})^*\right)\xi\end{aligned} \tag{3.12}$$

where

$$\xi := \int_{-\infty}^{0} dy\, e^{-iyz} f_-(y) \in \mathbb{C}^2 ee \tag{3.13}$$

Since

$$\|\xi\|_{\mathbb{C}^2} \le \frac{1}{\sqrt{2\Im\mathrm{m}(z)}} \|f_-\|_{L^2(\mathbb{R}_-,\mathbb{C}^2)} \tag{3.14}$$

we obtain

$$\begin{aligned}
&\|(K(V_n) - z)^{-1} - (K(V) - z)^{-1}\|_{\mathfrak{L}_1(\mathfrak{K})} \\
&\le \ \|(H(V_n) - z)^{-1} - (H(V) - z)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \\
&\quad + \frac{1}{\sqrt{2\Im\mathrm{m}(z)}} \|T_*(V_n)(\overline{z})^* - (T_*(V)(\overline{z})^*\|_{\mathfrak{L}_1(\mathbb{C}^2;\mathfrak{H})} \\
&\quad + \frac{1}{\sqrt{2\Im\mathrm{m}(z)}} \|T(V_n)(z) - T(V)(z)\|_{\mathfrak{L}_1(\mathbb{C}^2;\mathfrak{H})} \\
&\quad + \frac{1}{2\Im\mathrm{m}(z)} \|\Theta_{H(V_n)}(\overline{z})^* - \Theta_{H(V)}(\overline{z})^*\|_{\mathfrak{L}_1(\mathbb{C}^2)}.
\end{aligned} \tag{3.15}$$

Applying (3.1), (3.2), (3.4) and (3.6) we obtain (3.11). □

3.3 Eigenfunction expansions

Lemma 3.3 *Assume $V \in L^\infty$, $V_n \in L^\infty$, $n = 1, 2, \ldots$. If $V_n \stackrel{L^\infty}{\longrightarrow} V$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \sup_{\lambda\in\mathcal{C}} \|\psi_\pm^-(V_n)(\cdot,\lambda,\tau) - \psi_\pm^-(V)(\cdot,\lambda,\tau)\|_{L^\infty(\mathbb{R}_\pm,\mathbb{C}^2)} = 0 \tag{3.16}$$

and

$$\lim_{n\to\infty} \sup_{\lambda\in\mathcal{C}} \|\psi^-(V_n)(\cdot,\lambda,\tau) - \psi^-(V)(\cdot,\lambda,\tau)\|_{\mathfrak{H}} = 0, \tag{3.17}$$

$\tau = a, b$, for any compact set $\mathcal{C} \subseteq \mathbb{R}$.

Proof. By (2.42) one gets that $\psi_-^-(V)(x,\lambda,\tau) \equiv \psi_-^-(V_n)(x,\lambda,\tau)$ for $x \in \mathbb{R}_-$, $\lambda \in \mathbb{R}$, $n = 1, 2, \ldots$ and $\tau = a, b$. So the assertion (3.16) is obvious for the sign "-". Further, we find

$$\begin{aligned}
&\psi_+^-(V_n)(x,\lambda,\tau) - \psi_+^-(V)(x,\lambda,\tau) \\
&= \ \frac{1}{\sqrt{2\pi}} e^{ix\lambda} \left(\Theta_{H(V_n)}(\lambda)^* - \Theta_{H(V)}(\lambda)^*\right) e_\tau
\end{aligned} \tag{3.18}$$

for $x \in [a, b]$, $\lambda \in \mathbb{R}$, $n = 1, 2, \ldots$ and $\tau = a, b$ which yields

$$\begin{aligned}
&\left\|\psi_+^-(V_n)(\cdot,\lambda,\tau) - \psi_+^-(V)(\cdot,\lambda,\tau)\right\|_{L^\infty(\mathbb{R}_+,\mathbb{C}^2)} \\
&\le \ \frac{1}{\sqrt{2\pi}} \|\Theta_{H(V_n)}(\lambda)^* - \Theta_{H(V)}(\lambda)^*\|_{\mathcal{B}(\mathbb{C}^2)}
\end{aligned} \tag{3.19}$$

for $\lambda \in \mathbb{R}$, $n = 1, 2, \ldots$ and $\tau = a, b$. Applying (3.6) we prove (3.16). Furthermore, by (2.42) we get

$$\begin{aligned} &\psi^-(V_n)(x,\lambda,\tau) - \psi^-(V)(x,\lambda,\tau) \\ &\qquad = \frac{1}{\sqrt{2\pi}}\left(T_*(V_n)(\lambda)^* - T_*(V)(\lambda)^*\right) e_\tau \end{aligned} \tag{3.20}$$

for $x \in [a,b]$, $\lambda \in \mathbb{R}$, $n = 1, 2, \ldots$ and $\tau = a, b$. Hence

$$\begin{aligned} &\|\psi^-(V_n)(\cdot,\lambda,\tau) - \psi^-(V)(\cdot,\lambda,\tau)\|_{\mathfrak{H}} \\ &\qquad \leq \frac{1}{\sqrt{2\pi}}\|T_*(V_n)(\lambda)^* - T_*(V)(\lambda)^*\|_{\mathcal{B}(\mathbb{C}^2;\mathfrak{H})} \end{aligned} \tag{3.21}$$

for $\lambda \in \mathbb{R}$, $n = 1, 2, \ldots$ and $\tau = a, b$. Applying (3.4) we obtain (3.17). □

