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GLAUBER DYNAMICS OF THE RANDOM ENERGY MODEL

II. Aging below the critical temperature

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Abstract: We investigate the long-time behavior of the Glauber dynamics for the random energy model below the critical temperature. We establish that for a suitably chosen timescale that diverges with the size of the system, one can prove that a natural autocorrelation function exhibits aging. Moreover, we show that the long-time asymptotics of this function coincide with those of the so-called "REM-like trap model" proposed by Bouchaud and Dean. Our results rely on very precise estimates on the distribution of transition times of the process between different states of extremely low energy.

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1.Introduction and background

1.1. Introduction.

In this paper we continue the analysis of the Glauber dynamics of the random energy model that was started in [BBG1]. We refer the reader to the introduction of that paper for the general background of the problem.

We recall that we consider the following version of the REM. A spin configuration σ is a vertex of the hypercube $\mathcal{S}_N \equiv \{-1,1\}^N$. On an abstract probability space $(\Omega, \mathcal{F}, P)$ we define the family of i.i.d. standard normal random variables $\{X_\sigma\}_{\sigma\in\mathcal{S}_N}$. We set $E_\sigma \equiv [X_\sigma]_+ \equiv (X_\sigma \wedge 0)$. We define a random (Gibbs) probability measure on $\mathcal{S}_N$, $\mu_{\beta,N}$, by setting

$$\mu_{\beta,N}(\sigma) \equiv \frac{e^{\beta\sqrt{N}E_\sigma}}{Z_{\beta,N}} \tag{1.1}$$

where $Z_{\beta,N}$ is the normalizing partition function[4]. It is well-known [D1,D2]] that this model exhibits a phase transition at $\beta_c = \sqrt{2\ln 2}$. For $\beta \leq \beta_c$, the Gibbs measures is supported, asymptotically as $N \uparrow \infty$ on the set of states σ for which $E_\sigma \sim \sqrt{N}\beta$, and no single configuration has positive mass. For $\beta > \beta_c$, on the other hand, the Gibbs measure gives positive mass to the extreme elements of the order statistics of the family E_σ.

The dynamics we will consider is a *discrete time Glauber dynamics*. That is we construct a Markov chain $\sigma(t)$ with state space $\mathcal{S}_N$ and discrete time $t \in \mathbb{N}$ by prescribing transition probabilities $p_N(\sigma,\eta) = P[\sigma(t+1) = \eta | \sigma(t) = \sigma]$ by

$$p_N(\sigma,\eta) = \begin{cases} \frac{1}{N} e^{-\beta\sqrt{N}E_\sigma}, & \text{if } \|\sigma-\eta\|_2 = \sqrt{2} \\ 1 - e^{-\beta\sqrt{N}E_\sigma}, & \text{if } \sigma = \eta \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

Note that the dynamics is also random, i.e. the law of the Markov chain is a measure valued random variable on Ω that takes values in the space of Markov measures on the path space $\mathcal{S}_N^{\mathbb{N}}$. We will mostly take a pointwise point of view, i.e. we consider the dynamics for a given fixed realization of the disorder parameter $\omega \in \Omega$ (we persistently suppress the dependence on ω in the notation).

It is easy to see that this dynamics is reversible with respect to the Gibbs measure $\mu_{\beta,N}$. One also sees that it represents a nearest neighbor random walk on the hypercube with traps

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[4] The standard model has X_σ instead of E_σ. This modification has no effect on the equilibrium properties of the model, and will be helpful for setting up the dynamics.

of random depths (i.e. the probability to make a zero step is rather large when E_σ is large)[5].

1.2 Bouchaud's trap model.

In this sub-section we will explain the heuristics of the dynamics of the REM that was developed in several papers by Bouchaud and others [B,BD,BM,BCKM]. We will actually give a slightly varied form of this model that will fit better with the rigorous analysis we will present later. Understanding the trap model will provide a crucial guideline for the analysis of the full model later on.

The basic idea of Bouchaud can be explained as follows. As was explained in [BBG1], the Gibbs measure of the REM for $\beta > \sqrt{2\ln 2}$ is concentrated, asymptotically, on a countable set of states. Therefore we know that the Glauber dynamics for these temperatures will spend almost all of its time in these same states. This suggests, as we will do in the main part of the paper, to consider the dynamics on these states at appropriate time scales. Instead of doing this, Bouchaud proposes to *define* directly an *new* dynamics on these countably many states in the infinite volume limit[6] that he *expects* to behave in the same way as the real model.

Thus we start with the random measure $\tilde{\mu}_\beta$ defined in Eq. 1.12 of [BBG1]. We want to introduce a stochastic process on the support of this measure that leaves $\tilde{\mu}_\beta$ invariant. Obviously we can identify the support of this measure with the atoms of the Poisson point process $\mathcal{P}$ (defined in Section 1.2 of [BBG1]). The question is what the transition probabilities should be.

Bouchaud proposes the following: Starting at a state i with energy E_i, the process waits an exponential time of mean $\tau_0 \exp(\alpha E_i)$ (where α has the physical meaning of $\alpha = \beta/\beta_c$), and then jumps at random to any of the other states j with equal probability. Here τ_0 denotes a time-scale that will have to be chosen appropriately later. The problem is that while we would want the process to reach each state with equal probability, this makes no sense given that there are infinitely many states. Thus we have to introduce some cut-off procedure. Bouchaud proposes to allow jumps only to the M states of largest mass, and to take the limit as $M \uparrow \infty$ in the end. We find it more instructive to restrict our process to states whose energy is larger than E, where E is a parameter that will be taken to $-\infty$ later[7].

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[5] We have chosen this particular dynamics for technical reasons. To study e.g. the Metropolis algorithm would require some extra work, but we expect essentially the same results to hold.

[6] This is completely analogous to the procedure of Ruelle to define a model based on Poisson process as the infinite volume version of Derrida's REM rather then proving the convergence of Derrida's model to this limit.

[7] This has the advantage that via the parameter E we control explicitly the time-scale we consider, whereas

This is very convenient, since it amounts to replace the Poisson process $\mathcal{P}$ (from Section 1.2 of [BBG1]) by its restriction $\mathcal{P}_E$ to the half line $[E,\infty)$. Since our new Poisson process has a finite intensity measure, it has a very useful representation: Consider a random variable $n_E \in \mathbb{N}$ that is Poisson distributed with parameter $\int_E^\infty e^{-x}dx = e^{-E}$. Let $E_i, i \in \mathbb{N}$ be a family of i.i.d. real valued r.v., independent of n_E whose common distribution has density $e^E e^{-x} 1\!\!1_{x\geq E}$ with respect to Lebesgue measure. Then $\mathcal{P}_E$ is equal in distribution to

$$\sum_{i=1}^{n_E} \delta_{E_i} \tag{1.3}$$

Given a realization of $\mathcal{P}_E$, we can now define a Markov process on the random set $\mathcal{S}_E \equiv \{1,\dots,M(E)\}$. Let $Y_E(n), n \in \mathbb{N}$ be an discrete time Markov chain with state space $\mathcal{S}_E$. We will actually only consider the case where $Y_E(n)$ are i.i.d. random variables with some distribution q. Next we introduce, for each $i \in \mathbb{N}$, a family $T_n(i), n \in \mathbb{N}$ of i.i.d. random variables taking values in $\mathbb{R}_+$ and having an exponential distribution with rates $\tau_i \equiv \tau_0 \exp(\alpha E_i)$, i.e.

$$P[T_n(i) \leq t] \equiv F_i(t) = 1 - e^{-t/\tau_i} \tag{1.4}$$

Now we set

$$R_n \equiv \sum_{k=1}^{n} T_k(Y_E(k)) \tag{1.5}$$

and

$$r(t) = n \quad \text{if} \quad R_n \leq t < R_{n+1} \tag{1.6}$$

Finally, the Markov jump process is defined as

$$X_E(t) \equiv Y_E(r(t)) \;\; t \geq 0 \tag{1.7}$$

Observe that the random variables τ_i are the atoms of a Poisson point process $\mathcal{N}^*$ obtained from $\mathcal{P}$ by transformation with the map $\tau : E \to \tau_0 e^{\alpha E}$. A simple computation shows that $\mathcal{N}^*$ is a Poisson process with intensity measure $\nu^*(dx) = \alpha^{-1}\tau_0^{1/\alpha} x^{-(1+\alpha)/\alpha} dx$ (see [Rue]). We will also denote by $\mathcal{N}_E^*$ the transform of the restricted process $\mathcal{N}_E$ which is of course just the restriction of $\mathcal{N}^*$ to the half-line $[\tau_0 e^{-\alpha E}, \infty)$.

Let us note that in the case where $Y_E(n), n \in \mathbb{N}$ are i.i.d., the random variables $T_k(Y_E(k)), k \in \mathbb{N}$ are also i.i.d., and therefore $r(t)$ is a renewal process. Moreover, in the case when the distribution, q, of $Y_E(k)$ is of the form $q(Y_E(k) = i) = p(\tau_i)$, for some non-negative function

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otherwise this would be some non-trivial random variable.

p satisfying $\sum_{i=1}^{n_E} p(\tau_i) = 1$, the law of the renewal variable $T_k(Y_E(k))$ can be expressed in terms of the process $\mathcal{N}_E^*$ as

$$P[T_k(Y_E(k)) > t] \equiv 1 - F_E(t) = \sum_{i=1}^{n_E} q_i(1 - F_i(t)) = \int \mathcal{N}_E^*(ds)p(s)e^{-t/s} \tag{1.8}$$

The two point function that is used to characterize the "aging" phenomenon is the probability that during a time-interval $[t, t+s]$ the process does not jump, i.e.

$$\Pi_E(s,t) \equiv P\left[\forall_{u\in[t,t+s], X_E(u^-)=X_E(u)}\right] \tag{1.9}$$

(we set $f(u^-) \equiv \lim_{v\uparrow u} f(v)$). Here we assume that the initial distribution of the chain coincides with the jump distribution, i.e., $P(X_E(0) = i) = p(\tau_i)$.

The following theorem paraphrases the results on the asymptotic behaviour for this correlation function as found by Bouchaud and Dean [BD]

Proposition 1.1:*Define*

$$H_0(w) \equiv \frac{1}{\pi \,cosec\,(\pi/\alpha)} \int_w^\infty dx \frac{1}{(1+x)x^{1/\alpha}} \tag{1.10}$$

Then, for $\alpha > 0$,

$$\lim_{E\downarrow-\infty} \frac{\Pi_E(s,t)}{H_0(s/t)} = 1, \quad P\text{-}a.s. \tag{1.11}$$

Moreover, the asymptotic behavior of $H_0(t/s)$ *when* s/t *tends to zero or* ∞, *respectively, is readily evaluated:*

(i) If $(s/t) \downarrow 0$,

$$H_0(s/t) = 1 - \frac{1}{\pi \,cosec\,(\pi/\alpha)} \int_0^{s/t} dx \frac{1}{(1+x)x^{1/\alpha}} \sim 1 - \frac{(s/t)^{1-1/\alpha}}{(1-1/\alpha)\pi \,cosec\,(\pi/\alpha)} \tag{1.12}$$

(ii) If $(s/t) \uparrow \infty$,

$$H_0(s/t) \sim \frac{1}{\pi \,cosec\,(\pi/\alpha)} \int_{s/t}^\infty dx \frac{1}{x^{1+1/\alpha}} = \frac{(t/s)^{1/\alpha}}{(1/\alpha)\pi \,cosec\,(\pi/\alpha)} \tag{1.13}$$

In the remainder of this subsection we outline the proof of this theorem.

Lemma 1.2 *The function* $\Pi_E(s,t)$ *defined in (1.9) satisfies the equations*

$$\Pi_E(s,t) = 1 - F_E(s+t) + \int_0^t \Pi_E(,t-u)dF_E(u) \tag{1.14}$$

Proof: The proof of this lemma is elementary since $\Pi_E(s,t)$ is a function of the renewal process $r(t)$ alone. $\diamondsuit$

Remember that we study the solution of this equation in the limit when $E \downarrow -\infty$. For this it is important to make a choice of the time-scale τ_0. The choice $\tau_0 = e^{-\alpha E}$ is natural since in this way we will measure time at the scale of the fastest states[8]. Our first step will be to replace F_E by its limit[9]

$$F_\infty(t) \equiv 1 - \alpha^{-1} \int_1^\infty dx e^{-t/x} x^{-(1+\alpha)/\alpha} \tag{1.15}$$

which is no longer random. From now on we will only consider the case when q is the uniform measure, $q_i = \frac{1}{n_E}$. Let $\Pi_\infty(s,t)$ denote the unique solution of the equation

$$\Pi_\infty(s,t) = 1 - F_\infty(s+t) + \int_0^t \Pi_\infty(s,t-u) dF_\infty(u) \tag{1.16}$$

Lemma 1.4 *For all $s,t \geq 0$,*

$$\lim_{E\downarrow -\infty} \Pi_E(s,t) = \Pi_\infty(s,t), \quad \mathbb{P}\text{-}a.s \tag{1.17}$$

The limiting equation (1.16) is solved following standard procedures (see e.g. [Fe]). One defines the renewal function $M(t)$ that solves the equation

$$M(t) = F_\infty(t) + \int_0^t M(t-u) dF_\infty(u) \tag{1.18}$$

In terms of this function, the solution of (1.16) is then given as

$$\Pi_\infty(s,t) = 1 - F_\infty(s+t) + \int_0^t (1 - F_\infty(s+t-u)) dM(u) \tag{1.19}$$

Setting $f_\infty(t) \equiv F'_\infty(t)$,

$$f_\infty(t) = \alpha^{-1} \int_1^\infty e^{-t/x} x^{-(2\alpha+1)/\alpha} dx \tag{1.20}$$

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[8] Other choices may lead to completely different behaviors.

[9] In this introduction we will not justify the various passages to limits (which is also never done in the physics literature). Note however that these issues are treated in Section 5, and the results proven there can easily be used to justify everything that we will do in the present section.

Denote by g^* the Laplace transform of a function g, i.e. $g^*(u) = \int_0^\infty e^{-ut} g(t)$. Then

$$\begin{aligned} F^*_\infty(u) &= u^{-1} - \alpha^{-1} \int_1^\infty \frac{dx}{(ux+1)x^{1/\alpha}} \\ &= u^{-1} - \alpha^{-1} u^{(1-\alpha)/\alpha} \int_u^{u\infty} \frac{dx}{(1+x)x^{1/\alpha}} \end{aligned} \tag{1.21}$$

In the last expression, the integration is understood to be along a transformed path in the complex plane if u is complex. Note that[10]

$$\int_0^\infty \frac{dx}{(1+x)x^{1/\alpha}} = \Gamma(\alpha^{-1})\Gamma(1-\alpha^{-1}) = \frac{\pi}{\sin(\pi/\alpha)} = \pi \text{cosec}\,(\pi/\alpha) \tag{1.22}$$

Thus, when $u \to 0$, the integral in (1.21) converges to the constant $\pi\text{cosec}\,(\pi/\alpha)$. Similarly, we have that

$$f^*_\infty(u) = \alpha^{-1} \int_1^\infty \frac{1}{1+ux} x^{-(1+\alpha)/\alpha} dx \tag{1.23}$$

In particular, $f^*_\infty(0) = 1$, and

$$1 - f^*_\infty(u) = \alpha^{-1} \int_1^\infty \left(1 - \frac{1}{1+ux}\right) x^{-(1+\alpha)/\alpha} dx = \alpha^{-1} u^{1/\alpha} \int_u^{u\infty} \frac{dx}{(x+1)x^{1/\alpha}} \tag{1.24}$$

Taking the Laplace transform of (1.18) this implies that

$$M^*(u) = \frac{F^*_\infty(u)}{1 - f^*_\infty(u)} = \frac{1}{\alpha^{-1} u^{(1+\alpha)/\alpha} \int_u^{u\infty} \frac{dx}{(1+x)x^{1/\alpha}}} - u^{-1} \tag{1.25}$$

and, by classical results on the asymptotics of the inverse Laplace transform (see [Doe], Vol. 2, Section 7), this in turn implies that for $t \uparrow +\infty$,

$$M(t) \sim \frac{t^{1/\alpha}}{\pi \alpha^{-1} \Gamma(\alpha^{-1}) \text{cosec}\,(\pi/\alpha)} - 1 \tag{1.26}$$

Finally, we can compute the asymptotics of the solution of equation (1.16). Here we will directly make use of the fact that the Laplace transform of $\Pi_\infty(s,t)$ is given explicitly as

$$\Pi^*_\infty(u,s) = \frac{\alpha \int_1^\infty e^{-s/x} \frac{dx}{(ux+1)x^{1/\alpha}}}{1 - f^*_\infty(u)} \tag{1.27}$$

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[10] Performing the change of variable $x = y^{-1} - 1$, $\int_0^\infty \frac{dx}{(1+x)x^{1/\alpha}} = \int_0^1 \frac{dy}{(1-x)^{1/\alpha} x^{1-1/\alpha}}$ where one recognizes the Beta integral $\int_0^1 \frac{dy}{(1-y)^{\mu-1} y^{\nu-1}} = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$

we have already established the asymptotics of $1 - f^*_\infty(u)$ near $u = 0$. We still need to treat the numerator. It will be convenient to write

$$\begin{aligned}\alpha^{-1}\int_1^\infty e^{-s/x}\frac{dx}{(ux+1)x^{1/\alpha}} = \alpha^{-1}\int_1^\infty dx\int_{s/x}^\infty dve^{-v}\frac{1}{(ux+1)x^{1/\alpha}}\\ =\alpha^{-1}\int_0^\infty dve^{-v}\int_{s/v\wedge 1}^\infty dx\frac{1}{(ux+1)x^{1/\alpha}}\\ =\alpha^{-1}\int_0^\infty dve^{-v}\int_{s/v}^\infty dx\frac{1}{(ux+1)x^{1/\alpha}}\\ -\alpha^{-1}\int_s^\infty dve^{-v}\int_{s/v}^1 dx\frac{1}{(ux+1)x^{1/\alpha}}\end{aligned} \tag{1.28}$$

Now the first term can be conveniently represented as u^α times an explicit Laplace transform:

$$\alpha^{-1}\int_0^\infty dve^{-v}\int_{s/v}^\infty dx\frac{1}{(ux+1)x^{1/\alpha}} = \alpha^{-1}u^{1/\alpha}\int_0^{\infty/u} dve^{-uv}\int_{s/v}^{u\infty} dx\frac{1}{(x+1)x^{1/\alpha}} \tag{1.29}$$

Note that since all integrands vanish at infinity in the right-half plane, $0/u$ and $u\infty$ can be replaced with 0 and ∞, resp., i.e. the integration contours can be deformed to integrations along the real line. We will show that this term is the dominant one.

In fact, combining (1.24) with (1.28) we get from (1.27) that

$$\Pi^*_\infty(u,s) = \frac{\int_0^{\infty/u} dve^{-uv}\int_{s/v}^{u\infty} dx\frac{1}{(1+x)x^{1/\alpha}}}{\int_u^\infty \frac{dx}{(1+x)x^{1/\alpha}}} - \frac{\int_s^\infty dve^{-v}\int_{s/v}^1 dx\frac{1}{(u+1/x)x^{1/\alpha}}}{u^{1/\alpha}\int_u^\infty \frac{dx}{(1+x)x^{1/\alpha}}} \tag{1.30}$$

Now the integral in the denominator equals

$$\begin{aligned}\int_u^\infty \frac{dx}{(1+x)x^{1/\alpha}} &= \int_0^\infty \frac{dx}{(1+x)x^{1/\alpha}} - \int_0^u \frac{dx}{(1+x)x^{1/\alpha}}\\ &= \pi\mathrm{cosec}\,(\pi/\alpha) - u^{1-1/\alpha}\sum_{n=0}^\infty(-1)^n\frac{u^n}{n+1-1/\alpha}\end{aligned} \tag{1.31}$$

where the last sum is convergent for $|u| < 1$. Thus the leading singular (at $u = 0$) term from the first term in (1.30) is given by

$$\frac{\int_0^\infty dve^{-uv}\int_{s/v}^\infty dx\frac{1}{(1+x)x^{1/\alpha}}}{\pi\mathrm{cosec}\,(\pi/\alpha)} \tag{1.32}$$

which obviously is the Laplace transform of the function $H_0(s/t)$.

It remains to consider the second term in (1.30). Here the numerator converges to a constant as u tends to zero, in fact, at $u = 0$ it equals

$$\int_s^\infty dve^{-v}\int_{s/v}^1 dx\frac{1}{x^{1/\alpha}} = \frac{1}{1-1/\alpha}\int_s^\infty dye^{-y}\left[1-y^{1-1/\alpha}\right] \leq const.e^{-s} \tag{1.33}$$

Therefore the leading asymptotic of the second term is given by

$$Const. u^{-1/\alpha} e^{-s} \tag{1.34}$$

The inverse Laplace transform of the second term has therefore the leading asymptotic behavior

$$H_1(s,t) \sim Const. t^{1/\alpha-1} e^{-s} \tag{1.35}$$

Note that while the asymptotics in t looks the same as that of the second term of $H_0(s/t)$ in the case $s/t \downarrow 0$, due to the exponential decay in s, this term can be neglected if s is large. Thus we have now established the "aging" asymptotics found in Bouchaud.

1.3. The renewal equations. Statement of the main results.

Guided by Bouchaud's trap model, we can now construct the setup for the analysis of aging in the full REM dynamics. First of all the natural subset of states in $\mathcal{S}_N$ to play the rôle of the state space in the trap model is the set

$$T_N(E) \equiv \left\{ \sigma \in \mathcal{S}_N \middle| E_\sigma \geq u_N(E) \right\} \tag{1.36}$$

where (recall Section 1.1 of [BBG1])

$$u_N(x) \equiv \sqrt{2N \ln 2} + \frac{x}{\sqrt{2N \ln 2}} - \frac{1}{2} \frac{\ln(N \ln 2) + \ln 4\pi}{\sqrt{2N \ln 2}} \tag{1.37}$$

We will call the set $T_N(E)$ 'the top', and frequently suppress indices, writing $T = T(E) = T_N(E)$ whenever no confusion seems possible. Moreover, we will use the convention that $M \equiv |T_N(E)|$, and $d \equiv 2^M$.

The idea is clearly to observe the process only at its visits to T. The natural generalization of Bouchaud's correlation function $\Pi_E(s,t)$ is therefore the probability that the process does not jump from a state in the top to another state in the top during a time interval of the form $[n, n+m]$. There is some ambiguity how this should be defined precisely, but the following definition appears most convenient. To formulate it, let us introduce the following random times. For any $k \in \mathbb{N}$, let k_- denote the last time before k at which the process has visited the top, i.e.

$$k_- \equiv \sup \{ l < k \mid \sigma(l) \in T_N(E) \} \tag{1.38}$$

Now set

$$\Pi(n, m, N, E) \equiv \mathbb{P} \left[\forall_{k \in [n+1, n+m]} \sigma(k) \notin T_N(E) \backslash \sigma(k_-) \right] \tag{1.39}$$

Of course we still have to specify the initial distribution. To be as close as possible to Bouchaud, the natural choice is the uniform distribution on $T_N(E)$ that we will denote by π_E. However, we will also need to introduce the respective functions with starting point in an arbitrary state σ. Thus we set

$$\Pi_\sigma(m,m,N,E) \equiv \mathbb{P}\left[\forall_{k\in[n+1,n+m]}\,\sigma(k)\not\in T_N(E)\backslash\sigma(k_-) \mid \sigma(0)=\sigma\right] \tag{1.40}$$

and

$$\Pi(m,n,N,E) \equiv \frac{1}{|T_N(E)|}\sum_{\sigma\in T_N(E)}\Pi_\sigma(n,m,N,E) \tag{1.41}$$

We will also use vector notation and write $\underline{\Pi}(n,m,N,E)$ for the M dimensional vector with components $\Pi_\sigma(n,m,N,E)$, $\sigma\in T_N(E)$. We are now ready to state the main theorem of this paper.

