Infrared regular representation of the three dimensional massless Nelson model

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Abstract

We prove that in the Euclidean representation of the three dimensional massless Nelson model the $t = 0$ projection of the interacting measure is absolutely continuous with respect to a Gaussian measure with suitably adjusted mean. We also determine the Hamiltonian in the Fock space over this Gaussian measure space.

KEYWORDS: Nelson's scalar field model, infrared regular representation, ground state, Gibbs measure on path space, cluster expansion

1 Introduction

The Nelson model [6] of a spinless electron coupled to a scalar massless Bose field is infrared divergent in 3 space dimensions. In its algebraic version the ground state representation of the Nelson Hamiltonian is not unitarily equivalent to the Fock representation [3, 2]. From the point of view of functional integrals, which is the one taken here, the $t = 0$ projection of the interacting measure is singular with respect to the $t = 0$ projection of the free measure [5]. Such a result leaves open the possibility of an explicit representation of the $t = 0$ functional measure. In this letter we will settle the issue. In the last section we will explain the connection to the non-Fock representation.

The infrared divergence in Nelson's model can be seen already on the level of the classical action. The Euclidean action for the electron is

$$S_{\mathrm{p}}(\{q_t\}) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{q}_t^2 + V(q_t) \right) dt. \tag{1.1}$$

Here $t \mapsto q_t \in \mathbb{R}^3$ is a path of the electron. The electron's mass is set equal to one only for convenience. $V : \mathbb{R}^3 \to \mathbb{R}$ is a confining potential, which for technical reasons we require to be bounded from below, continuous and have the asymptotics $V(x) = C|x|^{2s} + o(|x|^{2s})$ for large $|x|$ with some constant $C > 0$ and exponent $s > 1$. The massless field $\xi_t : \mathbb{R}^3 \to \mathbb{R}$ has the action

$$S_{\mathrm{f}}(\{\xi_t\}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{1}{2} \left(\partial_t \xi_t(x)^2 + \nabla \xi_t(x)^2 \right) dx dt. \tag{1.2}$$

Finally, the electron is coupled locally and translation invariant as given by

$$S_{\mathrm{int}}(\{q_t, \xi_t\}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \varrho(x - q_t) \xi_t(x) dx dt. \tag{1.3}$$

Here ϱ is a form factor, assumed to be a sufficiently smooth function and of rapid decrease, normalized such that $\int_{\mathbb{R}^3} \varrho(x) dx = e$, where e is the charge of the particle. The case of interest is $e \neq 0$, since $e = 0$ implies $\hat{\varrho}(0) = 0$ ($\hat{\varrho}$ the Fourier transform of ϱ) and therefore regularizes the interaction in the infrared region. At a later stage we will need that $|e| \leq e^*$ with some sufficiently small e^*.

The infrared divergence can be deduced from the solution of the classical field equation which minimizes energy. An easy computation leads to the minimizer $q_t = q_{\mathrm{min}}$, where q_{min} corresponds to an absolute minimum of the potential, i.e. $V(x) \geq V(q_{\mathrm{min}})$, for all $x \in \mathbb{R}^3$, and $\xi_t = \xi_{\mathrm{min}}$ given by

$$\xi_{\mathrm{min}}(x) = \Delta^{-1} \varrho(x - q_{\mathrm{min}}). \tag{1.4}$$

In three dimensions $\xi_{\mathrm{min}}(x) \cong -e/(4\pi|x|)$ for large $|x|$. On the other hand, for the free measure $\varrho = 0$ and $\xi_{\mathrm{min}} = 0$. If the total action $S = S_{\mathrm{p}} + S_{\mathrm{f}} + S_{\mathrm{int}}$ is used for constructing the Euclidean path measure (formally like $\exp(-S)$), then basically one has to compare two Gaussian fields of the same covariance, one with mean $\Delta^{-1}\varrho(x)$ and the other with zero mean. Because of slow decay of the Coulomb potential in 3 dimensions, these two measures cannot have a density with respect to each other.

