THE BARGMANN TRANSFORM AND CANONICAL TRANSFORMATIONS

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ABSTRACT. This paper is about a relationship between the kernel of the Bargmann transform (BT) and the corresponding canonical transformation. We study this fact for a BT (introduced by L. Thomas and S. Wassell) when the configuration space is the 2-sphere S^2 and for a BT that we introduce for S^3. It is shown that the kernel of the BT is a power series of a function which is a generating function of the corresponding canonical transformation (CT) (a classical analog of the BT). We show in each case that our CT is a composition of other two CT involving the complex quadric in $\mathbf{C}^3$ or $\mathbf{C}^4$. We also study the quantizations of those other two CT by dealing with spaces of holomorphic functions on the mentioned quadric. We show reproducing kernels for those spaces. We also relate some of the quantizations with work of V. Bargmann[20] and V. Guillemin[21]. Since powers of the generating functions are coherent states for $L^2(S^2)$ or $L^2(S^3)$, we finally show that the studied BT are actually coherent states transforms.

I. INTRODUCTION

V. Bargmann[1] introduced the so called Bargmann Transform as a unitary transformation from $L^2(\mathbf{R}^n)$ onto a space $\mathcal{B}_n$ of analytic functions (of n complex variables) which are square integrable with respect to a Gaussian measure. The Bargmann transform is defined as the integral operator

$$\mathbf{B}_{\mathbf{R}^n}\Psi(\mathbf{z}) \equiv \frac{1}{\pi^{n/4}}\int_{\mathbf{R}^n} \exp(-\frac{1}{2}(\mathbf{z}^2+\mathbf{x}^2)+\sqrt{2}\mathbf{z}\bullet\mathbf{x})\Psi(\mathbf{x})d\mathbf{x} \tag{1}$$

(see section I for notation). The Bargmann transform is also called the "Segal-Bargmann Transform". (see[2] for a historical remark about Bargmann and Segal's work on this transform. I. Segal[3] studied the infinite dimensional case $n=\infty$).

The Bargmann space $\mathcal{B}_n$ provides a very adequate framework to represent and study the n-dimensional quantum harmonic oscillator (i. e. the addition of n one-dimensional harmonic oscillators with the same frequency). It is known[4] that the Hamiltonian of such a oscillator can be written as

$$\hat{H} = \sum_{j=1}^{n} a_j^\dagger a_j \tag{2}$$

where $a_j^\dagger$ and a_j, $j=1,2,\ldots,n$ are creation and annihilation operators given by

$$\begin{aligned} a_j^\dagger &\equiv \frac{1}{\sqrt{2}}(\hat{q}_j - \imath\hat{p}_j) = \frac{1}{\sqrt{2}}\left(q_j - \frac{\partial}{\partial q_j}\right) \\ a_j &\equiv \frac{1}{\sqrt{2}}(\hat{q}_j + \imath\hat{p}_j) = \frac{1}{\sqrt{2}}\left(q_j + \frac{\partial}{\partial q_j}\right) \end{aligned} \tag{3}$$

with $\hat{q}_j$ the operator of multiplication by the coordinate q_j and $\hat{p} = -\imath\frac{\partial}{\partial q_j}$.

The operators corresponding to $a_j^\dagger$, a_j and $\hat{H}$ in the Bargmann space $\mathcal{B}_n$ are multiplication by the complex variable z_j, partial derivative with respect to z_j and $\sum_{j=1}^n z_j\frac{\partial}{\partial z_j}$ respectively. In particular, the spectrum of the n-dimensional quantum harmonic oscillator is very easy to compute because in the Bargmann representation the monomials $z_1^{a_1}z_2^{a_2}\ldots z_n^{a_n}$ (with $a_j, j=1,2,\ldots,n$, nonnegative integer numbers) are the eigenfunctions of the operator $\sum_{j=1}^n z_j\frac{\partial}{\partial z_j}$.

From the viewpoint of Classical Mechanics, the problem of a n-dimensional harmonic oscillator can be studied in the phase space $\mathbf{R}^{2n} = \{(\mathbf{q},\mathbf{p})|\mathbf{x},\mathbf{p}\in\mathbf{R}^n\}$ with symplectic form $\omega_n = \sum_{j=1}^n dq_j\wedge dp_j$ and Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + q_j^2 - \frac{n}{2} \tag{4}$$

It is known[4] that this problem can also be described by introducing complex variables $z_j = \frac{1}{\sqrt{2}}(q_j - \imath p_j)$, $j=1,2,\ldots,n$. In this complex variables the Hamiltonian H is given by $\sum_{j=1}^n z_j\overline{z}_j$. This is the classical analog of equation(2).

A remarkable and fundamental idea for this paper is the following: as in reference[5], let us consider the argument of the kernel of the Bargmann transform in equation (1)

$$\Phi_n(\mathbf{q},\mathbf{z}) \equiv -\imath\left(-\frac{1}{2}(\mathbf{z}^2+\mathbf{q}^2)+\sqrt{2}\mathbf{z}\cdot\mathbf{q}\right) \tag{5}$$

It turns out that the function $\Phi_n(\mathbf{q},\mathbf{z})$ is a generating function of a canonical transformation (symplectomorphism) from the phase space $(\mathbf{R}^{2n},\omega_n)$ onto the phase space $(\mathbf{C}^n,\mu_n)$ where

μ_n *is the symplectic form* $\mu_n = -\imath \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j$. (see section III for details). Thus we shall say that $\mathcal{C}_n$ is the classical analog of the Bargmann transform (or that $\mathbf{B}_{\mathbf{R}^n}$ is the Quantization of $\mathcal{C}_n$) in the sense that $\mathbf{B}_{\mathbf{R}^n}$ is a unitary transformation between the Hilbert spaces $L^2(\mathbf{R}^n)$ and $\mathcal{B}_n$, and $\mathcal{C}_n$ is a diffeomorphism between the phase spaces $(\mathbf{R}^{2n}, \omega_n)$ and $(\mathbf{C}^n, \mu_n)$.

This paper concerns about the same relationship between the Bargmann transform and a corresponding canonical transformation when the configuration space $\mathbf{R}^n$ is replaced by the 2-sphere S^2 or the three sphere S^3 (here we mean by $\mathcal{S}^n$ the set of unit vectors in $\mathbf{R}^{n+1}$). For the S^2 case, L.Thomas and S. Wassell[6] introduced a Bargmann transform from $L^2(S^2)$ onto a proper subspace of $\mathcal{B}_2$. For the S^3 case, a Bargmann transform was introduced by C. Villegas-Blas[7]. This last transform goes from $L^2(S^3)$ onto a proper subspace of $\mathcal{B}_4$ determined by the kernel of an operator which carries the restriction in quantum mechanics of the classical fact that the Angular Momentum and Runge-Lenz vectors should be orthogonal for the Kepler problem. We show in both cases S^2 and S^3, that the kernel of the Bargmann transform is a power series in a function which plays the role of a generating function of a canonical transformation which in turn is the classical analog of the Bargmann transform.

In section II, we describe the Bargmann transform for the three cases $\mathbf{R}^n$, S^2 and S^3. The mentioned relationship between the Bargmann transform and corresponding canonical transformations is established in section III. For the S^2 and S^3 cases we do this by using spherical coordinates (a local chart). In order to define and extend our canonical transformations to the whole spaces $T^*S^2 - \{0\}$ and $T^*S^3 - \{0\}$ (we need to take off the zero section in both cases to avoid singularities) we use a key property of the generating functions. Namely, that these functions can be written as the inner product of the position vector $\mathbf{x}$ (in S^2 or S^3) and a complex vector $\alpha(\mathbf{z})$ in the quadric $\mathbf{Q}_3$ (for the S^2 case) and $\mathbf{Q}_4$ (for the S^3 case). Here, for n a positive integer, the quadric $\mathbf{Q}_n$ is the set of vectors in $\mathbf{C}^n$ such that the addition of the square of their components is zero. Note that a vector in $\mathbf{C}^n$ belongs to $\mathbf{Q}_n$ if and only if the norm of its real and imaginary parts is the same and these two vectors are orthogonal to each other. Thus $\mathbf{Q}_3 - \{0\}$ and $\mathbf{Q}_4 - \{0\}$ can be identified through symplectomorphisms ρ_2 and ρ_3 with $T^*S^2 - \{0\}$ and $T^*S^3 - \{0\}$ respectively. We remark here that ρ_2 (ρ_3) is not the restriction to $\mathbf{Q}_3 - \{0\}$ ($\mathbf{Q}_4 - \{0\}$) of the canonical symplectic form for the ambient $\mathbf{C}^3$ ($\mathbf{C}^4$).

Thus we are able to extend our canonical transformation as the inverse of a composition of two maps. For the S^2 case, one map ($\alpha(\mathbf{z})$) goes from $\mathbf{C}^2 - \{0\}$ onto $\mathbf{Q}_3 - \{0\}$ and the other one,ρ_2, goes from $\mathbf{Q}_3 - \{0\}$ onto $T^*S^2 - \{0\}$. For the S^3 case, we have a similar extension, but instead of $\mathbf{C}^2 - \{0\}$, we have a six real-dimensional phase space which is a submanifold of $\mathbf{C}^4$ and determined by reduction.

Once we have, for each case S^2 or S^3, the symplectomorphisms between the three phase spaces, a natural question to ask is about the description of their corresponding quantizations. This is done in section IV.

The fact that our generating functions for the S^2 and S^3 cases are inner products of position vectors and a vectors in the quadric has an important implication. The Bargmann transforms described in section II for S^2 and S^3 are *coherent states transforms*. That is, they can be written as an infinite sum of inner products with coherent states. This is in analogy with the Bargmann transform $\mathbf{B}_{\mathbf{R}^n}$ which acting on a function is the inner product of the function and a coherent state of the n-dimensional harmonic oscillator. The coherent states appearing here are well known[8]. They concentrate on geodesics of S^2 or S^3 in the semi-classical regime and they give a resolution of the identity for each eigenspace of the spherical Laplacian on S^2 or S^3 respectively. (these eigenspaces are irreducible representations of the groups SO(3) or SO(4)). Section V is devoted to describe the Bargmann transforms as coherent states transforms. These coherent states also help us to obtain reproducing kernels for the range of our Bargmann transforms for both the $\mathcal{S}^2$ and $\mathcal{S}^3$ cases. We also show reproducing kernels for spaces of holomorphic functions on the quadric $\mathbf{Q}_3$ and $\underline{\mathrm{Q}}_4$ related with the range and domain of the quantizations described in section IV. For the quantizations going from these spaces of holomorphic functions on the quadric onto $L^2(S^2)$ or $L^2(S^3)$, we relate our descriptions with work of V. Bargmann[14] and V. Guillemin[15].

We also include two appendices to prove that the kernel of the Bargmann transforms for both spheres $\mathcal{S}^2$ and $\mathcal{S}^3$ is a power series in a certain function.