Theorem 1 *Let* $\beta>\sqrt{2\ln 2}$. *Then there is a sequence* $c_N\sim\exp(\beta\sqrt{N}u_N(E))$ *such that for any* $\epsilon>0$

$$\lim_{t,s\uparrow\infty}\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty}P\left[\left|\frac{\Pi([c_Ns],[c_Nt],N,E)}{\Pi_\infty(s,t)}-1\right|>\epsilon\right]=0 \tag{1.42}$$

where $\Pi_\infty(s,t)$ *is the limiting correlation function of the trap model, defined in (1.17).*

Before closing the introduction, let us say a few words about the heuristics of this theorem and the difficulties we will have to expect. Let us recall from [BBG1]the notation, for $\sigma\in\mathcal{S}_N, I\subset\mathcal{S}_N$,

$$\tau_I^\sigma\equiv\inf\{n>0\mid\sigma(n)\in I,\sigma(0)=\sigma\} \tag{1.43}$$

for the first positive time the process starting in σ reaches the set I, Note that it is easy to derive a renewal equation for the quantities (1.40). Namely, the event in the probability in (1.40) occurs either

(i) if $\sigma(k)\not\in T(E)\backslash\sigma$, for all $k\in[0,n+m]$, or

(ii) if there is $0<l\le n$, s.t. $l=\inf\{k\le n\mid\sigma(k)\in T(E)\backslash\sigma\}$, and $\forall_{k\in[n+1,n+m]}\,\sigma(k)\not\in T_N(E)\backslash\sigma(k_-)$.

Since this decomposition is disjoint, it implies immediately the following system of renewal equations:

$$\begin{aligned}\Pi_\sigma(m,n,E) &= \mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} > m+n]\\ &+\sum_{k=1}^{n}\sum_{\sigma'\in T(E)\backslash\sigma}\mathbb{P}_\sigma[\tau^\sigma_{T(E)\backslash\sigma}=k, X_k=\sigma', X_l\not\in T(E)\backslash X_{l_-}, \forall n\leq l\leq m+n]\\ &= \mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} > m+n]+\sum_{k=1}^{n}\sum_{\sigma'\in T(E)\backslash\sigma}\mathbb{P}[\tau^\sigma_{\sigma'}=\tau^\sigma_{T(E)\backslash\sigma}=k]\Pi_{\sigma'}(m,n-k,E)\end{aligned} \tag{1.44}$$

The extra difficulty stems from the fact that the kernels $\mathbb{P}[\tau^\sigma_{\sigma'}=\tau^\sigma_{T(E)\backslash\sigma}=k]$ depend on both σ and σ', while in the trap model it is assumed that this quantity is independent of σ' for any value of k. Indeed, if we had the relation

$$\mathbb{P}[\tau^\sigma_{\sigma'}=\tau^\sigma_{T(E)\backslash\sigma}=k]=\frac{\pi_E(\sigma')}{1-\pi_E(\sigma)}\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}=k] \tag{1.45}$$

averaging (1.44) over σ would yield

$$\begin{aligned}\Pi(m,n,E) &= \sum_{\sigma\in T(E)}\pi_E(\sigma)\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}>m+n]\\ &+\sum_{k=1}^{n}\sum_{\sigma\in T(E)}\pi_E(\sigma)\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}=k]\sum_{\sigma'\in T(E)\backslash\sigma}\frac{\pi_E(\sigma')}{1-\pi_E(\sigma)}\Pi_{\sigma'}(m,n-k,E)\\ \\ &=\sum_{\sigma\in T(E)}\pi_E(\sigma)\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}>m+n]+\sum_{k=1}^{n}\sum_{\sigma\in T(E)}\pi_E(\sigma)\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}=k]\Pi(m,n-k,E)\\ &+\sum_{k=1}^{n}\sum_{\sigma\in T(E)}\frac{\pi_E(\sigma)}{1-\pi_E(\sigma)}\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}=k]\pi_E(\sigma)[\Pi(m,n-k,E)-\Pi_\sigma(m,n-k,E)]\end{aligned} \tag{1.46}$$

The last term is bounded by $|T(E)|^{-1}$ which tends to zero uniformly as $E\uparrow\infty$ and would be treated as an error term. If we ignore this term for a moment, (1.46) takes the desired form:

Setting

$$F_{N,E}(n)\equiv\sum_{\sigma\in T(E)}\pi_E(\sigma)\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}>n] \tag{1.47}$$

and

$$f_{N,E}(n)\equiv\sum_{\sigma\in T(E)}\pi_E(\sigma)\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma}=n] \tag{1.48}$$

Eq. (1.46) then becomes

$$\Pi(m,n,E)=F_{N,E}(m+n)+\sum_{k=1}^{n}f_{N,E}(k)\Pi(m,n-k,E) \tag{1.49}$$

which has the form of the equation in the trap model. Unfortunately, even though we will show in Section 2 that (1.45) is true (up to an negligible error) when summed over k, we have not been able to find an argument that would show that (1.45) was true pointwise. Thus the only way out appears to be to study the solution of the full system (1.44). This will require some substantial preparations and will be undertaken only in Section 5.

The remainder of this paper is devoted to proving Theorem 1. In the next section we recall some important results from [BBG1]. In Section 3 we prove the necessary refined estimates on the probability distributions appearing as kernels or inhomogeneous terms in the renewal system (1.44). Armed with these estimates, we will return to the analysis of the solution of this system in Section 4 where we prove Theorem 1.

2. Basic estimates.

We will briefly recall a number of estimates that were proven in [BBG1] and that we will use heavily in our analysis.

The first concerns various hitting probabilities.

Proposition 2.1: *Set* $M = |T(E)|$, $d = 2^M$ *and* $\delta(N) \equiv \left(\frac{d}{N}\right)^{1/2} \log N$. *There exists a subset* $\mathcal{E} \subset \Omega$ *with* $P(\mathcal{E}) = 1$, *such that for all* $\omega \in \mathcal{E}$, *for all* N *large enough, the following holds:*

For $\varepsilon > 0$ *a constant, define the sets*

$$\mathcal{B}_{\sqrt{\varepsilon N}}(\sigma) = \{\sigma' \in \mathcal{S}_N \mid \|\sigma' - \sigma\|_2 \leq \sqrt{\varepsilon N}\}, \quad \sigma \in \mathcal{S}_N \tag{2.1}$$

and

$$W_\varepsilon(I) \equiv \bigcap_{\sigma \in I} \mathcal{B}^c_{\sqrt{\varepsilon N}}(\sigma), \quad I \subseteq \mathcal{S}_N \tag{2.2}$$

Then,

i) For all $\varepsilon > 0$ *there exists a constant* $c > 0$ *such that, for all* $\eta \in T$ *and all* $\sigma \in W_\varepsilon(T)$,

$$\left| \mathbb{P}\left(\tau^\sigma_\eta < \tau^\sigma_{T\setminus\eta}\right) - \tfrac{1}{M} \right| \leq \tfrac{d}{NM}(1 + c\delta(N)) \tag{2.3}$$

ii) There exists a constant $c > 0$ *such that, for all* $\eta \in T$ *and* $\bar\eta \in T$ *with* $\eta \neq \bar\eta$,

$$\left| e^{\beta\sqrt{N}E_{\bar\eta}} \mathbb{P}\left(\tau^{\bar\eta}_\eta < \tau^{\bar\eta}_{T\setminus\eta}\right) - \tfrac{1}{M} \right| \leq \tfrac{d}{NM}(1 + c\delta(N)) \tag{2.4}$$

iii) There exists a constant $c > 0$ such that, for all $\eta \in T$ and $\bar\eta \in T$ with $\eta \neq \bar\eta$,

$$\left|\mathbb{P}\left(\tau_{\eta}^{\bar\eta} < \tau_{T\setminus\{\eta,\bar\eta\}}^{\bar\eta}\right) - \tfrac{1}{M-1}\right| \leq \tfrac{d}{N(M-1)}(1 + c\delta(N)) \tag{2.5}$$

iv) There exists a constant $c > 0$ such that, for all $\eta \in T$,

$$\left|e^{\beta\sqrt{N}E_\eta}\mathbb{P}\left(\tau_{T\setminus\eta}^{\eta} < \tau_{\eta}^{\eta}\right) - \left(1 - \tfrac{1}{M}\right)\right| \leq \left(1 - \tfrac{1}{M}\right)\tfrac{d}{N}(1 + c\delta(N)) \tag{2.6}$$

v) There exists a constant $c > 0$ such that, for all $\sigma \notin T$,

$$\left(1 - \tfrac{1}{M}\right)\left(1 - \tfrac{d}{N}(1 + c\delta(N))\right) \leq e^{\beta\sqrt{N}E_\sigma}\mathbb{P}\left(\tau_{T}^{\sigma} < \tau_{\sigma}^{\sigma}\right) \leq 1 \tag{2.7}$$

vi) For all $\varepsilon > 0$ there exists a constant $c > 0$ such that, for all $\sigma \notin T$ and all $\bar\sigma \in W_\varepsilon(T \cup \sigma)$,

$$\mathbb{P}\left(\tau_{\sigma}^{\bar\sigma} \leq \tau_{T}^{\bar\sigma}\right) \leq \tfrac{1}{M} + \tfrac{d}{NM}(1 + c\delta(N)) \tag{2.8}$$

The next statement (Theorem 1.4 of [BBG1]) gives sharp estimates on mean transition times.

Theorem 2.2: *Assume that $\alpha \equiv \beta/\sqrt{2\ln 2} > 1$. Then there exists a subset $\widetilde{\mathcal{E}} \subset \Omega$ with $P(\widetilde{\mathcal{E}}) = 1$, such that for all $\omega \in \widetilde{\mathcal{E}}$, for all N large enough, the following holds:*

i) For all $\eta \in T(E)$,

$$\mathbb{E}(\tau_{T(E)\setminus\eta}^{\eta}) = \frac{1}{1 - \frac{1}{M}}\left[e^{\beta\sqrt{N}E_\eta^+} + \mathcal{W}_{\beta,N,T(E)}\right](1 + O(1/N)) \tag{2.9}$$

ii) For all $\sigma \notin T(E)$,

$$\begin{aligned}\mathbb{E}(\tau_{T(E)}^{\sigma}) &\leq \frac{1}{1 - \frac{1}{M}}\left[e^{\beta\sqrt{N}E_\sigma^+} + \mathcal{W}_{\beta,N,T(E)}\right](1 + O(1/N))\\ \mathbb{E}(\tau_{T(E)}^{\sigma}) &\geq \frac{1}{1 - \frac{1}{M}}\left[e^{\beta\sqrt{N}E_\sigma^+} + \frac{1 - e^{E}(\alpha - 1)}{1 + 1/M}\mathcal{W}_{\beta,N,T(E)}\right](1 + O(1/N))\end{aligned} \tag{2.10}$$

iii) For all $\eta, \bar\eta \in T(E)$, $\eta \neq \bar\eta$,

$$\left|\mathbb{E}(\tau_{\bar\eta}^{\eta} \mid \tau_{\bar\eta}^{\eta} \leq \tau_{T(E)\setminus\eta}^{\eta}) - \mathbb{E}(\tau_{T(E)\setminus\eta}^{\eta})\right| \leq \frac{1}{1 - \frac{1}{M}}\mathcal{W}_{\beta,N,T(E)}O(1/N) \tag{2.11}$$

where

$$\mathcal{W}_{\beta,N,T(E)} \equiv \frac{e^{(\alpha-1)E+\beta\sqrt{N}u_N(0)}}{M(\alpha-1)}\left(1+\mathcal{V}_{N,E}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right) \tag{2.12}$$

and $\mathcal{V}_{N,E}$ *is a random variable of mean zero and variance one.*

We will also make use of the following simple corollary to this proposition:

Corollary 2.3: *Under the assumptions and with the notation of Theorem 2.2 we have:*

i) For all $\eta, \bar\eta \in T$, $\eta \neq \bar\eta$*:*

$$\left|\frac{1}{|T(E)\setminus\eta|}\sum_{\bar\eta\in T(E)\setminus\eta}\mathbb{E}(\tau^\eta_{\bar\eta} \mid \tau^\eta_{\bar\eta} \le \tau^\eta_{T(E)\setminus\eta}) - \mathbb{E}(\tau^\eta_{\bar\eta} \mid \tau^\eta_{\bar\eta} \le \tau^\eta_{T(E)\setminus\eta})\right| \le \frac{1}{1-\frac{1}{M}}\mathcal{W}_{\beta,N,T(E)}O(1/N) \tag{2.13}$$

ii) For all $\eta \in T(E)$,

$$0 < \mathbb{E}(\tau^\eta_{T(E)\setminus\eta}) - \mathbb{P}^{-1}(\tau^\eta_{T\setminus\eta} < \tau^\eta_\eta) \le \frac{1}{1-\frac{1}{M}}\mathcal{W}_{\beta,N,T(E)}(1+O(1/N)) \tag{2.14}$$

Proof of Corollary 2.3: The first assertion is an obvious consequence of the last assertion of Theorem 2.2. The second assertion simply follows from Eq. (3.8) of [BBG1] and is proven just as the first assertion of Theorem 2.2. ◊

Equipped with this information we proceed in the next section to analyse the Laplace transforms of the distribution functions of such transition times.

3. Estimates on Laplace transforms.

We will use the method of Laplace tranforms to solve the system of renewal equations (1.44). Doing so this will require precise control on the Laplace transforms of the distributions functions of the probability distributions appearing in these equations. In this section we derive the basic estimates on these Laplace transforms.

As in [BEGK1], Section 3, the first crucial step is an estimate of the maximal mean time to reach the set $T(E)$.

Lemma 3.1: *Define*

$$\Theta(E) \equiv \max_{\sigma\in\mathcal{S}_N}\mathbb{E}\tau^\sigma_{T(E)} \tag{3.1}$$

and

$$\widehat{\Theta}(E) \equiv (1-\tfrac{1}{|T(E)|})^{-1} e^{\beta\sqrt{N}u_N(0)+\alpha E}\left[1+\frac{e^{-E}}{|T(E)|(\alpha-1)}\left(1+\mathcal{V}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}\right)\right](1+O(1/N)) \tag{3.2}$$

where $\mathcal{V}$ is a random variable of mean zero and variance 1*. Then, under the assumptions of Theorem 2.2,*

$$\Theta(E) \leq \widehat{\Theta}(E) \tag{3.3}$$

Proof: For $\sigma \notin T(E)$, the bound $\mathbb{E}\tau^\sigma_{T(E)} \leq \widehat{\Theta}(E)$ follows immediately from the estimate from Theorem 2.2, i). If $\sigma \in T(E)$, the forward Kolmogorov equation shows that

$$\mathbb{E}\tau^\sigma_{T(E)} = \sum_{\sigma' \in T(E)} p_N(\sigma,\sigma') + \sum_{\sigma' \notin T(E)} p_N(\sigma,\sigma')(1+\mathbb{E}\tau^{\sigma'}_{T(E)}) \tag{3.4}$$

Using the previous result in (3.4) one sees that the same estimate holds in this case. ♢

We define, for $\sigma \in \mathcal{S}_N$, $I, J \subset \mathcal{S}_N$, and $u \in D \subset \mathbb{C}$,

$$G^\sigma_{I,J}(u) \equiv \mathbb{E}e^{u\tau^\sigma_I}\mathbb{1}_{\{\tau^\sigma_I \leq \tau^\sigma_J\}} \equiv \sum_{n=1}^{\infty}\mathbb{P}[\tau^\sigma_I = n \leq \tau^\sigma_J] \tag{3.5}$$

where D is chosen such that the right-hand side of (3.5) exists. Note that this is always the case for u s.t. $\Re(u) \leq 0$, but in fact, for x, I, J given, there will be some $u_0 \equiv u_0(\sigma, I, J) > 0$, s.t. $G^\sigma_{I,J}(u)$ exists for all u with $\Re(u) \leq u_0$.

Theorem 3.2: *For any $\sigma \in T(E)$, the Laplace transform $G^\sigma_{T(E)\setminus\sigma}(u)$ can be written as*

$$G^\sigma_{T(E)\setminus\sigma}(u) = \frac{a_\sigma}{1-(1-e^{-u})\mathbb{E}\tau^\sigma_{T(E)\setminus\sigma}b_\sigma} + R_\sigma(u) \tag{3.6}$$

where

$$a_\sigma = 1 + O(\widehat{\Theta}(E)/\mathbb{E}\tau^\sigma_{T(E)\setminus\sigma}) \tag{3.7}$$

$$b_\sigma = 1 + O(\widehat{\Theta}(E)/\mathbb{E}\tau^\sigma_{T(E)\setminus\sigma}) \tag{3.8}$$

and $R_\sigma(u)$ is analytic in the half-plane $\Re(u) < 1/\widehat{\Theta}(E)$, periodic with period 2π in the imaginary direction, and satisfies

(i) for all $|u| \leq a/\widehat{\Theta}(E)$,

$$|R_\sigma(u)| \leq C(a)\left(e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E)\right)^2 \tag{3.9}$$

and

(ii) for all u with $\Re(u) < (1-\epsilon)\widehat{\Theta}(E)$ and $|1-e^{-u}| \geq 2\epsilon^{-1}e^{-\beta\sqrt{N}E_\sigma}$

$$|R_\sigma(u)| \leq 2\frac{e^{-\beta\sqrt{N}E_\sigma}}{|1-e^{-u}|(1-\Re(u)\widehat{\Theta}(E))} \tag{3.10}$$

Moreover,

$$a_\sigma + R_\sigma(0) = 1 \tag{3.11}$$

This proposition allows in fact to prove very good estimates on the distribution function of $\tau^\sigma_{T(E)\backslash\sigma}$. Note first that if

$$\mathcal{L}(u) \equiv \sum_{n=0}^{\infty} e^{un}\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} > n] \tag{3.12}$$

then

$$\mathcal{L}(u) = \frac{G^\sigma_{T(E)\backslash\sigma}(u) - 1}{e^u - 1} \tag{3.13}$$

Corollary 3.3: *With the notation of Theorem 3.2, for any $\epsilon > 0$ and for any positive integer $n \in \mathbb{N}$,*

$$\begin{aligned}\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} = n] =& \frac{a_\sigma}{\mathbb{E}\tau^\sigma_{T(E)\backslash\sigma}b_\sigma}e^{-n/\mathbb{E}\tau^\sigma_{T(E)\backslash\sigma}b_\sigma} \\ &+ O\left(e^{-n(1-\epsilon)/\widehat{\Theta}(E)}e^{-\beta\sqrt{N}E_\sigma}\epsilon^{-1}\ln\left(\widehat{\Theta}(E)\epsilon\right)\right)\end{aligned} \tag{3.14}$$

and (for $n > 0$)

$$\mathbb{P}[\tau^\sigma_{T(E)\backslash\sigma} > n] = a_\sigma e^{-n/\mathbb{E}\tau^\sigma_{T(E)\backslash\sigma}b_\sigma} + O\left(e^{-n(1-\epsilon)/\widehat{\Theta}(E)}e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E)\epsilon^{-1}\right) \tag{3.15}$$

Proof of Theorem 3.2: Our analysis of the Laplace transforms will follow closely the strategy employed in [BEGK1], but some simplifications will occur due to the particular properties of the model at hand.

3.1 A priori estimates on Laplace transforms.

As in [BEGK1], Lemma 3.1 implies immediate control on the Laplace transforms $g_\sigma^{\sigma'}(u) \equiv G_{\sigma,T(E)}^{\sigma'}(u)$:

Lemma 3.4: *For all $\epsilon > 0$, and for all real $u \leq (1-\epsilon)/\widehat{\Theta}(E)$, for all $\sigma, \sigma' \in \mathcal{S}_N$,*

$$g_\sigma^{\sigma'}(u) \leq \frac{1}{1-u\widehat{\Theta}(E)} \leq \epsilon^{-1} \tag{3.16}$$

Proof: The proof is identical to the proof of Lemma 3.2 of [BEGK1]. Just note that if we set

$$v_u(\sigma') \equiv \begin{cases} g_\sigma^{\sigma'}(u), & \text{for } \sigma' \notin T(E) \cup \sigma \\ 1, & \text{for } \sigma' = \sigma \\ 0, & \text{for } \sigma' \in T(E) \backslash \sigma \end{cases} \tag{3.17}$$

then v_u is the unique solution of the Dirichlet problem

$$\begin{aligned} (1-e^u P_N) v_u(\sigma') = 0, &\quad \text{if} \quad \sigma' \notin T(E) \cup \sigma \\ v_u(\sigma) = 1 & \\ v_u(\sigma') = 0 &\quad \text{if} \quad \sigma' \in T(E) \backslash \sigma \end{aligned} \tag{3.18}$$

Setting $w_u(\sigma') \equiv v_u(\sigma') - v_0(\sigma')$, we see that w_u solves

$$\begin{aligned} (1-P_N) w_u(\sigma') = (1-e^{-u}) v_u(\sigma'), &\quad \text{if} \quad \sigma' \notin T(E) \cup \sigma \\ w_u(\sigma') = 0 &\quad \text{if} \quad \sigma' \in T(E) \cup \sigma \end{aligned} \tag{3.19}$$

The solution of (3.19) can be represented as

$$w_u(\sigma') = \mathbb{E} \sum_{t=1}^{\tau_{T(E)\cup\sigma}^{\sigma'}-1} (1-e^{-u}) v_u(X_t) \tag{3.20}$$

implying that

$$v_u(\sigma') = \mathbb{P}[\tau_\sigma^{\sigma'} = \tau_{T(E)\cup\sigma}^{\sigma'}] + (1-e^{-u}) \mathbb{E} \sum_{t=1}^{\tau_{T(E)\cup\sigma}^{\sigma'}-1} v_u(X_t) \tag{3.21}$$

Setting $S(u) \equiv \max_{\sigma' \notin T(E)\cup\sigma} v_u(\sigma')$, (3.21) implies

$$\begin{aligned} S(u) &\leq 1 + (1-e^{-u}) \max_{\sigma' \notin T(E)\cup\sigma} \mathbb{E}\tau_{T(E)\cup\sigma}^{\sigma'} S(u) \\ &\leq 1 + u\widehat{\Theta}(E) S(u) \end{aligned} \tag{3.22}$$

and hence

$$S(u) \leq \frac{1}{1-u\widehat{\Theta}(E)} \tag{3.23}$$

which proves the lemma.◊

This basic estimate can be improved in certain cases:

Lemma 3.5: *Let $\sigma \in T(E)$. Then, for u as in Lemma 3.4,*

(i)

$$G^{\sigma}_{T(E)\backslash\sigma,\sigma}(u) \leq e^{-\beta\sqrt{N}E_\sigma}\frac{e^u}{1-u\widehat{\Theta}(E)} \leq 2\epsilon^{-1}\mathbb{P}[\tau^{\sigma}_{T(E)} < \tau^{\sigma}_{\sigma}] \tag{3.24}$$

(ii)

$$G^{\sigma}_{\sigma,T(E)}(u) \leq e^u\left(1+e^{-\beta\sqrt{N}E_\sigma}\frac{1}{1-u\widehat{\Theta}(E)}\right) \leq 1+2\epsilon^{-1}\mathbb{P}[\tau^{\sigma}_{T(E)} < \tau^{\sigma}_{\sigma}] \tag{3.25}$$

Proof: Let us first prove (i). This goes essentially along the same lines as the proof of Lemma 3.4. Define

$$\psi_u(\sigma') \equiv \begin{cases} G^{\sigma'}_{T(E)\backslash\sigma,\sigma}(u), & \text{for } \sigma' \notin T(E)\cup\sigma \\ 1, & \text{for } \sigma' \in T(E)\backslash\sigma \\ 0, & \text{for } \sigma' = \sigma \end{cases} \tag{3.26}$$

and $\phi_u(\sigma') \equiv \psi_u(\sigma') - \psi_0(\sigma')$. Then ϕ_u solves

$$\begin{aligned} (1-P_N)\phi_u(\sigma') &= (1-e^{-u})\psi_u(\sigma'), \quad \text{if} \quad \sigma' \notin T(E) \\ \phi_u(\sigma') &= 0 \quad \text{if} \quad \sigma' \in T(E) \end{aligned} \tag{3.27}$$

Just as in the previous proof, we get first the uniform bound

$$\psi_u(\sigma') \leq \frac{1}{1-u\widehat{\Theta}(E)} \tag{3.28}$$

Now

$$G^{\sigma}_{T(E)\backslash\sigma,\sigma}(u) = \sum_{\sigma'\neq\sigma} p_N(\sigma,\sigma')e^u G^{\sigma'}_{T(E)\backslash\sigma,\sigma}(u) \leq \sum_{\sigma'\neq\sigma}\frac{1}{N}e^{-\beta\sqrt{N}E_\sigma}\frac{e^u}{1-u\widehat{\Theta}(E)} \tag{3.29}$$

Since $\mathbb{P}[\tau^{\sigma}_{T(E)} < \tau^{\sigma}_{\sigma}] \sim e^{-\beta\sqrt{N}E_\sigma}$, (i) is proven.

