On the Collapse of Tubes Carried by 3D Incompressible Flows

by

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0 Introduction

The 3-dimensional incompressible Euler equation ("3D Euler") is as follows:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + u \cdot \nabla_x\right) u = -\nabla_x p &&(x \in \mathbb{R}^3,\, t \geq 0)\\
&\nabla_x \cdot u = 0 &&(x \in \mathbb{R}^3,\, t \geq 0)\\
&u(x,0) = u^0(x) &&(x \in \mathbb{R}^3)
\end{aligned}$$

with u^0 a given, smooth, divergence-free, rapidly decreasing vector field on $\mathbb{R}^3$. Here, $u(x,t)$ and $p(x,t)$ are the unknown velocity and pressure for an ideal, incompressible fluid flow at zero viscosity. An outstanding open problem is to determine whether a 3D Euler solution can develop a singularity at a finite time T. A classic result of Beale-Kato-Majda [1] asserts that, if a singularity forms at time T, then the vorticity $\omega(x,t) = \nabla_x \times u(x,t)$ grows so rapidly that

$$\int_0^T \sup_x |\,\omega(x,t)\,|\, dt = \infty.$$

In [2], Constantin-Fefferman-Majda showed that, if the velocity remain bounded up to the time T of singularity formation, then the vorticity direction $\omega(x,t)/\,|\omega(x,t)|$ cannot remain uniformly Lipschitz continuous up to time T.

One scenario for possible formation of a singularity in a 3D Euler solution is a constricting vortex tube. Recall that a *vortex line* in a fluid is an arc on an integral curve of the vorticity $\omega(x,t)$ for fixed t, and a *vortex tube* is a tubular neighborhood in $\mathbb{R}^3$ arising as a union of vortex lines. In numerical simulations of 3D Euler solutions, one

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routinely sees that vortex tubes grow longer and thinner, while bending and twisting. If the thickness of a piece of a vortex tube becomes zero in finite time, then one has a singular solution of 3D Euler. It is not known whether this can happen.

Our purpose here is to adapt our work [3], [4] on two-dimensional flows to three dimensions, for application to 3D Euler. We introduce below the notion of a "regular tube". Under the mild assumption that

$$\int_0^T \sup_x |u(x,t)|\, dt < \infty,$$

we show that a regular tube cannot reach zero thickness at time T. In particular, for 3D Euler solutions, a vortex tube cannot reach zero thickness in finite time, unless it bends and twists so violently that no part of it forms a regular tube. This significantly sharpens the conclusion of [2] for possible singularities of 3D Euler solutions arising from vortex tubes. On the other hand, [2] applies to arbitrary singularities of 3D Euler solutions, while our results apply to "regular tubes".

Although we are mainly interested in 3D Euler solutions, our result is stated for arbitrary incompressible flows in 3 dimensions. The proof is simple and elementary. The main novelty for readers familiar with [3], [4] is that we can adapt the ideas of [3] to three dimensions, even though there is no scalar that plays the rôle of the stream function on $\mathbb{R}^2$.

1 Regular Tubes

Let $Q = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$ be a closed rectangular box (with I_j a bounded interval), and let $T > 0$ be given.

A *regular tube* is an open set $\Omega_t \subset Q$ parametrized by time $t \in [0,T)$, having the form

$$\Omega_t = \{(x_1, x_2, x_3) \in Q : \theta(x_1, x_2, x_3, t) < 0\} \tag{1}$$

with

$$\theta \in C^1(Q \times [0,T)), \tag{2}$$

and satisfying the following properties:

$$|\nabla_{x_1,x_2}\theta| \neq 0 \text{ for } (x_1,x_2,x_3,t) \in Q \times [0,T),\ \theta(x_1,x_2,x_3,t) = 0\,; \tag{3}$$

$$\Omega_t(x_3) := \{(x_1,x_2) \in I_1 \times I_2 : (x_1,x_2,x_3) \in \Omega_t\} \text{ is non} - \text{empty}, \tag{4}$$

for all $x_3 \in I_3$, $t \in [0,T)$;

$$\text{closure } (\Omega_t(x_3)) \subset \text{ interior } (I_1 \times I_2) \tag{5}$$

for all $x_3 \in I_3$, $t \in [0,T)$.