We remark that the case of the 3-sphere is physically relevant in both quantum and classical mechanics. The symmetries of the bound states of the hydrogen atom are given by the lie algebra of the group $SO(4)$ (see[7,9]). In this way we can explain the degeneracy of the negative energies for the hydrogen atom. In classical Mechanics, J. Moser[10] showed that the Hamiltonian flow for the Kepler problem (with negative energy) is equivalent, after a time reparametrization, to the geodesic flow on S^3. In this way, J. Moser is able to regularize the motion of the Kepler problem.

There are other Bargmann Transforms introduced by B. Hall[2,11,12] (for compact Lie groups), M. Stenzel[13] (for symmetric spaces of compact type), K. Ii[14] and R. Wada[15] (for n-spheres), but they are different from the transformations introduced in this paper. J. Rawnsley[16] also studies a transformation for the n-spheres, but his transformation is not unitary.

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II. BARGMANN TRANSFORM FOR $L^2(S^2)$ AND $L^2(S^3)$

The purpose of this section is to describe both the Bargmann transform for $L^2(S^2)$ (introduced by L.Thomas and S. Wassell[6]) and $L^2(S^3)$ (introduced by C. Villegas-Blas[7]). In both cases we need the usual Bargmann spaces introduced by Bargmann[1] in his transform for $L^2(\mathbf{R}^n)$. To that end, and as a reference for next sections, we first remind the reader what these spaces and the Bargmann transform are.

II.1 Bargmann Transform for $L^2(\mathbf{R}^n)$

In this section we make a review of important properties of the Bargmann Transform that we need for further sections in the paper. For a detailed explanation and proofs see references[1,2].

The Bargmann space $\mathcal{B}_n$ is the space of analytic functions of n complex variables $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ which are square integrable with respect to the Gaussian measure

$$d\nu_n(\mathbf{z}) \equiv \frac{1}{\pi^n} \exp(-|\mathbf{z}|^2) \prod_{j=1}^{n} dx_j dy_j \ , \tag{2.1}$$

where, $|\mathbf{z}|^2 \equiv |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2$ and $z_j = x_j + \imath y_j$, $x_j, y_j \in \mathbf{R}$, with $j = 1, 2, \ldots, n$.

The space $\mathcal{B}_n$ is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{B}_n} = \int_{\mathbf{C}^n} F(\mathbf{z}) \overline{G}(\mathbf{z}) d\nu_n(\mathbf{z}). \tag{2.2}$$

The set of all monomials $\frac{z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n}}{\sqrt{a_1! a_2! \ldots a_n!}}$, with $a_1, a_2, \ldots, a_n$ non-negative integer numbers, is an orthonormal basis of $\mathcal{B}_n$.

The coherent states in $\mathcal{B}_n$

$$\Psi_{\mathbf{a}}(\mathbf{z}) \equiv \exp(\overline{\mathbf{a}}\mathbf{z}) \quad \text{with} \quad \mathbf{a} \in \mathbf{C}^n \tag{2.3}$$

provide a reproducing kernel for the space $\mathcal{B}_n$. That is, for $\Phi \in \mathcal{B}_n$

$$\Phi(\mathbf{z}) = \langle \Phi, \Psi_{\mathbf{z}} \rangle_{\mathcal{B}_n} = \int_{\mathbf{C}^n} \exp(\mathbf{z}\overline{\mathbf{w}}) \Phi(\mathbf{w}) d\nu_n(\mathbf{w}) \ . \tag{2.4}$$

The Bargmann Transform for $L^2(\mathbf{R}^n)$ is defined as the operator $\mathbf{B}_{\mathbf{R}^n} : L^2(\mathbf{R}^n) \mapsto \mathcal{B}_n$ given by

$$\mathbf{B}_{\mathbf{R}^n} \Psi(\mathbf{z}) \equiv \frac{1}{\pi^{n/4}} \int_{\mathbf{R}^n} A(\mathbf{x}, \mathbf{z}) \Psi(\mathbf{x}) d\mathbf{x}, \quad \Psi \in L^2(\mathbf{R}^n) \tag{2.5}$$

where $d\mathbf{x}$ is the usual Lebesgue measure on $\mathbf{R}^n$ and

$$A(\mathbf{x}, \mathbf{z}) \equiv \exp(-\frac{1}{2}(\mathbf{z}^2 + \mathbf{x}^2) + \sqrt{2}\mathbf{z} \bullet \mathbf{x}) \ , \tag{2.6}$$

with $\mathbf{z}^2 = z_1^2 + z_2^2 + \ldots + z_n^2$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{x}^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ and $\mathbf{z} \bullet \mathbf{x} = z_1 x_1 + z_2 x_2 + \ldots + z_n x_n$. The Bargmann transform $\mathbf{B}_{\mathbf{R}^n}$ is a unitary operator (with $L^2(\mathbf{R}^n)$ endowed with the usual inner product).

The inverse $\mathbf{B}_{\mathbf{R}^n}^{-1}$ is given by

$$\mathbf{B}_{\mathbf{R}^n}^{-1} f(\mathbf{x}) = \lim_{M \mapsto \infty} \int_{|\mathbf{z}| \leq M} \overline{A(\mathbf{x}, \mathbf{z})} f(\mathbf{z}) d\nu_n(\mathbf{z}) \tag{2.7}$$

Here we need to take the limit because the integral over all the space $\mathbf{C}^n$ might not exist (note that, for fixed $\mathbf{x}$, $A(\mathbf{x},\mathbf{z}) \notin \mathcal{B}_n$).

II.2 Bargmann Transform for $L^2(S^2)$

Let us first define the space $L^2(S^n)$ for a given integer number n. Let $d\sigma_n$ be the usual normalized Lebesgue surface measure on S^n. Let us define $L^2(S^n)$ as the space of Lebesgue measurable functions on S^n which are square integrable with respect to the measure $d\sigma_n$.

The space $L^2(S^n)$ is a Hilbert space with inner product

$$\langle F, G\rangle_{S^n} = \int_{S^n} F(\mathbf{x})\overline{G}(\mathbf{x}) d\sigma_n(\mathbf{x}) \ , \tag{2.8}$$

In order to define the Bargmann transform for $L^2(S^2)$, We first remind the reader some well known facts about two different representations of the $SO(3)$ group of rotations of the space $\mathbf{R}^3$.

Let us consider the Laplacian operator on the 2-sphere S^2 denoted by Δ_{S^2}. We will actually consider the normalized Laplacian operator such that its spectrum is the set of numbers $\left\{(\ell+1/2)^2 | \ell = 0,1,2\ldots\right\}$

If we introduce spherical coordinates on the 2-sphere

$$x_1 = \sin(\theta)\cos(\phi), \quad x_2 = \sin(\theta)\sin(\phi), \quad x_3 = \cos(\theta) \quad 0 < \theta < \pi, \ \ 0 \le \phi \le 2\pi \ ,$$

the expression for the Laplacian on S^2 in spherical coordinates is

$$\Delta_{S^2} = -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} - \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} + \frac{1}{4} \tag{2.9}$$

The Laplacian Δ_{S^2} is in turn related to the addition of the square of the generators L_j of rotations in $\mathbf{R}^3$ around the plane perpendicular to the axis x_j for $j = 1,2,3$:

$$\Delta_{S^2} = L_1^2 + L_2^2 + L_3^2 + 1/4 \tag{2.10}$$

where

$$\begin{aligned} L_1 &= \imath\sin(\phi)\frac{\partial}{\partial\theta} + \imath\cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi} \ , \\ L_2 &= -\imath\cos(\phi)\frac{\partial}{\partial\theta} + \imath\cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi} \ , \\ L_3 &= -\imath\frac{\partial}{\partial\phi}. \end{aligned} \tag{2.11}$$

The operators L_j act on a dense subspace of $L^2(S^2)$ and provide a representation for the Lie algebra $so(3)$ of $SO(3)$. They also satisfy the commutation relations

$$[L_i, L_j] = \imath\epsilon_{ijk}L_k \qquad i,j,k = 1,2,3 \ , \tag{2.12}$$

where $\{i,j,k\}$ is a permutation of $\{1,2,3\}$. If this permutation is even (odd) then $\epsilon_{ijk} = 1$ ($\epsilon_{ijk} = -1$).

It is known that the eigenfunctions of the Laplacian Δ_{S^2} provide a basis for $L^2(S^2)$. Moreover, $L^2(S^2)$ is the direct sum of the eigenspaces of the Laplacian. For each $\ell = 0,1,2\ldots$, we shall denote by V_ℓ the eigenspace of the Laplacian Δ_{S^2} associated to the eigenvalue $(\ell+1/2)^2$.

The group $SO(3)$ acts on $L^2(S^2)$ as follows. Given $R \in SO(3)$, we define $T_R : L^2(S^2) \mapsto L^2(S^2)$ by

$$T_R\Psi(\mathbf{x}) = \Psi(R^{-1}\mathbf{x}), \qquad \mathbf{x} \in S^2 \ . \tag{2.13}$$

A particular and important basis of $L^2(S^2)$ is obtained by taking common normalized eigenfunctions of both operators Δ_{S^2} and L_3 (these two operators commute). These functions are known as "spherical harmonics" and we shall denote them by

$$Y_{\ell m} \ \text{ with } \ \ell = 0,1,2\dots \ \text{ and } \ m = -\ell, -\ell+1, \dots, \ell \ , \tag{2.14}$$

Moreover, if, for each $\ell = 0, 1, 2, \dots$, we let the group $\{T_R | R \in SO(3)\}$ to act on the space

$$V_\ell = span \, \{Y_{\ell m} | m = -\ell, -\ell+1, \dots, \ell\} \ , \tag{2.15}$$

then we obtain all the irreducible representations of $SO(3)$.

In the other hand, the group $SO(3)$ can be regarded as a subgroup of $SU(2)$ (the group of two by two unitary matrices with complex entries and determinant one). We can obtain irreducible representations for $SO(3)$ on base of the irreducible representations of $SU(2)$. An important representation of the $SU(2)$ group is obtained by letting this group to act on the Bargmann space $\mathcal{B}_2$ of analytic functions in two complex variables $\mathbf{z} = (z_1, z_2)$ in a similar way as we made the $SO(3)$ group to act on $L^2(S^2)$.

Let us denote by $\mathbf{S}$ the set of all numbers of the form $p/2$ with p a non negative integer number. For $s \in \mathbf{S}$, let us also denote by $\mathcal{W}_s$ the subspace of $\mathcal{B}_2$ of homogeneous polynomials in two complex variables (z_1, z_2) of degree $2s$. An orthonormal basis for $\mathcal{W}_s$ is given by the monomials:

$$y_{sm} = \frac{z_1^{s+m} z_2^{s-m}}{\sqrt{(s+m)!(s-m)!}} \ , \tag{2.16}$$

with $m = -s, -s+1, \dots, s-1, s$.