In the same way,

$$
\begin{aligned}
G^{\sigma}_{\sigma,T(E)}(u) &= e^{u} p_N(\sigma,\sigma) + e^{u} \sum_{\sigma'\neq\sigma} p_N(\sigma,\sigma') G^{\sigma'}_{\sigma,T(E)}(u) \\
&\leq \left(1 + e^{-\beta\sqrt{N}E_\sigma} \frac{1}{1-u\widehat{\Theta}(E)}\right) e^{u}
\end{aligned}
\tag{3.30}
$$

and this proves (ii).♢

Finally we turn to the Laplace transform of hitting times without extra exclusion sets. We set $G^{\sigma}_{I}(u) \equiv \mathbb{E}e^{u\tau^{\sigma}_{I}}$.

Proposition 3.6: *Let $\sigma \in T(E)$. Then, for $u = \rho/\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma}$, if $|\rho| \leq (1-\gamma)$, $\gamma > 0$,*

$$
\begin{aligned}
G^{\sigma}_{T(E)\setminus\sigma}\left(\rho/\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma}\right) = &\frac{1}{1-\rho\left[1+O(e^{-\alpha u_N^{-1}(E_\sigma)+E}) + \rho O\left((\widehat{\Theta}(E)/\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma})^2\right)\right]} \\
&\times \left(1+\rho O(\widehat{\Theta}(E)/\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma})\right)
\end{aligned}
\tag{3.31}
$$

Proof: As in the analogous analysis in [BEGK1], the starting point of our analysis is the renewal equation

$$
G^{\sigma}_{T(E)\setminus\sigma}(u) = \frac{G^{\sigma}_{T(E)\setminus\sigma,\sigma}(u)}{1-G^{\sigma}_{\sigma,T(E)}(u)}
\tag{3.32}
$$

It is reasonable to rewrite this as

$$
G^{\sigma}_{T(E)\setminus\sigma}(u) = \frac{\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} < \tau^{\sigma}_{\sigma}]}{1-G^{\sigma}_{\sigma,T(E)}(u)} + \frac{G^{\sigma}_{T(E)\setminus\sigma,\sigma}(u) - G^{\sigma}_{T(E)\setminus\sigma,\sigma}(0)}{1-G^{\sigma}_{\sigma,T(E)}(u)} \equiv (I) + (II)
\tag{3.33}
$$

Using Taylor-Lagrange formula with remainder to second order, we have

$$
\begin{aligned}
(I) &= \frac{\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} < \tau^{\sigma}_{\sigma}]}{\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} < \tau^{\sigma}_{\sigma}] - u\mathbb{E}\tau^{\sigma}_{\sigma}\mathbb{1}_{\{\tau^{\sigma}_{\sigma}\leq\tau^{\sigma}_{T(E)}\}} - u^2/2\frac{d^2}{du^2}G^{\sigma}_{\sigma,T(E)}(\tilde{u})} \\
&= \left[1-\rho\frac{\mathbb{E}\tau^{\sigma}_{\sigma}\mathbb{1}_{\{\tau^{\sigma}_{\sigma}\leq\tau^{\sigma}_{T(E)}\}}}{\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} < \tau^{\sigma}_{\sigma}]\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma}} - \frac{1}{2}\rho^2\frac{\frac{d^2}{du^2}G^{\sigma}_{\sigma,T(E)}(\tilde{u})}{\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} < \tau^{\sigma}_{\sigma}](\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma})^2}\right]^{-1}
\end{aligned}
\tag{3.34}
$$

We want to show that the coefficient of ρ in the denominator is essentially equal to one, while the coefficient of ρ^2 tends to zero. Differentiating the renewal equation (3.32) and evaluating at $u=0$ gives

$$
\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}] = \mathbb{E}[\tau^{\sigma}_{T\setminus\sigma}] - \frac{\mathbb{E}\tau^{\sigma}_{\sigma}\mathbb{1}_{\{\tau^{\sigma}_{\sigma}\leq\tau^{\sigma}_{T}\}}}{1-\mathbb{P}[\tau^{\sigma}_{\sigma}\leq\tau^{\sigma}_{T}]}
\tag{3.35}
$$

which implies immediately that

$$\frac{\mathbb{E}\tau_\sigma^\sigma \mathbb{1}_{\{\tau_\sigma^\sigma \leq \tau_{T(E)}^\sigma\}}}{\mathbb{P}[\tau_{T(E)\setminus\sigma}^\sigma < \tau_\sigma^\sigma]\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma} \leq 1 \tag{3.36}$$

Moreover,

$$\mathbb{E}\tau_\sigma^\sigma \mathbb{1}_{\{\tau_\sigma^\sigma \leq \tau_{T(E)}^\sigma\}} \geq \mathbb{P}[\tau_\sigma^s = 1] = 1 - e^{\beta\sqrt{N}E_\sigma} \tag{3.37}$$

while by (2.9) of Theorem 2.2 and (2.6) of Proposition 2.1, the denominator in (3.36) is bounded from above by

$$1 + e^{-\alpha u_N^{-1}(E_\sigma)}\mathcal{V}e^{-E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}} \tag{3.38}$$

Thus

$$\frac{\mathbb{E}\tau_\sigma^\sigma \mathbb{1}_{\{\tau_\sigma^\sigma \leq \tau_{T(E)}^\sigma\}}}{\mathbb{P}[\tau_{T(E)\setminus\sigma}^\sigma < \tau_\sigma^\sigma]\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma} \geq \frac{1 - e^{-\beta\sqrt{N}E_\sigma}}{1 + e^{-\alpha u_N^{-1}(E_\sigma)}\mathcal{V}e^{E/2}\frac{\alpha-1}{\sqrt{2\alpha-1}}} \tag{3.39}$$

Next we turn to the coefficient of ρ^2. By (3.30) we can write

$$G_{\sigma,T(E)}^\sigma(u) = e^u p_N(\sigma,\sigma) + f(u) \tag{3.40}$$

where $f(u)$ is analytic in the half-plane $\Re(u) < 1/\widehat{\Theta}(E)$ and satisfies

$$f(u) \leq e^{-\beta\sqrt{N}E_\sigma}\frac{e^u}{1 - u\widehat{\Theta}(E)} \tag{3.41}$$

By Cauchy's integral formula, this implies that for $\Re(u) < (1-\gamma)/\widehat{\Theta}(E)$,

$$|f''(u)| \leq e^{-\beta\sqrt{N}E_\sigma}\frac{C\gamma^{-1}}{(\widehat{\Theta}(E)^{-1} - \Re(u))^2} \tag{3.42}$$

with some universal numerical constant C. Thus for $u = \lambda/\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma \leq (1-\gamma)/\widehat{\Theta}(E)$, $\gamma > 0$, we get

$$\left|f''\left(\lambda/\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma\right)\right| \leq e^{-\beta\sqrt{N}E_\sigma}C\gamma^{-3}\widehat{\Theta}(E)^2 \tag{3.43}$$

Therefore, under the same condition,

$$\left|\frac{\frac{d^2}{du^2}G_{\sigma,T(E)}^\sigma(\tilde{u})}{\mathbb{P}[\tau_{T(E)\setminus\sigma}^\sigma < \tau_\sigma^\sigma](\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma)^2}\right| \leq \frac{e^{-\beta\sqrt{N}E_\sigma}C\gamma^{-2}}{\mathbb{P}[\tau_{T(E)\setminus\sigma}^\sigma < \tau_\sigma^\sigma]}\frac{\widehat{\Theta}(E)^2}{(\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma)^2} \leq 2C\gamma^{-3}\frac{\widehat{\Theta}(E)^2}{(\mathbb{E}\tau_{T(E)\setminus\sigma}^\sigma)^2} \tag{3.44}$$

which is small if $u_N^{-1}(E_\sigma) \gg E$.

Finally we turn to the term (II). While the denominator is the same as in (I), the numerator can now be written as

$$G_{T(E)\setminus\sigma,\sigma}^\sigma(u) - G_{T(E)\setminus\sigma,\sigma}^\sigma(0) = u\frac{d}{du}G_{T(E)\setminus\sigma,\sigma}^\sigma(\tilde{u}) \tag{3.45}$$

This can be bounded in the same way as before, using the Cauchy estimates under the same assumptions on u (with a different constant C), by

$$\left|\frac{d}{du}G^{\sigma}_{T(E)\backslash\sigma,\sigma}(\tilde{u})\right| \leq e^{-\beta\sqrt{N}E_\sigma}C\gamma^{-2}\widehat{\Theta}(E) \tag{3.46}$$

This shows that (II) can be estimated as a small fraction of (I). This concludes the proof of the proposition.◇

3.2. Analyticity properties.

Let us note first that all Laplace transform that we are considering can be identified with meromorphic function that are given as the solutions of Dirichlet problems of the same type as (3.18). Note also that trivially all these functions are periodic with period 2π in the imaginary direction. Equation (3.32) allows to derive more precise estimates on our Laplace transform when we have obtained so far. Note that both Laplace transform on the left hand side of (3.32) are analytic in the half plane $\Re(u) < 1/\widehat{\Theta}$. This implies that that the only singularities of $G^{\sigma}_{\mathbb{T}(E)\backslash\sigma(u)}$ in that half-plane are poles at those values of u for which the denominator vanishes, i.e.

$$1 = G^{\sigma}_{\sigma,T(E)}(u) \tag{3.47}$$

By inspection of the proof of Proposition 3.6, there is only one solution of this equation in the strip $-\pi \leq \Im(u) \leq \pi$, $u_\sigma = \rho/\mathbb{E}\tau^{\sigma}_{T(E)\backslash\sigma}$, where ρ satisfies

$$1-\rho = \rho O(e^{-\alpha u_N^{-1}(E_\sigma)+\alpha E}) + \rho^2 O\big((\widehat{\Theta}(E)/\mathbb{E}\tau^{\sigma}_{T(E)\backslash\sigma})^2\big) \tag{3.48}$$

This implies the existence of a solution $\rho_0 = 1 + O\big((\widehat{\Theta}(E)/\mathbb{E}\tau^{\sigma}_{T(E)\backslash\sigma})^2\big)$.

This implies that the function $G^{\sigma}_{T(E)\backslash\sigma}(u)$ has simple poles at $u_\sigma(mod{+}i2\pi)$, and all other poles satisfy $\Re(u) \geq \widehat{\Theta}(E)^{-1}$, $\Im(u) = 0$ or $\Im(u) = \pi$. Moreover, Proposition 3.6 implies that the residue at u_σ equals

$$\operatorname{res}_{u_\sigma} = \frac{G^{\sigma}_{T(E)\backslash\sigma}(u_\sigma)}{\frac{d}{du}G^{\sigma}_{\sigma,T(E)}(u_\sigma)} = \frac{1}{\mathbb{E}\tau^{\sigma}_{T(E)\backslash\sigma}}\Big(1 + O\big((\widehat{\Theta}(E)/\mathbb{E}\tau^{\sigma}_{T(E)\backslash\sigma})^2\big)\Big) \tag{3.49}$$

This allows in particular to extend the validity of the renewal equation (3.32) to the entire domain of analyticity of this function. This will prove very helpful in obtaining good bounds. As a first observation, we note that the domain of validity of (3.31) can be immediately extended to the set $\rho < \mathbb{E}\tau^{\sigma}_{T(E)\backslash\sigma}/\widehat{\Theta}(E)$.

We will now estimate the difference between $G^\sigma_{T(E)\backslash\sigma}(u)$ and the contribution from the pole at u_σ. We set

$$R_\sigma(u) = G^\sigma_{T(E)\backslash\sigma}(u) + \frac{G^\sigma_{T(E)\backslash\sigma,T(E)}(u_\sigma)}{(u-u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T(E)}(u_\sigma)} \tag{3.50}$$

We first give a uniform estimate of the modulus of R_σ on the disk $|u| \leq a/\widehat{\Theta}(E)$, $a < 1$. Note that a straightforward computation and the use of Taylor expansion to first order shows that

$$\begin{aligned} &R_\sigma(u) \\ &= \frac{G^\sigma_{T(E)\backslash\sigma,T(E)}(u)(u-u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T(E)}(u_\sigma) - G^\sigma_{T(E)\backslash\sigma,T(E)}(u_\sigma)(G^\sigma_{\sigma,T(E)}(u) - G^\sigma_{\sigma,T(E)}(u_\sigma))}{(1-G^\sigma_{\sigma,T(E)}(u))(u-u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T(E)}(u_\sigma)} \\ &= \frac{\frac{d}{du}G^\sigma_{\sigma,T(E)}(u_\sigma)\frac{d}{du}G^\sigma_{T(E)\backslash\sigma,T(E)}(\tilde{u}) - \frac{1}{2}G^\sigma_{T(E)\backslash\sigma,T(E)}(u_\sigma)\frac{d^2}{du^2}G^\sigma_{\sigma,T(E)}(\hat{u})}{\frac{d}{du}G^\sigma_{\sigma,T(E)}(u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T(E)}(u')} \end{aligned} \tag{3.51}$$

where $\tilde{u}, \hat{u}, u'$ are somewhere on the ray between u_σ and u. From (3.30) and the Cauchy bounds used as in (3.42) we get that

$$\left|\frac{d}{du}G^\sigma_{\sigma,T(E)}(u) - p_N(\sigma,\sigma)e^u\right| \leq C\frac{e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E)}{1-\Re(u)\widehat{\Theta}(E)} \tag{3.52}$$

$$\left|\frac{d}{du}G^\sigma_{T(E)\backslash\sigma,T(E)}(u)\right| \leq C\frac{e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E)}{1-\Re(u)\widehat{\Theta}(E)} \tag{3.53}$$

$$\left|\frac{d^2}{du^2}G^\sigma_{\sigma,T(E)}(u)\right| \leq e^{\Re(u)} + C\frac{e^{-\beta\sqrt{N}E_\sigma}(\widehat{\Theta}(E))^2}{(1-\Re(u)\widehat{\Theta}(E))^2} \tag{3.54}$$

and by Lemma 3.5,

$$\left|G^\sigma_{T(E)\backslash\sigma,T(E)}(u)\right| \leq C\frac{e^{-\beta\sqrt{N}E_\sigma}}{1-\Re(u)\widehat{\Theta}(E)} \tag{3.55}$$

Combining these estimates, we see that indeed on $|u| \leq a/\widehat{\Theta}(E)$,

$$|R_\sigma(u)| \leq C(a)\left(e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E)\right)^2 \tag{3.56}$$

as desired.

It remains to estimate $G^\sigma_{T(E)\backslash\sigma}(u)$ for

$$(1/\mathbb{E}\tau^\sigma_{T(E)\backslash\sigma}) < \Re(u) < 1/\widehat{\Theta}(E)$$

To do so, we rely on (3.32). We will use (3.24) to bound the numerator *uniformly* in the imaginary part of u, while the denominator will provide extra decay in the imaginary direction. Note that by (3.30),

$$\begin{aligned}\left|G^{\sigma}_{\sigma,T(E)}(u)-p_N(\sigma,\sigma)e^{u}\right| &\leq e^{-\beta\sqrt{N}E_\sigma}|e^{u}|\max_{\sigma'\sim\sigma}|G^{\sigma'}_{\sigma,T(E)}(u)|\\ &\leq \frac{e^{-\beta\sqrt{N}E_\sigma}|e^{u}|}{1-\Re(u)\widehat{\Theta}(E)}\end{aligned} \tag{3.57}$$

Therefore

$$|G^{\sigma}_{\sigma,T(E)}(u)-1|\geq |e^{u}||1-e^{-u}|-e^{-\beta\sqrt{N}E_\sigma}\left(1-\frac{1}{1-\Re(u)\widehat{\Theta}(E)}\right) \tag{3.58}$$

Combining this estimate with (i) of Lemma 3.5, we arrive at the bound, valid for $\Re(u)<(1-\epsilon)/\widehat{\Theta}(E)$ and $|1-e^{-u}|\geq 2\epsilon^{-1}e^{-\beta\sqrt{N}E_\sigma}$

$$|G^{\sigma}_{T(E)\setminus\sigma}(u)|\leq 2\frac{e^{-\beta\sqrt{N}E_\sigma}}{(1-u\widehat{\Theta}(E))|1-e^{-u}|} \tag{3.59}$$

Combining these observations we arrive at the assertion of Theorem 3.2. ◊◊

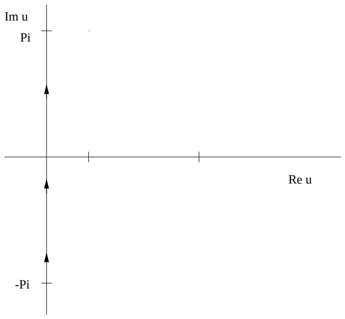
Finally we prove Corollary 3.3.

Proof of Corollary 3.3: We give only the proof of (3.14), the proof of (3.15) being completely analogous.

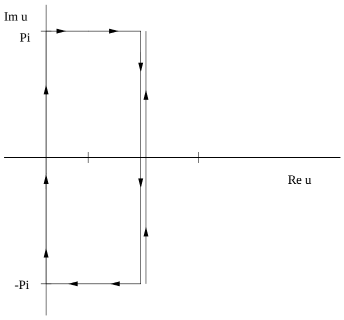
Note that by the Laplace inversion formula [Doe],

$$\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma}=n]=\frac{1}{2\pi i}\int_{-i\pi}^{i\pi}e^{-un}G^{\sigma}_{T(E)\setminus\sigma}(u)du \tag{3.60}$$

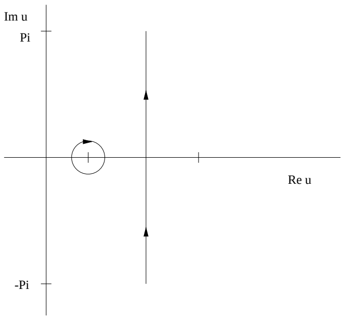
where the integration is along the imaginary axis. Inserting (3.6) into (3.60), in the first two terms the integration contour can be modified to any circle enclosing the point $1/\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma}b_\sigma$, and the integral yields, by Cauchy's theorem, the residue of $e^{-un}\frac{a_\sigma}{1-(1-e^{-u})\mathbb{E}\tau^{\sigma}_{T(E)\setminus\sigma}b_\sigma}$ at this point. In the integral over the remainder term $R_\sigma(u)$, we shift the contour by $(1-\epsilon)/\widehat{\Theta}(E)$ along the positive real axis and use the uniform bound (3.10) along the integration contour. This gives the claimed estimate. ◊



Integration contour



Deformed integration contour



Deformed integration contour

4. The renewal equations.

4.1 Introduction.

We have now all ingredients needed to study the system of renewal equations (1.44) established in Section 1.4. As usual, to solve (1.44) we pass to Laplace transforms, solve the ensuing linear system, and then transform back. We set

$$\Pi^*_\sigma(m,u,E) \equiv \sum_{n=0}^{\infty} e^{nu}\Pi_\sigma(m,n,E) \tag{4.1}$$

for $u \in \mathbb{C}$ whenever this sum converges. Let us define

$$F^*_\sigma(m,u) \equiv \sum_{n=0}^{\infty} e^{nu}\mathbb{P}[\tau^\sigma_{T(E)\setminus\sigma} > m+n] \tag{4.2}$$

Then it follows from (1.44) that for any $\sigma \in T(E)$,

$$\Pi^*_\sigma(m,u,E) = F^*_\sigma(m,u) + \sum_{\sigma'\in T(E)\setminus\sigma} G^\sigma_{\sigma',T(E)\setminus\sigma}(u)\Pi^*_{\sigma'}(m,u,E) \tag{4.3}$$

Let us denote by $K^*_E(u)$ the $|T(E)| \times |T(E)|$ matrix with elements[11]

$$(K^*_E(u))_{\sigma,\sigma'} \equiv \begin{cases} G^\sigma_{\sigma',T(E)\setminus\sigma}(u), & \text{if } \sigma \neq \sigma' \\ 0, & \text{if } \sigma = \sigma' \end{cases} \tag{4.4}$$

Then clearly the solution of equation (4.3) can be written as[12]

$$\underline{\Pi}^*(m,u,E) = \left([\mathbb{1} - K^*_E(u)]^{-1} K^*_E(u) + \mathbb{1}\right)\underline{F}^*(m,u) \tag{4.5}$$

---

[11] We will often write $K^*_{\sigma,\sigma'}(u)$ instead of $(K^*_E(u))_{\sigma,\sigma'}$ whenever no confusion is possible

[12] The reason for separating the $\mathbb{1}$ in this representation is that the operator $\left[\mathbb{1} - K^*_E(u)\right]^{-1} K^*_E(u)$ has better decay properties at infinity than the $\left[\mathbb{1} - K^*_E(u)\right]^{-1}$ itself. This is important for computing the inverse Laplace transforms.

where $\underline{\Pi}^*$ and $\underline{F}^*$ denote the vectors with components Π^*_σ, and F^*_σ.

The matrix

$$M^*_E(u) \equiv \left[\mathbb{1} - K^*_E(u)\right]^{-1} K^*_E(u) \tag{4.6}$$

is known as the Laplace transform of the resolvent of the system of renewal equations.

Our task is to compute the inverse Laplace transform of the right hand side of (4.5). This requires estimates in the complex u-plane. We will separate this analysis in two steps. First, we establish a priori bounds on the norm of M^*_E in a suitable domain. Next we will perform a suitable perturbation analysis that is valid in a small neighborhood of $u = 0$ only. Then we show that the dominant part of the contribution from the Laplace-inversion formula comes from this region and is thus explicitly computed, while the remainder is controlled by our a priori bounds.

4.2. Bounds on the resolvent.

In the sequel we will always work with the matrix norm

$$\|K\| \equiv \max_{\sigma\in T(E)} \sum_{\sigma'\in T(E)} |K_{\sigma,\sigma'}| \tag{4.7}$$

Note that $\|\,.\,\|$ is an operator norm in $L_\infty(\mathbb{C}^M)$ equipped with the supremum norm, i.e. $\|KF\|_\infty \leq \|K\|\,\|F\|_\infty$. This norm serves our purposes, and moreover will turn out to be particularly well suited to the matrices that we need to deal with.

We will begin by deriving estimates on the matrices $K^*_E(u)$. It follows from the results of Section 3 that

Lemma 4.1: *Considered as a function $\mathbb{C} \to L(\mathbb{C}^M, \mathbb{C}^M)$, $K^*_E(u)$ is*

(i) Periodic with period 2π in the imaginary direction.

(ii) Meromorphic in $\mathbb{C}$ with poles only on the positive real axis and its 2π translates.