For example, a thin tubular neighborhood of a curve Γ forms a regular tube, provided the tangent vector Γ' stays transverse to the (x_1,x_2) plane.

Let $u(x,t) = (u_k(x,t))_{1\le k\le 3}$ be a C^1 velocity field defined on $Q \times [0,T)$. We say that the regular tube Ω_t *moves with the velocity field* u, if we have

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla_x\right)\theta = 0 \text{ whenever } (x,t) \in Q \times [0,T), \theta(x,t) = 0\,. \tag{6}$$

It is well-known that a vortex tube arising from a 3D Euler solution moves with the fluid velocity.

2 Statement of the Main Result

Theorem: *Let $\Omega_t \subset Q (t \in [0,T))$ be a regular tube that moves with a C^1, divergence free velocity field $u(x,t)$.*

If

$$\int_0^T \sup_{x\in Q} |u(x,t)|\,dt < \infty \tag{7}$$

then

$$\liminf_{t\to T-} \text{ Vol}(\Omega_t) > 0. \tag{8}$$

3 Calculus Formulas for Regular Tubes

Let Ω_t be a regular tube, as in (1)$\cdots$(5). Recall that

$$\Omega_t(x_3) = \{(x_1, x_2) \in I_1 \times I_2 : \theta(x_1, x_2, x_3, t) < 0\}. \tag{9}$$

Define also

$$S_t(x_3) = \{(x_1, x_2) \in \text{ interior } (I_1 \times I_2) : \theta(x_1, x_2, x_3, t) = 0\} \text{ for } x_3 \in I_3,\ t \in [0, T). \tag{10}$$

Also, for intervals $I \subset I_3$, and for $t \in [0, T)$, define

$$\Omega_t(I) = \{(x_1, x_2, x_3) \in Q : x_3 \in I \text{ and } \theta(x_1, x_2, x_3, t) < 0\}, \text{ and} \tag{11}$$

$$S_t(I) = \{(x_1, x_2, x_3) \in Q : x_3 \in I \text{ and } (x_1, x_2) \in S_t(x_3)\}. \tag{12}$$

Let ν denote the outward-pointing unit normal to $S_t(I_3)$, and let $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, 0)$, where $(\tilde{\nu}_1, \tilde{\nu}_2)$ is the outward-pointing unit normal to $S_t(x_3)$. Thus, ν and $\tilde{\nu}$ are continuous vector-valued functions, defined on $\mathcal{S} = \{(x_1, x_2, x_3, t) \in Q \times [0, T) : (x_1, x_2, x_3) \in S_t(I_3)\}$. Define also scalar-valued functions $\sigma, \tilde{\sigma}$ on $\mathcal{S}$ by requiring that

$$\left(\frac{\partial}{\partial t} + \sigma\nu \cdot \nabla_x\right)\theta = \left(\frac{\partial}{\partial t} + \tilde{\sigma}\tilde{\nu} \cdot \nabla_x\right)\theta = 0 \text{ for } x \in S_t(I_3). \tag{13}$$

Again, σ and $\tilde{\sigma}$ are well-defined and continuous on $\mathcal{S}$, thanks to (3). Let F be any continuous function on Q. We will establish the following elementary formulas.

$$\frac{d}{dt}\left[\int_{\Omega_t(x_3)} F(x_1, x_2, x_3)\, d\,(\text{Area})\right] = \int_{(x_1,x_2)\in S_t(x_3)} F(x_1, x_2, x_3)\,\tilde{\sigma}(x_1, x_2, x_3, t)\, d\,(\text{length}) \tag{14}$$

$$\int_{S_t(I)} F d(\text{Area}) = \int_{x_3 \in I}\left\{\int_{S_t(x_3)} \frac{F}{\nu \cdot \tilde{\nu}}\, d\,(\text{length})\right\} dx_3\,. \tag{15}$$