When we restrict the action of $SU(2)$ to the spaces $\mathcal{W}_s$, we get all the irreducible representations of $SU(2)$. In particular, when we take $s = \ell$ a non negative integer number, we obtain all the irreducible representations of $SO(3)$.

In order to define a Bargmann Transform for $L^2(S^2)$, the description above of two type of irreducible representations of $SO(3)$ suggests to define the following assignment:

$$Y_{\ell m} \longmapsto y_{\ell m} \quad \text{for all } \ \ell = 0,1,2,\dots \ \text{ and } \ m = -\ell, -\ell+1, \dots, \ell \ . \tag{2.17}$$

Thus we define the Bargmann Transform $\mathbf{B}_{S^2}$ for $L^2(S^2)$ as the linear extension of this assignment. Notice that $\mathbf{B}_{S^2}$ is a unitary operator from $L^2(S^2)$ onto the closed subspace $\mathcal{B}_{S^2}$ of $\mathcal{B}_2$ generated by the monomials with even degree, that is $\mathcal{B}_{S^2} = \oplus \mathcal{W}_\ell$ (where the direct sum is taken over all non negative integer numbers).

The Bargmann transform $\mathbf{B}_{S^2}$ can be written as an integral operator

$$\mathbf{B}_{S^2}\Psi(\mathbf{z}) = \int_{\mathbf{x} \in S^2} K_{S^2}(\mathbf{x}, \mathbf{z}) \Psi(\mathbf{x}) d\sigma_2(\mathbf{x}) \ , \tag{2.18}$$

with kernel

$$K_{S^2}(\mathbf{x}, \mathbf{z}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \overline{Y}_{\ell m}(\mathbf{x}) y_{\ell m}(\mathbf{z}) \ . \tag{2.19}$$

This kernel has the property of being a power series of the function

$$\Phi_{S^2}(\mathbf{x}, \mathbf{z}) = \sum_{m=-1}^{m=1} \overline{Y}_{1m}(\mathbf{x}) y_{1m}(\mathbf{z}) \tag{2.20}$$

in the following way

$$K_{S^2}(\mathbf{x},\mathbf{z}) = \sum_{\ell=0}^{\infty} \frac{\sqrt{2\ell+1}}{\ell!} \left(\frac{1}{\sqrt{3}}\right)^{\ell} (\Phi_{S^2}(\mathbf{x},\mathbf{z}))^{\ell} \ , \tag{2.21}$$

with the following explicit expression for $\Phi_{S^2}(\mathbf{x},\mathbf{z})$:

$$\Phi_{S^2}(\mathbf{x},\mathbf{z}) = \frac{\sqrt{3}}{2}\left(-\sin(\theta)\exp(-\imath\phi)z_1^2 + 2\cos(\theta)z_1 z_2 + \sin(\theta)\exp(\imath\phi)z_2^2\right) \ . \tag{2.22}$$

We give a proof of this fact in appendix A.

The Bargmann transform $\mathbf{B}_{S^2}$ intertwines $L_1, L_2, L_3, \Delta_S^2$ with the following operators

$$\begin{aligned} L_1' &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1}\right) \\ L_2' &= \frac{\imath}{2}\left(z_1\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_1}\right) \\ L_3' &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right) \\ \Delta_{S^2}' &= \left(\frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right) + \frac{1}{2}\right)^2 \end{aligned} \tag{2.23}$$

respectively.

Notice that the operator $L_0 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}$ is the addition of Hamiltonians of two independent harmonic oscillators $z_j\frac{\partial}{\partial z_j}$, $j = 1,2$ with the same frequency.

II.3 Bargmann Transform for $L^2(S^3)$

In analogy with the case $L^2(S^2)$, let us consider the spherical normalized Laplacian on S^3. In spherical coordinates

$$\begin{aligned} x_1 &= \sin(\lambda)\sin(\theta)\cos(\phi), \qquad x_2 = \sin(\lambda)\sin(\theta)\sin(\phi), \\ x_3 &= \sin(\lambda)\cos(\theta), \qquad\qquad\quad x_4 = \cos(\lambda) \ , \end{aligned}$$

where $0 < \theta, \lambda < \pi$ and $0 \le \phi \le 2\pi$.

The expression for Δ_{S^3} is

$$\Delta_{S^3} = -\frac{\partial^2}{\partial\lambda^2} - 2\cot(\lambda)\frac{\partial}{\partial\lambda} + \frac{\Delta_2}{\sin^2(\lambda)} + 1 \ , \tag{2.24}$$

where

$$\Delta_2 = \frac{-1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{-1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2} \ . \tag{2.25}$$

Note that we are taking Δ_{S^3} such that its spectrum is the set $\{n^2 \mid n \text{ is a positive integer}\}$.

Let us also consider generators for rotations on the (μ,ν) plane with $\mu,\nu = 1,2,3,4$, given by restrictions to S^3 of the following operators

$$E_{\mu\nu} = \frac{1}{\imath}\left(x_\mu\frac{\partial}{\partial x_\nu} - x_\nu\frac{\partial}{\partial x_\mu}\right) \quad \mu,\nu = 1,2,3,4 \ . \tag{2.26}$$

There are six linearly independent operators among the operators $E_{\mu\nu}$. These operators provide a representation of the Lie algebra $so(4)$. It is well known that this Lie algebra describes the symmetries of the hydrogen atom problem[7,9]. This can be shown by considering restrictions of the Angular Momentum $\mathbf{L} = (L_1, L_2, L_3)$ and Runge-Lenz $\mathbf{A} = (A_1, A_2, A_3)$ operators to eigenspaces of the Hamiltonian of the hydrogen atom problem. In order to keep track of this fact, we rename the operators $L_1 = E_{23}$, $L_2 = E_{31}$, $L_3 = E_{12}$, $A_j = E_{4j}$ $j = 1, 2, 3$.

The expressions for the operators $\mathbf{L}$ and $\mathbf{A}$ in spherical coordinates are:

$$\begin{aligned}
L_1 &= \imath\left(\sin(\phi)\frac{\partial}{\partial\theta} + \cot(\theta)\cos(\phi)\frac{\partial}{\partial\phi}\right),\\
L_2 &= -\imath\left(\cos(\phi)\frac{\partial}{\partial\theta} - \cot(\theta)\sin(\phi)\frac{\partial}{\partial\phi}\right),\\
L_3 &= -\imath\frac{\partial}{\partial\phi},\\
A_1 &= -\imath\cot(\lambda)\left(\cos(\theta)\cos(\phi)\frac{\partial}{\partial\theta} - \frac{\sin(\phi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\right) - \imath\sin(\theta)\cos(\phi)\frac{\partial}{\partial\lambda},\\
A_2 &= -\imath\cot(\lambda)\left(\cos(\theta)\sin(\phi)\frac{\partial}{\partial\theta} + \frac{\cos(\phi)}{\sin(\theta)}\frac{\partial}{\partial\phi}\right) - \imath\sin(\theta)\sin(\phi)\frac{\partial}{\partial\lambda},\\
A_3 &= -\imath\left(\cos(\theta)\frac{\partial}{\partial\lambda} - \sin(\theta)\cot(\lambda)\frac{\partial}{\partial\theta}\right).
\end{aligned} \tag{2.27}$$

Note that if we denote by $L^2 = L_1^2 + L_2^2 + L_3^2$ and $A^2 = A_1^2 + A_2^2 + A_3^2$ then we have

$$\Delta_{S^3} = L^2 + A^2 + 1\,. \tag{2.28}$$

Since Δ_{S^3}, L^2 and L_3 commute, we can find common normalized eigenfunctions of these operators. We shall call these functions hyper-spherical harmonics[17] and denote them by $Y_{n\ell m}$ with $n = 1, 2, 3\ldots$, $\ell = 0, 1, \ldots, n-1$, $m = -\ell, -\ell+1, \ldots \ell$. For each n, we have n^2 linearly independent hyper-spherical harmonics, which is exactly the multiplicity of the corresponding eigenvalues of Δ_{S^3} and the Hamiltonian of the hydrogen atom[7,9]. The n^2-dimensional vector spaces generated by $Y_{n\ell m}$ (one space for each n) provide all the irreducible representations of the group SO(4).

In order to find a Hilbert space of analytic functions related to $L^2(S^3)$, we notice that the commutation relations satisfied by $\mathbf{L}$ and $\mathbf{A}$ are:

$$\begin{aligned}
&[L_i, L_j] = \imath\epsilon_{ijk}L_k,\\
&[L_i, A_j] = \imath\epsilon_{ijk}A_k,\\
&[A_i, A_j] = \imath\epsilon_{ijk}L_k.
\end{aligned} \tag{2.29}$$

Now let us define the operators

$$\mathbf{C} = \frac{\mathbf{L}+\mathbf{A}}{2}, \qquad \mathbf{D} = \frac{\mathbf{L}-\mathbf{A}}{2}\,. \tag{2.30}$$

The commutation relation for these operators are:

$$\begin{aligned}
[C_i, C_j] &= \imath\epsilon_{ijk}C_k,\\
[D_i, D_j] &= \imath\epsilon_{ijk}D_k,\\
[C_i, D_j] &= 0.
\end{aligned} \tag{2.31}$$

Thus we obtain $so(4) = su(2) \times su(2)$.

Notice that the operators $\mathbf{L}$ and $\mathbf{A}$ satisfy the relation

$$\mathbf{L} \bullet \mathbf{A}\Psi = (L_1A_1 + L_2A_2 + L_3A_3)\Psi = 0\,. \tag{2.32}$$

This relation implies:

$$C^2\Psi = D^2\Psi = \left(\frac{L^2 + A^2}{4}\right)\Psi \; . \tag{2.33}$$

Since the eigenvalues of the operators C^2 and D^2 are of the form $u(u+1)$ and $v(v+1)$ (with $u, v \in \mathbf{S}$) respectively, then we must have $u(u+1) = v(v+1)$. This equation implies $u = v$.

Since $\Delta_{S^3} = L^2 + A^2 + 1$ and the eigenvalues of Δ_{S^3} are n^2, then we have $n^2 = 4u(u+1)+1$, which in turn implies $u = v = \frac{n-1}{2}$. Since n is a positive integer, $\mathbf{S}$ is the set of permitted values of u and v (including $\frac{1}{2}, \frac{3}{2}, \ldots$).

From the discussion above, we want to consider two copies of the representation of su(2) on the Bargmann space $\mathcal{B}_2$ of analytical functions of two complex variables described above.

Let us consider the Bargmann space $\mathcal{B}_4 = \mathcal{B}_2 \otimes \mathcal{B}_2$ of analytical functions of four complex variables (z_1, z_2, z_3, z_4). For each positive integer n, let us also consider the n^2-dimensional subspaces $\mathcal{M}_n$ of $\mathcal{B}_4$ given by

$$\mathcal{M}_n = \mathcal{W}_{\frac{n-1}{2}} \otimes \mathcal{W}_{\frac{n-1}{2}} \; , \tag{2.34}$$

with $\mathcal{W}_{\frac{n-1}{2}}$ the vector space of homogeneous polynomials of degree $n-1$ in two complex variables.