(iii) For $\sigma \neq \sigma' \in T(E)$,

$$K^*_{\sigma,\sigma'}(u) = \frac{G^\sigma_{\sigma',T(E)}(u)}{1 - G^\sigma_{\sigma,T(E)}(u)} \tag{4.8}$$

The following observation will be extremely useful:

Lemma 4.2: *For any $u \in \mathbb{C}$ for which $G^{\sigma}_{T(E)\backslash\sigma,T(E)}(u)$ is finite,*

$$\sum_{\sigma'\in T(E)\backslash\sigma} G^{\sigma}_{\sigma',T(E)}(u) = G^{\sigma}_{T(E)\backslash\sigma,T(E)}(u) \tag{4.9}$$

Proof: it is enough to prove (4.9) for u in the negative imaginary half plane. Now

$$\mathbb{1}_{\tau^{\sigma}_{T(E)\backslash\sigma}\leq\tau^{\sigma}_{T(E)}} = \sum_{\sigma'\in T(E)\backslash\sigma} \mathbb{1}_{\tau^{\sigma}_{\sigma'}\leq\tau^{\sigma}_{T(E)}} \tag{4.10}$$

Thus

$$\begin{aligned} G^{\sigma}_{T(E)\backslash\sigma,T(E)}(u) =& \mathbb{E}e^{u\tau^{\sigma}_{T(E)\backslash\sigma}}\mathbb{1}_{t^{\sigma}_{T(E)\backslash\sigma}\leq\tau^{\sigma}_{T(E)}} = \mathbb{E}\sum_{\sigma'\in T(E)\backslash\sigma} e^{u\tau^{\sigma}_{T(E)\backslash\sigma}}\mathbb{1}_{\tau^{\sigma}_{\sigma'}\leq\tau^{\sigma}_{T(E)}} \\ =& \sum_{\sigma'\in T(E)\backslash\sigma} \mathbb{E}e^{u\tau^{\sigma}_{\sigma'}}\mathbb{1}_{\tau^{\sigma}_{\sigma'}\leq\tau^{\sigma}_{T(E)}} = \sum_{\sigma'\in T(E)\backslash\sigma} G^{\sigma}_{\sigma',T(E)}(u) \end{aligned} \tag{4.11}$$

♢

An immediate, but important consequence of Lemma 4.2 is that

$$\|K^{*}_{E}(0)\| = 1 \tag{4.12}$$

The first step towards control in the complex plane will be to show that $\|K^{*}_{E}(u)\|$ decreases down from zero along the imaginary axis in the strip $\Im(u) \in [-\pi,\pi]$.

Lemma 4.3: *Let $v \in [-\pi,\pi]$ and set*

$$\bar{\Theta} \equiv e^{\beta\sqrt{N}u_N(0)+\alpha E} \tag{4.13}$$

Recall $M = |T(E)|$ and $d = 2^M$. Then (for N large enough),

$$\|K^{*}_{E}(iv)\| \leq \frac{1}{\sqrt{2(1-\cos v)\bar{\Theta}^2\left(1-O(\bar{\Theta}^{-1})\right)+1-\frac{4}{M-1}\left(1+O(d/N)\right)}} \tag{4.14}$$

Before proving the lemma, we will note the obvious consequence that

Corollary 4.4: *Under the assumptions and notations of Lemma 4.3,*

(i) If $\bar{\Theta}|v| > \frac{3}{\sqrt{M-1}}$, then $\|K^{*}_{E}(iv)\| < 1$.

(ii) For any $0 < \epsilon < 1$, *if*

$$2(1-\cos v) \geq \bar{\Theta}^{-2}\left[\frac{\epsilon}{1-\epsilon}\frac{2-\epsilon}{1-\epsilon} + \frac{9}{(m-1)(1-\epsilon)}\right](1+O(d/N)) \tag{4.15}$$

then $\|K_E^*(iv)\| \leq 1-\epsilon$.

(iii) Under the same assumptions as in (i),

$$\|M_E^*(iv)\| \leq \frac{1}{\sqrt{1+2\bar{\Theta}^2(1-\cos v)(1-O(\bar{\Theta}^{-1}))} - 1 - \frac{4}{M-1}(1+O(d/N))} \tag{4.16}$$

Proof: To bound the norm of K_E^*, we use simply that

$$\begin{aligned}\sum_{\sigma' \in T\setminus\sigma} |K^*_{\sigma,\sigma'}(iv)| &= \frac{\sum_{\sigma'\in T\setminus\sigma} |G^{\sigma}_{\sigma',T}(iv)|}{|1-G^{\sigma}_{\sigma,T}(iv)|} \\ &\leq \frac{\mathbb{P}[\tau^{\sigma}_{T\setminus\sigma} \leq \tau^{\sigma}_{T}]}{|1-G^{\sigma}_{\sigma,T}(iv)|}\end{aligned} \tag{4.17}$$

Thus the key point is to bound the denominator from below. Now

$$\begin{aligned}\Im G^{\sigma}_{\sigma,T(E)}(iv) &= \sum_{n=1}^{\infty} \sin(vn)\mathbb{P}[\tau^{\sigma}_{\sigma} = \tau^{\sigma}_{T(E)} = n] \\ &= \sin(v)p_N(\sigma,\sigma) + \sum_{\sigma'\notin T(E)} p_N(\sigma,\sigma')\sum_{n=1}^{\infty}\sin(v(n+1))\mathbb{P}[\tau^{\sigma'}_{\sigma} = \tau^{\sigma'}_{T(E)} = n] \\ &\equiv p_N(\sigma,\sigma)\sin v + d_{\sigma}(v)\end{aligned} \tag{4.18}$$

where

$$|d_{\sigma}(v)| \leq e^{-\beta\sqrt{N}E_{\sigma}} \sum_{\sigma'\sim\sigma}\frac{1}{N}\mathbb{P}[\tau^{\sigma'}_{\sigma} = \tau^{\sigma'}_{T(E)}] \leq 2\frac{e^{-\beta\sqrt{N}E_{\sigma}}}{|T(E)|}(1+O(|T(E)|/N)) \tag{4.19}$$

where we used the bound (2.3) from Proposition 2.1.

$$\Re\left(1-G^{\sigma}_{\sigma,T(E)}(iv)\right) = p_N(\sigma,\sigma)(1-\cos v) + c_{\sigma}(v) \tag{4.20}$$

where

$$\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} = \tau^{\sigma}_{T(E)}] \leq c_{\sigma}(v) \leq \mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} = \tau^{\sigma}_{T(E)}] + 2\frac{e^{-\beta\sqrt{N}E_{\sigma}}}{|T(E)|}(1+O(|T(E)|/N)) \tag{4.21}$$

Thus we have that

$$\begin{aligned} |1-G^{\sigma}_{\sigma,T}(iv)| \geq & \sqrt{(p_N(\sigma,\sigma)\sin v)^2 + \left(p_N(\sigma,\sigma)(1-\cos v) + \mathbb{P}[\tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]\right)^2} \\ & - |d_\sigma(v)| - |c_\sigma(v) - \mathbb{P}[\tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]| \end{aligned} \tag{4.22}$$

To simplify the notation, set $p_N \equiv p_N(\sigma,\sigma)$, $\mathbb{P}_\sigma \equiv \mathbb{P}[\tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]$. Let

$$Y \equiv (p_N \sin v)^2 + (p_N(1-\cos v) + \mathbb{P}_\sigma)^2 = 2p_N(1-\cos v)(p_N + \mathbb{P}_\sigma) + \mathbb{P}_\sigma^2 \tag{4.23}$$

Thus we have in fact that

$$|1-G^{\sigma}_{\sigma,T}(iv)| \geq \sqrt{2p_N(1-\cos v)(p_N+\mathbb{P}_\sigma) + \mathbb{P}_\sigma^2} - \frac{4}{M} e^{-\beta\sqrt{N}E_\sigma}(1+O(M/N)) \tag{4.24}$$

which together with (4.17) gives that

$$\begin{aligned} \sum_{\sigma'\in T\setminus\sigma} |K^*_{\sigma,\sigma'}(iv)| & \leq \frac{\mathbb{P}_\sigma}{\sqrt{2p_N(1-\cos v)(p_N+\mathbb{P}_\sigma)+\mathbb{P}_\sigma^2} - \frac{4}{M} e^{-\beta\sqrt{N}E_\sigma}(1+O(M/N))} \\ & = \frac{1}{\sqrt{2p_N\mathbb{P}_\sigma^{-2}(1-\cos v)(p_N+\mathbb{P}_\sigma)+1} - \frac{4}{M}\mathbb{P}_\sigma^{-1} e^{-\beta\sqrt{N}E_\sigma}(1+O(M/N))} \end{aligned} \tag{4.25}$$

Now recall from Proposition 2.1, (iii), that

$$\frac{1}{1-\frac{1}{M}}(1-O(d/N)) \leq \mathbb{P}_\sigma^{-1} e^{-\beta\sqrt{N}E_\sigma} \leq \frac{1}{1-\frac{1}{M}}(1+O(d/N)) \tag{4.26}$$

It follows readily that

$$p_N + \mathbb{P}_\sigma = 1 - e^{-\beta\sqrt{N}E_\sigma} + \mathbb{P}_\sigma \geq 1 - \frac{e^{-\beta\sqrt{N}E_\sigma}}{M}(1-O(d/N)) \tag{4.27}$$

and hence

$$1 > p_N(p_N+\mathbb{P}_\sigma) \geq 1 - e^{-\beta\sqrt{N}E_\sigma}(1+1/M)(1+O(d/N)) \tag{4.28}$$

Since by definition of T, $\min_{\sigma\in T(E)} \sqrt{N}E_\sigma \geq u_N(E)$, this implies

$$\min_{\sigma\in T} p_N(p_N+\mathbb{P}_\sigma) \geq 1 - \bar{\Theta}^{-1}(1+1/M)(1+O(d/N)) \tag{4.29}$$

and

$$\|K^*_E(iv)\| \leq \frac{1}{\sqrt{\bar{\Theta}^2 2(1-\cos v)(1-\bar{\Theta}^{-1}(1+1/M)+1} - \frac{4}{M-1}(1+O(d/N))} \tag{4.30}$$

which proves the Lemma.♢

The proof of Corollary 4.4 is an exercise in simple algebra and is left to the reader.

Next we use these results to extend similar bounds somewhat into the positive imaginary half plane. The important point permitting this is that we will need to Taylor-expand in the real part of u only Dirichlet Green's functions with exclusion set T and these are analytic up to $\Re(u) \approx 1/\widehat{\Theta}$. Let us first fix some notation.

Notation: As before the letter $u \in \mathbb{C}$ denotes a complex number. Its real and imaginary parts will always be called w and v:

$$u = w + iv \tag{4.31}$$

For given $u \in \mathbb{C}$, we will denote by $z \in \mathbb{C}$ the number

$$z = \widehat{\Theta}(E)u \tag{4.32}$$

The real and imaginary parts of z will always be called r and s:

$$z = r + is \tag{4.33}$$

Thus

$$\begin{aligned} r &= \widehat{\Theta}(E)w \\ s &= \widehat{\Theta}(E)v \end{aligned} \tag{4.34}$$

To simplify the notation the dependence on u of z (or on w, resp. v, of r, resp. s) will never be made explicit. No confusion should arise from this as, up until Section 4.7[13], the letters u, w, v and z, r, s will be used exclusively according to the relations specified above.

For ready reference we make the following definitions.

Definition 4.5: *Let $0 < C_1, C_2 < \infty$, and $0 < \gamma < 1$ be numerical constants. With the above notation we define the sets:*

$$\begin{aligned} D_1(C_1) &\equiv \left\{ u \in \mathbb{C} \,:\, \sqrt{r^2 + s^2} \geq C_1/\sqrt{M} \right\} \\ D_2(C_2, \gamma) &\equiv \left\{ u \in \mathbb{C} \,:\, 0 \leq r < \min\left(\frac{\gamma s^2}{C_2\sqrt{1+s^2}}, 1 - \gamma \right),\ \ v \in [-\pi, \pi] \right\} \\ D_3 &\equiv \{ u \in \mathbb{C} \,:\, -1 \leq r < 0,\ |s| < 1 \} \\ D_4 &\equiv \{ u \in \mathbb{C} \,:\, |r| < 1,\ |s| < 1 \} \end{aligned} \tag{4.35}$$

---

[13] There, the letter s will retrieve the initial meaning it was given in Theorem 1.

Lemma 4.6: *There exist constants* $0 < C, C' < \infty$ *such that, for all* $0 < \gamma < 1$ *and all* $u \in D_2(C', \gamma)$,

$$\|K^*_E(u)\| \leq \frac{1 + C\gamma^{-1}r}{\sqrt{1 + \widehat{\Theta}^2 2(1 - \cos v)(1 - O(\bar{\Theta}^{-1}))} - \frac{4}{M-1}(1 + O(d/N)) - C'\gamma^{-1}r} \tag{4.36}$$

Proof: As in the proof of Lemma 4.3, we begin by writing the analogue of (4.17) and again we bound the numerator by the value obtained when putting its imaginary part equal to zero. This yields

$$\begin{aligned} \|K^*_E(u)\| \leq & \frac{\sum_{\sigma' \in T \backslash \sigma} |G^\sigma_{\sigma',T}(w)|}{|1 - G^\sigma_{\sigma,T}(w + iv)|} \\ = & \frac{G^\sigma_{T\backslash\sigma,T}(w)}{|1 - G^\sigma_{\sigma,T}(w + iv)|} \end{aligned} \tag{4.37}$$

We now Taylor expand both the numerator and the denominator. Note that we will only be interested in $w \leq (1 - \gamma)/\widehat{\Theta}$. For the numerator we will use (3.45) together with the bound (3.46) to write, for $0 \leq w \leq (1 - \gamma)/\widehat{\Theta}$,

$$G^\sigma_{T\backslash\sigma,T}(w) \leq \mathbb{P}[\tau^\sigma_{T\backslash\sigma} \leq \tau^\sigma_T] + Cw\gamma^{-1}\widehat{\Theta}e^{-\beta\sqrt{N}E_\sigma} \tag{4.38}$$

On the other hand, from (3.30) and the Cauchy bound we get that, again for $0 \leq \tilde{w} \leq (1 - \gamma)/\widehat{\Theta}$,

$$\left| \frac{d}{dw} G^\sigma_{\sigma,T(E)}(iv + \tilde{w}) \right| \leq |e^{\tilde{w}}| + C\gamma^{-1}e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E) \leq C'\gamma^{-1} \tag{4.39}$$

This implies again

$$|1 - G^\sigma_{\sigma,T}(iv + w)| \geq |1 - G^\sigma_{\sigma,T}(iv)| - w\gamma^{-1}C' \tag{4.40}$$

As we already have bounded the first term on the right in the proof of Lemma 4.3, we readily arrive at

$$\begin{aligned} & \sum_{\sigma' \in T \backslash \sigma} |K^*_{\sigma,\sigma'}(u)| \\ & \leq \frac{1 + Cw\widehat{\Theta}\gamma^{-1}e^{-\beta\sqrt{N}E_\sigma}\mathbb{P}_\sigma^{-1}}{\sqrt{1 + 2\mathbb{P}_\sigma^{-2}p_N(1 - \cos v)(p_N + \mathbb{P}_\sigma)} - \frac{4e^{-\beta\sqrt{N}E_\sigma}}{M\mathbb{P}_\sigma}(1 + O(M/N)) - C'\gamma^{-1}\mathbb{P}_\sigma^{-1}w} \end{aligned} \tag{4.41}$$

Proceeding from there on exactly as in the proof of Lemma 4.3 we then get, using relation (4.34),

$$\sum_{\sigma' \in T \backslash \sigma} |K^*_{\sigma,\sigma'}(u)| \leq \frac{1 + C\gamma^{-1}r}{\sqrt{1 + \mathbb{P}_\sigma^{-2} 2(1 - \cos v)(1 - O(\bar{\Theta}^{-1}))} - \frac{4}{M-1}(1 + O(d/N)) - \frac{C'\gamma^{-1}r}{\mathbb{P}_\sigma\widehat{\Theta}}} \tag{4.42}$$

Since we need to take the maximum over all $\sigma \in T$, it is important to restrict r as a function of v in such a way that the maximum will be taken on by the σ that maximises $\mathbb{P}_\sigma$. Some elementary algebra shows that this will be the case provided that

$$\frac{\left(2(1-\cos v)\widehat{\Theta}^2(1-O(\bar{\Theta}^{-1}))\right)^2}{1+2(1-\cos v)\widehat{\Theta}^2(1-O(\bar{\Theta}^{-1}))} \geq \left(C'\gamma^{-1}r\right)^2 \tag{4.43}$$

or

$$r \leq \frac{2(1-\cos v)\widehat{\Theta}^2(1-O(\bar{\Theta}^{-1}))}{\gamma^{-1}C'\sqrt{1+2(1-\cos v)\widehat{\Theta}^2(1-O(\bar{\Theta}^{-1}))}} \tag{4.44}$$

Since this is a serious condition only if v is very small we see, using relation (4.34), that this condition reduces to

$$r < \min\left(\frac{\gamma s^2}{C'\sqrt{1+s^2}}, 1-\gamma\right) \tag{4.45}$$

On this domain we can thus estimate the norm of K_E^* by

$$\|K_E^*(u)\| \leq \frac{1+C\gamma^{-1}r}{\sqrt{1+\widehat{\Theta}^2 2(1-\cos v)(1-O(\bar{\Theta}^{-1}))} - \frac{4}{M-1}(1+O(d/N)) - C'\gamma^{-1}r} \tag{4.46}$$

This proves the lemma.$\diamondsuit$

As in the case of Lemma 4.3, we get as an immediate corollary an upper bound on the norm of the resolvent.

Corollary 4.7: *For all* $0 < \gamma < 1$ *there exists a constant* $0 < L < \infty$ *(depending on* C, C' *and* γ*) such that, for all* $u \in D_1(4) \cap D_2(L, \gamma)$*,*

$$\|K_E^*(u)\| < 1 \tag{4.47}$$

and

$$\|M_E^*(u)\| \leq \frac{1+C\gamma^{-1}r}{\sqrt{1+\widehat{\Theta}^2 2(1-\cos v)(1-O(\bar{\Theta}^{-1}))} - 1 - \frac{4}{M-1}(1+O(d/N)) - (C+C')\gamma^{-1}r} \tag{4.48}$$

Finally we will need an estimate on $\|M_E^*(u)\|$ in the case when $|u|$ is very small and $w \leq 0$ that shows that there, the negative real part helps to depress $\|K_E^*(u)\| < 1$ down from one.

Lemma 4.8: *For M large enough,*

(i) for all $u \in D_3$,

$$\|K_E^*(u)\| \leq \frac{1}{\sqrt{1+r^2+s^2} - \frac{5}{M}} \tag{4.49}$$

(i) for all $u \in D_1(4) \cap D_3$, $\|K_E^*(u)\| < 1$ and

$$\|M_E^*(u)\| \leq \frac{1}{\sqrt{1+r^2+s^2} - 1 - \frac{5}{M}} \tag{4.50}$$

Proof: The proof of this estimate goes quite along the lines of the proof of the previous lemmas. However, to simplify things, we bound the Green function in the numerator of (4.37) by its value at zero and, instead of using (4.40) in the denominator, we go back to the estimates (4.18) and (4.20) which we modify slightly to yield, for $w \leq 0$,

$$\begin{aligned}
\Im G^{\sigma}_{\sigma,T(E)}(iv+w) &= \sum_{n=1}^{\infty} e^{nw} \sin(vn) \mathbb{P}[\tau^{\sigma}_{\sigma} = \tau^{\sigma}_{T(E)} = n] \\
&= e^{w} \sin(v) p_N(\sigma,\sigma) + \sum_{\sigma' \notin T(E)} p_N(\sigma,\sigma') \sum_{n=1}^{\infty} e^{wn} \sin(v(n+1)) \mathbb{P}[\tau^{\sigma'}_{\sigma} = \tau^{\sigma'}_{T(E)} = n] \\
&\leq p_N(\sigma,\sigma) e^{w} \sin v + d_\sigma(v)
\end{aligned} \tag{4.51}$$

with $d_\sigma(v)$ from (4.18). Similarly,

$$\Re\left(1 - G^{\sigma}_{\sigma,T(E)}(iv+w)\right) = p_N(\sigma,\sigma)(1 - e^{w}\cos v) + c_\sigma(v) \tag{4.52}$$

with $c_\sigma(v)$ from (4.20). On the other hand

$$|\mathbb{P}_\sigma + p_N(1-e^u)|^2 = \mathbb{P}_\sigma^2 + 2p_N(1-\cos v)(p_N + \mathbb{P}_\sigma) - 2\cos v p_N(e^w - 1)(p_N + \mathbb{P}_\sigma) + p_N^2(e^{2w} - 1) \tag{4.53}$$

For w small, we can expand e^w to second order and, using that $w \leq 0$, we get

$$\begin{aligned}
&|\mathbb{P}_\sigma + p_N(1-e^u)|^2 \\
&= \mathbb{P}_\sigma^2 + 2p_N(1-\cos v)(p_N + \mathbb{P}_\sigma) - 2wp_N[p_N - \cos v(p_N + \mathbb{P}_\sigma)] \\
&\quad + + w^2 p_N[2p_N - \cos v(p_N + \mathbb{P}_\sigma)] + O(w^3) \\
&= \mathbb{P}_\sigma^2 + 2p_N(1-\cos v)(p_N + \mathbb{P}_\sigma)(1-w) - 2wp_N(1-p_N) \\
&\quad + w^2 p_N[2p_N - \cos v(p_N + \mathbb{P}_\sigma)] + O(w^3) \\
&\geq \mathbb{P}_\sigma^2 + v^2 + w^2 + O(w^3)
\end{aligned} \tag{4.54}$$

Thus

$$\sum_{\sigma'\in T\setminus\sigma} |K^*_{\sigma,\sigma'}(u)| \leq \frac{1}{\sqrt{1+\mathbb{P}_\sigma^{-2}\widehat{\Theta}^2(s^2+r^2)} - \frac{5}{M}} \tag{4.55}$$

and since this is clearly monotone in $\mathbb{P}_\sigma$, it follows that

$$\|K^*_E(u)\| \leq \frac{1}{\sqrt{1+s^2+w^2} - \frac{5}{M}} \tag{4.56}$$

and hence, for $u\in D_1(4)$, $\|K^*_E(u)\| < 1$ and

$$\|M^*_E(u)\| \leq \frac{1}{\sqrt{1+s^2+w^2} - 1 - \frac{5}{M}} \tag{4.57}$$

◊

4.3. Perturbative estimates for small u.

The a priori bounds obtained in the last subsection will suffice to show that the contributions from u away from zero in the Laplace inversion formula are sub-dominant. In the neighborhood of zero we have to proceed more carefully and extract the dominant contribution to the resolvent, while estimating the remainders. This will be done by decomposing $K^*_E(u)$ in a suitable way, the idea being that the leading term should allow exact computations; in fact, we will want this term to be a matrix with constant columns. To this end note that for $\sigma\neq\sigma'$, by Taylor's formula,

$$\begin{aligned} K^*_{\sigma,\sigma'}(u) =& \frac{1}{1-G^\sigma_{\sigma,T}(u)}\left(G^\sigma_{\sigma',T}(0) + u\frac{d}{du}G^\sigma_{\sigma',T}(0) + \frac{u^2}{2}\frac{d^2}{du^2}G^\sigma_{\sigma',T}(\tilde{u})\right) \\ =& \frac{1}{1-G^\sigma_{\sigma,T}(u)}\left(\mathbb{P}[\tau^\sigma_{\sigma'}\leq\tau^\sigma_T] + u\mathbb{E}\tau^\sigma_{\sigma'}\mathbb{1}_{\{\tau^\sigma_{\sigma'}\leq\tau^\sigma_T\}} + \frac{u^2}{2}\frac{d^2}{du^2}G^\sigma_{\sigma',T}(\tilde{u})\right) \end{aligned} \tag{4.58}$$

where $\tilde{u}$ is on the ray between 0 and u. The idea is of course that since u is small, the quadratic term is a small perturbation[14] while the constant and linear terms are essentially independent of σ', the deviations being treatable as perturbations as well.