Proposition 3.4 *Let $V \in L^\infty$, $V_n \in L^\infty$, $n = 1, 2, \ldots$. If $V_n \xrightarrow{L^\infty} V$ as $n \to \infty$, then*

$$s - \lim_{n\to\infty} \Phi_-(V_n) = \Phi_-(V). \tag{3.22}$$

Proof. Because the operators $\Phi_-(V_n)$ $(n \in \mathbb{N})$ and $\Phi_-(V)$ are unitary, it is enough to verify $w - \lim_{n\to\infty} \Phi_-(V_n) = \Phi_-(V)$. Let $\vec{g} \in \mathfrak{K}$ and $\hat{h} \in L^2(\mathbb{R}, \mathbb{C}^2)$ with compact supports. By (2.46) we find

$$\begin{aligned} &\left(\Phi_-(V)\vec{g}, \hat{h}\right)_{L^2(\mathbb{R},\mathbb{C}^2)} \\ &\quad = \sum_{\tau=a,b}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_-} dx\ \left\langle g_-(x), \psi_-^-(V)(x,\lambda,\tau)\right\rangle_{\mathbb{C}^2}\overline{\hat{h}^\tau(\lambda)} \\ &\quad + \sum_{\tau=a,b}\int_{\mathbb{R}} d\lambda \int_a^b dx\ \left(g(x), \psi^-(V)(x,\lambda,\tau)\right)_{\mathbb{C}}\overline{\hat{h}^\tau(\lambda)} \\ &\quad + \sum_{\tau=a,b}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_+} dx\ \left\langle g_+(x), \psi_+^-(V)(x,\lambda,\tau)\right\rangle_{\mathbb{C}^2}\overline{\hat{h}^\tau(\lambda)}. \end{aligned} \tag{3.23}$$

Using this formula one gets

$$\begin{aligned} &\left((\Phi_-(V_n) - \Phi_-(V))\vec{g}, \hat{h}\right)_{L^2(\mathbb{R},\mathbb{C}^2)} \\ &= \sum_{\tau=a,b}\int_{\mathbb{R}} d\lambda \int_a^b dx\ \left(g(x), (\psi^-(V_n)(x,\lambda,\tau) - \psi^-(V)(x,\lambda,\tau))\right)_{\mathbb{C}}\overline{\hat{h}^\tau(\lambda)} \\ &\quad + \sum_{\tau=a,b}\int_{\mathbb{R}} d\lambda \int_{\mathbb{R}_+} dx\ \left\langle g_+(x), (\psi_+^-(V_n)(x,\lambda,\tau) - \psi_+^-(V)(x,\lambda,\tau))\right\rangle_{\mathbb{C}^2}\overline{\hat{h}^\tau(\lambda)}. \end{aligned} \tag{3.24}$$

From (3.24) we get the estimate

$$\left|\left((\Phi_-(V_n) - \Phi_-(V))\vec{g}, \hat{h}\right)_{L^2(\mathbb{R},\mathbb{C}^2)}\right| \tag{3.25}$$
$$\leq \|g\|_{\mathfrak{H}} \sum_{\tau=a,b} \int_{\mathbb{R}} d\lambda \, \|\psi^-(V_n)(\cdot,\lambda,\tau) - \psi^-(V)(\cdot,\lambda,\tau))\|_{\mathfrak{H}} |\hat{h}^\tau(\lambda)|$$
$$+\|g_+\|_{L^1(\mathbb{R}_+,\mathbb{C}^2)} \sum_{\tau=a,b} \int_{\mathbb{R}} d\lambda \, \|\psi_+^-(V_n)(\cdot,\lambda,\tau) - \psi_+^-(V)(\cdot,\lambda,\tau))\|_{L^\infty(\mathbb{R}_+,\mathbb{C}^2)} |\hat{h}^\tau(\lambda)|.$$

Applying Lemma 3.3 we prove

$$\lim_{n\to\infty} \left((\Phi_-(V_n) - \Phi_-(V))\vec{g}, \hat{h}\right)_{L^2(\mathbb{R},\mathbb{C}^2)} = 0 \tag{3.26}$$

for $\vec{g} \in \mathfrak{K}$ and $\hat{h} \in L^2(\mathbb{R}, \mathbb{C}^2)$ with compact supports. Since both sets are dense in $\mathfrak{K}$ and $L^2(\mathbb{R}, \mathbb{C}^2)$, respectively, and $\Phi_-(V_n)$, $n = 1, 2, \ldots$, are isometric operators the convergence (3.26) implies the weak convergence. □

3.4 Carrier densities

Proposition 3.5 *Let $V \in L^\infty$, $V_n \in L^\infty$, $n = 1, 2, \ldots$. If $\varrho(\cdot)$ satisfies the condition (2.56) and $V_n \xrightarrow{L^\infty} V$ as $n \to \infty$, then $u_{\hat{\varrho}}(V_n) \xrightarrow{L^1} u_{\hat{\varrho}}(V)$ as $n \to \infty$, i.e.*

$$\lim_{n\to\infty} \int_a^b dx \, |u_{\hat{\varrho}}(V_n)(x) - u_{\hat{\varrho}}(V)(x)| = 0. \tag{3.27}$$

In particular, the carrier density operator $\mathcal{N}_{\hat{\varrho}}(\cdot) : L^\infty \longrightarrow L^1$ is continuous.