The functions

$$y_{nm_1m_2} = \frac{z_1^{\frac{n-1}{2}+m_1} z_2^{\frac{n-1}{2}-m_1} z_3^{\frac{n-1}{2}+m_2} z_4^{\frac{n-1}{2}-m_2}}{\sqrt{(\frac{n-1}{2}+m_1)!(\frac{n-1}{2}-m_1)!(\frac{n-1}{2}+m_2)!(\frac{n-1}{2}-m_2)!}} \; , \tag{2.35}$$

with $-\frac{n-1}{2} \leq m_1, m_2 \leq \frac{n-1}{2}$ provide a basis for the space $\mathcal{M}_n$.

Since we want to assign an analytic function (of four complex variables) to the hyperspherical harmonics $Y_{n\ell m}$ and these functions are in turn common eigenfunctions of Δ_{S^3}, L^2 and L_3, then we want actually to consider the following functions

$$y_{n\ell m}(z_1, z_2, z_3, z_4) = \sum_{m_1+m_2=m} C(\frac{n-1}{2}, m_1, \frac{n-1}{2}, m_2; \ell, m)\, y_{nm_1m_2} \; , \tag{2.36}$$

where $C(\frac{n-1}{2}, m_1, \frac{n-1}{2}, m_2; \ell, m)$ are Clebsh-Gordon coefficients[18].

Let us define the Bargmann transform for $L^2(S^3)$ as the linear extension of the previous assignment. Thus

$$\mathbf{B}_{S^3} : L^2(S^3) \mapsto \mathcal{F}_4 \tag{2.37}$$

where $\mathcal{F}_4$ is the closed subspace of $\mathcal{B}_4$ generated by the spaces $\mathcal{M}_n$, that is $\mathcal{F}_4 = \oplus_{n=1}^{\infty} \mathcal{M}_n$.

The Bargmann transform $\mathbf{B}_{S^3}$ is a unitary operator and intertwines the operators L_1, L_2, L_3, A_1, A_2, A_3 with the following operators:

$$\begin{aligned}
L'_1 &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_4} + z_2\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_3}\right), \\
L'_2 &= \frac{1}{2\imath}\left(z_1\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_4} - z_2\frac{\partial}{\partial z_1} - z_4\frac{\partial}{\partial z_3}\right), \\
L'_3 &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3} - z_2\frac{\partial}{\partial z_2} - z_4\frac{\partial}{\partial z_4}\right), \\
A'_1 &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1} - z_3\frac{\partial}{\partial z_4} - z_4\frac{\partial}{\partial z_3}\right), \\
A'_2 &= \frac{1}{2\imath}\left(z_1\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_3} - z_2\frac{\partial}{\partial z_1} - z_3\frac{\partial}{\partial z_4}\right), \\
A'_3 &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_4} - z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3}\right)
\end{aligned} \tag{2.38}$$

respectively.

The Laplacian on S^3 is intertwined with the operator

$$\Delta'_{S^3} = \left(\frac{1}{2}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4}\right) + 1\right)^2 \tag{2.39}$$

Thus the Laplacian Δ'_{S^3} is basically the square of the addition of four harmonic oscillators with the same frequency.

The expression for $\mathbf{L} \bullet \mathbf{A}$ in the space $\mathcal{F}_4$ is given by the following operator

$$\mathcal{L} \equiv z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4} \tag{2.40}$$

The restriction $(\mathbf{L} \bullet \mathbf{A})\,\Psi = 0$ is already included in the space $\mathcal{F}_4$, as the following proposition establishes it:

Proposition 2.1 *The range of the Bargmann Transform* $\mathbf{B}_{S^3}$ *is equal to the kernel of the operator* $\mathcal{L}$*, that is*

$$\mathcal{F}_4 = \{f \in \mathcal{B}_4 | \mathcal{L}f = 0\} \equiv \mathrm{Ker}\mathcal{L} \tag{2.41}$$

Proof: Let f in $\mathcal{F}_4$. Then

$$f = \sum_{n=0}^{\infty} \sum_{m_1,m_2} A_{nm_1m_2} y_{nm_1m_2} \tag{2.42}$$

where the convergence is in the norm of $\mathcal{B}_4$. Since norm-convergence in $\mathcal{B}_4$ implies point-wise convergence (this is because $\mathcal{B}_4$ has a reproducing kernel and the Schwartz inequality) then f is an analytic function with the following power series

$$f = \sum_{n=0}^{\infty} \sum_{m_1,m_2} A_{nm_1m_2} \frac{z_1^{\frac{n-1}{2}+m_1} z_2^{\frac{n-1}{2}-m_1} z_3^{\frac{n-1}{2}+m_2} z_4^{\frac{n-1}{2}-m_2}}{\sqrt{(\frac{n-1}{2}+m_1)!(\frac{n-1}{2}-m_1)!(\frac{n-1}{2}+m_2)!(\frac{n-1}{2}-m_2)!}} \tag{2.43}$$

Since f is analytic, we can differentiate its power series termwise and obtain

$$\mathcal{L}f = \sum_{n=0}^{\infty} \sum_{m_1,m_2} \frac{A_{nm_1m_2}\mathcal{L}\left(z_1^{\frac{n-1}{2}+m_1} z_2^{\frac{n-1}{2}-m_1} z_3^{\frac{n-1}{2}+m_2} z_4^{\frac{n-1}{2}-m_2}\right)}{\sqrt{(\frac{n-1}{2}+m_1)!(\frac{n-1}{2}-m_1)!(\frac{n-1}{2}+m_2)!(\frac{n-1}{2}-m_2)!}} \tag{2.44}$$

Since $\mathcal{L}\left(z_1^{\frac{n-1}{2}+m_1} z_2^{\frac{n-1}{2}-m_1} z_3^{\frac{n-1}{2}+m_2} z_4^{\frac{n-1}{2}-m_2}\right) = 0$ for all permitted values of n, m_1, m_2, we conclude $f \in \mathrm{Ker}\mathcal{L}$.

Conversely, suppose $f \in \mathrm{Ker}\mathcal{L}$. Since

$$\left\{\frac{z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4}}{\sqrt{a_1!a_2!a_3!a_4!}} \mid a_j \text{ is a nonnegative integer number}, \ j = 1,2,3,4.\right\}$$

is an orthonormal basis of $\mathcal{B}_4$ then f can be written as

$$f = \sum_{\mathbf{a}} A_{\mathbf{a}} \frac{z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4}}{\sqrt{a_1!a_2!a_3!a_4!}} \tag{2.45}$$

with $\mathbf{a} = (a_1, a_2, a_3, a_4)$ Thus we have

$$0 = \mathcal{L}f = \sum_{\mathbf{a}} A_{\mathbf{a}}(a_1 + a_2 - a_3 - a_4) \frac{z_1^{a_1} z_2^{a_2} z_3^{a_3} z_4^{a_4}}{\sqrt{a_1!a_2!a_3!a_4!}} \tag{2.46}$$

Now consider the case when $\mathbf{b} = (b_1, b_2, b_3, b_4)$ is such that $b_1 + b_2 - b_3 - b_4 \neq 0$. By taking the inner product of $\mathcal{L}f$ with $z_1^{b_1} z_2^{b_2} z_3^{b_3} z_4^{b_4}$ we obtain $A_{\mathbf{b}} = 0$. Thus f must be a linear combination of monomials $z_1^{b_1} z_2^{b_2} z_3^{b_3} z_4^{b_4}$ where $b_1 + b_2 - b_3 - b_4 = 0$. But each one of these monomials belongs to some space $\mathcal{M}_n$, therefore $f \in \mathcal{F}_4$. □

The Bargmann transform $\mathbf{B}_{S^3}$ is an integral operator

$$\mathbf{B}_{S^3}\Psi(\mathbf{z}) = \int_{x\in S^3} K_{S^3}(\mathbf{x},\mathbf{z})\Psi(\mathbf{x})d\sigma_3(\mathbf{x}) \tag{2.47}$$

with kernel

$$K_{S^3}(\mathbf{x},\mathbf{z}) = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} \overline{Y}_{n\ell m}(\mathbf{x}) y_{n\ell m}(\mathbf{z}) \tag{2.48}$$

As in the case of the 2-sphere, K_{S^3} can be written as a power series of a function

$$\Phi_{S^3} = \sum_{\ell=0}^{1}\sum_{m=-\ell}^{m=\ell} \overline{Y}_{2\ell m}(\mathbf{x}) y_{2\ell m}(\mathbf{z}) \tag{2.49}$$

in the following way

$$K_{S^3} = \sum_{n=1}^{\infty} \frac{\sqrt{2n}}{(n-1)!}\frac{1}{2^{n/2}}\left(\Phi_{S^3}\right)^{n-1} \tag{2.50}$$

with the following explicit expression for $\Phi_{S^3}(\mathbf{x},\mathbf{z})$:

$$\begin{aligned}\Phi_{S^3}(\mathbf{x},\mathbf{z}) \quad = \quad & \sqrt{2}\,[z_1 z_4(\cos(\lambda) - \imath\cos(\theta)\sin(\lambda)) - z_2 z_3(\cos(\lambda) + \imath\cos(\theta)\sin(\lambda)) \\ & + \imath z_1 z_3 \sin(\theta)\sin(\lambda)\exp(-\imath\phi) - \imath z_2 z_4 \sin(\theta)\sin(\lambda)\exp(\imath\phi)]\end{aligned} \tag{2.51}$$

The proof of this last fact is given in appendix B.

III. THE KERNEL OF THE BARGMANN TRANSFORM AND CANONICAL TRANSFORMATIONS

In this section we describe a beautiful relationship between the Bargmann Transform and a corresponding canonical transformation which plays the role of its classical analog. This fact occurs in the three cases described in the previous section. Namely, we have shown that the kernel of the Bargmann transform is a power series of a certain function. This function is in turn a generating function of the mentioned canonical transformation. This will give us symplectomorphisms mapping the phase space of a particle moving in $\mathbf{R}^n$, S^2 or S^3 onto a phase space described by complex coordinates.

We start by describing this relationship in the case of the BT for $L^2(\mathbf{R}^n)$ and then for the cases $L^2(S^2)$ and $L^2(S^3)$.

III.1 Canonical transformation for a particle moving in $\mathbf{R}^n$

In this case the relationship between the kernel of the Bargmann transform and a corresponding canonical transformation is well known (see [5,19]).

Following S. Graffi and A. Parmeggiani[5] , let us consider the function

$$\Phi_n(\mathbf{q},\mathbf{z}) = -\imath\left(-\frac{1}{2}(\mathbf{z}^2+\mathbf{q}^2)+\sqrt{2}\mathbf{z}\cdot\mathbf{q}\right) \tag{3.1}$$

which appears in the argument of the exponential function which in turn gives the kernel $A(\mathbf{q},\mathbf{z})$ of $\mathbf{B}_{\mathbf{R}^n}$.