To check (14) and (15), we may use a partition of unity to reduce to the case in which F is supported in a small neighborhood U. Also, we may restrict attention to a small time interval J. In a small enough $U \times J$, we may assume that $S_t(I_3)$ is given by the graph of a C^1 function ψ, thanks to (3). Thus, without loss of generality, we may suppose that, in $U \times J$, we have

$$(x_1, x_2, x_3) \in \Omega_t(I_3) \text{ if and only if } x_1 < \psi(x_2, x_3, t), \text{ and} \tag{16}$$

$$(x_1, x_2, x_3) \in S_t(I_3) \text{ if and only if } x_1 = \psi(x_2, x_3, t). \tag{17}$$

¿From (16), (17) we have also

$$(x_1, x_2) \in \Omega_t(x_3) \text{ if and only if } x_1 < \psi(x_2, x_3, t), \text{ and} \tag{18}$$

$$(x_1, x_2) \in S_t(x_3) \text{ if and only if } x_1 = \psi(x_2, x_3, t)\,. \tag{19}$$

In view of (16) $\cdots$ (19), we have

$$\nu = \left(1, -\frac{\partial}{\partial x_2}\psi, -\frac{\partial \psi}{\partial x_3}\right) \Big/ \sqrt{1 + \left(\frac{\partial}{\partial x_2}\psi\right)^2 + \left(\frac{\partial}{\partial x_3}\psi\right)^2} \text{ and} \tag{20}$$

$$\tilde{\nu} = \left(1, -\frac{\partial}{\partial x_2}\psi, 0\right) \Big/ \sqrt{1 + \left(\frac{\partial}{\partial x_2}\psi\right)^2} \text{ on } \mathcal{S} \cap (U \times J). \tag{21}$$

¿From (10), (17), we have $\theta(\psi(x_2, x_3, t), x_2, x_3, t) = 0$ on $\mathcal{S} \cap (U \times J)$. Differentiating in t, we obtain

$$\left(\frac{\partial}{\partial t} + \left(\frac{\partial \psi}{\partial t}(x_2, x_3, t), 0, 0\right) \cdot \nabla_x\right) \theta = 0 \text{ on } \mathcal{S} \cap (U \times J).$$

Subtracting this from (13), we find that $\left[\left(\frac{\partial \psi}{\partial t}(x_2, x_3, t), 0, 0\right) - \sigma\nu\right]$ is orthogonal to $\nabla_x \theta$, hence also to ν. This yields the formula

$$\sigma = \left(\frac{\partial \psi}{\partial t}(x_2, x_3, t), 0, 0\right) \cdot \nu = \left(\frac{\partial \psi}{\partial t}\right) \Big/ \sqrt{1 + \left(\frac{\partial \psi}{\partial x_2}\right)^2 + \left(\frac{\partial \psi}{\partial x_3}\right)^2}\,. \tag{22}$$

Similarly, subtracting the two equations (13), we learn that $(\sigma\nu - \tilde{\sigma}\tilde{\nu})$ is orthogonal to $\nabla_x\theta$, hence also to ν. Therefore,

$$\sigma = \tilde{\sigma}\,(\nu \cdot \tilde{\nu}) \text{ on } \mathcal{S}\,. \tag{23}$$

¿From (20) $\cdots$ (23), we obtain

$$\tilde{\sigma} = \left(\frac{\partial\psi}{\partial t}\right) \Big/ \sqrt{1 + \left(\frac{\partial\psi}{\partial x_2}\right)^2} \text{ on } \mathcal{S} \cap (U \times J)\,. \tag{24}$$