Our arguments also offer a hint of how to improve on this situation. We reshuffle the field action by adding a term dependent on a function u to be chosen below:

$$S_{\mathrm{f}}^{u}(\{\xi_t\}) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\frac{1}{2}\left(\partial_t\xi_t(x)^2 + (\nabla\xi_t(x))^2\right) + \xi_t(x)u(x)\right)dxdt. \tag{1.5}$$

For convenience, we represent this auxiliary function as the convolution

$$u(x) = (\varrho * h)(x) \equiv \int_{\mathbb{R}^3} e^{ik\cdot x}\hat{\varrho}(k)\hat{h}(k)dk \tag{1.6}$$

where $\hat{h}$ denotes the Fourier transform of h, of which we require that

1. $\hat{h}$ is a real-valued, even, bounded, and sufficiently smooth function;
2. $\hat{h}(0) = 1$.

The first condition is more for simplicity than necessity, the second is essential. The density $\exp(-S_{\mathrm{f}}^{u}(\{\xi_t\}))$ provides the free Gaussian measure. Keeping the electron's action and the total action unchanged, we obtain a reshuffled interaction of the form

$$S_{\mathrm{int}}^{u}(\{q_t,\xi_t\}) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\left(\varrho(x-q_t) - (\varrho * h)(x)\right)\xi_t(x)dxdt. \tag{1.7}$$

The function u is chosen in such a way that the free Gaussian's mean asymptotically agrees with the solution of minimal energy of the classical field equation. This then makes up for the possibility of the $t = 0$ interacting measure to be absolutely continuous with respect to the $t = 0$ projection of the above Gaussian measure. In the remainder of the letter this argument will be made sound.

2 Infrared regular representation

First we translate the actions to well defined measures on path space. S_{p} is associated with the stationary $P(\phi)_1$-process q_t on $C(\mathbb{R},\mathbb{R}^3)$ defined by the stochastic differential equation

$$dq_t = (\nabla \log\psi_0)dt + dB_t \tag{2.1}$$

(dB_t denotes Brownian motion), with invariant measure $d\mathsf{N}^0 = \psi_0^2(x)dx$. Here ψ_0 is the ground state of the Schrödinger operator $(-1/2)\Delta + V(q)$ generating the $P(\phi)_1$-process. The path measure of this process will be denoted by $\mathcal{N}^0$. The paths $Q = \{q_t : t \in \mathbb{R}\}$ are $\mathcal{N}^0$-almost surely continuous and q_t is a time-reversible Markov process. Moreover, for our class of potentials V the semigroup of this process is intrinsically ultracontractive [4].

The field $\{\xi_t(f) : f \in \mathcal{S}(\mathbb{R}^3), t \in \mathbb{R}\}$ with $f \in \mathcal{S}(\mathbb{R}^3)$, i.e. the Schwartz space over $\mathbb{R}^3$, is in its turn described by the Gaussian measure $\mathcal{G}$ on $C(\mathbb{R},\mathcal{S}'(\mathbb{R}^3))$ resulting from the (modified) Euclidean action S_{f}^{u}. This Gaussian measure has mean

$$\mathbb{E}_{\mathcal{G}}[\xi_t(f)] = -\int_{\mathbb{R}^3}\frac{\hat{\varrho}(k)\hat{f}(k)\hat{h}(k)}{|k|^2}dk \equiv \int_{\mathbb{R}^3} f(x)\gamma(x)dx \tag{2.2}$$

(i.e., represented in position space in terms of the function γ), and covariance

$$\mathrm{cov}_{\mathcal{G}}\,(\xi_s(f_1),\xi_t(f_2)) = \int_{\mathbb{R}^3} \frac{\hat{f}_1(k)\hat{f}_2^*(k)}{2|k|} e^{-|k||t-s|} dk. \tag{2.3}$$