Let us consider now a particle moving in $\mathbf{R}^n$. The phase space in this case is $T^*\mathbf{R}^n$ with coordinates $(\mathbf{q},\mathbf{p})$ and symplectic form $\omega_n = \sum_{j=1}^n dq_j\wedge dp_j$.

Now consider the function Φ_n as a generating function of a canonical transformation mapping the variables $(\mathbf{q},\mathbf{p})$ to the variables $(\mathbf{z},\mathbf{w})\in T^*(\mathbf{C}^n)$ via the following equations:

$$\begin{aligned} p_j &= -\frac{\partial\Phi_n}{\partial q_j} \quad j=1,2,\ldots,n \\ w_j &= \imath\frac{\partial\Phi_n}{\partial z_j} \quad j=1,2,\ldots,n \end{aligned} \tag{3.2}$$

Note that by considering these equations we guarantee that,

$$\sum_{j=1}^n dq_j\wedge dp_j = -\imath\sum_{j=1}^n dz_j\wedge dw_j \tag{3.3}$$

The expressions for $(\mathbf{z},\mathbf{w})$ in terms of $(\mathbf{q},\mathbf{p})$ are

$$\begin{aligned} z_j &= \frac{1}{\sqrt{2}}(q_j-\imath p_j) \\ w_j &= \frac{1}{\sqrt{2}}(q_j+\imath p_j) \end{aligned} \tag{3.4}$$

In this way we obtain that w_j must be the complex conjugate of z_j. This fact suggests to define the following submanifold of $T^*(\mathbf{C}^n)$

$$\mathcal{D}_n = \{(\mathbf{z},\mathbf{w})\in\mathbf{C}^{2n} | w=\overline{\mathbf{z}}\} \tag{3.5}$$

Since $T^*(\mathbf{C}^n)$ is a symplectic manifold with symplectic form $\kappa_n = -\imath\sum_{j=1}^n dz_j\wedge dw_j$, then we can consider $\mathcal{D}_n$ as a symplectic manifold with symplectic form τ_n given by the restriction of κ_n to $\mathcal{D}_n$. Notice that we can regard the symplectic manifold $(\mathcal{D}_n,\tau_n)$ as the symplectic manifold $(\mathbf{C}_n,\mu_n)$ where $\mu_n = -\imath\sum_{j=1}^n dz_j\wedge d\overline{z}_j$ and $(\mathbf{z},\overline{\mathbf{z}})\mapsto(\mathbf{z})$ being the symplectomorphism between $(\mathcal{D}_n,\tau_n)$ and $(\mathbf{C}_n,\mu_n)$.

Thus we obtain the following

Theorem 3.1 *The function Φ_n is a generating function of a linear canonical transformation which is a symplectomorphism $\mathcal{C}_n$ from $(T^*\mathbf{R}^n, \omega_n)$ onto $(\mathbf{C}_n, \mu_n)$ given by the equations $z_j = \frac{1}{\sqrt{2}}(q_j - \imath p_j) \quad j = 1, 2, \ldots, n$ The transformation $\mathcal{C}_n$ maps the Hamiltonian $H = \frac{1}{2}\sum_{j=1}^n q_j^2 + p_j^2$ to the Hamiltonian $H' = \sum_{j=1}^n z_j \overline{z}_j$*

This theorem gives the classical analog of the Bargmann Transform $\mathbf{B}_{\mathbf{R}^n}$ in the sense that $\mathbf{B}_{\mathbf{R}^n}$ is a unitary operator from $L^2(\mathbf{R}^n)$ onto the Bargmann space $\mathcal{B}_n$ and intertwines the quantum Hamiltonian of the harmonic oscillator $\hat{H} = \frac{1}{2}\left(\sum_{j=1}^n -\frac{\partial^2}{\partial q_j^2} + q_j^2 - 1\right)$ with the operator $\hat{H}' = \sum_{j=1}^n z_i \frac{\partial}{\partial z_i}$.

III.2 Canonical transformation for a particle moving on $\mathcal{S}^2$

The description above for a particle moving in $\mathbf{R}^n$ suggests to consider the function $\frac{-\imath}{\sqrt{3}}\Phi_{S^2}(\mathbf{x}, \mathbf{z})$ (the factor $\frac{1}{\sqrt{3}}$ is just to scale variables in a convenient way) as a generating function of a canonical transformation $\mathcal{C}_{S^2}$ which maps the phase space $T^*S^2 - \{0\}$ (i.e. T^*S^2 minus the zero section) onto a submanifold of $T^*\mathbf{C}^2$.

Thus let us consider the equations

$$
\begin{aligned}
p_\theta &= -\frac{\partial}{\partial\theta}\frac{-\imath}{\sqrt{3}}\Phi_{S^2}, \qquad p_\phi = -\frac{\partial}{\partial\phi}\frac{-\imath}{\sqrt{3}}\Phi_{S^2} \\
w_j &= \imath\frac{\partial}{\partial z_j}\frac{-\imath}{\sqrt{3}}\Phi_{S^2} \qquad j = 1, 2
\end{aligned}
\tag{3.6}
$$

Since

$$
\Phi_{S^2}(\mathbf{x}, \mathbf{z}) = \frac{\sqrt{3}}{2}\left(-\sin(\theta)\exp(-\imath\phi)z_1^2 + 2\cos(\theta)z_1 z_2 + \sin(\theta)\exp(\imath\phi)z_2^2\right)
$$

the equations above take the following explicit expression:

$$
\begin{aligned}
p_\theta &= -\frac{\imath}{2}\left(\cos(\theta)\exp(-\imath\phi)z_1^2 + 2\sin(\theta)z_1 z_2 - \cos(\theta)\exp(\imath\phi)z_2^2\right) \\
p_\phi &= \frac{-1}{2}\left(\sin(\theta)\exp(-\imath\phi)z_1^2 + \sin(\theta)\exp(\imath\phi)z_2^2\right) \\
w_1 &= -\sin(\theta)\exp(-\imath\phi)z_1 + \cos(\theta)z_2 \\
w_2 &= \cos(\theta)z_1 + \sin(\theta)\exp(\imath\phi)z_2
\end{aligned}
\tag{3.7}
$$

Note that by working with spherical coordinates for S^2, we are dealing with a specific local chart of S^2 and hence with a local chart for $T^*S^2 - \{0\}$. Later on we shall define the canonical transformation $\mathcal{C}_{S^2}$ by extending the relationship between $(\theta, \phi, p_\theta, p_\phi)$ and (z_1, z_2, w_1, w_2) (determined by equation (3.7) to the whole space $T^*S^2 - \{0\}$.

In order to identify the submanifold of $T^*\mathbf{C}^2$ where $T^*S^2 - \{0\}$ is mapped to, we need to find how the variables (w_1, w_2) are related to (z_1, z_2). To do this, we shall use the condition that the variables $\theta, \phi, p_\theta, p_\phi$ are real variables. Thus from the first two equations (3.7) and the reality condition $\theta = \overline{\theta}$, $\phi = \overline{\phi}$, $p_\theta = \overline{p}_\theta$, $p_\phi = \overline{p}_\phi$, we obtain

$$
\begin{aligned}
\sin(\theta) &= \frac{\left|\overline{z}_1^2 - z_2^2\right|}{|z_1|^2 + |z_2|^2}, \\
\cos(\theta) &=^{+}_{-} \frac{\overline{z}_1\overline{z}_2 + z_1 z_2}{|z_1|^2 + |z_2|^2}, \\
\exp(\imath\phi) &=^{+}_{-} \frac{\overline{z}_2^2 - z_1^2}{\left|\overline{z}_2^2 - z_1^2\right|},
\end{aligned}
\tag{3.8}
$$

If we substitute equations (3.8) in the last two equations of (3.7), we obtain:

$$w_1 =^{+}_{-} \overline{z}_1 \ \text{ and } \ w_2 =^{+}_{-} \overline{z}_2 \,. \tag{3.9}$$

We shall restrict ourselves to consider the plus sign in equations (3.8) and (3.9) in such a way that $(w_1, w_2) = (\overline{z}_1, \overline{z}_2)$. It can be shown, by expressing (z_1, z_2) in terms of $(\theta, \phi, p_\theta, p_\phi)$, that given $(\theta, \phi, p_\theta, p_\phi)$, there are four solutions for (z_1, z_2) [1]. Two of these solutions correspond to consider the plus sign in equations (3.8), (3.9) and the other two correspond to the minus sign. The two solutions corresponding to the plus sign are related to each other by a minus sign (i.e. if (z_1, z_2) is a solution of equations (3.7), with $(\theta, \phi, p_\theta, p_\phi)$ given, then $(-z_1, -z_2)$ is a solution too). Thus we need to identify (z_1, z_2) with $-(z_1, z_2)$ in order to make our canonical transformation $\mathcal{C}_{S^2}$ a function. In this way, we see that $\mathcal{C}_{S^2}$ takes values in the following submanifold of $T^*\mathbf{C}^2$:

$$\mathcal{D}_2 = \left\{(z_1, z_2, w_1, w_2) \in \mathbf{C}^4 | w_1 = \overline{z}_1, \ \ w_2 = \overline{z}_2\right\} / \{1, -1\} \,. \tag{3.10}$$

Note that the symplectic manifold $(\mathcal{D}_2, \tau_2)$ can be regarded as $\mathbf{C}^2/\{1,-1\}$ with symplectic form $\mu_2 = -\imath\,(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2)$.

Once we have chosen the plus sign in equations (3.8) and (3.9), we find the following expressions for p_θ, p_ϕ in terms of $(z_1, z_2, \overline{z}_1, \overline{z}_2)$:

$$p_\theta = \left(\frac{z_1 z_2 - \overline{z}_1 \overline{z}_2}{2\imath}\right)\left(\frac{|z_1|^2 + |z_2|^2}{z_1^2 - \overline{z}_2^2}\right), \qquad p_\phi = \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right)$$

$$\text{and} \quad H = p_\theta^2 + \frac{1}{\sin^2(\theta)} p_\phi^2 = \left(\frac{|z_1|^2 + |z_2|^2}{2}\right)^2 . \tag{3.11}$$

Note that we have a singularity in equations (3.8) when the energy H is zero. Thus we need to take off the zero section of T^*S^2 (which we shall denote by $\{0\}$) and the origin of the manifold $\mathbf{C}_2/\{1,-1\}$. (denoted by $\{0\}$ too). The other singularity in the expression for p_θ corresponds to the point $z_1^2 = \overline{z}_2^2$, which in turn is equivalent to the equation $\sin(\theta) = 0$ (north and south poles). We actually don't have such a singularity since we are taking $0 < \theta < \pi$ and it won't appear in our extension of our canonical transformation.

III.2.1 Canonical transformation for S^2 and quadric.