Let us first establish a bound on the second order contribution. The notation and definitions of the present are the same as in the previous one (recall in particular Definition 4.5).

---

[14] It will become clear only later why we expand to second order and are not content with the first order as before.

Lemma 4.9: *Denote by* $K_E^{*(2)}$ *the matrix with entries*

$$K^{*,(2)}_{\sigma,\sigma'}(u) = \begin{cases} \frac{\frac{1}{2}u^2\frac{d^2}{du^2}G^{\sigma}_{\sigma',T}(\tilde{u})}{1-G^{\sigma}_{\sigma,T}(u)}, & \textit{if } \sigma \neq \sigma' \\ 0, & \textit{if } \sigma = \sigma' \end{cases} \tag{4.59}$$

For $0 < \gamma < 1$*, let the constant* L *be chosen such that*

$$\left\{u \in \mathbb{C} \mid r \leq s^2/4\right\} \subseteq D_2(L,\gamma) \cap D_4 \tag{4.60}$$

Then, there exists a constant $C > 0$ *such that for all for* $u \in D_2(L,\gamma) \cap D_4$ *and* N *large enough,*

$$\|K_E^{*(2)}(u)\| \leq \frac{\gamma^{-2}C(s^2+r^2)}{\sqrt{1+(s^2+r^2)/2} - 5/M} \tag{4.61}$$

Remark: The assumption (4.60) is made for convenience only as it allows to simplify the expressions of our estimates.

Remark: Note also that the bound (4.61) simply behaves, for small $\widehat{\Theta}u$, like $\gamma^{-2}C(s^2+r^2)$.

Proof: To bound the denominator we proceed as in the proofs of Lemma 4.6 and 4.8 with the difference that, for $r > 0$, the bound (4.54) becomes, using that $r \leq s^2/4$,

$$|\mathbb{P}_{\sigma} + p_N(1-e^u)|^2 \geq \mathbb{P}_{\sigma}^2 + (v^2+w^2)/2 + O(w^3) \tag{4.62}$$

For the numerator we use that

$$\sum_{\sigma' \in T \setminus \sigma} \left| \frac{d^2}{du^2} G^{\sigma}_{\sigma',T}(\tilde{u}) \right| \leq \sum_{\sigma' \in T \setminus \sigma} \frac{d^2}{du^2} G^{\sigma}_{\sigma',T}(\Re\tilde{u}) = \frac{d^2}{du^2} G^{\sigma}_{T\setminus\sigma,T}(\Re\tilde{u}) \tag{4.63}$$

and, since $\Re\tilde{u} \leq (1-\gamma)/\widehat{\Theta}$, bound the last quantity in the r.h.s proceeding as in the proof of Proposition 3.6 (see the treatment of the term (II) therein). ♢

What remains of K_E^* after subtraction of $K_E^{*(2)}$ is almost of the desired form (i.e. has almost constant columns); however, a few cosmetic changes need to be made: first, the matrix elements

$$K^{*(1)}_{\sigma,\sigma'}(u) \equiv \frac{1}{1-G^{\sigma}_{\sigma,T}(u)} \mathbb{P}[\tau^{\sigma}_{\sigma'} \leq \tau^{\sigma}_{T}]\left(1 + u\mathbb{E}[\tau^{\sigma}_{\sigma'} | \tau^{\sigma}_{\sigma'} \leq \tau^{\sigma}_{T}]\right), \quad \sigma \neq \sigma' \tag{4.64}$$

have to be replaced by their leading, σ'-independent part

$$K^{*(0)}_{\sigma,\sigma'}(u) \equiv \frac{1}{1-G^{\sigma}_{\sigma,T}(u)} \left(\tfrac{1}{M}\mathbb{P}[\tau^{\sigma}_{T\setminus\sigma} < \tau^{\sigma}_{T}]\left(1 + u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]\right)\right), \quad \sigma \neq \sigma' \tag{4.65}$$

As shown in the next lemma, this replacement can be done at the cost of error terms of order at most $O(1/N)$.

Lemma 4.10: *Denote by* $K_E^{*(0)}$ *and* $K_E^{*(1)}$ *the matrix with off-diagonal entries given respectively by (4.65) and (4.64) and zero diagonals. Then, under the assumptions and with the notation of Lemma 4.9 and Proposition 2.2 we have, for* N *large enough,*

$$\|K^{*(0)}(u) - K^{*(1)}(u)\| \leq \frac{1+3\sqrt{s^2+r^2}}{\sqrt{1+(s^2+r^2)/2-5/M}}O(1/N) \tag{4.66}$$

Second, since the matrix $K^{*(0)}(u)$ has zero diagonal, we still have to compare it to the matrix $\mathcal{K}^{*(0)}$ with entries

$$\mathcal{K}^{*(0)}_{\sigma,\sigma'}(u) \equiv \frac{1}{1-G^{\sigma}_{\sigma,T}(u)}\left(\tfrac{1}{M}\mathbb{P}[\tau^{\sigma}_{T\setminus\sigma} < \tau^{\sigma}_{T}]\left(1+u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma}|\tau^{\sigma}_{T\setminus\sigma}=\tau^{\sigma}_{T}]\right)\right), \quad \forall\sigma,\sigma'\in T(E) \tag{4.67}$$

This involves controlling the norm of the diagonal matrix $\mathcal{K}^{*(0)}(u) - K^{*(0)}(u)$:

Lemma 4.11: *Let* $\mathcal{K}^{*(0)}$ *be the matrix defined in (4.67). Under the assumptions and with the notation of Lemma 4.10 we have, for* N *large enough,*

$$\|\mathcal{K}^{*(0)}(u) - K^{*(0)}(u)\| \leq \frac{1+\sqrt{s^2+r^2}}{\sqrt{1+(s^2+r^2)/2-5/M}}O(1/(M-1)) \tag{4.68}$$

Proof of Lemma 4.10: For $\sigma,\sigma'\in T$, $\sigma\neq\sigma'$, let $\kappa_{\sigma,\sigma'}(u)$ be defined through

$$K^{*(0)}_{\sigma,\sigma'}(u) - K^{*(1)}_{\sigma,\sigma'}(u) = \frac{\kappa_{\sigma,\sigma'}(u)}{1-G^{\sigma}_{\sigma,T}(u)} \tag{4.69}$$

Since the denominator in (4.69) has already been dealt with in Lemma 4.9, what we need is an upper bound on $|\kappa_{\sigma,\sigma'}(u)|$. Appropriately sorting out the different terms contributing to $\kappa_{\sigma,\sigma'}(u)$ we may write,

$$\begin{aligned} |\kappa_{\sigma,\sigma'}(u)| \leq & \left|\mathbb{P}[\tau^{\sigma}_{\sigma'} \leq \tau^{\sigma}_{T}] - \frac{1}{M}e^{-\beta\sqrt{N}E_\sigma}\right|\left(1+|u|\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma}|\tau^{\sigma}_{T\setminus\sigma}=\tau^{\sigma}_{T}]\right) \\ & +|u|\mathbb{P}[\tau^{\sigma}_{\sigma'} \leq \tau^{\sigma}_{T}]\left|\mathbb{E}[\tau^{\sigma}_{\sigma'} \mid \tau^{\sigma}_{\sigma'} \leq \tau^{\sigma}_{T}] - \mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} \mid \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]\right| \end{aligned} \tag{4.70}$$

Plugging in the estimates of Proposition 2.1, ii),

$$\begin{aligned} |\kappa_{\sigma,\sigma'}(u)| \leq & \frac{e^{-\beta\sqrt{N}E_\sigma}}{M}\Bigg[\left(1+|u|\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma}|\tau^{\sigma}_{T\setminus\sigma}=\tau^{\sigma}_{T}]\right)O(1/N) \\ & +|u|\left|\mathbb{E}[\tau^{\sigma}_{\sigma'} \mid \tau^{\sigma}_{\sigma'} \leq \tau^{\sigma}_{T}] - \mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} \mid \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]\right|(1+O(1/N))\Bigg] \end{aligned} \tag{4.71}$$

and we are left to bound the expected transition time $\mathbb{E}[\tau^\sigma_{T\setminus\sigma}|\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T]$, together with the difference $\left|\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]-\mathbb{E}[\tau^\sigma_{T\setminus\sigma}\mid\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T]\right|$. To deal with the latter, first observe that differentiating the renewal equation $G^\sigma_{\sigma',T\setminus\sigma}(u)=\frac{G^\sigma_{\sigma',T}(u)}{1-G^\sigma_{\sigma,T}(u)}$, we have

$$\begin{aligned}\frac{d}{du}G^\sigma_{\sigma',T}(0)&=(1-\mathbb{P}[\tau^\sigma_\sigma\leq\tau^\sigma_T])\mathbb{E}\tau^\sigma_{\sigma'}1\!\!1_{\{\tau^\sigma_{\sigma'}\leq\tau^\sigma_{T\setminus\sigma}\}}-\mathbb{P}[\tau^\sigma_{\sigma'}\leq\tau^\sigma_{T\setminus\sigma}]\mathbb{E}\tau^\sigma_\sigma1\!\!1_{\{\tau^\sigma_\sigma\leq\tau^\sigma_T\}}\\&=\mathbb{P}[\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]\left(\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_{T\setminus\sigma}]-\frac{\mathbb{E}\tau^\sigma_\sigma1\!\!1_{\{\tau^\sigma_\sigma\leq\tau^\sigma_T\}}}{1-\mathbb{P}[\tau^\sigma_\sigma\leq\tau^\sigma_T]}\right)\end{aligned}\tag{4.72}$$

implying that

$$\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]=\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_{T\setminus\sigma}]-\frac{\mathbb{E}\tau^\sigma_\sigma1\!\!1_{\{\tau^\sigma_\sigma\leq\tau^\sigma_T\}}}{1-\mathbb{P}[\tau^\sigma_\sigma\leq\tau^\sigma_T]}\tag{4.73}$$

and, since the last term in the r.h.s. is σ'-independent, we can express our conditional expectation in the following, remarkably useful form:

$$\begin{aligned}\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]=&\frac{1}{|T\setminus\sigma|}\sum_{\sigma'\in T\setminus\sigma}\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]\\&+\left\{\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_{T\setminus\sigma}]-\frac{1}{|T\setminus\sigma|}\sum_{\sigma'\in T\setminus\sigma}\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_{T\setminus\sigma}]\right\}\end{aligned}\tag{4.74}$$

Next observe that by (4.9), $\sum_{\sigma'\in T\setminus\sigma}\mathbb{P}[\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]=\mathbb{P}[\tau^\sigma_{T\setminus\sigma}\leq\tau^\sigma_T]$, as well as

$$\sum_{\sigma'\in T\setminus\sigma}\mathbb{E}\tau^\sigma_{\sigma'}1\!\!1_{\{\tau^\sigma_{\sigma'}\leq\tau^\sigma_T\}}=\mathbb{E}\tau^\sigma_{T\setminus\sigma}1\!\!1_{\{\tau^\sigma_{T\setminus\sigma}\leq\tau^\sigma_T\}}\tag{4.75}$$

hold ((4.75) is obtained by differentiating (4.9) and setting $u=0$). On the other hand, using (2.4) from Proposition 2.1, the first term in the r.h.s of (4.74) may thus be rewritten as

$$\begin{aligned}\frac{1}{|T\setminus\sigma|}\sum_{\sigma'\in T\setminus\sigma}\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]&=\sum_{\sigma'\in T\setminus\sigma}\frac{\mathbb{E}\tau^\sigma_{\sigma'}1\!\!1_{\{\tau^\sigma_{\sigma'}\leq\tau^\sigma_T\}}}{\mathbb{P}[\tau^\sigma_{T\setminus\sigma}\leq\tau^\sigma_T]}\left(\frac{1}{|T\setminus\sigma|}\sum_{\sigma''\in T\setminus\sigma}\frac{\mathbb{P}[\tau^\sigma_{\sigma''}\leq\tau^\sigma_T]}{\mathbb{P}[\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]}\right)\\&=\mathbb{E}[\tau^\sigma_{T\setminus\sigma}\mid\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T](1+O(1/N))\end{aligned}\tag{4.76}$$

Since the term in braces in the last line of (4.74) was estimated in Corollary 2.3, inserting (2.13) and (4.76) in (4.74), we obtain that, under the assumptions and with the notation of Proposition 2.2,

$$\left|\mathbb{E}[\tau^\sigma_{\sigma'}\mid\tau^\sigma_{\sigma'}\leq\tau^\sigma_T]-\mathbb{E}[\tau^\sigma_{T\setminus\sigma}\mid\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T]\right|\leq O(1/N)\left(\mathbb{E}[\tau^\sigma_{T\setminus\sigma}\mid\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T]+(1-\tfrac{1}{M})^{-1}\mathcal{W}_{\beta,N,T}\right)\tag{4.77}$$

Therefore, collecting (4.77) and (4.71),

$$|\kappa_{\sigma,\sigma'}(u)| \leq \frac{e^{-\beta\sqrt{N}E_\sigma}}{M}\left[1+|u|\left(2\mathbb{E}[\tau^\sigma_{T\setminus\sigma} | \tau^\sigma_{T\setminus\sigma} = \tau^\sigma_T] + (1-\tfrac{1}{M})^{-1}\mathcal{W}_{\beta,N,T}\right)\right]O(1/N) \tag{4.78}$$

and we are left to bound the term $\mathbb{E}[\tau^\sigma_{T\setminus\sigma} | \tau^\sigma_{T\setminus\sigma} = \tau^\sigma_T]$ from above. To do so, we proceed as in (4.72), (4.73), but this time using (3.35) and the fact that $\mathbb{E}\tau^\sigma_\sigma \mathbb{1}_{\{\tau^\sigma_\sigma \leq \tau^\sigma_T\}} \geq \mathbb{P}[\tau^\sigma_\sigma = 1] = 1 - e^{-\beta\sqrt{N}E_\sigma}$, we obtain that

$$\begin{aligned} \mathbb{E}[\tau^\sigma_{T\setminus\sigma} | \tau^\sigma_{T\setminus\sigma} = \tau^\sigma_T] &\leq \mathbb{E}[\tau^\sigma_{T\setminus\sigma}] - \frac{1}{\mathbb{P}(\tau^\sigma_{T\setminus\sigma} < \tau^\sigma_\sigma)} + \frac{1}{e^{\beta\sqrt{N}E_\sigma}\mathbb{P}(\tau^\sigma_{T\setminus\sigma} < \tau^\sigma_\sigma)} \\ &\leq \frac{1}{1-\frac{1}{M}}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)(1+O(1/N)) \end{aligned} \tag{4.79}$$

where the second line follows from the bound (2.14) of Corollary 2.3 together with the estimate (2.6) of Proposition 2.1. Inserting this bound in (4.78) yields,

$$|\kappa_{\sigma,\sigma'}(u)| \leq \frac{e^{-\beta\sqrt{N}E_\sigma}}{M}\left[1+3|u|(1-\tfrac{1}{M})^{-1}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)\right]O(1/N) \tag{4.80}$$

Thus

$$\begin{aligned} \|K^{*(0)}(u) - K^{*(1)}(u)\| &\leq \max_{\sigma\in T}\frac{\sum_{\sigma'\in T\setminus\sigma}|\kappa_{\sigma,\sigma'}(u)|}{|1-G^\sigma_{\sigma,T}(u)|} \\ &\leq \max_{\sigma\in T}\frac{(1-\frac{1}{M})e^{-\beta\sqrt{N}E_\sigma}}{|1-G^\sigma_{\sigma,T}(u)|}\left[1+3|u|(1-\tfrac{1}{M})^{-1}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)\right]O(1/N) \end{aligned} \tag{4.81}$$

and observing that, by assertion (v) of Proposition 2.1,

$$(1-\tfrac{1}{M})e^{-\beta\sqrt{N}E_\sigma} = G^\sigma_{T\setminus\sigma,T}(0)(1+O(1/N)) \tag{4.82}$$

we finally arrive at

$$\|K^{*(0)}(u) - K^{*(1)}(u)\| \leq \max_{\sigma\in T}\frac{G^\sigma_{T\setminus\sigma,T}(0)}{|1-G^\sigma_{\sigma,T}(u)|}\left[1+3|u|(1-\tfrac{1}{M})^{-1}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)\right]O(1/N) \tag{4.83}$$

From there on, the proof proceeds exactly as the proofs of Lemma 4.6, 4.8 and 4.9, yielding

$$\|K^{*(0)}(u) - K^{*(1)}(u)\| \leq \frac{1+3\sqrt{s^2+r^2}(1-\frac{1}{M})^{-1}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)\widehat{\Theta}^{-1}}{\sqrt{1+(s^2+r^2)/2-5/M}}O(1/N) \tag{4.84}$$

which, since $(1-\frac{1}{M})^{-1}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)\widehat{\Theta}^{-1} \leq 1$, gives (4.68), proving Lemma 4.10. ♢

Proof of Lemma 4.11: By definition of $\mathcal{K}^{*(0)}(u)$ and $K^{*(0)}(u)$,

$$\begin{aligned}\|\mathcal{K}^{*(0)}(u) - K^{*(0)}(u)\| &= \max_{\sigma\in T} |\mathcal{K}^{*(0)}_{\sigma,\sigma}(u)| \\ &\leq \frac{1}{M-1} \max_{\sigma\in T} \frac{(1-\frac{1}{M})^{-1} e^{-\beta\sqrt{N}E_\sigma}}{|1-G^{\sigma}_{\sigma,T}(u)|} \left(1 + |u|\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}]\right)\end{aligned} \tag{4.85}$$

Eq. (4.79) then yields the bound

$$\|\mathcal{K}^{*(0)}(u) - K^{*(0)}(u)\| \leq \frac{1}{M-1} \max_{\sigma\in T} \frac{(1-\frac{1}{M}) e^{-\beta\sqrt{N}E_\sigma}}{|1-G^{\sigma}_{\sigma,T}(u)|} \left[1 + |u|(1-\tfrac{1}{M})^{-1}\left(1+\mathcal{W}_{\beta,N,T(E)}\right)\right] O(1/N) \tag{4.86}$$

which, up to some constants, is identical to that of (4.81). From there on the proof follows that of Lemma 4.10.♢

Let us introduce the decomposition

$$K^*(u) \equiv \mathcal{K}^{*(0)}(u) + \mathcal{K}^{*(1)}(u) \tag{4.87}$$

and note that $\mathcal{K}^{*(1)}(u)$ can be written in the form

$$\mathcal{K}^{*(1)}(u) \equiv (K^{*(0)}(u) - \mathcal{K}^{*(0)}(u)) + (K^{*(1)}(u) - K^{*(0)}(u)) + K^{*(2)}(u) \tag{4.88}$$

The following corollary then is an immediate consequence of the previous three lemmata.

Corollary 4.12: *Under the assumptions and with the notation of Lemma 4.9 and Lemma 4.10 we have, for N large enough,*

$$\|\mathcal{K}^{*(1)}(u)\| \leq \frac{\gamma^{-2}C(s^2+r^2) + (1+3\sqrt{s^2+r^2}) \max\left(O(1/(M-1)), O(1/N)\right)}{\sqrt{1+(s^2+r^2)/2} - 5/M} \tag{4.89}$$

The leading contribution to $K^*(u)$ thus comes from the matrix $\mathcal{K}^{*(0)}(u)$ whose spectrum is easily analysed. In particular, $\mathcal{K}^{*(0)}(u)$ has a unique non zero eigenvalue of algebraic multiplicity one, denoted by $\lambda(u)$, and given by:

$$\lambda(u) \equiv \sum_{\sigma\in T} \mathcal{K}^{*(0)}_{\sigma,\sigma'}(u) \tag{4.90}$$

The corresponding left eigenvector is proportional to $(1, 1, \ldots, 1)$. Similarly, defining

$$M^{*(0)}(u) \equiv [\mathbb{1} - \mathcal{K}^{*(0)}(u)]^{-1} \mathcal{K}^{*(0)}(u) \tag{4.91}$$

we decompose the Laplace transform of the resolvent (defined in (4.6)) into

$$M^*(u) \equiv M^{*(0)}(u) + M^{*(1)}(u) \tag{4.92}$$

It obviously follows from the previous results that $M^{*(0)}(u)$ has two eigenvalues, 0 and $\lambda(u)[1-\lambda(u)]^{-1}$, the latter having algebraic multiplicity one. We will have to show that the matrix $M^{*(1)}(u)$ has small norm, and this smallness should be inferred from that of $\|\mathcal{K}^{*(1)}(u)\|$. To make this explicit we want to use the following result:

Lemma 4.13: *Set*

$$\begin{aligned} R(u) &\equiv [\mathbb{1} - \mathcal{K}^{*(0)}(u)]^{-1} \\ \rho(u) &\equiv \max\left(|1-\lambda(u)|^{-1}, 1\right) \end{aligned} \tag{4.93}$$

Then,

$$M^{*(1)}(u) = R(u)\mathcal{K}^{*(1)}(u)R(u)\frac{1}{\mathbb{1} - R(u)\mathcal{K}^{*(1)}(u)} \tag{4.94}$$

and, if $\|R(u)\mathcal{K}^{*(1)}(u)\| < 1$,

$$\|M^{*(1)}(u)\| \leq \frac{\|\mathcal{K}^{*(1)}(u)\|\rho(u)^2}{1 - \|\mathcal{K}^{*(1)}(u)\|\rho(u)} \tag{4.95}$$

Proof: Observe that, using the decomposition(4.87), $[\mathbb{1} - K^*(u)]^{-1}$ can be written in the form

$$\frac{1}{\mathbb{1} - K^*(u)} = R(u) + R(u)\mathcal{K}^{*(1)}(u)\frac{1}{\mathbb{1} - K^*(u)} \tag{4.96}$$

Thus

$$\begin{aligned} M^*(u) &= M^{*(0)}(u) + R(u)\mathcal{K}^{*(1)}(u) + R(u)\mathcal{K}^{*(1)}(u)\frac{1}{\mathbb{1} - K^*(u)}K^*(u) \\ &= M^{*(0)}(u) + R(u)\mathcal{K}^{*(1)}(u)\frac{1}{\mathbb{1} - K^*(u)} \end{aligned} \tag{4.97}$$

Eq. (4.94) then results from (4.97) together with the identity

$$\frac{1}{\mathbb{1} - K^*(u)} = R(u)\frac{1}{\mathbb{1} - R(u)\mathcal{K}^{*(1)}(u)} \tag{4.98}$$

We now turn to the proof of (4.96). It follows from the spectral properties of $\mathcal{K}^{*(0)}(u)$ that

$$\|[\mathbb{1} - \mathcal{K}^{*(0)}(u)]^{-1}\| = \max\left(|1-\lambda(u)|^{-1}, 1\right) \equiv \rho(u) \tag{4.99}$$

Eq. (4.94) then yields the bound

$$\|M^{*(1)}(u)\| \leq \rho(u)^2 \|\mathcal{K}^{*(1)}(u)\| \, \|[\mathbb{1} - R(u)\mathcal{K}^{*(1)}(u)]^{-1}\| \tag{4.100}$$

and (4.95) follows from the fact that, if $\|R(u)\mathcal{K}^{*(1)}(u)\| < 1$, then

$$\|[\mathbb{1} - R(u)\mathcal{K}^{*(1)}(u)]^{-1}\| \leq [1 - \|R(u)\mathcal{K}^{*(1)}(u)\|]^{-1} \leq [1 - \rho(u)\|\mathcal{K}^{*(1)}(u)\|]^{-1} \tag{4.101}$$

The lemma is proven. ♢

At this stage we see that to fully control the behavior of both $M^{*(0)}(u)$ and $M^{*(1)}(u)$ in a small neighborhood of the origin requires a precise control of $1 - \lambda(u)$. Observe that

$$1 - \lambda(u) = \frac{1}{|T|} \sum_{\sigma \in T} \left[1 - \frac{G^{\sigma}_{T\setminus\sigma,T}(0)}{1 - G^{\sigma}_{\sigma,T}(u)} \left(1 + u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}] \right) \right] \tag{4.102}$$

so that $1 - \lambda(u)$ takes the form of a sum over σ. The evaluation of such sums is a rather involved question whose treatment is the object of the next subsection. The analysis of $M^{*(0)}(u)$ and $M^{*(1)}(u)$ will then be brought to a close in Section 4.5. As for the present section, we conclude it with the analysis of the summands of (4.102).