Proof. By (2.59) we have the representation

$$\int_a^b dx \, (u_{\hat{\varrho}}(V_n)(x) - u_{\hat{\varrho}}(V)(x))h(x) = \mathrm{tr}\,((\varrho(V_n) - \varrho(V))M(h)) \tag{3.28}$$

for each $h \in L^\infty$ and $n = 1, 2, \ldots$. Since

$$\mathrm{tr}\,(\varrho(V)M(h)) = \mathrm{tr}\left(\varrho(V)(K(V)+i)(K(V)+i)^{-1}M(h)\right). \tag{3.29}$$

one gets

$$\begin{aligned} &\mathrm{tr}\,((\varrho(V_n) - \varrho(V))M(h)) \\ &\quad = \mathrm{tr}\left(\varrho(V_n)(K(V_n)+i)\left\{(K(V_n)+i)^{-1} - (K(V)+i)^{-1}\right\}M(h)\right) \\ &\quad\quad +\mathrm{tr}\left((\varrho(V_n)(K(V_n)+i) - \varrho(V)(K(V)+i))(K(V)+i)^{-1}M(h)\right) \end{aligned} \tag{3.30}$$

for $n = 1, 2, \ldots$. By

$$\|\varrho(V_n)(K(V_n)+i)\|_{\mathcal{B}(\mathfrak{K})} \leq C_{\hat{\varrho}} \tag{3.31}$$

we obtain

$$\begin{aligned}\left|\operatorname{tr}\left(\varrho(V_n)(K(V_n)+i)\left\{(K(V_n)+i)^{-1}-(K(V)+i)^{-1}\right\}M(h)\right)\right| \\ \le C_{\hat\varrho}\left\|(K(V_n)+i)^{-1}-(K(V)+i)^{-1}\right\|_{\mathfrak{L}_1(\mathfrak{K})}\|h\|_{L^\infty}\end{aligned} \tag{3.32}$$

for $n=1,2,\ldots$. Setting

$$k(\lambda) := \varrho(\lambda)(\lambda+i), \quad \lambda\in\mathbb{R}, \tag{3.33}$$

and denoting by $\hat{k}$ the multiplication operator induced by $k(\cdot)$ on $L^2(\mathbb{R},\mathbb{C}^2)$ we find that

$$\varrho(V_n)(K(V_n)+i) = \Phi_-(V_n)^{-1}\hat{k}\Phi_-(V_n), \tag{3.34}$$

$n=1,2,\ldots$, and

$$\varrho(V)(K(V)+i) = \Phi_-(V)^{-1}\hat{k}\Phi_-(V). \tag{3.35}$$

Hence the representation

$$\begin{aligned}&\varrho(V_n)(K(V_n)+i)-\varrho(V)(K(V)+i) \\ &\quad = (\Phi_-(V_n)^{-1}-\Phi_-(V)^{-1})\hat{k}\Phi_-(V_n)+\Phi_-(V)\hat{k}(\Phi_-(V_n)-\Phi_-(V))\end{aligned} \tag{3.36}$$

is valid. From (3.36) we deduce the estimate

$$\begin{aligned}&\left|\operatorname{tr}\left((\varrho(V_n)(K(V_n)+i)-\varrho(V)(K(V)+i))(K(V)+i)^{-1}M(h)\right)\right| \\ &\quad\le C_{\hat\varrho}\|(\Phi_-(V_n)-\Phi_-(V))P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})}\|h\|_{L^\infty} \\ &\qquad + C_{\hat\varrho}\|(\Phi_-(V_n)-\Phi_-(V))(K(V)+i)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\|_{\mathfrak{L}_1(\mathfrak{H})}\|h\|_{L^\infty}.\end{aligned} \tag{3.37}$$

Taking into account (3.30), (3.32) and (3.37) we finally get the estimate

$$\begin{aligned}&|\operatorname{tr}((\varrho(V_n)-\varrho(V))M(h))| \\ &\quad\le C_{\hat\varrho}\Big\{\left\|(K(V_n)+i)^{-1}-(K(V)+i)^{-1}\right\|_{\mathfrak{L}_1(\mathfrak{K})} \\ &\qquad + \|(\Phi_-(V_n)-\Phi_-(V))P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \\ &\qquad + \|(\Phi_-(V_n)-\Phi_-(V))(K(V)+i)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\|_{\mathfrak{L}_1(\mathfrak{H})}\Big\}\|h\|_{L^\infty}.\end{aligned} \tag{3.38}$$