We denote the $t = 0$ stationary distribution of this Markov process by G; it is itself a Gaussian measure with mean (2.2) and covariance (2.3) taken with $s = t$. For easing the argument below, we introduce the centred field

$$\eta_t(f) = \xi_t(f) - \mathbb{E}_{\mathcal{G}}[\xi_t(f)], \quad f \in \mathcal{S}(\mathbb{R}^3); \tag{2.4}$$

as an auxiliary variable; this obviously has then a Gaussian distribution with zero mean and the same covariance as (2.3) above. Note that u introduces a shift in the mean of the unmodified Gaussian measure $\mathcal{G}_0$ on the same space (which with $h \equiv 0$ would be 0). We denote by G_0 the $t = 0$ distribution of $\mathcal{G}_0$. This shift induces the unitary map

$$\mathcal{U} : L^2(\mathcal{S}'(\mathbb{R}^3), d\mathsf{G}) \to L^2(\mathcal{S}'(\mathbb{R}^3), d\mathsf{G}_0), \quad (\mathcal{U}F)(f) = F(f-\gamma), \tag{2.5}$$

with $F : \mathcal{S}'(\mathbb{R}^3) \to \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R}^3)$ and γ given by (2.2), which we introduce here for later use.

The non-interacting joint particle-field process is set up on $C(\mathbb{R}, \mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3))$ and described by the path measure $\mathcal{P}^0 = \mathcal{N}^0 \times \mathcal{G}$ with $t = 0$ distribution $\mathsf{P}^0 = \mathsf{N}^0 \times \mathsf{G}$. The interacting system can be described by taking $\mathcal{P}^0$ as reference process and modifying it with the density given by

$$\frac{d\mathcal{P}_T}{d\mathcal{P}^0}(q,\xi) = \frac{1}{Z_T} \exp\left(-\int_{-T}^{T} S^u_{\mathrm{int}}(\{q_t,\xi_t\})dt\right) \tag{2.6}$$

Here

$$Z_T = \int \exp\left(-\int_{-T}^{T} S^u_{\mathrm{int}}(\{q_t,\xi_t\})dt\right) d\mathcal{P}^0 \tag{2.7}$$

is the normalizing partition function. To have a more explicit formula instead of (2.7) we first make the Gaussian part of the integration to obtain

$$\begin{aligned} Z_T &= \mathbb{E}_{\mathcal{N}^0_T}\left[\mathbb{E}_{\mathcal{G}}\left[\exp\left(-\int_{-T}^{T}\int_{\mathbb{R}^3} \hat{\xi}_t(k)\hat{\varrho}(k)(e^{ik\cdot q_t} - \hat{h}(k))dkdt\right)\right]\right] \\ &= \mathbb{E}_{\mathcal{N}^0_T}[Z^Q_T] \end{aligned}$$

where

$$\begin{aligned} Z^Q_T = \int \exp&\left(-\int_{-T}^{T}\int_{\mathbb{R}^3} \hat{\eta}_t(k)\hat{\varrho}(k)(e^{ik\cdot q_t} - \hat{h}(k))dkdt\right) \times \\ &\times \exp\left(\int_{-T}^{T}\int_{\mathbb{R}^3} \hat{\gamma}(k)\hat{\varrho}(k)(e^{ik\cdot q_t} - \hat{h}(k))dkdt\right) d\mathcal{G}_0 \\ = \exp&\left(-\frac{1}{2}\int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2\hat{h}(k)}{|k|}dk \int_{-\infty}^{\infty} ds \int_{-T}^{T} dte^{-|k||t-s|}(e^{ik\cdot q_t} - \hat{h}(k))\right) \times \\ \times \exp&\left(\frac{1}{4}\int_{-T}^{T}\int_{-T}^{T} \frac{|\hat{\varrho}(k)|^2}{|k|} dk(e^{ik\cdot q_s} - \hat{h}(k))(e^{-ik\cdot q_t} - \hat{h}(k))e^{-|k||t-s|}dtds\right). \end{aligned} \tag{2.8}$$

Notice that in the above expression we passed from integration with respect to $\mathcal{G}$ to integration with respect to $\mathcal{G}_0$. From this we get that

$$Z_T^Q = C_T\, \mathcal{Z}_T^Q \tag{2.9}$$

with $C_T > 0$ independent of Q and

$$\begin{aligned}\mathcal{Z}_T^Q = \exp\left(-\int_{-T}^{T}\int_{-T}^{T} W(q_s - q_t, s-t)dtds\right) \times \\ \exp\left(-\int_{-T}^{T} dt \int_{|s|>T} ds \int_{\mathbb{R}^3} W(q_t - q, t-s)h(q)dq\right) \times \\ \times \exp\left(-\int_{-T}^{T} ds \int_{|t|>T} dt \int_{\mathbb{R}^3} W(q_s - q, t-s)h(q)dq\right).\end{aligned} \tag{2.10}$$