In this section we identify the equations giving $(\theta, \phi, p_\theta, p_\phi)$ as functions of $(z_1, z_2, \overline{z}_1, \overline{z}_2)$ with the composition of two maps involving the quadric in $\mathbf{C}^3$ defined by:

$$\mathbf{Q}_3 \equiv \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbf{C}^3 / \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0\} \,. \tag{3.12}$$

One of the maps goes from $\mathbf{C}^2/\{1,-1\} - \{0\}$ onto $\mathbf{Q}_3 - \{0\}$ and the other one from $\mathbf{Q}_3 - \{0\}$ onto $T^*S^2 - \{0\}$. In this manner, we shall extend our canonical transformation $\mathcal{C}_{S^2}$ as the inverse of such a composition.

[1] It actually can be shown that, if (z_1, z_2) is a solution then $(-z_1, -z_2)$, $(\overline{z}_2, -\overline{z}_1)$ and $(-\overline{z}_2, \overline{z}_1)$ are solutions too. If we take real and imaginary parts of z_1 and z_2, we see that the 4×4 matrices taking (z_1, z_2) to $(-z_1, -z_2)$, $(\overline{z}_2, -\overline{z}_1)$ and $(-\overline{z}_2, \overline{z}_1)$ are

$$\mathbf{A_2} = \begin{pmatrix} -\mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \mathbf{A_3} = \begin{pmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{pmatrix}, \quad \mathbf{A_4} = \begin{pmatrix} 0 & -\mathbf{K} \\ \mathbf{K} & 0 \end{pmatrix},$$

respectively (where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are 2×2 matrices). These three matrices together with the identity matrix are a representation of the cyclic group C_4 of four elements(the same group given by the four complex numbers $1, -1, \imath, -\imath$). Thus, for $(\theta, \phi, p_\theta, p_\phi)$ given, the four corresponding solutions to the equations determined by the canonical transformation are related to each other by elements of the group C_4

The generating function $\frac{-\imath}{\sqrt{3}}\Phi_{S^2}$ of our canonical transformation $\mathcal{C}_{S^2}$ can be written as a inner product:

$$\frac{-\imath}{\sqrt{3}}\Phi_{S^2} = -\imath(\alpha(\mathbf{z}) \bullet \mathbf{x}), \tag{3.13}$$

where

$$\begin{aligned} \mathbf{x} &= (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \\ \text{and } \alpha(\mathbf{z}) &= \frac{1}{2}\left(-z_1^2 + z_2^2, \imath z_1^2 + \imath z_2^2, 2z_1z_2\right) . \end{aligned} \tag{3.14}$$

Here we are taking the usual inner product of two vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ in $\mathbf{C}^3$:

$$\alpha \bullet \beta \equiv \alpha_1\overline{\beta}_1 + \alpha_2\overline{\beta}_2 + \alpha_3\overline{\beta}_3 \tag{3.15}$$

Notice the relevant fact that $\alpha(\mathbf{z})$ is in the quadric $\mathbf{Q}_3$. This suggests that $\mathbf{Q}_3$ might have an important role to play related to our canonical transformation $\mathcal{C}_{S^2}$. Thus let us define the following map:

$$\begin{aligned} \rho_2 : \tilde{\mathbf{C}}^2 - \{0\} &\mapsto \mathbf{Q}_3 - \{0\} \\ \rho_2(z_1, z_2) &= \frac{1}{2}(-z_1^2 + z_2^2, \imath z_1^2 + \imath z_2^2, 2z_1z_2) \end{aligned} \tag{3.16}$$

where we define $\tilde{\mathbf{C}}^2$ as the set of equivalence classes of $\mathbf{C}^2$ identifying (z_1, z_2) with $(-z_1, -z_2)$.

Now being in the quadric $\mathbf{Q}_3 - \{0\}$, we can go to $T^*S^2 - \{0\}$ through the following map

$$\begin{aligned} \sigma_2 : \mathbf{Q}_3 - \{0\} &\mapsto T^*S^2 - \{0\} \\ \sigma_2(\alpha_1, \alpha_2, \alpha_3) &= \left(\frac{\Re(\alpha)}{|\Re(\alpha|)}, -\Im(\alpha)\right) \end{aligned} \tag{3.17}$$

here $\Re(\alpha)$ and $\Im(\alpha)$ are the real and imaginary part of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ respectively.

Thus the composition map is:

$$\begin{aligned} \sigma_2 \circ \rho_2 : \tilde{\mathbf{C}}^2 - \{0\} &\mapsto T^*S^2 - \{0\} \\ \sigma_2 \circ \rho_2(z_1, z_2) &= \left(\frac{\Re(-z_1^2 + z_2^2)}{|z_1|^2 + |z_2|^2}, \frac{\Re(\imath z_1^2 + \imath z_2^2)}{|z_1|^2 + |z_2|^2}, \frac{\Re(2z_1z_2)}{|z_1|^2 + |z_2|^2}, \right. \\ &\quad \left. -\frac{1}{2}\Im(-z_1^2 + z_2^2), -\frac{1}{2}\Im(\imath z_1^2 + \imath z_2^2), -\frac{1}{2}\Im(2z_1z_2)\right) \end{aligned} \tag{3.18}$$

In the other hand, since a point in T^*S^2 can be described in spherical coordinates by

$$\begin{aligned} &\left(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), cos(\theta), \cos(\theta)\cos(\phi)p_\theta - \frac{\sin(\phi)}{\sin(\theta)}p_\phi, \right. \\ &\qquad \left. \cos(\theta)\sin(\phi)p_\theta + \frac{\cos(\phi)}{\sin(\theta)}p_\phi, -\sin(\theta)p_\theta\right) \end{aligned} \tag{3.19}$$

then, we find after some computation (by using equations (3.7) and (3.8)) that this point in T^*S^2 expressed in terms of $(z_1, z_2, \overline{z}_1, \overline{z}_2$ is exactly the same point that the composition $\sigma_2 \circ \rho_2$ assigns to (z_1, z_2). This conclusion allows us to extend our canonical transformation $\mathcal{C}_{S^2}$ (without referring to spherical coordinates) as the map $(\sigma_2 \circ \rho_2)^{-1}$. Let us state our conclusions in the following theorem:

Theorem 3.2 *The function* $\frac{-\imath}{\sqrt{3}}\Phi_{S^2}$ *is a generating function of a transformation that can be extended as a canonical transformation* $\mathcal{C}_{S^2}$ *which makes the following triangle of canonical transformations to commute:*

$$\begin{tikzcd}
\tilde{\mathbf{C}}^2 - \{0\} \arrow[d, "\rho_2"'] & T^*S^2 - \{0\} \arrow[l, "\mathcal{C}_{S^2}"'] \\
\mathbf{Q}_3 - \{0\} \arrow[ur, "\sigma_2"'] &
\end{tikzcd}$$

where (a) The symplectic form ν_2 *on* T^*S^2 *is given by the restriction to* T^*S^2 *of the canonical symplectic form* ω_3 *on* $T^*\mathbf{R}^3$*,*
(b) The symplectic form μ_2 *on* $\tilde{\mathbf{C}}^2 - \{0\}$ *is given by*

$$\mu_2 = -\imath\,(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2)\ , \tag{3.20}$$

(c) The symplectic form q_3 *on* $\mathbf{Q}_3 - \{0\}$ *is given by:*

$$q_3 = \frac{1}{2\imath|\Re\alpha|}\sum_{j=1}^{3} d\alpha_j \wedge d\overline{\alpha}_j - \frac{1}{8\imath|\Re\alpha|^3}\sum_{j,k=1}^{3}(\alpha_j + \overline{\alpha}_j)\,(\alpha_k + \overline{\alpha}_k)\,d\alpha_j \wedge d\overline{\alpha}_k \tag{3.21}$$

which is not the restriction to $\mathbf{Q}_3 - \{0\}$ *of the symplectic form* $\sum_{j=1}^{3} d\alpha_j \wedge d\overline{\alpha}_j$ *defined on the ambient* $\mathbf{C}^3$*.*

The transformation $\mathcal{C}_{S^2}$ *maps the components* (L_1, L_2, L_3) *of the angular momentum and the energy* H *as follows*

$$\begin{aligned}
L_1 &= -\sin(\phi)p_\theta - \frac{\cos(\theta)\cos(\phi)}{\sin(\theta)}p_\phi = \frac{1}{2}(z_1\overline{z}_2 + \overline{z}_1 z_2)\ , \\
L_2 &= \cos(\phi)p_\theta - \frac{\cos(\theta)\sin(\phi)}{\sin(\theta)}p_\phi = \frac{-\imath}{2}(z_1\overline{z}_2 - \overline{z}_1 z_2)\ , \\
L_3 &= p_\phi = \frac{1}{2}(|z_1|^2 - |z_2|^2)\ , \\
H &= \frac{1}{2}\left(p_\theta^2 + \frac{1}{\sin^2(\theta)}p_\phi^2\right) = \frac{1}{2}\left(\frac{|z_1|^2 + |z_2|^2}{2}\right)^2
\end{aligned} \tag{3.22}$$

which is the classical analog of what the Bargmann Transform $\mathbf{B}_{S^2}$ *does in quantum mechanics (see equation (2.23)).*

III.3 Canonical Transformation for a particle moving on S^3.

In analogy with the cases where the configuration space is $\mathbf{R}^n$ or S^2, we want to consider the function $\frac{-\imath}{\sqrt{2}}\Phi_{S^3}$ as a generating function of a canonical transformation $\mathcal{C}_{S^3}$ which maps $T^*S^3 - \{0\}$ onto a space involving four complex variables and their conjugates (we used four complex variables when we defined the Bargmann transform for $L^2(S^3)$). Notice that $T^*S^3 - \{0\}$ is a six (real) dimensional manifold, whereas to determine four complex variables we need eight real parameters. Thus we expect to have some restrictions between the four complex variables. The first restriction we expect is the following (in analogy with equation (2.32) in quantum mechanics):

$$|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2\ . \tag{3.23}$$

This equation is actually contained in the equations determined by the generating function $\frac{-\imath}{\sqrt{2}}\Phi_{S^3}$ as we show it below. The second restriction comes from a S^1 invariance of those equations.