Since h is arbitrary we obtain from (3.28) the estimate

$$\begin{aligned}&\int_a^b dx\ |(u_{\hat\varrho}(V_n)(x)-u_{\hat\varrho}(V)(x))| \\ &\quad\le C_{\hat\varrho}\Big\{\left\|(K(V_n)+i)^{-1}-(K(V)+i)^{-1}\right\|_{\mathfrak{L}_1(\mathfrak{K})} \\ &\qquad + \|(\Phi_-(V_n)-\Phi_-(V))P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-i)^{-1}\|_{\mathfrak{L}_1(\mathfrak{H})} \\ &\qquad + \|(\Phi_-(V_n)-\Phi_-(V))(K(V)+i)^{-1}P^{\mathfrak{K}}_{\mathfrak{H}}\|_{\mathfrak{L}_1(\mathfrak{H})}\Big\}.\end{aligned} \tag{3.39}$$

The first addend of the r.h.s goes to zero by (3.11) as $n\to\infty$. Since $P^{\mathfrak{K}}_{\mathfrak{H}}(K(V)-i)^{-1}$ is a trace class operator one gets by (3.22) that the second addend of the r.h.s tends to zero too. Similarly one proves that the third attend tends to zero. This proves (3.27). □

4 Dissipative Schrödinger-Poisson systems

In this section we intend to regard the dissipative Schrödinger Poisson system as a whole. Let us introduce some further notations:

Definition 4.1 We denote the real part of $W^{1,2}$ by $W^{1,2}_{\mathbb{R}}$ and that of $C[a,b]$ by $C_{\mathbb{R}}[a,b]$. Let $\Gamma \subset \partial\Omega$ be the (possibly empty) set of Dirichlet points. We define

$$W^{1,2}_{\Gamma} := W^{1,2}_{\mathbb{R}} \cap \{\psi \;:\; \psi(\Gamma) \subset \{0\}\}.$$

By $W^{-1,2}_{\Gamma}$ we denote the dual space of $W^{1,2}_{\Gamma}$ and by $< \cdot, \cdot >_1$ the dual pairing between $W^{1,2}_{\Gamma}$ and $W^{-1,2}_{\Gamma}$. The embedding constants from $W^{1,2}_{\mathbb{R}}$ into $C_{\mathbb{R}}[a,b]$ and into L^∞ are denoted by ε_c and ε_∞, respectively. The embedding constant from L^1 into $W^{-1,2}_{\Gamma}$ will be denoted by ε_1.

Following [9] the important ingredients of dissipative Schrödinger-Poisson systems are two dissipative Schrödinger-type operators $H^{\pm}(V_{\pm})$, cf. (1.3), for electrons (sign "-") and holes (sign "+") and Poisson's equation (1.1). The dissipative Schrödinger-type operators are determined by the 'effective' masses $m_{\pm}$, the boundary coefficients $\kappa^{\pm}_a, \kappa^{\pm}_b$ and the potentials $V^{\pm}$ which are of the form

$$V_{\pm} = V_0^{\pm} \pm \varphi, \tag{4.1}$$

where $V_0^{\pm}$ are external potentials representing the band-edge offsets and φ is a potential which is determined by Poisson's equation. To formulate Poisson's equation one needs the dielectric permittivity ϵ, the doping profile C, the function k and the function φ_Γ, which represents the boundary values given on Γ and the inhomogeneous boundary conditions of third kind in $\partial\Omega \setminus \Gamma$, cf. (1.2).

Assumptions 4.2 Throughout section 4 we always assume that the following assumptions are fulfilled:

($A_1^{\pm}$) The 'effective' masses $m_{\pm}$ are positive and obey $m_{\pm}$, $\frac{1}{m_{\pm}} \in L^\infty$. As above, we use the convention $\|m_{\pm}\| := \|m_{\pm}\|_{L^\infty}$.

($A_2^{\pm}$) The boundary coefficients $\kappa^{\pm}_a, \kappa^{\pm}_b$ are from the upper half plane $\mathbb{C}_+$.

($A_3^{\pm}$) The external potentials $V_0^{\pm}$ belong to L^∞.

($A_4^{\pm}$) The matrix valued-functions $\varrho_{\pm}(\cdot) \in L^\infty(\mathbb{R}, \mathbb{C}^2)$ satisfy (2.56).

(A_5) The doping profile C belongs to $W^{-1,2}_{\Gamma}$.

(A_6) The dielectric permittivity ϵ is positive and obeys ϵ, $\frac{1}{\epsilon} \in L^\infty$. We set $\tilde{\epsilon} := \max\{1, \|\frac{1}{\epsilon}\|_{L^\infty}\}$.

(A_7) The set Γ is not empty, or at least one of the numbers $k(x)$, $x \in \{a,b\} \setminus \Gamma$, is strictly positive.

(A_8) The function φ_Γ is from the set $W^{1,2}_{\mathbb{R}}$.