Here

$$W(q,t) = -\frac{1}{4}\int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2}{|k|} e^{-|k||t|}\ \cos(k\cdot q)\ dk \tag{2.11}$$

This implies that the marginal distribution $\mathcal{N}_T$ of $\mathcal{P}_T$ for the particle paths satisfies

$$\frac{d\mathcal{N}_T}{d\mathcal{N}^0} = \frac{\mathcal{Z}_T^Q}{\mathcal{Z}_T}. \tag{2.12}$$

with $\mathcal{Z}_T = \int \mathcal{Z}_T^Q d\mathcal{N}^0$. The $t=0$ single time distribution of $\mathcal{N}_T$ will be denoted by N_T.

Lemma 2.1 *There is some $e^* > 0$ such that for all $|e| \le e^*$ the weak local limit* $\lim_{T\to\infty} \mathcal{N}_T$ $= \mathcal{N}$ *exists and is a Gibbs probability measure on $C(\mathbb{R}, \mathbb{R}^3)$. Moreover, $\mathcal{N}$ does not depend on h, i.e. it coincides with the limiting measure constructed for $h \equiv 0$.*

Proof: By using cluster expansion, the Gibbs measure for $h \equiv 0$ can be shown to exist for W at sufficiently weak couplings $|e| \le e^*$. Here we have an interaction which differs from the cases studied in [4] by the last two factors in (2.10). However, they appear as extra energies coming from interactions between pieces of paths lying inside $[-T,T]$ with constant boundary conditions in $\mathbb{R} \setminus [-T,T]$ weighted by h, to which our previous arguments extend directly. $\square$

For any fixed path $Q = \{q_t : t \in \mathbb{R}\} \in C(\mathbb{R}, \mathbb{R}^3)$ consider now the conditional distribution $\mathcal{P}_T^Q = \mathcal{P}_T(\,\cdot\,|\{q_t\} = Q)$ as a probability measure on $C(\mathbb{R}, \mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3))$. Since the interaction is linear in ξ_t, $\mathcal{P}_T^Q$ is itself a Gaussian measure with covariance (2.3) and mean

$$\mathbb{E}_{\mathcal{P}_T^Q}[\xi_t(f)] = \int_{\mathbb{R}^3} \hat{f}(k)\hat{g}_T^{Q,t}(k)dk \tag{2.13}$$

where

$$\begin{aligned}\hat{g}_T^{Q,t}(k) &= \hat{\gamma}(k) - \frac{\hat{\varrho}(k)}{2|k|}\int_{-T}^{T} e^{-|k||t-\tau|}\left(e^{ik\cdot q_\tau} - \hat{h}(k)\right)d\tau \\ &= -\frac{\hat{\varrho}(k)}{2|k|}\int_{-T}^{T} e^{-|k||t-\tau|}e^{ik\cdot q_\tau}d\tau - \frac{\hat{\varrho}(k)\hat{h}(k)}{2|k|}\int_{|\tau|>T} e^{-|k||t-\tau|}d\tau.\end{aligned} \tag{2.14}$$

Lemma 2.2 *There is some $e^* > 0$ such that for all $|e| \le e^*$ the weak local limit $\lim_{T\to\infty} \mathcal{P}_T = \mathcal{P}$ exists and is a probability measure on $C(\mathbb{R}, \mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3))$. Moreover, $\mathcal{P}$ coincides with the path measure for $h \equiv 0$.*

Proof: By using (2.14) it is immediate that $\lim_{T\to\infty} \hat{g}_T^{Q,t} = \hat{g}^{Q,t}$ exists and coincides with the mean for $\mathcal{P}^Q$ with $h \equiv 0$. Thus we have also that $\mathcal{P}_T^Q \to \mathcal{P}^Q$ exists. Moreover, since $\mathcal{P}_T(E \times F) = \int_F \mathcal{P}_T^Q(E) d\mathcal{N}_T$, for all $T > 0$, where E and F are appropriate measurable sets, we also have the claim of the lemma. Hence the joint particle-field path measure does not depend on h, i.e. on how the reshuffling of the free field and interaction energies is performed. □

$\mathcal{P}$ and $\mathcal{P}^0$ are Markov processes whose $t = 0$ distributions we denote below by P resp. P^0.