Lemma 4.14: *Recall that* $u = z/\widehat{\Theta}(E)$ *and set*

$$z_\sigma \equiv \left(1 - \tfrac{1}{M}\right) e^{-\beta\sqrt{N}E_\sigma} \widehat{\Theta}(E) \tag{4.103}$$

If u *belongs to the set*

$$D_\delta \equiv \left\{ u \in \mathbb{C} \mid r < s^2/4 \,,\, |z| \leq \delta \right\}, \quad 0 < \delta < 1 \tag{4.104}$$

then, for N *large enough,*

$$\left| 1 - \frac{G^{\sigma}_{T\setminus\sigma,T}(0)}{1 - G^{\sigma}_{\sigma,T}(u)} \left(1 + u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}] \right) - \frac{z}{z - z_\sigma} \right| \leq C(\delta)|z| \tag{4.105}$$

for some constant $0 < C(\delta) < \infty$ *that only depends on* δ.

Proof: Let us write

$$\begin{aligned} & 1 - \frac{G^{\sigma}_{T\setminus\sigma,T}(0)}{1 - G^{\sigma}_{\sigma,T}(u)} \left(1 + u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}] \right) \\ &= \left(1 - \frac{G^{\sigma}_{T\setminus\sigma,T}(0)}{1 - G^{\sigma}_{\sigma,T}(u)} \right) \left(1 + u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}] \right) - u\mathbb{E}[\tau^{\sigma}_{T\setminus\sigma} | \tau^{\sigma}_{T\setminus\sigma} = \tau^{\sigma}_{T}] \end{aligned} \tag{4.106}$$

Recall that we denote by u_σ the smallest real number that solves the equation $G^\sigma_{\sigma,T}(u) = 1$. We will first look at the term in round brackets.

$$\begin{aligned}
1 - \frac{G^\sigma_{T\setminus\sigma,T}(0)}{1 - G^\sigma_{\sigma,T}(u)} &= \frac{G^\sigma_{\sigma,T}(0) - G^\sigma_{\sigma,T}(u)}{1 - G^\sigma_{\sigma,T}(u)} \\
&= -\frac{G^\sigma_{\sigma,T}(0) - G^\sigma_{\sigma,T}(u)}{(u - u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)} \\
&+ \left(G^\sigma_{\sigma,T}(0) - G^\sigma_{\sigma,T}(u)\right)\left(\frac{1}{1 - G^\sigma_{\sigma,T}(u)} + \frac{1}{(u - u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)}\right) \\
&= \frac{u}{u - u_\sigma} - \frac{G^\sigma_{\sigma,T}(0) - G^\sigma_{\sigma,T}(u) + u\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)}{(u - u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)} \\
&+ \frac{\left(G^\sigma_{\sigma,T}(0) - G^\sigma_{\sigma,T}(u)\right)\left((u - u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma) + 1 - G^\sigma_{\sigma,T}(u)\right)}{(1 - G^\sigma_{\sigma,T}(u))(u - u_\sigma)\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)} \\
&= \frac{u}{u - u_\sigma}(1 + \widetilde{R}_\sigma(u)) + \widehat{R}_\sigma(u)
\end{aligned} \tag{4.107}$$

$\widetilde{R}_\sigma$ and $\widehat{R}_\sigma$ being defined through

$$\begin{aligned}
\widetilde{R}_\sigma(u) &\equiv \frac{(\tilde{u} - u_\sigma)\frac{d^2}{du^2}G^\sigma_{\sigma,T}(\hat{u})}{\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)} \\
\widehat{R}_\sigma(u) &\equiv u\frac{d}{du}G^\sigma_{\sigma,T}(\tilde{u})\frac{\frac{1}{2}\frac{d^2}{du^2}G^\sigma_{\sigma,T}(\check{u})}{\frac{d}{du}G^\sigma_{\sigma,T}(u')\frac{d}{du}G^\sigma_{\sigma,T}(u_\sigma)}
\end{aligned} \tag{4.108}$$

where $\tilde{u}$ is on the ray between 0 and u, $\hat{u}$ on the ray between $\tilde{u}$ and u_σ, and both $\check{u}$ and u' are on the ray between u and u_σ, and u'.

The various first and second derivatives entering the expressions of $\widehat{R}_\sigma(u)$ and $\widetilde{R}_\sigma(u)$ can be bounded with the help of (3.52) and (3.54). We then get that on the disk $|u| \leq \delta/\widehat{\Theta}(E)$, $0 < \delta < 1$,

$$|\widehat{R}_\sigma(u)| \leq c(\delta) z_\sigma |z| \tag{4.109}$$

where z_σ is defined in (4.103) and $0 < c(\delta) < \infty$ only depends on δ. Similarly, using that $\tilde{u}$ is on the ray between 0 and u,

$$|\widetilde{R}_\sigma(u)| \leq c'(\delta) z_\sigma (|z| + |\widehat{\Theta}(E) u_\sigma|) \tag{4.110}$$

for some $0 < c'(\delta) < \infty$. Recall from Section 3.2 (formula (3.48)) that $u_\sigma \approx \frac{1}{\mathbb{E}\tau^\sigma_{T(E)\setminus\sigma}}$; however, inspecting the proof of Proposition 3.6 (see also (2.9)) an alternative representation is

$$u_\sigma = G^\sigma_{T(E)\setminus\sigma,T(E)}(0)(1 + O(e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E))) \tag{4.111}$$

and this will be even more convenient here as, using (4.82), we then have

$$\widehat{\Theta}(E)u_\sigma = z_\sigma(1+O(z_\sigma)) \tag{4.112}$$

The bound (4.110) thus becomes

$$|\widetilde{R}_\sigma(u)| \leq c''(\delta)z_\sigma(|z|+z_\sigma) \tag{4.113}$$

We now come to the main contribution to the r.h.s. of (4.107), namely to the term $u/(u-u_\sigma)$. Using (4.112) we can write

$$\frac{u}{u-u_\sigma} = \frac{z}{z-z_\sigma} + \overline{R}_\sigma(z) \tag{4.114}$$

where

$$\overline{R}_\sigma(z) \equiv \frac{z(u_\sigma\widehat{\Theta}(E)-z_\sigma)}{(z-z_\sigma)(z-u_\sigma\widehat{\Theta}(E))} = \frac{zO(z_\sigma^2)}{(z-z_\sigma)(z-z_\sigma(1+O(z_\sigma)))} \tag{4.115}$$

To bound this term we use that on the set $\left\{z\in\mathbb{C} \mid r < s^2/4\right\}$:

$$|z-z_\sigma| \geq z_\sigma \tag{4.116}$$

and

$$|z-z_\sigma(1+O(z_\sigma))| \geq \begin{cases} z_\sigma(1+O(z_\sigma)), & \text{if } \ z_\sigma(1+O(z_\sigma)) \leq 2 \\ 2\sqrt{z_\sigma(1+O(z_\sigma))-1}, & \text{otherwise} \end{cases} \tag{4.117}$$

Therefore, for $z\in D_\delta$,

$$|\overline{R}_\sigma(z)| \leq \frac{|z|O(z_\sigma^2)}{|z-z_\sigma|\,|z-z_\sigma(1+O(z_\sigma))|} \leq c|z| \tag{4.118}$$

for some constant $c>0$.

Inserting (4.114) in (4.107), and plugging the resulting expression in (4.106), we may now write

$$1-\frac{G^\sigma_{T\setminus\sigma,T}(0)}{1-G^\sigma_{\sigma,T}(u)}\left(1+u\mathbb{E}[\tau^\sigma_{T\setminus\sigma}|\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T]\right) = I^0_\sigma(u)+I^1_\sigma(u) \tag{4.119}$$

where

$$\begin{aligned} I^0_\sigma(u) &\equiv \left(\frac{z}{z-z_\sigma}+\overline{R}_\sigma(z)\right)(1+\widetilde{R}_\sigma(u))+\widehat{R}_\sigma(u) \\ I^1_\sigma(u) &\equiv z\frac{\mathbb{E}[\tau^\sigma_{T\setminus\sigma}|\tau^\sigma_{T\setminus\sigma}=\tau^\sigma_T]}{\widehat{\Theta}(E)}\left[\left(\frac{z}{z-z_\sigma}+\overline{R}_\sigma(z)\right)(1+\widetilde{R}_\sigma(u))+\widehat{R}_\sigma(u)-1\right] \end{aligned} \tag{4.120}$$

Assume that $z \in D_\delta$. Since $\mathbb{E}[\tau^\sigma_{T\setminus\sigma} | \tau^\sigma_{T\setminus\sigma} = \tau^\sigma_T]\widehat{\Theta}(E)^{-1} \leq 1$, it readily follows from the estimates (4.109), (4.113), (4.118), and the bound

$$\left|\frac{z}{z - z_\sigma}\right| = \left|1 + \frac{z_\sigma}{z - z_\sigma}\right| \leq 2 \tag{4.121}$$

which, by (4.116), holds for all $z \in D_\delta$, that

$$|I^1_\sigma(u)| \leq C'(\delta)|z| \tag{4.122}$$

for some constant $C'(\delta) > 0$. To treat the term $I^0_\sigma(u)$ note that using in turn (4.113) and (4.116),

$$\left|\frac{z}{z - z_\sigma}\widetilde{R}_\sigma(u)\right| \leq c''(\delta)\frac{z_\sigma|z|}{|z - z_\sigma|}(|z| + z_\sigma) \leq c''(\delta)|z|(|z| + z_\sigma) \tag{4.123}$$

Therefore,

$$\left|I^0_\sigma(u) - \frac{z}{z - z_\sigma}\right| \leq C''(\delta)|z| \tag{4.124}$$

for some constant $C''(\delta) > 0$. Combining (4.119) together with (4.123) and (4.124) yields (4.105). This concludes the proof of Lemma 4.13.$\diamondsuit$

4.4 Poisson convergence.

Finally we need to control the convergence of various integral functions of the variables z_σ. We will do this in a general setting first and then apply this to the various occurrences later on.

Note first that by (4.103) and (3.2),

$$\begin{aligned} z_\sigma &= (1 - 1/M)e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}(E) \\ &= e^{-\alpha(u_N^{-1}(E_\sigma) - E)}\left(1 + \frac{e^{-E}}{|T(E)|(\alpha - 1)}\left(1 + \mathcal{V}_{N,E}e^{E/2}\frac{\alpha - 1}{\sqrt{2\alpha - 1}}\right)\right)(1 + O(1/N)) \\ &\equiv \frac{1}{e^{\alpha(u_N^{-1}(E_\sigma) - E)}\tau_{E,N}} \end{aligned} \tag{4.125}$$

only depends on σ through $u_N^{-1}(E_\sigma)$. As has been explained in Section 1, the point process

$$\mathcal{N}^*_{N,E} \equiv \sum_{\sigma \in \{-1,1\}^N} \delta_{\exp\{\alpha(-E + u_N^{-1}(E_\sigma))\}} = \sum_{\sigma \in \{-1,1\}^N} \delta_{1/(z_\sigma \tau_{N,E})} \tag{4.126}$$

converges weakly to the Poisson point process $\mathcal{N}^*_E$ on $[1, \infty)$ with intensity measure $\alpha^{-1}e^E x^{-1-1/\alpha}dx$.

We will now show how to make use of the convergence of our point processes to Poisson point processes in the analysis of the asymptotic behavior of our functions as both N and E

tend to infinity. As a first example we will explain how to control the behavior of the random coefficients $\tau_{N,E}$.

Lemma 4.15: *Set* $\tau_\infty \equiv \frac{\alpha-1}{\alpha}$. *Then,*

$$\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty}\tau_{N,E}=\tau_\infty, \quad \textit{in Probability.} \tag{4.127}$$

Proof: $\tau_{N,E}$ depends on two random variables, $\mathcal{V}_{N,E}$ (defined in Eq. (3.2) of [BBG1]) and $|T(E)|$. Let us first look at $\mathcal{V}_{N,E}$. We want to show that $\mathcal{V}_{N,E}e^{E/2}$ tends to zero. By Chebychev's inequality of order four, we have that

$$\mathbb{P}[|\mathcal{V}_{N,E}e^{+E/2}|>\epsilon]\leq\frac{\mathbb{E}\mathcal{V}_{N,E}^4}{\epsilon^4 e^{-2E}} \tag{4.128}$$

But (see [BKL], Lemma 3.3, where however the normalisation of $\mathcal{V}_N$ is different) the moments of the random variable $\mathcal{V}_{N,E}$ converge, as $N\uparrow\infty$, and in particular

$$\lim_{N\uparrow\infty}\mathbb{E}\mathcal{V}_{N,E}^4=\frac{(2\alpha-1)^2}{4\alpha-1}e^{-E}+3 \tag{4.129}$$

Therefore, there exists N_0, such that for all $N>N_0$, and for $-E$ large enough,

$$\mathbb{P}[|\mathcal{V}_{N,E}e^{E/2}|>\epsilon]\leq\frac{4e^E}{\epsilon^4} \tag{4.130}$$

Next we note that $|T(E)|=\int_E^\infty\mathcal{N}_N(dx)$ converges, as $N\uparrow\infty$, to a Poisson random variable with parameter e^E. In particular,

$$\lim_{N\uparrow\infty}\mathbb{P}[|e^E|T(E)|-1|>\epsilon]=\sum_{n=0}^{e^{-E}(1-\epsilon)}\frac{e^{-nE}}{n!}e^{-e^{-E}}+\sum_{n=e^{-E}(1+\epsilon)}^{\infty}\frac{e^{-nE}}{n!}e^{-e^{-E}}\leq Ce^{-E}e^{-\epsilon^2e^{-E}} \tag{4.131}$$

Combining these two observations proves the lemma. ◇

Remark: Note that we actually prove that $\tau_{N,E}$ converges, as $N\uparrow\infty$, to a random variable τ_E which in turn, as $E\downarrow-\infty$, converges to a constant. This latter convergence can easily be shown to take place almost surely. However, it is not correct that the joint convergence takes place almost surely. It may be possible to show that almost sure convergence holds along certain diagonal limits $N\uparrow\infty$ with $E=E_N$ depending on N in a suitable way. Due to the generally rather slow convergence of extremal distributions, proving such a statement rigorously would require a considerable extra effort and is not guaranteed to succeed.

The next lemma is an immediate application of the weak convergence of the point process $\mathcal{N}^*_{N,E}$:

Lemma 4.16: *Let g be a bounded continuous function on $\mathbb{R}^+$, such that $\left|\int_0^\infty \frac{dx}{x^{1+1/\alpha}} g(x)\right| < +\infty$, and let X_N be a family of positive random variables that converge in distribution to the positive random variable X. Then for any $b > 0$,*

(i) $\int_b^\infty \mathcal{N}^*_{N,E}(dx) g(xX_N)$ converges, as $N \uparrow \infty$, to the random variable $\int_b^\infty \mathcal{N}^*_E(dx) g(xX)$.

(ii) If X_E is a family of random variables such that, as $E \downarrow -\infty$, $X_E \to a \in \mathbb{R}^+$ almost surely, then

$$\lim_{E \downarrow -\infty} e^{+E} \int_1^\infty \mathcal{N}^*_E(dx) g(xX_E) = \alpha^{-1} \int_1^\infty \frac{dx}{x^{1+1/\alpha}} g(xa), \quad a.s. \tag{4.132}$$

(iii) If g is a complex valued function on $\mathbb{C}$, and if for some domain $B \subset \mathbb{C}$, for all $x \in \mathbb{R}^+$, $z \in B$, $g(zx)$ is bounded, and for all $z \in B$,

$$\left| \int_0^\infty \frac{dx}{x^{1+1/\alpha}} g(zx) \right| < \infty \tag{4.133}$$

holds, then

$$\lim_{E \downarrow -\infty} P\left[\lim_{N \uparrow \infty} \sup_{z \in B} \left| e^E \int_1^\infty \mathcal{N}^*_E(dx) g(zxX_E) - (az)^{1/\alpha} \alpha^{-1} \int_{az}^\infty \frac{dx}{x^{1+1/\alpha}} g(x) \right| > \epsilon \right] = 0 \tag{4.134}$$

Proof: (i) is a standard result that follow from the equivalence of convergence in distribution of a r.v. and almost sure convergence of a sequence of r.v. having the same distribution. To prove (ii), recall that by definition of the Poisson process $\mathcal{N}^*_E$,

$$\int_1^\infty \mathcal{N}^*_E(dx) g(x) = \sum_{i=1}^{n_E} g(x_i) \tag{4.135}$$

where n_E is a Poisson random variable with mean e^E, and $x_i, i \in \mathbb{N}$ are i.i.d. random variables such that

$$P[x_i \leq a] \equiv \alpha^{-1} \int_1^a \frac{dx}{x^{1+1/\alpha}} \tag{4.136}$$

Note that first by continuity $g(xX_E) - g(xa)$ converge to zero and since g is integrable w.r.t. the law of x_i, $g(xX_E) - g(xa) \downarrow 0$ as a random variable. On the other hand, it follows from our assumptions that $g(x_i)$ are bounded random variables. In particular, their moment generating functions $\mathbb{E}e^{\lambda g(x_i)}$ is finite for all λ. Therefore standard arguments imply that there exists a constant c such that

$$P\left[|n_E^{-1}\sum_{i=1}^{n_E}(g(x_i)-\mathbb{E}g(x_i))|>\epsilon\right]\leq 2\mathbb{E}_{n_E}\exp\left(-\frac{\epsilon^2 n_E}{c\mathrm{var}^2(g)}\right) \tag{4.137}$$

where $\mathbb{E}_{n_E}$ denotes expectation with respect to the Poisson variable n_E and

$$\mathrm{var}^2(g)\equiv\alpha^{-1}\int_1^\infty\frac{dx}{x^{1+1/\alpha}}\left(g(x)-\alpha^{-1}\int_1^\infty\frac{dx}{x^{1+1/\alpha}}g(x)\right)^2 \tag{4.138}$$

is, by our assumptions on g, finite. Together with the exponential estimate on the concentration of the Poisson variable n_E (4.131), this yields

$$P\left[\left|n_E^{-1}\int_1^\infty\mathcal{N}_E^*(dx)g(x)-\alpha^{-1}\int_1^\infty\frac{dx}{x^{1+1/\alpha}}g(x)\right|\geq\epsilon\right]\leq 2\exp\left(-\frac{\epsilon^2 e^{-E}}{2c\mathrm{var}^2(g)}\right)+Ce^{-E}e^{-e^{-E/4}} \tag{4.139}$$

From this (ii) follows immediately. To prove (iii), note that (ii) also holds if g takes complex values by simply considering real and imaginary part separately. By a simple change of variables we have, for $s\leq 1$,

$$\mathbb{E}g(s\cdot)=s^{1/\alpha}\alpha^{-1}\int_s^\infty\frac{dx}{x^{1+1/\alpha}}g(x) \tag{4.140}$$

and

$$\mathrm{var}^2(g(s\cdot))=s^{1/\alpha}\alpha^{-1}\int_s^\infty\frac{dx}{x^{1+1/\alpha}}\left(g(x)-s^{1/\alpha}\alpha^{-1}\int_s^\infty\frac{dx}{x^{1+1/\alpha}}g(x)\right)^2 \tag{4.141}$$

If (4.133) holds, this implies that $\mathbb{E}g(s\cdot)\leq Cs^{1/\alpha}$ and $\mathrm{var}^2(g(s\cdot))\leq Cs^{1/\alpha}$ for small s. Thus, for s small, we get from (4.139) that for some finite constant C_g depending on g

$$P\left[\left|e^E\int_1^\infty\mathcal{N}_E^*(dx)g(sx)-\alpha^{-1}\int_1^\infty g(sx)\right|\geq s^{1/(2\alpha)}\epsilon\right]\leq 2\exp\left(-\frac{\epsilon^2 e^{-E}}{2C_g}\right)+Ce^{-E}e^{-e^{-E/4}} \tag{4.142}$$

Remark: This means that fluctuations are at most of order $s^{1/(2\alpha)}e^{E/2}$ which is less than the mean as long as $s>e^E$. This should be taken as a sign that on time scales larger than e^{-E} self-averaging no longer takes place.

The uniformity of the convergence in z claimed under (iii) follows from the exponential estimate (4.142) and the continuity of g by standard arguments. This concludes the proof of the lemma.$\diamondsuit$

As first and main application of this lemma we obtain the

Corollary 4.17: *Uniformly in* $\Re(z) < \max(|\Im(z)|, 1/2)$,

$$\begin{aligned}&\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty}\frac{1}{|T(E)|}\sum_{\sigma\in T(E)}\frac{z}{z-z_\sigma}\\ &\qquad=\alpha^{-1}\int_1^\infty\frac{dx}{x^{1+1/\alpha}}\frac{xz\tau_\infty}{xz\tau_\infty-1},\quad \textit{in Probability.}\end{aligned}\tag{4.143}$$

Moreover, on the same set,

$$\alpha^{-1}\int_1^\infty\frac{dx}{x^{1+1/\alpha}}\frac{xz\tau_\infty}{xz\tau_\infty-1}=(-z\tau_\infty)^{1/\alpha}\pi\, cosec\,(\pi/\alpha)+O(|z|)\tag{4.144}$$

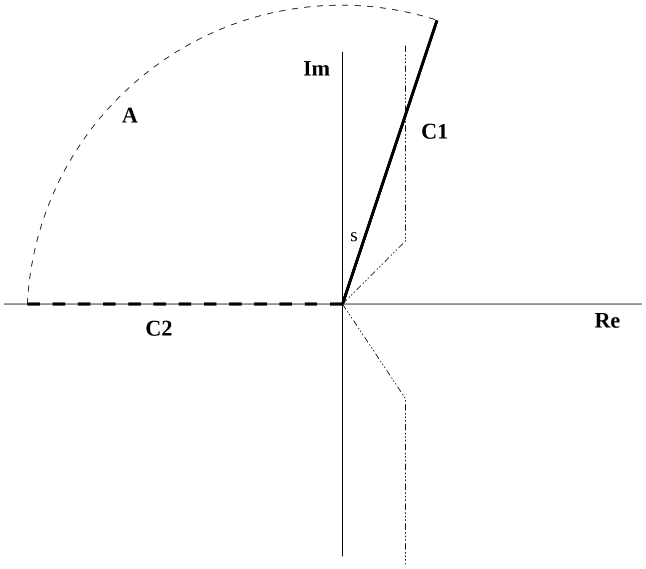
for $|z|$ *small.*

Proof: To get (4.143), just check that the hypothesis of Lemma 4.16 are satisfied. To prove (4.144), note first that

$$\begin{aligned}\left|\int_0^1\frac{dx}{x^{1+1/\alpha}}\frac{xz\tau_\infty}{xz\tau_\infty-1}\right|&\leq\int_0^1\frac{dx}{x^{1+1/\alpha}}\frac{|z|\tau_\infty x}{\sqrt{2}}\\ &=\frac{|z|\tau_\infty}{\sqrt{2}}\frac{1}{1-1/\alpha}\end{aligned}\tag{4.145}$$

where we used that $|(a+ib-1)|^{-1}\leq[(a-1)^2+a^2]^{-1/2}\leq 2^{-1/2}$, if $a\leq|b|$. Thus it remains to compute the integral from zero to infinity. To do this we change variables from x to $xz\tau_\infty$. This turns the integral into an integration over a path **C1** in the complex plane which is the straight line from zero passing through s to infinity. Since $\frac{z}{z-1}$ is analytic in the complex plane with the positive real axis removed, the integration path **C1** can be rotated to the negative real axis **C2** without changing the integral, since the integral along the arc **A** at infinity vanishes. (see figure). In fact

$$\begin{aligned}\int_0^\infty\frac{dx}{x^{1+1/\alpha}}\frac{xz\tau_\infty}{xz\tau_\infty-1}&=z^{1/\alpha}\tau_\infty^{1/\alpha}\int_0^{\infty z}\frac{dx}{x^{1+1/\alpha}}\frac{x}{x-1}=z^{1/\alpha}\tau_\infty^{1/\alpha}\int_0^{-\infty}\frac{dx}{x^{1+1/\alpha}}\frac{x}{x-1}\\ &=(-z)^{1/\alpha}\tau_\infty^{1/\alpha}\int_0^\infty\frac{dx}{x^{1+1/\alpha}}\frac{x}{x+1}\end{aligned}\tag{4.146}$$



Rotation of the integration contour to the real axis

This proves the lemma.$\diamondsuit$

We can now collect the results obtained in this and the previous subsection to control the asymptotics of the eigenvalue $\lambda(u)$.