Each Schrödinger-type operator $H^\pm(V_\pm)$ corresponds a minimal self-adjoint dilation $K^\pm(V_\pm)$. In accordance with section 2.5 the functions $\varrho_\pm(\cdot)$ define steady states $\varrho_\pm(V_\pm)$, i.e. no-negative self-adjoint operators which commute with $K^\pm(V_\pm)$. By section 2.6 one can introduce carrier densities $u^\pm_{\hat{\varrho}_\pm}(V_\pm)$ for electrons and holes. Notice that the electron carrier density $u^-_{\hat{\varrho}_-}(V_-)$ is determined by the electron quantities m_-, κ^-_a, κ^-_b and V^-_0 while the hole carrier density $u^+_{\hat{\varrho}_+}(V_+)$ by the hole quantities m_+, κ^+_a, κ^+_b and V^+_0. The corresponding carrier density operators are denoted by $\mathcal{N}^\pm_{\hat{\varrho}_\pm}(\cdot)$.

4.1 Rigorous definition

At first we will give a rigorous definition of Poisson's equation and afterwards define what we will call a solution of the dissipative Schrödinger Poisson system.

Definition 4.3 We define the Poisson operator $\mathcal{P} : W^{1,2}_{\mathbb{R}} \longrightarrow W^{-1,2}_{\Gamma}$ as usual by

$$< \mathcal{P}v, \varsigma >_1 = \int_a^b dx\, \epsilon\, \frac{dv}{dx}\frac{d\varsigma}{dx} + \sum_{x\in\{a,b\}\setminus\Gamma} k(x)v(x)\varsigma(x), \qquad v \in W^{1,2}_{\mathbb{R}},\ \varsigma \in W^{1,2}_{\Gamma}. \tag{4.2}$$

The restriction of $\mathcal{P}$ to the subspace $W^{1,2}_{\Gamma}$ will be denoted by $\mathcal{P}_0$.

We have

$$| < \mathcal{P}v, \varsigma >_1 | \le \left(\|\epsilon\|_{L^\infty} + \sum_{x\in\{a,b\}\setminus\Gamma} k(x)\, \varepsilon_c^2 \right) \|v\|_{W^{1,2}_{\mathbb{R}}} \|\varsigma\|_{W^{1,2}_{\Gamma}}.$$

Hence $\mathcal{P}$ is continuous. Furthermore we get

$$\|\varphi\|^2_{W^{1,2}_{\Gamma}} \le (1+\gamma_k)\left(\int_a^b |\varphi'(x)|^2\, dx + \sum_{x\in\{a,b\}\setminus\Gamma} k(x)|\varphi(x)|^2 \right), \quad \text{for all } \varphi \in W^{1,2}_{\Gamma}, \tag{4.3}$$

with

$$\gamma_k := \sup_{0\neq\psi\in W^{1,2}_{\Gamma}} \frac{\int_a^b \psi^2\, dx}{\int_a^b \left(\frac{d\psi}{dx}\right)^2 dx + \sum_{x\in\{a,b\}\setminus\Gamma} k(x)|\psi(x)|^2}. \tag{4.4}$$

Because the case of purely homogeneous Neumann conditions is excluded by (A_7), the constant γ_k is indeed finite. Thus we get by (4.3)

$$\|\varphi\|^2_{W^{1,2}_{\Gamma}} \le \tilde{\epsilon}(1+\gamma_k)| < \mathcal{P}_0\varphi, \varphi >_1 |.$$

Therefore we get by the Lax-Milgram lemma that the inverse of $\mathcal{P}_0$ exists and its norm does not exeed $\tilde{\epsilon}(1+\gamma_k)$.

We denote by $\tilde{\varphi}_\Gamma$ the form

$$v \longmapsto \int_a^b dx\; \epsilon(x)\varphi_\Gamma'(x)v'(x), \qquad v \in W_\Gamma^{1,2}.$$

Definition 4.4 Assume $u^\pm \in L^1$. We say $\varphi \in W_{\mathbb{R}}^{1,2}$ satisfies Poisson's equation (1.1) iff $\varphi - \varphi_\Gamma \in W_\Gamma^{1,2}$ satisfies

$$\mathcal{P}_0(\varphi - \varphi_\Gamma) = D + q\,u^+ - q\,u^-, \tag{4.5}$$

where $D := qC - \tilde{\varphi}_\Gamma$.

Of course, the right hand side of (4.5) is to be understood in $W_\Gamma^{-1,2}$ by embedding $u^+,\, u^- \in L^1 \hookrightarrow W_\Gamma^{-1,2}$.

Definition 4.5 We say a triple $(\varphi, u^+, u^-) \in W_{\mathbb{R}}^{1,2} \times L^1 \times L^1$ satisfies the dissipative Schrödinger Poisson system if φ satisfies Poisson's equation as well as $u^+ = u^+_{\hat{\varrho}_+}(V_0^+ + \varphi)$ and $u^- = u^-_{\hat{\varrho}_-}(V_0^- - \varphi)$.

4.2 Existence of solutions and a priori estimates

The aim of this section is to prove that the dissipative Schrödinger Poisson system always admits a solution and to investigate these solutions. At first in accordance with [5] we define a mapping whose fixed points exactly determine the solutions of the dissipative Schrödinger Poisson system.