Thus let us consider the canonical transformation generated by $\frac{-\imath}{\sqrt{2}}\Phi_{S^3}$ through the following equations:

$$\begin{aligned} p_\theta = -\frac{\partial}{\partial\theta}\frac{-\imath}{\sqrt{2}}\Phi_{S^3}, \quad p_\phi &= -\frac{\partial}{\partial\phi}\frac{-\imath}{\sqrt{2}}\Phi_{S^3}, \quad p_\lambda = -\frac{\partial}{\partial\lambda}\frac{-\imath}{\sqrt{2}}\Phi_{S^3}, \\ w_j &= \imath\frac{\partial}{\partial z_j}\frac{-\imath}{\sqrt{2}}\Phi_{S^3} \quad j = 1,2,3,4. \end{aligned} \tag{3.24}$$

Since

$$\begin{aligned} \Phi_{S^3}(\mathbf{x},\mathbf{z}) &= \sqrt{2}\,[z_1z_4(\cos(\lambda) - \imath\cos(\theta)\sin(\lambda)) - z_2z_3(\cos(\lambda) + \imath\cos(\theta)\sin(\lambda)) \\ &\quad + \imath z_1z_3\sin(\theta)\sin(\lambda)\exp(-\imath\phi) - \imath z_2z_4\sin(\theta)\sin(\lambda)\exp(\imath\phi)]\ , \end{aligned}$$

then we have

$$\begin{aligned} p_\theta &= -z_1z_4\sin(\theta)\sin(\lambda) - z_2z_3\sin(\theta)\sin(\lambda) - \\ &\quad -z_1z_3\cos(\theta)\sin(\lambda)\exp(-\imath\phi) + z_2z_4\cos(\theta)\sin(\lambda)\exp(\imath\phi)\ , \\ p_\phi &= \imath\sin(\theta)\sin(\lambda)\,[z_1z_3\exp(-\imath\phi) + z_2z_4\exp(\imath\phi)]\ , \\ p_\lambda &= -\imath z_1z_4(\sin(\lambda) + \imath\cos(\theta)\cos(\lambda)) - \imath z_2z_3(-\sin(\lambda) + \imath\cos(\theta)\cos(\lambda)) \\ &\quad -z_1z_3\sin(\theta)\cos(\lambda)\exp(-\imath\phi) + z_2z_4\sin(\theta)\cos(\lambda)\exp(\imath\phi)\ , \\ w_1 &= z_4(\cos(\lambda) - \imath\cos(\theta)\sin(\lambda)) + \imath z_3\sin(\theta)\sin(\lambda)\exp(-\imath\phi)\ , \\ w_2 &= -z_3(\cos(\lambda) + \imath\cos(\theta)\sin(\lambda)) - \imath z_4\sin(\theta)\sin(\lambda)\exp(\imath\phi)\ , \\ w_3 &= -z_2(\cos(\lambda) + \imath\cos(\theta)\sin(\lambda)) + \imath z_1\sin(\theta)\sin(\lambda)\exp(-\imath\phi)\ , \\ w_4 &= z_1(\cos(\lambda) - \imath\cos(\theta)\sin(\lambda)) - \imath z_2\sin(\theta)\sin(\lambda)\exp(\imath\phi)\ . \end{aligned} \tag{3.25}$$

As in the case for S^2, let us state the reality condition:

$$\begin{aligned} \theta = \overline{\theta}, \quad \phi &= \overline{\phi}, \quad \lambda = \overline{\lambda}, \\ p_\theta = \overline{p}_\theta, \quad p_\phi &= \overline{p}_\phi, \quad p_\lambda = \overline{p}_\lambda\ . \end{aligned} \tag{3.26}$$

This condition implies:

$$\begin{aligned} \sin(\theta) &= \frac{|\ z_1z_3 + \overline{z}_2\overline{z}_4\ |}{\sqrt{(|z_1|^2+|z_2|^2)\,(|z_3|^2+|z_4|^2) - (\Re(z_1z_4 - z_2z_3))^2}}, \\ \cos(\theta) &= \frac{\pm\Im(z_1z_4 + z_3z_2)}{\sqrt{(|z_1|^2+|z_2|^2)\,(|z_3|^2+|z_4|^2) - (\Re(z_1z_4 - z_2z_3))^2}}, \\ \exp(\imath\phi) &= \pm\frac{\imath\,|\ z_2z_4 + \overline{z}_1\overline{z}_3\ |}{z_2z_4 + \overline{z}_1\overline{z}_3}, \\ \sin(\lambda) &= \frac{\sqrt{(|z_1|^2+|z_2|^2)\,(|z_3|^2+|z_4|^2) - (\Re(z_1z_4 - z_2z_3))^2}}{\sqrt{(|z_1|^2+|z_2|^2)\,(|z_3|^2+|z_4|^2)}}, \\ \cos(\lambda) &= \pm\frac{\Re(z_1z_4 - z_2z_3)}{\sqrt{(|z_1|^2+|z_2|^2)\,(|z_3|^2+|z_4|^2)}}\ . \end{aligned} \tag{3.27}$$

These equations in turn imply:

$$(w_1, w_2, w_3, w_4) = \pm(\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4) \tag{3.28}$$

We shall choose the plus sign wherever we have the choice between either plus or minus sign in the above equations. Thus we have the relation $(w_1, w_2, w_3, w_4) = (\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4)$. Using this relation in the last four equations we obtain the expected restriction

$$|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 \, . \tag{3.29}$$

This restriction will appear as part of the definition of the manifold where our canonical transformation $\mathcal{C}_{S^3}$ should take values in.

In the other hand, the expressions for $p_\theta, p_\phi, p_\lambda$ in terms of (z_1, z_2, z_3, z_4) are:

$$\begin{aligned}
p_\theta &= -\frac{\Re(z_1 z_4)\left(|z_3|^2 + |z_2|^2\right) + \Re(z_2 z_3)\left(|z_1|^2 + |z_4|^2\right)}{|z_1 z_3 + \overline{z}_2 \overline{z}_4|}, \\
p_\phi &= \frac{1}{2}\left(|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2\right), \\
p_\lambda &= \frac{\left(|z_3|^2 + |z_4|^2\right)\Im(z_1 z_4 + \overline{z}_2 \overline{z}_3)}{\sqrt{\left(|z_3|^2 + |z_4|^2\right)^2 - \left(\Re(z_1 z_4 - z_2 z_3)\right)^2}}
\end{aligned} \tag{3.30}$$

Now notice the following relevant property of equations (3.25): Suppose the point $(\theta, \phi, \lambda, p_\theta, p_\phi, p_\lambda)$ is given. If (z_1, z_2, z_3, z_4) is a solution of equations (3.25); then, for any real number ψ, the vector $(z_1 \exp(\imath\psi), z_2 \exp(\imath\psi), z_3 \exp(-\imath\psi), z_4 \exp(-\imath\psi))$ is a solution too. Thus we define the following equivalence relation $\sim$ in $\mathbf{C}^4$ in order to consider the invariance of equations (3.25):

$$\begin{gathered}
(z_1, z_2, z_3, z_4) \sim (z_1\prime, z_2\prime, z_3\prime, z_4\prime) \text{ iff } \exists\, \psi \in \mathbf{R} \text{ such that} \\
(z_1\prime, z_2\prime, z_3\prime, z_4\prime) = (z_1 \exp(\imath\psi), z_2 \exp(\imath\psi), z_3 \exp(-\imath\psi), z_4 \exp(-\imath\psi))
\end{gathered} \tag{3.31}$$

We shall denote by $[(z_1, z_2, z_3, z_4)]$ the equivalence class of (z_1, z_2, z_3, z_4).

Thus our canonical transformation $\mathcal{C}_{S^3}$ should take values in the following six (real) dimensional set:

$$\tilde{\mathbf{C}}^4 \equiv \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 \mid |z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2\}/\sim \tag{3.32}$$

We shall regard the manifold $\tilde{\mathbf{C}}^4 - \{0\}$ as a symplectic manifold endowed with the following symplectic form:

$$\mu_3 = -\imath\left(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 + dz_3 \wedge d\overline{z}_3 + dz_4 \wedge d\overline{z}_4\right) \tag{3.33}$$

III.3.1 Canonical transformation for S^3 and quadric.

Now we want to extend the assignment $(\theta, \phi, \lambda, p_\theta, p_\phi) \mapsto [(z_1, z_2, z_3, z_4)]$ given by equations (3.25) to the whole space $T^*S^3 - \{0\}$. To that end, let us write the generating function $\frac{-\imath}{\sqrt{2}}\Phi_{S^3}$ as a inner product:

$$\frac{-\imath}{\sqrt{2}}\Phi_{S^3} = -\imath(\alpha(\mathbf{z}) \bullet \mathbf{x}) \tag{3.34}$$

where

$$\begin{aligned}
\alpha(\mathbf{z}) &= (\imath z_1 z_3 - \imath z_2 z_4, z_1 z_3 + z_2 z_4, -\imath z_1 z_4 - \imath z_2 z_3, z_1 z_4 - z_2 z_3) \text{ and} \\
\mathbf{x} &= (\sin(\lambda)\sin(\theta)\cos(\phi), \sin(\lambda)\sin(\theta)\sin(\phi), \sin(\lambda)\cos(\theta), \cos(\lambda))
\end{aligned} \tag{3.35}$$

Now notice that, for all $(z_1, z_2, z_3, z_4) \in \mathbf{C}^4$, the vector $\alpha(\mathbf{z})$ belongs to the quadric

$$\mathbf{Q}_4 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbf{C}^4 / \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 0\} \tag{3.36}$$

This suggests to define the following map

$$\begin{aligned} \rho_3 : \tilde{\mathbf{C}}^4 - \{0\} &\mapsto \mathbf{Q}_4 - \{0\} \\ \rho_3([z_1, z_2, z_3, z_4]) &= (\imath z_1 z_3 - \imath z_2 z_4, z_1 z_3 + z_2 z_4, -\imath z_1 z_4 - \imath z_2 z_3, z_1 z_4 - z_2 z_3) \end{aligned} \tag{3.37}$$

with (z_1, z_2, z_3, z_4) a representative of the equivalent class $[z_1, z_2, z_3, z_4]$. From now on we shall denote $\rho_3([z_1, z_2, z_3, z_4])$ just by $\rho_3(z_1, z_2, z_3, z_4)$.

Let us also define the map

$$\begin{aligned} \sigma_3 : \mathbf{Q}_4 - \{0\} &\mapsto T^* S^3 - \{0\} \\ \sigma_3(\alpha) &= \left(\frac{\Re\alpha}{|\Re\alpha|}, -\Im\alpha \right) \end{aligned} \tag{3.38}$$

As in the case for S^2, we shall extend our assignment $(\theta, \phi, \lambda, p_\theta, p_\phi) \mapsto [(z_1, z_2, z_3, z_4)]$ determined by equations (3.25) to the whole space $T^*S^3 - \{0\}$. For this purpose, we first check that if we consider a point in $T^*S^3 - \{0\}$ given in spherical coordinates by

$$\begin{aligned} &(\sin(\lambda)\sin(\theta)\cos(\phi), \sin(\lambda)\sin(\theta)\sin(\phi), \sin(\lambda)\cos(\theta), \cos(\lambda), \\ &\cos(\lambda)\sin(\theta)\cos(\phi)p_\lambda + \cos(\theta)\cos(\phi)p_\theta/\sin(\lambda) - \frac{\sin(\phi)p_\phi}{\sin(\lambda)\sin(\theta)}, \\ &\cos(\lambda)\sin(\theta)\sin(\phi)p_\lambda + \cos(\theta)\sin(\phi)p_\theta/\sin(\lambda) - \frac{\cos(\phi)p_\phi}{\sin(\lambda)\sin(\theta)}, \\ &\qquad \cos(\lambda)\cos(\theta)p_\lambda - \sin(\theta)p_\theta/\sin(\lambda), -\sin(\lambda)p_\lambda) \end{aligned} \tag{3.39}$$

and we use equations (3.27), then we obtain a map which assigns to (z_1, z_2, z_3, z_4) the same element in T^*S^3 as the evaluation of the function $\sigma_3 \circ \rho_3$ in (z_1, z_2, z_3, z_4). This allows us to extend our canonical transformation $\mathcal{C}_{S^3}$ as the map $(\sigma_3 \circ \rho_3)^{-1}$.