Now we can read off (14) and (15). In fact, (18) gives

$$\frac{d}{dt}\left[\int_{\Omega_t(x_3)} F\,d\,(\text{Area})\right] = \frac{d}{dt}\left[\iint_{x_1<\psi(x_2,x_3,t)} F dx_1\,dx_2\right] = \int F(\psi, x_2, x_3)\,\frac{\partial\psi}{\partial t}\,dx_2,$$

and (19), (24) yield

$$\int_{S_t(x_3)} F\tilde{\sigma}\,d\,(\text{length}) = \int F\tilde{\sigma}\cdot\sqrt{1+\left(\frac{\partial\psi}{\partial x_2}\right)^2}\,dx_2 = \int F(\psi, x_2, x_3)\,\frac{\partial\psi}{\partial t}\,dx_2,$$

proving (14). Similarly, (17) gives

$$\int_{S_t(I)} F\,d\,(\text{Area}) = \int F(\psi, x_2, x_3)\cdot\sqrt{1+\left(\frac{\partial}{\partial x_2}\psi\right)^2+\left(\frac{\partial}{\partial x_3}\psi\right)^2}\,dx_2\,dx_3$$

while (20) and (21) imply

$$\nu\cdot\tilde{\nu} = \sqrt{1+\left(\frac{\partial}{\partial x_2}\psi\right)^2} \Big/ \sqrt{1+\left(\frac{\partial}{\partial x_2}\psi\right)^2+\left(\frac{\partial}{\partial x_3}\psi\right)^2},$$

so that (19) yields

$$\int\limits_{x_3\in I}\left\{\int\limits_{S_t(x_3)}\frac{F}{\nu\cdot\tilde{\nu}}\,d\,(\text{length})\right\}dx_3 = \int\limits_{x_3\in I}\left\{\int\left(\frac{F}{\nu\cdot\tilde{\nu}}\right)\sqrt{1+\left(\frac{\partial}{\partial x_2}\psi\right)^2}\,dx_2\right\}dx_3$$

$$= \int F(\psi,x_2,x_3)\cdot\sqrt{1+\left(\frac{\partial}{\partial x_2}\psi\right)^2+\left(\frac{\partial}{\partial x_3}\psi\right)^2}\,dx_2\,dx_3.$$

This completes the proof of (15). Note that (23) allows us to rewrite (15) in the form

$$\int\limits_{S_t(I)}F\,d(\text{Area}) = \int\limits_{x_3\in I}\left\{\int\limits_{S_t(x_3)}F\,\frac{\tilde{\sigma}}{\sigma}\,d\,(\text{length})\right\}dx_3\,. \tag{25}$$

4 Proof of the Theorem

We retain the notation of the previous sections. We will define a time-dependent interval

$$J_t = [A(t),B(t)]\subset I_3 \tag{26}$$

and establish an obvious formula for the time derivative of Vol $\Omega_t(J_t)$. We assume that the endpoints $A(t)$, $B(t)$ are C^1 functions of t. We have

$$\text{Vol}\,\Omega_t(J_t) = \int\limits_{x_3\in J_t}\text{Area}\,\Omega_t(x_3)dx_3, \text{so that}$$

$$\frac{d}{dt}\text{Vol}\,\Omega_t(J_t) = B'(t)\,\text{Area}\,\Omega_t(B(t)) - A'(t)\,\text{Area}\,\Omega_t(A(t)) + \int\limits_{x_3\in J_t}\frac{\partial}{\partial t}\text{Area}\,\Omega_t(x_3)\,dx_3\,.$$

Applying (14) with $F\equiv 1$, we find that

$$\frac{d}{dt}\mathrm{Vol}\,\Omega_t(J_t) = B'(t)\,\mathrm{Area}\,\Omega_t(B(t)) - A'(t)\,\mathrm{Area}\,\Omega_t(A(t)) + \int\limits_{x_3\in J_t} \left\{ \int\limits_{S_t(x_3)} \tilde{\sigma}\, d(\mathrm{length}) \right\} dx_3\,.$$

In view of (25) (with $F \equiv \sigma$ on $\mathcal{S}$), this is equivalent to

$$\frac{d}{dt}\,\mathrm{Vol}\,\Omega_t(J_t) = B'(t)\,\mathrm{Area}\,\Omega_t(B(t)) \;-\; A'(t)\,\mathrm{Area}\,\Omega_t(A(t)) + \int\limits_{S_t(J_t)} \sigma\, d(\mathrm{Area})\,. \tag{27}$$