By $\mathcal{J} : L^1 \times L^1 \longrightarrow W_{\mathbb{R}}^{1,2}$ we denote the map which assigns to $(u^+, u^-) \in L^1 \times L^1$ the solution of Poisson's equation. Obviously, the map $\mathcal{J}$ is continuous. Further, we define $\Psi : L^\infty \longrightarrow W_{\mathbb{R}}^{1,2}$ by

$$\Psi : V \longrightarrow \big(\mathcal{N}^+_{\hat{\varrho}_+}(V_0^+ + V), \mathcal{N}^-_{\hat{\varrho}_-}(V_0^- - V)\big) \longrightarrow \mathcal{J}\big(\mathcal{N}^+_{\hat{\varrho}_+}(V_0^+ + V), \mathcal{N}^-_{\hat{\varrho}_-}(V_0^- - V)\big).$$

Since the map $\mathcal{J} : L^1 \times L^1 \longrightarrow W_{\mathbb{R}}^{1,2}$ is continuous and by Proposition 3.5 the maps $\mathcal{N}^\pm(\cdot) : L^\infty \longrightarrow L^1$ are also continuous, the map $\Psi : L^\infty \longrightarrow W_{\mathbb{R}}^{1,2}$ is continuous, too. Let $E_\infty : W_{\mathbb{R}}^{1,2} \longrightarrow L^\infty$ denote the embedding operator of $W_{\mathbb{R}}^{1,2}$ into L^∞. With Ψ we associate the map $\Psi_\infty : L^\infty \longrightarrow L^\infty$,

$$\Psi_\infty := E_\infty \Psi,$$

which is also continuous. Moreover, since E_∞ is compact the map Ψ_∞ is also compact.

Lemma 4.6 *An element $\varphi \in L^\infty$ is a fixed point of Ψ_∞ if and only if the triple*

$$(\varphi, u^+, u^-) = (\varphi, u^+_{\hat{\varrho}_+}(V_0^+ + \varphi), u^-_{\hat{\varrho}_-}(V_0^- - \varphi))$$

satisfies the dissipative Schrödinger Poisson system.

The proof is obvious. To prove the central results of this chapter, we also need the following technical

Lemma 4.7 *Let $\sigma_1, \sigma_2, \sigma_3$ be three strictly positive numbers and let x_0 be the smallest positive root of the polynomial $p : x \mapsto x^4 - \sigma_1 x^2 - \sigma_2 x - \sigma_3$. Then for all $x > x_0$ one has $p(x) > 0$. In particular, p does not admit other positive roots.*

Proof. It is clear that at least one positive root must exist. Then one has for $x = tx_0$ with $t > 1$:

$$\begin{aligned} p(x) = t^4 x_0^4 - \sigma_1 t^2 x_0^2 - \sigma_2 t x_0 - \sigma_3 \\ &= t^4(\sigma_1 x_0^2 + \sigma_2 x_0 + \sigma_3) - \sigma_1 t^2 x_0^2 - \sigma_2 t x_0 - \sigma_3 \\ &= \sigma_1 t^2 (t^2 - 1) x_0^2 + \sigma_2 t (t^3 - 1) x_0 + \sigma_3 (t^4 - 1) > 0. \end{aligned}$$

This shows that a positive root larger than x_0 does not exists. □

Let $c_\pm$ be the constants defined by (2.21). We specify $\sigma_1, \sigma_2, \sigma_3$ to

$$\sigma_1 := 8\sqrt{2} q(b-a)\varepsilon_1 \varepsilon_\infty \tilde{\epsilon}(1+\gamma_k) \cdot \left(C_{\hat{\varrho}_+} \sqrt{\|m_+\|} + C_{\hat{\varrho}_-} \sqrt{\|m_-\|} \right), \tag{4.6}$$

$$\sigma_2 := 16\, q \mathfrak{g}_1 \varepsilon_1 \varepsilon_\infty \tilde{\epsilon}(1+\gamma_k) \cdot \left(C_{\hat{\varrho}_+} p_+ + C_{\hat{\varrho}_-} p_- \right), \tag{4.7}$$

$$\begin{aligned} \sigma_3 := {} & \varepsilon_\infty \cdot \|\varphi_\Gamma\|_{W^{1,2}_{\mathbb{R}}} + \varepsilon_\infty \tilde{\epsilon}(1+\gamma_k) \cdot \|D\|_{W^{-1,2}_\Gamma} + \\ & + q\varepsilon_1 \varepsilon_\infty \tilde{\epsilon}(1+\gamma_k) \cdot (C_{\hat{\varrho}_+} r_+ + C_{\hat{\varrho}_-} r_-) \end{aligned} \tag{4.8}$$

where

$$p_\pm := ((\alpha_a^\pm)^2 + (\alpha_b^\pm)^2)^{1/2} \|(H_0^\pm)^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H}, W^{1,2})}, \tag{4.9}$$

$$\begin{aligned} r_\pm := {} & 8 + 4\sqrt{2}\,(b-a)\sqrt{\|m_\pm\|} \cdot \sqrt{8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty}} + \\ & + 8\sqrt{2}\mathfrak{g}_1((\alpha_a^\pm)^2 + (\alpha_b^\pm)^2)^{1/2} \|(H_0^\pm)^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H}, W^{1,2})} (8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty})^{1/4}. \end{aligned} \tag{4.10}$$

Theorem 4.8 *The following statements are true:*

(i) The mapping $\Psi_\infty : L^\infty \longrightarrow L^\infty$ always admits a fixed point.