In the modified function space representation Nelson's Hamiltonian is H on the Hilbert space $\mathcal{H}^0 = L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), d\mathsf{P}^0)$ acting like

$$(F, e^{-tH} G)_{\mathcal{H}^0} = \mathbb{E}_{\mathcal{P}^0} \left[F(q_0, \xi_0) G(q_t, \xi_t) \exp(- \int_0^t \mathcal{L}_{\rm int}^u(\{q_s, \xi_s\}) ds) \right], \quad t > 0 \tag{2.15}$$

with suitable F and G, and where $\mathcal{L}_{\rm int}^u(\{q_t, \xi_t\}) = \int_{\mathbb{R}^3} (\varrho(x - q_t) - (\varrho * h)(x)) \xi_t(x) dx$ is the Lagrangian corresponding to (1.7). The bottom of the spectrum of H will be denoted by E_0. We define the Euclidean Hamiltonian $H_{\rm euc}$ acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), d\mathsf{P})$ as the self-adjoint operator generating the symmetric contracting semigroup T_t given by

$$(F, T_t G)_{\mathcal{H}} = \mathbb{E}_{\mathcal{P}}[F(q_0, \xi_0) G(\xi_t, q_t)], \quad t > 0. \tag{2.16}$$

Theorem 2.3 *Suppose $|e| \le e^*$ with some $e^* > 0$. Then H has a unique strictly positive ground state Ψ_0 corresponding to the lowest eigenvalue E_0, and $H - E_0$ is unitarily equivalent with $H_{\rm euc}$.*

Proof: From [5] we know that the existence of a strictly positive ground state of H implies that $H - E_0$ and $H_{\rm euc}$ are unitarily equivalent. Moreover, by Theorem 4.1 of the same reference we also know that for proving existence of the ground state it suffices to show that

$$\liminf_{T\to\infty} \int \left(\frac{d\mathsf{P}_T}{d\mathsf{P}^0} \right)^{1/2} d\mathsf{P}^0 \ > \ 0. \tag{2.17}$$

We have

$$\frac{d\mathsf{P}_T^Q}{d\mathsf{G}}(\xi) = \exp\left(\int_{\mathbb{R}^3} (\hat{\xi}(k) - \hat{\gamma}(k)) \hat{\varrho}(k) dk \int_{-T}^{T} (e^{ik\cdot q_t} - \hat{h}(k)) e^{-|k||t|} dt \right) \times$$
$$\exp\left(-\frac{1}{4} \int_{\mathbb{R}^3} dk \int_{-T}^{T} \int_{-T}^{T} \frac{|\hat{\varrho}(k)|^2}{|k|} (e^{ik\cdot q_t} - \hat{h}(k))(e^{-ik\cdot q_s} - \hat{h}(k)) e^{-|k|(|t|+|s|)} dt ds \right).$$

Moreover,

$$\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q, \xi) = \mathbb{E}_{\mathcal{N}_T} \left[\frac{d\mathsf{P}_T^Q}{d\mathsf{G}}(\xi) | q_0 = q \right] \frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q). \tag{2.18}$$