Now we bring in the hypothesis that Ω_t moves with a divergence-free C^1 velocity field u. From (6) and (13), we see that $(\sigma\nu - u)\cdot\nabla_x\theta = 0$ on $S_t(J_t)$. Thus $(\sigma\nu - u)$ is orthogonal to ν, so that $\sigma = u\cdot\nu$ on $S_t(J_t)$, and (27) may be rewritten as

$$\frac{d}{dt}\,\mathrm{Vol}\,\Omega_t(J_t) = B'(t)\,\mathrm{Area}\,\Omega_t(B(t)) \;-\; A'(t)\,\mathrm{Area}\,\Omega_t(A(t)) + \int\limits_{S_t(J_t)} u\cdot\nu d\,(\mathrm{Area})\,. \tag{28}$$

On the other hand, since $u = (u_1, u_2, u_3)$ is divergence-free, the divergence theorem yields

$$0 = \int\limits_{\Omega_t(J_t)} (\nabla_x\cdot u)\, d(\mathrm{Vol}) = \int\limits_{S_t(J_t)} u\cdot\nu\, d(\mathrm{Area}) \;+ \int\limits_{\Omega_t(B(t))} u_3\, d(\mathrm{Area}) \;- \int\limits_{\Omega_t(A(t))} u_3 d\,(\mathrm{Area})\,.$$

Hence, (28) may be rewritten in the form

$$\frac{d}{dt}\,\mathrm{Vol}\,\Omega_t(J_t) = \int\limits_{\Omega_t(B(t))} [B'(t) - u_3(x,t)]\, d\,(\mathrm{Area}) \;- \int\limits_{\Omega_t(A(t))} [A'(t) - u_3(x,t)]\, d(\mathrm{Area})\,. \tag{29}$$

This is our final formula for the time derivative of Vol $\Omega_t(J_t)$. It is intuitively clear.

We now pick the time-dependent interval $J_t = [A(t), B(t)] \subset I_3$. Let $I_3 = [a, b]$, and let $t_0 \in (0, T)$ be a time to be picked below. We define

$$B(t) = b - \int\limits_t^T \max_{x\in Q}\, |u(x,\tau)|\, d\tau\,, \tag{30}$$

and

$$A(t) = a + \int_t^T \max_{x \in Q} |u(x,\tau)| \, d\tau \,. \tag{31}$$

We are assuming that $u(x,\tau)$ is continuous on $Q \times [0,T)$, and that $\int_0^T \max_{x\in Q} |u(x,\tau)| d\tau < \infty$. It follows that $A(t), B(t)$ are C^1 functions on $[0,T)$, and that

$$a \le A(t) < B(t) \le b \text{ for } t \in [t_0, T) \,, \tag{32}$$

provided we pick t_0 close enough to T. We pick t_0 so that (32) holds. Thus, $\Omega_t(J_t) \subset Q$ for $t \in [t_0, T)$. Immediately from (30), (31), we obtain

$$B'(t) = \; -A'(t) = \max_{x \in Q} |u(x,t)| \; \ge \max_{x \in \Omega_t(A(t)) \cup \Omega_t(B(t))} |u_3(x,t)| \tag{33}$$

(recall $u = (u_1, u_2, u_3)$).

¿From (29) and (33) we see at once that

$$\frac{d}{dt} \operatorname{Vol} \Omega_t(J_t) \; \ge \; 0 \text{ for } t \in [t_0, T) \,. \tag{34}$$

On the other hand, (4) and (32) show that $\operatorname{Vol} \Omega_{t_0}(J_{t_0}) \; > \; 0$.

Consequently,

$$\liminf_{t \to T-} \operatorname{Vol} \Omega_t \; \ge \; \liminf_{t \to T-} \operatorname{Vol} \Omega_t(J_t) \; \ge \; \operatorname{Vol} \Omega_{t_0}(J_{t_0}) \; > \; 0 \,.$$

The proof of our theorem is complete. ■

5 References

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