(ii) If x_0 is the (unique) positive root of the polynomial $p : x \longrightarrow x^4 - \sigma_1 x^2 - \sigma_2 x - \sigma_3$, then for any fixed point V of Ψ_∞ the inequality

$$\|V\|_{L^\infty} \le x_0^4 \tag{4.11}$$

holds.

Proof. One has

$$\begin{aligned} \|\mathcal{J}(u^+, u^-)\|_{W^{1,2}_{\mathbb{R}}} &\le \|\varphi_\Gamma\|_{W^{1,2}_{\mathbb{R}}} + \|\mathcal{P}_0^{-1}(D + qu^+ - qu^-)\|_{W^{1,2}_\Gamma} \\ &\le \|\varphi_\Gamma\|_{W^{1,2}_{\mathbb{R}}} + \tilde{\epsilon}(1+\gamma_k) \cdot \|D + qu^+ - qu^-\|_{W^{-1,2}_\Gamma} \\ &\le \|\varphi_\Gamma\|_{W^{1,2}_{\mathbb{R}}} + \tilde{\epsilon}(1+\gamma_k) \cdot \left[\|D\|_{W^{-1,2}_\Gamma} + q\varepsilon_1 \left(\|u^+\|_{L^1} + \|u^-\|_{L^1} \right) \right], \end{aligned}$$

which implies

$$\begin{aligned} &\|E_\infty \mathcal{J}(u^+,u^-)\|_{L^\infty} \\ &\quad\leq \quad \varepsilon_\infty \|\varphi_\Gamma\|_{W^{1,2}_{\mathbb{R}}} \;+\; \varepsilon_\infty \tilde{\epsilon}(1+\gamma_k)\cdot\Big[\|D\|_{W_\Gamma^{-1,2}} + q\varepsilon_1\big(\|u^+\|_{L^1} + \|u^-\|_{L^1}\big)\Big]. \end{aligned} \tag{4.12}$$

Since $u^\pm = \mathcal{N}^\pm_{\hat{\varrho}\pm}(V_\pm)$ one gets from (2.61) the estimate

$$\begin{aligned} \|u^\pm\|_{L^1} \leq C_{\hat{\varrho}\pm}\cdot\Big[&8 + 4\sqrt{2}\,(b-a)\sqrt{\|m_\pm\|}\cdot\sqrt{8c_\pm + 5 + 4\|V_\pm\|_{L^\infty}} \\ &+\; 8\sqrt{2}\mathfrak{g}_1\big((\alpha_a^\pm)^2 + (\alpha_b^\pm)^2\big)^{1/2}\|(H_0^\pm)^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}(8c_\pm + 5 + 4\|V_\pm\|_{L^\infty})^{1/4}\Big]. \end{aligned}$$

By $V_\pm = V_0^\pm \pm V$ we obtain

$$\begin{aligned} &\|u^\pm\|_{L^1} \leq \\ &\quad C_{\hat{\varrho}\pm}\cdot\Big[8 + 4\sqrt{2}\,(b-a)\sqrt{\|m_\pm\|}\cdot\sqrt{8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty} + 4\|V\|_{L^\infty}} \\ &+8\sqrt{2}\mathfrak{g}_1\big((\alpha_a^\pm)^2 + (\alpha_b^\pm)^2\big)^{1/2}\|(H_0^\pm)^{-1/2}\|^{1/2}_{\mathcal{B}(\mathfrak{H},W^{1,2})}(8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty} + 4\|V\|_{L^\infty})^{1/4}\Big]. \end{aligned} \tag{4.13}$$

Using the estimates

$$\sqrt{8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty} + 4\|V\|_{L^\infty}} \leq \sqrt{8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty}} + 2\sqrt{\|V\|_{L^\infty}}$$

and

$$(8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty} + 4\|V\|_{L^\infty})^{1/4} \leq (8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty})^{1/4} + \sqrt{2}\|V\|^{1/4}_{L^\infty}$$

we get

$$\|u^\pm\|_{L^1} \leq \sigma_1^\pm\|V\|^{1/2}_{L^\infty} + \sigma_2^\pm\|V\|^{1/4}_{L^\infty} + \sigma_3^\pm, \tag{4.14}$$

where

$$\begin{aligned} \sigma_1^\pm &:= 8\sqrt{2}\,C_{\hat{\varrho}\pm}\cdot(b-a)\sqrt{\|m_\pm\|}, \\ \sigma_2^\pm &:= 16\,\sqrt{2}\,\mathfrak{g}_1\,C_{\hat{\varrho}\pm}\,p_\pm, \\ \sigma_3^\pm &:= C_{\hat{\varrho}\pm}\cdot\,r_\pm, \end{aligned}$$

where $p_\pm$ and $r_\pm$ are defined by (4.9) and (4.10), respectively. Inserting (4.14) into (4.12) we obtain

$$\|E_\infty \mathcal{J}(u^+,u^-)\|_{L^\infty} \leq \sigma_1\|V\|^{1/2}_{L^\infty} + \sigma_2\|V\|^{1/4}_{L^\infty} + \sigma_3$$

where $\sigma_1, \sigma_2, \sigma_3$ are the constants defined under (4.6) - (4.8). Hence we get the estimate

$$\|\Psi_\infty(V)\|_{L^\infty} \leq \sigma_1\|V\|^{1/2}_{L^\infty} + \sigma_2\|V\|^{1/4}_{L^\infty} + \sigma_3. \tag{4.15}$$