Corollary 4.18: *with* $u = z\widehat{\Theta}(E)^{-1}$, *on the domain* D_δ *defined in (4.104),*

$$\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty}(1-\lambda(u)) = (-z\tau_\infty)^{1/\alpha}\pi\, cosec\,(\pi/\alpha) + O(|z|), \quad \textit{in Probability.} \tag{4.147}$$

Proof: it suffices to combine formula (4.102), the estimate (4.105) of Lemma 4.14, and Corollary 4.17. $\diamondsuit$

Remark: The diligent reader (if any) who has reached this point will be relieved to finally see some formulas familiar from the trap-model emerge.

Having this result, we can now also estimate the norm of the error term $M^{*(1)}(u)$.

Corollary 4.19: *with* $u = z\widehat{\Theta}(E)^{-1}$, *on the domain* D_δ *defined in (4.104),*

$$\limsup_{E\downarrow-\infty}\limsup_{N\uparrow\infty}\|M^{*(1)}(u)\| \leq C(d)|z|^{2(1-1/\alpha)}, \quad \textit{in Probability.} \tag{4.148}$$

Proof: This follows from Lemma 4.13 and Corollary 4.18.$\diamondsuit$

Remark: We can only now appreciate why we expanded to second order in (4.58). It is crucial to have the norm of $\mathcal{K}^{*(1)}(u)$ bounded by something of order $|z|^2$ to obtain an estimate that tends to zero in the corollary above.

4.5 Controlling the inhomogeneous term.

Our next step is to establish control over the inhomogeneous term $F^*_\sigma(m,u)$ defined in (4.2). To do so we use the Markov property to represent

$$\begin{aligned}\mathbb{P}[\tau^\sigma_{T(E)\setminus\sigma} > m+n] &= \sum_{\sigma'\notin T(E)\setminus\sigma} \mathbb{P}[\sigma(m)=\sigma', \tau^\sigma_{T(E)\setminus\sigma} > m]\mathbb{P}[\tau^{\sigma'}_{T(E)\setminus\sigma} > n] \\ &= \sum_{\sigma'\notin T(E)} \mathbb{P}[\sigma(m)=\sigma', \tau^\sigma_{T(E)\setminus\sigma} > m]\mathbb{P}[\tau^{\sigma'}_{T(E)\setminus\sigma} > n] \\ &\quad + \mathbb{P}[\sigma(m)=\sigma, \tau^\sigma_{T(E)\setminus\sigma} > m]\mathbb{P}[\tau^{\sigma}_{T(E)\setminus\sigma} > n]\end{aligned} \tag{4.149}$$

Inserting this relation into (4.2) we obtain that

$$\begin{aligned}F^*_\sigma(m,u) = &\sum_{\sigma'\notin T(E)} \mathbb{P}[\sigma(m)=\sigma', \tau^\sigma_{T(E)\setminus\sigma} > m]\mathcal{L}^{\sigma'}_\sigma(u) \\ &+ \mathbb{P}[\sigma(m)=\sigma, \tau^\sigma_{T(E)\setminus\sigma} > m]\mathcal{L}^{\sigma}_\sigma(u)\end{aligned} \tag{4.150}$$

where $\mathcal{L}^\sigma_\sigma(u)$ is given by (3.13) and

$$\mathcal{L}^{\sigma'}_\sigma(u) = \frac{G^{\sigma'}_{T(E)\setminus\sigma}(u) - 1}{e^u - 1} \tag{4.151}$$

Thus, using that

$$G^{\sigma'}_{T(E)\setminus\sigma}(u) = G^{\sigma'}_{T(E)\setminus\sigma,T(E)}(u) + G^{\sigma'}_{\sigma,T(E)}(u)G^{\sigma}_{T(E)\setminus\sigma}(u) \tag{4.152}$$

we get

$$\begin{aligned}F^*_\sigma(m,u) = \frac{1}{e^u-1}\Bigg[&\mathbb{P}[\sigma(m)=\sigma, \tau^\sigma_{T(E)\setminus\sigma} > m]\left(G^{\sigma}_{T(E)\setminus\sigma}(u) - 1\right) \\ &+ \sum_{\sigma'\notin T(E)} \mathbb{P}[\sigma(m)=\sigma', \tau^\sigma_{T(E)\setminus\sigma} > m]\left(G^{\sigma'}_{T(E)\setminus\sigma,T(E)}(u) + G^{\sigma'}_{\sigma,T(E)}(u)G^{\sigma}_{T(E)\setminus\sigma}(u) - 1\right)\Bigg]\end{aligned} \tag{4.153}$$

As by now usual, we will need a rather crude bound for u away from the origin complemented by a finer estimate for very small values of $|u|$. The former follows from the next lemma.

Lemma 4.20: *Assume that* $\Re(u) \leq \frac{1}{2}\widehat{\Theta}^{-1}$. *Then*

$$|F^*_\sigma(m,u)| \leq \frac{2}{|e^u-1|}\mathbb{P}[\tau^\sigma_{T(E)\setminus\sigma} > m]\left(\left|G^{\sigma}_{T(E)\setminus\sigma}(u)\right| + 2\right) \tag{4.154}$$

Proof: By Lemma 3.4, under the condition on u, $\left|G^{\sigma'}_{\sigma,T(E)}(u)\right| = \left|g^{\sigma'}_{\sigma}(u)\right| \leq 2$ (in fact $\leq 2/(M-1)$). Similarly, $\left|G^{\sigma'}_{T(E)\backslash\sigma,T(E)}(u)\right| \leq 2$. Inserting this into (4.153) and noting that $\sum_{\sigma'\notin T(E)\backslash\sigma}\mathbb{P}[\sigma(m)=\sigma',\tau^{\sigma}_{T(E)\backslash\sigma}>m] = \mathbb{P}[\tau^{\sigma}_{T(E)\backslash\sigma}>m]$ one arrives readily at the claimed bound.♢

Bounds for $|u| \ll 1$.

As was the case for the resolvent, we have to identify more precisely the leading term of the inhomogeneous term for the contribution to the inversion integral for u very close to the origin. We begin with the m-dependent probabilities in (4.153).

Lemma 4.21: *There is a finite positive constant C such that, with b_σ as in (3.8),*

$$\left|\mathbb{P}\left[\tau^{\sigma}_{T(E)\backslash\sigma}>m,\sigma(m)=\sigma\right]-p_N(\sigma,\sigma)^m\right| \leq Cme^{-\beta\sqrt{N}E_\sigma}e^{-m/\mathbb{E}\tau^{\sigma}_{T\backslash\sigma}b_\sigma} \tag{4.155}$$

Proof: Note that $p_N(\sigma,\sigma)^m$ is the probability of the event that $\sigma(k)$ remains at σ during the entire period from time zero to time m which is a subset of the event $\{\tau^{\sigma}_{T(E)\backslash\sigma}>m,\sigma(m)=\sigma\}$. In what remains, there must be a first time when $\sigma(k)\neq\sigma$. Thus

$$\begin{aligned}
&\left|\mathbb{P}\left[\tau^{\sigma}_{T(E)\backslash\sigma}>m,\sigma(m)=\sigma\right]-p_N(\sigma,\sigma)^m\right|\\
&\leq\sum_{k=1}^{m-1}p_N(\sigma,\sigma)^{k-1}\sum_{\sigma'\sim\sigma}p_N(\sigma,\sigma')\mathbb{P}\left[\tau^{\sigma'}_{T(E)\backslash\sigma}>m-k,\sigma(m-k)=\sigma\right]\\
&\leq(1-p_N(\sigma,\sigma))\sum_{k=1}^{m-1}p_N(\sigma,\sigma)^{k-1}\max_{\sigma'\sim\sigma}\mathbb{P}\left[\tau^{\sigma'}_{T(E)\backslash\sigma}>m-k\right]
\end{aligned}\tag{4.156}$$

The probability in the last line is similar to the probabilities estimated in Corollary 3.3, except that the starting point is now σ' instead of σ. However using the decomposition (4.152), one verifies easily that following the same lines as in the proof of that corollary, one obtains the estimate

$$\mathbb{P}\left[\tau^{\sigma'}_{T(E)\backslash\sigma}>m-k\right]\leq Ce^{-(m-k)/\mathbb{E}\tau^{\sigma'}_{T\backslash\sigma}b_\sigma} \tag{4.157}$$

which is all we will need here. Inserting this estimate into (4.156) and using that, by Proposition 2.2 (together with the remark that follows it),

$$p_N(\sigma,\sigma)^k=(1-e^{-\beta\sqrt{N}E_\sigma})^k\leq e^{-ke^{-\beta\sqrt{N}E_\sigma}}\leq e^{-k/\mathbb{E}\tau^{\sigma}_{T\backslash\sigma}} \tag{4.158}$$

the bound (4.155) follows directly. ♢

Remark: Let us note that the bound (4.155) is really effectively smaller than the dominant term, if E_σ is "deep" within the top, even though we concede a little of the exponential decay when replacing $e^{\beta\sqrt{N}E_\sigma}$ by $\mathbb{E}^\sigma_{T\setminus\sigma}$. The point is that this error will tend to zero, while the prefactor of the exponential tends to zero as well. Since it will be the σ with exceptionally large E_σ that contribute to the long time behavior, this will do the job.

Lemma 4.22: *There exists a finite positive constant C such that*

(i) If $\mathbb{E}\tau^\sigma_{T\setminus\sigma} > \widehat{\Theta}$, then

$$\mathbb{P}\left[\tau^\sigma_{T(E)\setminus\sigma} > m, \sigma(m) \not\in T(E)\right] \leq \frac{e^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta}}{1-\widehat{\Theta}/\mathbb{E}\tau^\sigma_{T\setminus\sigma}} e^{-m/\mathbb{E}\tau^\sigma_{T\setminus\sigma} b_\sigma} \tag{4.159}$$

(ii) If $\mathbb{E}\tau^\sigma_{T\setminus\sigma} \leq \widehat{\Theta}$, then

$$\mathbb{P}\left[\tau^\sigma_{T(E)\setminus\sigma} > m, \sigma(m) \not\in T(E)\right] \leq \frac{m}{\widehat{\Theta}} e^{-m/\widehat{\Theta}} \tag{4.160}$$

Proof: Note that if the event $\{\tau^\sigma_{T(E)\setminus\sigma} > m, \sigma(m) \not\in T(E)\}$ occurs, then there exists a last time $m-k < m$ when the process visits the σ. This gives us the bound

$$\begin{aligned} &\mathbb{P}\left[\tau^\sigma_{T(E)\setminus\sigma} > m, \sigma(m) \not\in T(E)\right] \\ &\leq \sum_{k=1}^{m-1} \mathbb{P}\left[\tau^\sigma_{T(E)\setminus\sigma} > m-k,\right] \sum_{\sigma'\sim\sigma} p_N(\sigma,\sigma')\mathbb{P}\left[\tau^{\sigma'}_{T(E)} > k-1\right] \\ &\leq (1-p_N(\sigma,\sigma)) \sum_{k=1}^{m-1} e^{-(m-k)/\mathbb{E}\tau^\sigma_{T\setminus\sigma}-(k-1)/\widehat{\Theta}(E)} \end{aligned} \tag{4.161}$$

In case (i) we can extract $e^{-m/\mathbb{E}\tau^\sigma_{T\setminus\sigma}}$ from the sum and oversum the remaining geometric series to get (4.159), while in the latter case we simply bound the exponential terms by their maximum and retain that there are only m terms in the sum. This proves the lemma. ♢

Next we want to deal with the Laplace transforms appearing in (4.153). Concerning the first line, we are already in good position, since we have the estimates needed for $G^\sigma_{T(E)\setminus\sigma}(u) - 1$ (see Proposition 3.6). The second term has, as we have seen, a prefactor that is of lower order in the m behavior, but we have to show that the u-dependent coefficient is not more singular than that of the first term. To this end we rewrite

$$\begin{aligned} &G^{\sigma'}_{T(E)\setminus\sigma,T(E)}(u) + G^{\sigma'}_{\sigma,T(E)}(u) G^{\sigma}_{T(E)\setminus\sigma}(u) - 1 \\ &= G^{\sigma'}_{T(E)\setminus\sigma,T(E)}(u) + G^{\sigma'}_{\sigma,T(E)}(u) \left(G^{\sigma}_{T(E)\setminus\sigma}(u) - 1 \right) + G^{\sigma'}_{\sigma,T(E)}(u) - 1 \\ &= G^{\sigma'}_{T(E)}(u) - 1 + G^{\sigma'}_{\sigma,T(E)}(u) \left(G^{\sigma}_{T(E)\setminus\sigma}(u) - 1 \right) \end{aligned} \tag{4.162}$$

It will suffice to use that, for $\Re u < \frac{1}{2}\widehat{\Theta}^{-1}$,

$$|G^{\sigma'}_{T(E)}(u) - 1| \leq |u|\widehat{\Theta} \tag{4.163}$$

and that $G^{\sigma'}_{\sigma,T(E)}(u)$ is bounded and analytic.

4.6 Laplace inversion 1. The error terms.

After this preparation we are now ready to attack the Laplace inversion of the function $\Pi^*(u, m, E)$ given in principle by (4.5). Recall that we are interested in computing

$$\Pi(n, m, E) \equiv \frac{1}{|T(E)|} \sum_{\sigma \in T(E)} \Pi_\sigma(n, m, E) \equiv (\mathbb{1}, \underline{\Pi}(n, m, E)) \tag{4.164}$$

Setting

$$\Pi^0(n, m, E) \equiv \Pi(n, m, E) - (\mathbb{1}, \underline{F}(n, m)) \tag{4.165}$$

(4.5) and the inversion formula for Laplace transforms we can write

$$\Pi^0(n, m, E) = \frac{1}{2\pi i} \int_{-i\pi}^{i\pi} du e^{-un} \left(\mathbb{1}, M^*_E(u) \underline{F}^*(m, u)\right) \tag{4.166}$$

The notation of Section 4.2 (see (4.31)-(4.34)) are again brought into force in the present section; recall in particular that $z = \widehat{\Theta} u$. The first step of the analysis consists in deforming the contour of integration to the contour $\mathcal{C}$ consisting of the three parts

$$\mathcal{A} \equiv \left\{ u \in \mathbb{C} \,:\, \Re z = 1/2,\, |\Im z| \in [1/\sqrt{2\kappa}, \pi\widehat{\Theta}] \right\} \tag{4.167}$$

$$\mathcal{B} \equiv \left\{ u \in \mathbb{C} \,:\, \Re z \in [1/\tilde{t}, 1/2],\, \Re z = \kappa |\Im z|^2 \right\} \tag{4.168}$$

and

$$\mathcal{D} \equiv \mathcal{D}_1 \cup \mathcal{D}_2 \tag{4.169}$$

where

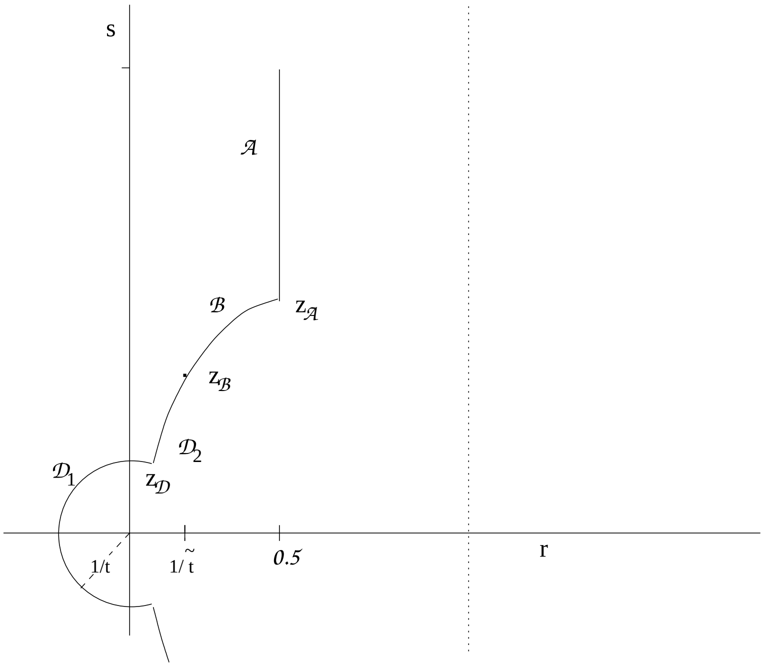
$$\begin{aligned} \mathcal{D}_1 &\equiv \left\{ u \in \mathbb{C} \,:\, |z| = 1/t\,,\, \Re z < c|\Im z|^2 \right\} \\ \mathcal{D}_2 &\equiv \left\{ u \in \mathbb{C} \,:\, \Re z \in [\sqrt{1/(4\kappa^2) + 1/t^2} - 1/(2\kappa), 1/\tilde{t}],\, \Re z = \kappa|\Im z|^2 \right\} \end{aligned} \tag{4.170}$$

Here t and κ are positive parameters that are assumed to be chosen such that $\mathcal{C}$ lies in the domain of validity of Corollary 4.7 and Lemma 4.8, (ii), namely in

$$(D_1(4) \cap D_2(L,\gamma)) \cup D_3, \quad \text{for some fixed} \quad \tfrac{1}{2} \leq \gamma < 1 \tag{4.171}$$

(Note that this essentially only imposes a constraint on κ, which has to be taken small enough compared with γ/L.) In what follows, t must be thought of as very large compared with one. At this stage no constraint is imposed on the parameter $\tilde{t}$; it will be chosen as $\tilde{t} = t^{\eta}$, for suitable $0 < \eta < 1$, later. For future reference let us define the points:

$$\begin{array}{lll} z_A = r_A + i s_A, & z_B = r_B + i s_B, & z_D = r_D + i s_D \\ r_A = 1/2, & r_B = 1/\tilde{t}, & r_D = \sqrt{1/(4\kappa^2) + 1/t^2} - 1/(2\kappa) \\ s_A = 1/\sqrt{2\kappa}, & s_B = 1/\sqrt{\kappa \tilde{t}}, & s_D = ((\sqrt{1 + (2\kappa/t)^2} - 1)/2\kappa^2)^{1/2} \end{array}$$



The contour C in the variables r and s

We expect the main contribution to the integral to come from the part $\mathcal{D}$ of the integration. Thus we show first how to bound the two other contributions. From now on the letter c will denote a positive constant whose value may change from line to line.

Lemma 4.23: *Let $\mathcal{A}$ be defined in (4.167). Then*

$$\left| \int_{\mathcal{A}} du e^{-un} \left(\mathbb{1}, M_E^*(u) \underline{F}^*(m,u) \right) \right| \leq c e^{-n/(2\widehat{\Theta})} \tag{4.172}$$

Proof: Calling $I_{\mathcal{A}}$ the left hand side of (4.172) we clearly have

$$\begin{aligned} I_{\mathcal{A}} &\leq 2e^{-n/(2\widehat{\Theta})} \int_{s_A/\widehat{\Theta}}^{\pi} dv \left| \left(\mathbb{1}, M_E^*(1/(2\widehat{\Theta}) + iv) \underline{F}^*(m, 1/(2\widehat{\Theta}) + iv) \right) \right| \\ &\leq 2e^{-n/(2\widehat{\Theta})} \widehat{\Theta}^{-1} \int_{1/\sqrt{2\kappa}}^{\pi\widehat{\Theta}} ds \left\| M_E^*((1/2 + is)/\widehat{\Theta}) \right\| \left\| \underline{F}^*(m, (1/2 + is)/\widehat{\Theta}) \right\|_{\infty} \end{aligned} \tag{4.173}$$

Under our assumption on κ, $\left\| M_E^*((1/2 + is)/\widehat{\Theta}) \right\|$ can be bounded as in (4.48) of Corollary 4.7. Since $\cos(s/\widehat{\Theta})$ is monotone decreasing on $[1/\sqrt{2\kappa}, \pi\widehat{\Theta}]$, we may add to our previous requirement on κ that it is chosen small enough so that

$$\frac{\sqrt{1 - O(\bar{\Theta}^{-1})}}{2} \sqrt{1 + \widehat{\Theta}^2 2(1 - \cos(1/(\widehat{\Theta}\sqrt{2\kappa})))} \geq 1 + \frac{4}{M-1}(1 + O(d/N)) + \frac{C + C'}{2\gamma} \tag{4.174}$$

The bound (4.48) then yields

$$\left\| M_E^*((1/2 + is)/\widehat{\Theta}) \right\| \leq \frac{1 + C\gamma^{-1}}{\sqrt{1 - O(\bar{\Theta}^{-1})}} \frac{1}{\sqrt{1 + \widehat{\Theta}^2 2(1 - \cos(s/\widehat{\Theta}))}} \tag{4.175}$$

To bound $\left\| \underline{F}^*(m, (1/2 + is)/\widehat{\Theta}) \right\|_{\infty}$ we use Lemma 4.20 together with the fact that on $\mathcal{A}$, by the estimates of Proposition 3.2, $|G^{\sigma}_{T(E)\setminus\sigma}(u)| \leq c$, to get that

$$\left\| \underline{F}^*(m, (1/2 + is)/\widehat{\Theta}) \right\|_{\infty} \leq c \left| e^{(1/2+is)/\widehat{\Theta}} - 1 \right|^{-1} \tag{4.176}$$

Set $\rho = \exp(1/(2\widehat{\Theta}))$ and $v = s/\widehat{\Theta}$. Then $\left|\rho e^{iv} - 1\right|^2 = (1-\rho)^2 + 2\rho(1 - \cos v)$, and since $\rho > 1 + 1/(2\widehat{\Theta})$,

$$\left\| \underline{F}^*(m, (1/2 + is)/\widehat{\Theta}) \right\|_{\infty} \leq \frac{c\widehat{\Theta}}{\sqrt{1/4 + \widehat{\Theta}^2 2(1 - \cos(s/\widehat{\Theta}))}} \tag{4.177}$$

Inserting (4.175) and (4.177) in (4.173) we get

$$\begin{aligned} I_{\mathcal{A}} &\leq 2ce^{-n/(2\widehat{\Theta})} \int_{1/\sqrt{2\kappa}}^{\pi\widehat{\Theta}} ds \frac{1}{\sqrt{1 + \widehat{\Theta}^2 2(1 - \cos(s/\widehat{\Theta}))}\sqrt{1/4 + \widehat{\Theta}^2 2(1 - \cos(s/\widehat{\Theta}))}} \\ &\leq 2ce^{-n/(2\widehat{\Theta})} \int_{1/\sqrt{2\kappa}}^{\pi\widehat{\Theta}} ds \frac{1}{1/4 + \widehat{\Theta}^2 2(1 - \cos(s/\widehat{\Theta}))} \end{aligned} \tag{4.178}$$