By using that for some $c > 0$

$$\frac{1}{c} \le \frac{d\mathsf{N}_T}{d\mathsf{N}^0} \le c \tag{2.19}$$

obtained similarly as in [4], in combination with Jensen's inequality we obtain

$$\begin{aligned}
&\int \left(\frac{d\mathsf{P}_T}{d\mathsf{P}^0}(q,\xi) \right)^{1/2} d\mathsf{P}^0(q) \qquad (2.20)\\
&= \int \mathbb{E}_{\mathcal{N}_T} \left[\left(\frac{d\mathsf{P}^Q}{d\mathsf{G}} \right)^{1/2} |q = q_0 \right] \left(\frac{d\mathsf{N}_T}{d\mathsf{N}^0}(q) \right)^{1/2} d\mathsf{P}^0 \\
&\ge \frac{1}{\sqrt{c}} \int \exp \left(\mathbb{E}_{\mathcal{N}_T} \left[\frac{1}{2} \int_{\mathbb{R}^3} (\hat{\xi}(k) - \hat{\gamma}(k)) \hat{\varrho}(k) \int_{-T}^{T} e^{-|k||t|} (e^{ik\cdot q_t} - \hat{h}(k)) dt dk | q = q_0 \right] \right) \\
&\quad \times \exp \left(-\frac{1}{8} \int_{\mathbb{R}^3} dk \int_{-T}^{T} \int_{-T}^{T} \frac{|\hat{\varrho}(k)|^2}{|k|} m_T(k;s,t) e^{-|k|(|t|+|s|)} dt ds \right) d\mathsf{P}^0 \\
&\ge \frac{1}{\sqrt{c}} \exp \left(-\frac{c}{8} \int_{\mathbb{R}^3} dk \int_{-T}^{T} \int_{-T}^{T} \frac{|\hat{\varrho}(k)|^2}{|k|} m_T(k;s,t) e^{-|k|(|t|+|s|)} dt ds \right),
\end{aligned}$$

where

$$m_T(k;s,t) = \mathbb{E}_{\mathcal{N}_T} \left[(e^{ik\cdot q_t} - \hat{h}(k))(e^{-ik\cdot q_s} - \hat{h}(k)) \right]. \tag{2.21}$$

We will use two different estimates of this function. One is the uniform bound

$$|m_T(k;s,t)| \le c_1, \tag{2.22}$$

and the other is

$$|m_T(k;s,t)| \le |k|^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} p_{s,t}^T(q_1,q_2)(|q_1| + b)(|q_2| + b) d\mathsf{N}^0(q_1) d\mathsf{N}^0(q_2) \tag{2.23}$$

for $|t - s| \ge 1$, where $p_{s,t}^T$ is the probability density with respect to $d\mathsf{N}^0(q_1)d\mathsf{N}^0(q_2)$ of the measure $\mathcal{N}_T$ with $q_t = q_1$, $q_s = q_2$, and $b, c_1 > 0$ are some constants. Using the intrinsic hypercontractivity of the reference measure and the cluster expansion of [4] we have in this case that

$$|p_{s,t}^T(q_1,q_2)| \le c_2 \tag{2.24}$$

with some $c_2 > 0$, moreover by the exponential fall-off of the stationary measure of the reference process

$$\int_{\mathbb{R}^3} (|q| + b) d\mathsf{N}^0(q) = c_3 < \infty, \tag{2.25}$$

with some $c_3 > 0$. Thus we estimate the exponent in (2.20) by writing

$$\begin{aligned}
&\int_{\mathbb{R}^3} dk \int_{-T}^{T} \int_{-T}^{T} \frac{|\hat{\varrho}(k)|^2}{|k|} m_T(k;s,t) e^{-|k|(|t|+|s|)} ds dt \\
&\quad \le \int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2}{|k|} \left(c_1 \int_{\substack{-T \le s,t \le T \\ |t-s| \le 1}} e^{-|k|(|t|+|s|)} ds dt + c_2 c_3^2 |k|^2 \int_{\substack{|t|,|s| \le T \\ |t-s| \ge 1}} e^{-|k|(|t|+|s|)} ds dt \right) dk \\
&\quad \le \int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2}{|k|} \left(\frac{A_1}{|k|} + A_2 \right) dk < \infty
\end{aligned}$$

with some constants $A_1, A_2 > 0$. This then yields a non-zero uniform lower bound on (2.17), and completes the proof of the theorem. $\square$

3 Hamiltonian in the new representation

The Hamiltonian H of the infrared regular representation is defined through (2.15). For having a more concrete expression the natural strategy is to build the Fock space $\mathcal{F}$ over $L^2(\mathcal{S}'(\mathbb{R}^3), d\mathsf{G})$ and to transform H unitarily to an operator acting on $L^2(\mathbb{R}^3, dq) \otimes \mathcal{F}$. We do this in three steps.