If x_0 is the (unique) positive root of the polynomial $x \longrightarrow x^4 - \sigma_1 x^2 - \sigma_2 x - \sigma_3$ and $\|V\|_{L^\infty} \leq x_0^4$, then by (4.15) one obtains

$$\|\Psi_\infty(V)\|_{L^\infty} \leq \sigma_1\big(\|V\|^{1/4}_{L^\infty}\big)^2 + \sigma_2\|V\|^{1/4}_{L^\infty} + \sigma_3 \leq \sigma_1 x_0^2 + \sigma_1 x_0 + \sigma_3 = x_0^4.$$

This means, that Ψ_∞ maps the ball $\{V : \|V\|_{L^\infty} \le x_0^4\}$ continuously into itself. Since Ψ_∞ is compact the image of this ball under Ψ_∞ is precompact in L^∞. Thus, by Schauder's fixed point theorem Ψ_∞ must have a fixed point. This proves the first assertion.

Assume that the second assertion is false and a fixed point V satisfying $\|V\|_{L^\infty} > x_0^4$ exists. Then (4.15) would give

$$\left(\|V\|_{L^\infty}^{1/4}\right)^4 = \|V\|_{L^\infty} = \|\Psi_\infty(V)\|_{L^\infty} \le \sigma_1\left(\|V\|_{L^\infty}^{1/4}\right)^2 + \sigma_2\|V\|_{L^\infty}^{1/4} + \sigma_3.$$

Because $\|V\|_{L^\infty}^{1/4} > x_0$ this contradicts Lemma 4.7. $\square$

Now we can state the main result of the paper:

Theorem 4.9 *Under the assumption $(A_1^\pm)$-(A_8) the following statements are true:*

(i) the dissipative Schrödinger Poisson system always admits a solution and

(ii) any solution (φ, u^+, u^-) of the dissipative Schrödinger Poisson system satisfies the a priori estimates

$$\|\varphi\|_{L^\infty} \le x_0^4 \quad \textit{and} \quad \|u^\pm\|_{L^1} \le C_{\hat{\varrho}_\pm}\hat{r}_\pm, \tag{4.16}$$

where x_0 is the unique positive root of the polynomial $x \longrightarrow x^4 - \sigma_1 x^2 - \sigma_2 x - \sigma_3$ with coefficients given by (4.6)-(4.10) and

$$\hat{r}_\pm := \Big[8 + 4\sqrt{2}\,(b-a)\sqrt{\|m_\pm\|}\cdot\sqrt{8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty} + x_0^4}$$
$$+ \quad 8\sqrt{2}\,\mathfrak{g}_1\big((\alpha_a^\pm)^2 + (\alpha_b^\pm)^2\big)^{1/2}\|(H_0^\pm)^{-1/2}\|_{\mathcal{B}(\mathfrak{H},W^{1,2})}^{1/2}(8c_\pm + 5 + 4\|V_0^\pm\|_{L^\infty} + x_0^4)^{1/4}\Big].$$

Proof. The first assertion follows from Lemma 4.6 and Theorem 4.8. The first inequality of (4.16) is implied by (4.11) while the second estimates are obtained from (4.13) and the first inequality. $\square$

5 Remarks

Let us comment the results.

(i) Theorem 4.9 shows that the dissipative Schrödinger-Poisson system always admits a solution, if the assumptions $(A_1^\pm)$-(A_8) are satisfied.

(ii) Solutions (φ, u^+, u^-) of dissipative Schrödinger-Poisson systems admit bounds which only depend on the inputs $m_\pm$, $\kappa_a^\pm, \kappa_b^\pm$, $V_0^\pm$, C, ϵ, φ_Γ, k, and the steady state $\varrho_\pm(\cdot)$. In particular, the occuring number x_0 may be directly calculated from the data by Cardano's formula.

(iii) In contrast to self-adjoint Schrödinger-Poisson systems dissipative ones allow in general non-trivial currents $j^{\pm}_{\hat{\varrho}_\pm}$ which are independent from $x \in [a, b]$ provided the steady states $\varrho_\pm(\cdot)$ obey

$$\int_{\mathbb{R}} d\lambda \, \mathrm{tr}(\varrho_\pm(\lambda)) < \infty, \tag{5.1}$$

cf. [9].

(iv) The last fact gives the possibility to couple dissipative Schrödinger-Poisson systems to drift diffusion models which acts outside the interval $[a, b]$ via a current continuity condition. In a forthcoming paper we show that this is really possible and, moreover, the coupled system admits a solution.

(v) The problem remains open under which conditions the solution, guaranteed by Theorem 4.9, is unique.

(vi) The present paper solves the dissipative Schrödinger-Poisson system in one dimension. The 2D- and 3D-problems remain open.

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