To evaluate the last integral above, we split the integration interval into $[1/\sqrt{2\kappa},\, \pi\widehat{\Theta}/4]$ and $[\pi\widehat{\Theta}/4,\, \pi\widehat{\Theta}]$. On the first of these intervals, $\widehat{\Theta}^2 2(1-\cos(s/\widehat{\Theta}))$ is well approximated by s^2 so that

$$\int_{1/\sqrt{2\kappa}}^{\pi\widehat{\Theta}/4} ds\, \frac{1}{1/4+\widehat{\Theta}^2 2(1-\cos(s/\widehat{\Theta}))} \leq c\int_{1/\sqrt{2\kappa}}^{\pi\widehat{\Theta}/4} \frac{ds}{1+s^2} \leq c' \tag{4.179}$$

We then use that on the remaining interval $\widehat{\Theta}^2 2(1-\cos(s/\widehat{\Theta})) > \widehat{\Theta}^2$ so that

$$\int_{\subset\widehat{\Theta}/4}^{\pi\widehat{\Theta}} ds\, \frac{1}{1/4+\widehat{\Theta}^2 2(1-\cos(s/\widehat{\Theta}))} \leq \frac{c}{\widehat{\Theta}} \tag{4.180}$$

Inserting (4.179) and (4.180) in (4.178) yields the claim of the lemma. ♢

Lemma 4.24: *Let $\mathcal{B}$ be defined in (4.168). If $t = n/\widehat{\Theta}(E)$ and $\tilde{t} = t^\eta$ then, for all $0<\eta<1$,*

$$\left|\int_{\mathcal{B}} due^{-un}\left(\mathbb{1}, M_E^*(u)\underline{F}^*(m,u)\right)\right| \leq ct^\eta \exp(-t^{1-\eta}) \tag{4.181}$$

Proof: It will be enough to use norm estimates, that is, calling $I_{\mathcal{B}}$ the left hand side of (4.181),

$$\begin{aligned} I_{\mathcal{B}} &\leq \int_{\mathcal{B}} |du| e^{-n\Re u} \left\|M_E^*(u)\right\| \left\|\underline{F}^*(m,u)\right\|_\infty \\ &= \widehat{\Theta}^{-1}\int_{\widehat{\Theta}\mathcal{B}} |dz| e^{-t\Re z} \left\|M_E^*(z/\widehat{\Theta})\right\| \left\|\underline{F}^*(m,z/\widehat{\Theta})\right\|_\infty \\ &\leq 2c\widehat{\Theta}^{-1}\int_{s_B}^{s_A} ds\, e^{-\kappa s^2 t}\left\|M_E^*((\kappa s^2+is)/\widehat{\Theta})\right\| \left\|\underline{F}^*(m,(\kappa s^2+is)/\widehat{\Theta})\right\|_\infty \end{aligned} \tag{4.182}$$

As in the proof of the previous lemma we use (4.48) to write the bound

$$\begin{aligned} &\left\|M_E^*((\kappa s^2+is)/\widehat{\Theta})\right\| \\ &\leq \frac{1+C\gamma^{-1}\kappa s^2}{\sqrt{1+\widehat{\Theta}^2 2(1-\cos(s/\widehat{\Theta}))(1-O(\bar{\Theta}^{-1})})-1-\frac{4}{M-1}(1+O(d/N))-(C+C')\gamma^{-1}\kappa s^2} \end{aligned} \tag{4.183}$$

Using this time that on the integration interval, $\widehat{\Theta}^2 2(1-\cos(s/\widehat{\Theta})) \geq s^2(1-1/(6\kappa\widehat{\Theta})^2)$, and that for $0<x<1$, $\sqrt{1+x} \geq 1+x/2$, we get (for κ small enough, t small enough compared with M, and M, N large) that the denominator in the r.h.s. of (4.183) is greater than $s^2/4$. Since the numerator is bounded above by a constant, we may write

$$\left\|M_E^*((\kappa s^2+is)/\widehat{\Theta})\right\| \leq cs^{-2} \tag{4.184}$$

Turning to the term $\left\|\underline{F}^*(m,(\kappa s^2+is)/\widehat{\Theta})\right\|_\infty$ observe that, proceeding as we did to derive (4.176) we obtain,

$$\left\|\underline{F}^*(m,(\kappa s^2+is)/\widehat{\Theta})\right\|_\infty \le c\left|e^{(\kappa s^2+is)/\widehat{\Theta}}-1\right|^{-1} \tag{4.185}$$

Now, with $\rho=\exp((\kappa s^2)/\widehat{\Theta})>1$ and $v=s/\widehat{\Theta}$, $\left|\rho e^{iv}-1\right|^2=(1-\rho)^2+2\rho(1-\cos v)\ge 2(1-\cos v)$. Combining this with the bound established on the line following (4.183), (4.185) becomes

$$\left\|\underline{F}^*(m,(1/2+is)/\widehat{\Theta})\right\|_\infty \le cs^{-1}\widehat{\Theta} \tag{4.186}$$

Collecting (4.182), (4.184) and (4.186), we arrive at

$$I_{\mathcal{B}}\le c\int_{s_B}^{s_A} ds\, e^{-\kappa s^2 t}s^{-3}\le c\int_{1/\sqrt{\tilde t}}^{1/\sqrt{2}} ds\, e^{-s^2 t}s^{-3}\le ce^{-t/\tilde t}\int_{1/\sqrt{\tilde t}}^{1/\sqrt{2}} ds\, s^{-3}\le c\tilde t e^{-t/\tilde t} \tag{4.187}$$

Thus, choosing $\tilde t=t^\eta$, $0<\eta<1$, concludes the proof of the lemma. ◇

We now consider the error term resulting from the $M^{*(1)}(u)$ part of the resolvent on the part $\mathcal{D}$ of the integration contour.

Lemma 4.25: *If* $t=n/\widehat{\Theta}(E)$ *then, for all* $0<\delta<1/2$,

$$\begin{aligned}&\limsup_{E\downarrow-\infty}\limsup_{N\uparrow\infty}\left|\int_{\mathcal{D}} due^{-nu}\left(1\!\!1, M^{*(1)}(u)\underline{F}^*(m,u)\right)\right|\\ &\le ct^{-2(1-1/\alpha)}+\frac{c}{2(1-1/\alpha)}t^{-2\delta(1-1/\alpha)}+ct^{-(1-2\delta)}\exp(-t^{1-2\delta})\end{aligned} \tag{4.188}$$

Proof: Again, it will be enough to use norm estimates, that is

$$\left|\int_{\mathcal{D}} due^{-nu}\left(1\!\!1, M^{*(1)}(u)\underline{F}^*(m,u)\right)\right|\le\int_{\mathcal{D}}|du|e^{-n\Re u}\|M^{*(1)}(u)\|\|\underline{F}^*(m,u)\|_\infty \tag{4.189}$$

To bound $\|\underline{F}^*(m,u)\|_\infty$ we proceed as in the previous two lemmata and use Lemma 4.20 together with the fact that on $\mathcal{D}$, by the estimates of Proposition 3.2, $|G^\sigma_{T(E)\backslash\sigma}(u)|\le c$, to establish that

$$\|\underline{F}^*(m,u)\|_\infty\le c|e^u-1|^{-1}\le c|u|^{-1} \tag{4.190}$$

Hence

$$\int_{\mathcal{D}}|du|e^{-n\Re u}\|M^{*(1)}(u)\|\|\underline{F}^*(m,u)\|_\infty\le c\int_{\widehat{\Theta}\mathcal{D}}|dz|e^{-t\Re z}\|M^{*(1)}(z/\widehat{\Theta})\||z|^{-1} \tag{4.191}$$

and[15] by Corollary 4.19

$$\begin{aligned}\limsup_{E\downarrow-\infty}\limsup_{N\uparrow\infty}\int_{\widehat{\Theta}\mathcal{D}}&|dz|e^{-t\Re z}\|M^{*(1)}(z/\widehat{\Theta})\||z|^{-1}\\ \leq& c\int_{\widehat{\Theta}\mathcal{D}}|dz|e^{-t\Re z}|z|^{1-2/\alpha}\equiv cI_{\widehat{\Theta}\mathcal{D}}\end{aligned}\tag{4.192}$$

We now decompose $I_{\widehat{\Theta}\mathcal{D}}$ as $I_{\widehat{\Theta}\mathcal{D}}=I_{\widehat{\Theta}\mathcal{D}_1}+I_{\widehat{\Theta}\mathcal{D}_2}$ according to (4.170). Clearly

$$I_{\widehat{\Theta}\mathcal{D}_1}\leq\int_0^{2\pi}d\theta\, t^{-(1-2/\alpha)-1}=2\pi t^{-(2-2/\alpha)}\tag{4.193}$$

To bound $I_{\widehat{\Theta}\mathcal{D}_2}$ we first observe that

$$I_{\widehat{\Theta}\mathcal{D}_2}\leq\int_{s_D}^{s_B}ds\,e^{-s^2\kappa t}\sqrt{(2\kappa s)^2+1}\left(\sqrt{(\kappa s^2)^2+s^2}\right)^{1-2/\alpha}\leq c\int_{\sqrt{\kappa}s_D}^{\sqrt{\kappa}s_B}ds\,e^{-s^2t}s^{1-2/\alpha}\tag{4.194}$$

and since for t large, $s_D\approx 1/t$,

$$I_{\widehat{\Theta}\mathcal{D}_2}\leq c\int_{\sqrt{\kappa}/t}^{1}ds\,e^{-s^2t}s^{1-2/\alpha}\tag{4.195}$$

Introducing a number $0<\delta<1/2$, we then split the last integral above into

$$J_1\equiv\int_{\sqrt{\kappa}/t}^{1/t^\delta}ds\,e^{-s^2t}s^{1-2/\alpha}\quad\text{and}\quad J_2\equiv\int_{1/t^\delta}^{1}ds\,e^{-s^2t}s^{1-2/\alpha}\tag{4.196}$$

As no exponential decay is to be gained in J_1, we simply write

$$J_1\leq\int_{\sqrt{\kappa}/t}^{1/t^\delta}ds\,s^{1-2/\alpha}=\frac{1}{2(1-1/\alpha)}(t^{-2\delta(1-1/\alpha)}-\kappa^{1-1/\alpha}t^{-2(1-1/\alpha)})\tag{4.197}$$

To deal with J_2 we distinguish two cases: if $1-2/\alpha>0$, then

$$J_2\leq\int_{1/t^\delta}^{1}ds\,e^{-s^2t}=\frac{1}{2}\int_{1/t^{2\delta}}^{1}dx\,\frac{e^{-xt}}{\sqrt{x}}\leq t^\delta\int_{1/t^{2\delta}}^{1}dx\,e^{-xt}\leq t^{-(1-\delta)}\exp(-t^{1-2\delta})\tag{4.198}$$

while if $1-2/\alpha\leq 0$,

$$J_2\leq t^{\delta(2/\alpha-1)}\int_{1/t^\delta}^{1}ds\,e^{-s^2t}\leq t^{\delta(2/\alpha-1)-(1-\delta)}\exp(-t^{1-2\delta})\leq t^{-(1-2\delta)}\exp(-t^{1-2\delta})\tag{4.199}$$

---

[15] The appearance of $\widehat{\Theta}$ after the limit has been taken in the inequality below may look confusing. Observe however that, for all N,E, the rescaled contour $\widehat{\Theta}\mathcal{D}$ does not depend on N and E so that this notation is formally correct.

We have thus obtained that

$$I_{\widehat{\Theta}\mathcal{D}_2} \leq \frac{c}{2(1-1/\alpha)} t^{-2\delta(1-1/\alpha)} + ct^{-(1-2\delta)} \exp(-t^{1-2\delta}) \tag{4.200}$$

which, together with (4.193), yields the claim of the lemma.$\diamondsuit$

4.7 Laplace inversion 2. The main contributions.

Warning: In this last section we abandon the notation $s = \Im(z)$ introduced in (4.33). The letter s now takes back its initial meaning and designates the rescaled time variable $s \equiv m/\widehat{\Theta})$ of Theorem 1.

We are now moving towards the principle contributions. Note that

$$\left(\mathbb{1}, M^{*(0)}(u)\underline{F}^*(m,u)\right) = \frac{\lambda(u)}{1-\lambda(u)} (\mathbb{1}, \underline{F}^*(m,u)) \equiv h_{N,E}(u,m) \tag{4.201}$$

We will proof the following result which together with the estimates on the error terms will imply our main theorem.

Proposition 4.26: *For u on $\mathcal{C}$, we have that*

$$\lim_{E\downarrow-\infty} \lim_{N\uparrow\infty} h_{N,E}(u,m) = H_0^*(s,z)(1 + O(|z|^{1-1/\alpha}, |z|^{1/\alpha})) + O(z^{-1/\alpha} e^{-s/\tau_\infty}) \tag{4.202}$$

where $H_0^*(s,u) \equiv \int_0^\infty dte^{zt} \int_{s/t}^\infty \frac{dx}{x^{1/\alpha}(1+x)}$ is the Laplace transform of the function H_0 defined in (1.10).

Proof: The analysis of $(\mathbb{1}, \underline{F}^*(m,u))$ is in spirit and even detail very similar to that of $M^*(u)$, except that it is considerably simpler. Note that using (4.153), Lemma 4.21, Lemma 4.22, equation (4.162), and the estimate (4.163), the leading term in this expression is

$$(\mathbb{1}, \underline{F}^*(m,u)) \approx \frac{1}{|T(E)|} \sum_{\sigma\in T(E)} p_N(\sigma,\sigma)^m \frac{G^\sigma_{T(E)\backslash\sigma}(u) - 1}{e^u - 1} \tag{4.203}$$

Note that from (4.107) we get furthermore that

$$\frac{G^\sigma_{T(E)\backslash\sigma}(u) - 1}{e^u - 1} = \widehat{\Theta}(E) \frac{1}{z_\sigma - z} (1 + R(u)) \tag{4.204}$$

where the remainder $R(u)$ is of the same type as those appearing in the proof of Lemma 4.14. Thus we obtain

Lemma 4.27: *With the notation of Lemma 4.14,*

$$\left| p_N(\sigma,\sigma)^m \frac{G^\sigma_{T(E)\setminus\sigma}(u)-1}{e^u-1} - e^{-me^{-\beta\sqrt{N}E_\sigma}} \frac{\widehat{\Theta}(E)}{z_\sigma - z} \right| \leq C\widehat{\Theta}(E)|z| \tag{4.205}$$

Proof: Essentially contained in the proof of Lemma 4.14.◇

Next we can now prove the analogue of Corollary 4.17.

Lemma 4.28: *Set* $s \equiv m/\widehat{\Theta}$*. Then, uniformly on* $\Re z < \max(\Im z, 1/2)$*, and* $\Re(u) \leq |\Im u|$*,*

$$\begin{aligned} &\lim_{E\downarrow -\infty}\lim_{N\uparrow\infty} \frac{1}{|T(E)|}\sum_{\sigma\in T(E)} e^{-me^{-\beta\sqrt{N}E_\sigma}} \frac{1}{z_\sigma - z} \\ &= \alpha^{-1}\tau_\infty \int_1^\infty e^{-s/(x\tau_\infty)} \frac{dx}{(1-zx\tau_\infty)x^{1/\alpha}} \quad \textit{in Probability.} \end{aligned} \tag{4.206}$$

Moreover,

$$\begin{aligned} &\alpha^{-1}\tau_\infty \int_1^\infty e^{-s/(x\tau_\infty)} \frac{dx}{(1-zx\tau_\infty)x^{1/\alpha}} \\ &= (-z\tau_\infty)^{1/\alpha}\left(z^{-1}\pi\, cosec\,(\pi/\alpha) - \int_0^\infty dt e^{zt}\int_0^{s/t} \frac{dx}{x^{1/\alpha}(1+x)}\right) + O(e^{-s/\tau_\infty}) \end{aligned} \tag{4.207}$$

Proof: Observing that, by (4.125), $me^{-\beta\sqrt{N}E_\sigma} = se^{-\beta\sqrt{N}E_\sigma}\widehat{\Theta} = sz_\sigma(1-1/M)^{-1}$, (4.206) is proven like (4.143) of Corollary 4.17. To prove (4.207), it will be convenient to extend the integration in (4.206) all the way to zero, as in the proof of Corollary 4.17. One can easily estimate the difference, namely

$$\left|\int_0^1 e^{-s/(x\tau_\infty)} \frac{dx}{(1-zx\tau_\infty)x^{1/\alpha}}\right| \leq \frac{1}{\sqrt{2}}\int_0^1 e^{-s/(x\tau_\infty)}\frac{dx}{x^{1/\alpha}} \leq \frac{e^{-s/\tau_\infty}}{\tau_\infty\sqrt{2}}\min\left(s^{-1}, \frac{1}{1-1/\alpha}\right) \tag{4.208}$$

In the extended integral we again change variables and rotate the integration contour to the negative real axis to get that

$$\begin{aligned} \int_0^\infty e^{-s/(x\tau_\infty)} \frac{dx}{(1-zx\tau_\infty)x^{1/\alpha}} &= (z\tau_\infty)^{1/\alpha-1}\int_0^{z\infty} e^{-sz/x}\frac{dx}{(1-x)x^{1/\alpha}} \\ &= -(-z\tau_\infty)^{1/\alpha-1}\int_0^\infty e^{+sz/x}\frac{dx}{(1+x)x^{1/\alpha}} \end{aligned} \tag{4.209}$$

According to whether $\Re(z)$ is positive or negative, we can represent

$$\begin{aligned} e^{+sz/x} &= \int_{-zs/x}^{z\infty} e^{-t}dt = z\int_{-s/x}^{+\infty} e^{-zt}dt \quad \text{or} \\ e^{+sz/x} &= \int_{-zs/x}^{-z\infty} e^{-t}dt = -z\int_{s/x}^{+\infty} e^{+zt}dt \quad \text{respectively} \end{aligned} \tag{4.210}$$

Inserting these representation into (4.209) and changing the order of integration in the resulting double integrals gives in both cases

$$\begin{aligned} &\int_0^\infty e^{-s/(x\tau_\infty)} \frac{dx}{(1-zx\tau_\infty)x^{1/\alpha}} \\ &= \tau_\infty^{-1}(-z\tau_\infty)^{1/\alpha}\left((z)^{-1}\alpha\pi\text{cosec}\,(\pi/\alpha) - \int_0^\infty dte^{zt}\int_0^{s/t}\frac{dx}{x^{1/\alpha}(1+x)}\right) \end{aligned} \tag{4.211}$$

♢

We can now combine the asymptotics for $1-\lambda(u)$ obtained in Corollary 4.18 with the preceding result. This shows that

$$\begin{aligned} &\lim_{E\downarrow-\infty}\lim_{N\uparrow\infty}\frac{\lambda(u)}{1-\lambda(u)}\frac{1}{|T(E)|}\sum_{\sigma\in T(E)} e^{-me^{-\sqrt{N}E_\sigma}}\frac{\widehat{\Theta}(E)}{z_\sigma - z} \\ =&z^{-1} - \frac{\int_0^\infty dte^{zt}\int_0^{s/t}\frac{dx}{x^{1/\alpha}(1+x)}}{\pi\text{cosec}\,(\pi/\alpha)}\left(1+O(|z|^{1-1/\alpha},|z|^{1/\alpha})\right) + O\left(z^{-1/\alpha}e^{-s/\tau_\infty}\right) \end{aligned} \tag{4.212}$$

The leading term is readily identified as the Laplace transform of

$$H_0(s/t) \equiv 1 - \frac{\int_0^{s/t}\frac{dx}{x^{1/\alpha}(1+x)}}{\pi\text{cosec}\,(\pi/\alpha)} \tag{4.213}$$

which we recognise as precisely the function that appeared as the leading asymptotic contribution in the trap model in Theorem 1.1. The bounds on the error terms then follow from simply estimating the corrections uniformly on $\mathcal{C}$. ♢

The last step before completing the proof of Theorem 1 is now to consider the contribution $F_{N,E}(n+m)$. We leave it to the reader to show that the leading asymptotics of this term is given by

$$\alpha^{-1}\int_1^\infty dxe^{-(t+s)/x}x^{-1-1/\alpha} \leq \frac{1}{\alpha(t+s)^{1/\alpha}}\int_0^\infty dxe^{-1/x}x^{-1-1/\alpha} \tag{4.214}$$

which is sub-dominant as s and t tend to infinity. Collecting all the estimates of this section concludes the proof of the main theorem.♢♢

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References

[B] J.P. Bouchaud, "Weak ergodicity breaking and aging in disordered systems", *J. Phys. I (France)*, **2**, 1705 (1992).

[BBG1] G. Ben Arous, A. Bovier, and V. Gayrard, "Glauber dynamics of the random energy model. I. Metastable motion on extreme states", (2001).

[BCKM] J.P. Bouchaud, L. Cugliandolo, J. Kurchan, M. Mézard, "Out-of-equilibrium dynamics in spi-glasses and other glassy systems", in *Spin-glasses and random fields* (A.P. Young, Ed.). World Scientific, Singapore (1998).

[BD] J.P. Bouchaud and D. Dean,"Aging on Parisi's tree" *J. Phys. I (France)*, **5**, 265 (1995).

[BDG] G. Ben Arous, A. Dembo and A. Guionnet, "Aging of spherical spin glasses", Probab. Theor. Rel. Fields **120**, 1-67 (2001).

[BEGK1] A. Bovier, M. Eckhoff, V. Gayrard and M. Klein, "Metastability in stochastic dynamics of disordered mean field models", *Prob. Theor. Rel. Fields* **119**, 99-161 (2001).

[BEGK2] A. Bovier, M. Eckhoff, V. Gayrard and M. Klein, "Metastability and low lying spectra in reversible Markov chains", to appear in *Commun. Math. Phys.* (2001).

[BKL] A. Bovier, I. Kurkova, M. Löwe, "Fluctuations of the free energy in the REM and the p-spin SK models", to appear in Ann. Probab. (2001).

[BM] J.P. Bouchaud, C. Monthus, "Models of traps and glass phenomenology", *J. Phys. A-Math. Gen.* **29**, (14) 3847-3869 (1996).

[BMR] J.P. Bouchaud, P. Maass, B. Rinn, "Multiple scaling regimes in simple aging models", *Phys. Rev. Letts.* **84**, (23) 5403-5406 (2000).

[CD] L. Cugliandolo and D. Dean, "Full dynamical solution for a spherical spin-glass model", *J. Phys. A - Math. Gen.* **28**, (15) 4213-4234 (1995).

[D1] B. Derrida, Random energy model: "Limit of a family of disordered models", *Phys. Rev. Letts.* **45**, 79-82 (1980).

[D2] B. Derrida, Random energy model: "An exactly solvable model of disordered disordered systems", *Phys. Rev. B* **24**, 2613-2626 (1981).

[Doe] G. Doetsch, *Handbuch der Laplace-Transformation*, Vol II, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe Band 15, Birkhäuser Verlag, Basel, 1955.

[Fe] W. Feller, *An introduction to probability theory and its applications*, Vol II, Wiley series in probability and mathematical statistics, New York, 1971.

[Li] T.M. Liggett, *Interacting particle systems*, Springer, Berlin, 1985.

[LLR] M.R. Leadbetter, G. Lindgren, H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer, Berlin-Heidelberg-New York, 1983.

[Ru] D. Ruelle, "A mathematical reformulation of Derrida's REM and GREM", *Commun. Math. Phys.* **108**, 225-239 (1987).