First we map $L^2(\mathbb{R}^3, d\mathsf{N}^0)$ to $L^2(\mathbb{R}^3, dq)$ through the similarity transformation $R : \phi(q) \mapsto \psi_0(q)\phi(q)$. Thereby the particle Hamiltonian becomes

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V(q). \tag{3.1}$$

Next we deal with the Hamiltonian for the free field. First $\mathcal{U}$ defined by (2.5) is used to map $L^2(\mathcal{S}'(\mathbb{R}^3), d\mathsf{G})$ into $L^2(\mathcal{S}'(\mathbb{R}^3), d\mathsf{G}_0)$. This map induces a transformation between the shifted, $\mathcal{G}$, and non-shifted, $\mathcal{G}_0$, Gaussian processes, and keeps their generator the same. Then by the Wiener-Itô transform $\mathcal{W} : L^2(\mathcal{S}'(\mathbb{R}^3), d\mathsf{G}^0) \to \mathcal{F}$ this is further mapped into Fock space (for details see e.g. [7]). Finally we transform the interaction. By (2.15) in $L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), dq \times d\mathsf{G})$ the interaction is multiplication by

$$\int_{\mathbb{R}^3} \hat{\varrho}(k)\hat{\xi}(k) \left(e^{ik\cdot q} - \hat{h}(k)\right) e^{-ik\cdot x} dk \tag{3.2}$$

which under $\mathcal{U}$ goes over to the multiplication operator

$$\int_{\mathbb{R}^3} \hat{\varrho}(k)(\hat{\xi}(k) - \hat{\gamma}(k)) \left(e^{ik\cdot q} - \hat{h}(k)\right) e^{-ik\cdot x} dk \tag{3.3}$$

on $L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), dq \times d\mathsf{G}_0)$. Consider now

$$\mathcal{V} = (1 \otimes \mathcal{W})(1 \otimes \mathcal{U})(R \otimes 1) \tag{3.4}$$

as an isometry from $L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), d\mathsf{P}^0)$ to $L^2(\mathbb{R}^3, dq) \otimes \mathcal{F}$. Then

$$\begin{aligned} \mathcal{V}H\mathcal{V}^{-1} &= \left(-\frac{1}{2}\Delta + V(q)\right) \otimes 1 + 1 \otimes \int_{\mathbb{R}^3} |k| a^*(k) a(k) dk \\ &+ \int_{\mathbb{R}^3} \frac{\hat{\varrho}(k)}{\sqrt{2|k|}} \left((e^{ik\cdot q} - \hat{h}(k)) \otimes a(k) + (e^{-ik\cdot q} - \hat{h}(k)) \otimes a^*(k)\right) dk \\ &- \int_{\mathbb{R}^3} \frac{|\hat{\varrho}(k)|^2}{k^2} \hat{h}(k) \left(e^{ik\cdot q} - \hat{h}(k)\right) \otimes 1 \, dk \\ &\equiv H_{\mathrm{N}}^{\mathrm{ren}}. \end{aligned} \tag{3.5}$$

The standard Nelson Hamiltonian H_{N} corresponds to setting $\hat{h} \equiv 0$ in (3.5). We see that the subtraction of $\hat{h}$ regularizes the interaction at small k and makes $H_{\mathrm{N}}^{\mathrm{ren}}$ a well defined Hamiltonian having a ground state in $L^2(\mathbb{R}^3, dq) \otimes \mathcal{F}$. In an operator theoretic context, Arai proposed $H_{\rm N}^{\rm ren}$ of the form (3.5) for the special case $\hat{h} \equiv 1$ and proved that under similar assumptions on the potential as ours $H_{\rm N}^{\rm ren}$ has a ground state [1].

Obviously, for obtaining the ground state expectation of an operator A of physical interest, one can use $H_{\rm N}^{\rm ren}$ only in conjunction with the transformation $A \mapsto \mathcal{V}A\mathcal{V}^{-1}$. Thus expectations of A obtained in the standard way by first introducing the cut-off Hamiltonian, taking ground state expectations with it, and subsequently removing the cut-off are identical with computing averages of $\mathcal{V}A\mathcal{V}^{-1}$ by using $H_{\rm N}^{\rm ren}$. As proved above, the infrared limit is thus correctly taken care of.

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