Let us state our conclusions in the following

Theorem 3.3 *The function $\frac{-\imath}{\sqrt{2}}\Phi_{S^3}$ generates a transformation which can be extended as a canonical transformation $\mathcal{C}_{S^3}$ that makes the following triangle of canonical transformations to commute:*

$$\begin{tikzcd} \tilde{\mathbf{C}}^4 - \{0\} \arrow[d, "\rho_3"'] & T^*S^3 - \{0\} \arrow[l, "\mathcal{C}_{S^3}"'] \\ \mathbf{Q}_4 - \{0\} \arrow[ur, "\sigma_3"'] & \end{tikzcd}$$

where (a) The symplectic form ν_3 on T^*S^3 is given by the restriction to T^*S^3 of the canonical symplectic form ω_4 on $T^*\mathbf{R}^4$,
(b) The symplectic form μ_3 on $\tilde{\mathbf{C}}^4 - \{0\}$ is given by

$$\mu_3 = -\imath \left(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 dz_3 \wedge d\overline{z}_3 + dz_4 \wedge d\overline{z}_4 \right) \ , \tag{3.40}$$

(c) The symplectic form q_4 *on* $\mathbf{Q}_4 - \{0\}$ *is given by:*

$$q_4 = \frac{1}{2\imath|\Re\alpha|}\sum_{j=1}^{4} d\alpha_j \wedge d\overline{\alpha}_j - \frac{1}{8\imath|\Re\alpha|^3}\sum_{j,k=1}^{4}(\alpha_j+\overline{\alpha}_j)(\alpha_k+\overline{\alpha}_k)\, d\alpha_j \wedge d\overline{\alpha}_k \tag{3.41}$$

which is not the restriction to $\mathbf{Q}_4 - \{0\}$ *of the symplectic form* $\sum_{j=1}^{4} d\alpha_j \wedge d\overline{\alpha}_j$ *defined on the ambient* $\mathbf{C}^4$*. The components of the angular momentum and Runge-Lenz vectors are:*

$$\begin{aligned}
L_1 &= \frac{1}{2}(z_1\overline{z}_2 + z_3\overline{z}_4 + z_2\overline{z}_1 + z_4\overline{z}_3)\,,\\
L_2 &= \frac{1}{2\imath}(z_1\overline{z}_2 + z_3\overline{z}_4 - z_2\overline{z}_1 - z_4\overline{z}_3)\,,\\
L_3 &= \frac{1}{2}(z_1\overline{z}_1 + z_3\overline{z}_3 - z_2\overline{z}_2 - z_4\overline{z}_4)\,,\\
A_1 &= \frac{1}{2}(z_1\overline{z}_2 + z_2\overline{z}_1 - z_3\overline{z}_4 - z_4\overline{z}_3)\,,\\
A_2 &= \frac{1}{2\imath}(z_1\overline{z}_2 + z_4\overline{z}_3 - z_2\overline{z}_1 + z_3\overline{z}_4)\,,\\
A_3 &= \frac{1}{2}(z_1\overline{z}_1 + z_4\overline{z}_4 - z_2\overline{z}_2 + z_3\overline{z}_3)\,.
\end{aligned}\tag{3.42}$$

The expressions for $L^2 = L_1^2 + L_2^2 + L_3^2$ *,* $A^2 = A_1^2 + A_2^2 + A_3^2$ *and the energy* H *are:*

$$\begin{aligned}
L^2 &= \frac{1}{4}(z_1\overline{z}_1 + z_2\overline{z}_2)^2 + \frac{1}{4}(z_3\overline{z}_3 + z_4\overline{z}_4)^2\\
&\qquad + \frac{1}{2}\left(4\Re(z_1 z_4 \overline{z}_2\overline{z}_3) + (|z_2|^2 - |z_1|^2)(|z_4|^2 - |z_3|^2)\right)\ ,\\
A^2 &= \frac{1}{4}(z_1\overline{z}_1 + z_2\overline{z}_2)^2 + \frac{1}{4}(z_3\overline{z}_3 + z_4\overline{z}_4)^2\\
&\qquad - \frac{1}{2}\left(4\Re(z_1 z_4 \overline{z}_2\overline{z}_3) + (|z_2|^2 - |z_1|^2)(|z_4|^2 - |z_3|^2)\right)\ ,\\
H &= \frac{1}{2}\left(p_\lambda^2 + \frac{p_\theta^2}{\sin^2(\lambda)} + \frac{p_\phi^2}{\sin^2(\lambda)\sin^2(\theta)}\right) = \frac{1}{2}\left(\frac{z_1\overline{z}_1 + z_2\overline{z}_2 + z_3\overline{z}_3 + z_4\overline{z}_4}{2}\right)^2 .
\end{aligned}\tag{3.43}$$

IV. QUANTIZATIONS OF THE MAPS ρ_2, σ_2 and ρ_3, σ_3.

Once we have the commuting triangles of canonical transformations in theorems (3.2) and (3.3), and the analogy between $\mathbf{B}_{S^2}$, $\mathbf{B}_{S^3}$ and their corresponding canonical transformations $\mathcal{C}_{S^2}, \mathcal{C}_{S^3}$, a natural question to ask is about the description of unitary transformations which play the role of quantizations of the maps ρ_2, σ_2 and ρ_3, σ_3 (for the case of S^2 and S^3 respectively). Such a description is the goal of this section.

IV.1 Quantizations of ρ_2 and σ_2

Let $d\mu_2(\alpha)$ be the measure on $\mathbf{Q}_3 - \{0\}$ obtained from the Gaussian measure

$$d\nu_2(\mathbf{z}) = \frac{1}{\pi^2} \exp(-|\mathbf{z}|^2) \prod_{j=1}^{2} dx_j dy_j \tag{4.1}$$

through the map ρ_2 (here we are actually regarding the map ρ_2 as a function from $\mathbf{C}^2$ onto $\mathbf{Q}_3$).

Given $\ell \geq 0$ an integer number, let $\mathcal{X}_\ell$ be the complex vector space of homogeneous polynomials of order ℓ in three complex variables $(\alpha_1, \alpha_2, \alpha_3)$ with the restriction $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$. The space $\mathcal{X}_\ell$ has dimension $2\ell + 1$ and a particular basis for it can be obtained by considering the monomials $y_{\ell m} = \frac{z_1^{\ell+m} z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}$ and the map ρ_2^{-1}. That is, we shall consider the following functions as a basis for $\mathcal{X}_\ell$:

$$\Upsilon_{\ell m}(\alpha) = y_{\ell m}(\mathbf{z}(\alpha)) \quad m = -\ell, -\ell + 1 \ldots \ell, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \ , \tag{4.2}$$

with $\mathbf{z}(\alpha)$ determined by the equations:

$$z_1^2 = -\alpha_1 - \imath\alpha_2, \ \ z_2^2 = \alpha_1 - \imath\alpha_2, \ \ z_1 z_2 = \alpha_3 \ . \tag{4.3}$$

Note that the functions $y_{\ell m}$ depend on $\mathbf{z} = (z_1, z_2)$ through integer powers of z_1^2, z_2^2 and $z_1 z_2$, so equations(4.3) are enough in order to express $y_{\ell m}$ in terms of α_1, α_2 and α_3.

Explicitly, the functions $\Upsilon_{\ell m}$ are given by

$$\Upsilon_{\ell m}(\alpha_1, \alpha_2, \alpha_3) = \frac{(-\mathrm{sgn}(m)\alpha_1 - \imath\alpha_2)^{|m|} \alpha_3^{\ell - |m|}}{\sqrt{(\ell+m)!(\ell-m)!}} \ , \tag{4.4}$$

with $sgn(m) = 1$ if $m \geq 0$ and $sgn(m) = -1$ if $m < 0$.

Moreover, for each $\ell \geq 0$, the set of functions $\{\Upsilon_{\ell m} | m = -\ell, -\ell + 1, \ldots, \ell\}$ is an orthonormal basis of $\mathcal{X}_\ell$ with inner product:

$$\langle F, G \rangle = \int_{\alpha \in \mathbf{Q}_3} F(\alpha) \overline{G(\alpha)} d\mu_2(\alpha) \ . \tag{4.5}$$

Let $\mathcal{E}_3$ be the Hilbert space generated by all the spaces $\mathcal{X}_\ell$ with the inner product defined as in equation (4.5).

Thus the quantization of the map ρ_2 is the unitary transformation $\mathbf{U}_{\rho_2} : \mathcal{B}_{S^2} \mapsto \mathcal{E}_3$ given by

$$\mathbf{U}_{\rho_2} \Psi(\alpha) = \Psi(\mathbf{z}(\alpha)), \quad \Psi \in \mathcal{B}_{S^2} \ . \tag{4.6}$$

where $\mathbf{z}(\alpha)$ is determined by equations 4.3. Notice that all the functions in the space $\mathcal{B}_{S^2}$ are infinite linear combinations of the functions $y_{\ell m}$, so the right hand side of equation(4.6) makes sense.

The quantization of σ_2 can be obtained in the following way. Let us consider the unitary transformation $\mathbf{U}_{\sigma_2} : \mathcal{E}_3 \mapsto L^2(S^2)$ given by

$$\mathbf{U}_{\sigma_2}\Psi(\mathbf{x}) = \lim_{M\mapsto\infty} \int_{|\alpha|\le M} \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \overline{\Upsilon_{\ell m}}(\alpha) Y_{\ell m}(\mathbf{x}) \right) \Psi(\alpha) d\mu_2(\alpha), \quad \Psi \in \mathcal{E}_3 \ . \tag{4.7}$$

By using the analog arguments to those given in Appendix A, the kernel of $\mathbf{U}_{\sigma_2}$ can be written as a power series of the function:

$$\begin{aligned} \Phi_{\sigma_2}(\alpha, \mathbf{x}) &= \overline{\Upsilon_{1,1}(\alpha)} Y_{1,1}(\mathbf{x}) + \overline{\Upsilon_{1,0}(\alpha)} Y_{1,0}(\mathbf{x}) + \overline{\Upsilon_{1,-1}(\alpha)} Y_{1,-1}(\mathbf{x}) \\ &= \sqrt{3}\mathbf{x} \bullet \alpha, \end{aligned} \tag{4.8}$$

Since

$$\sum_{m=-\ell}^{\ell} \overline{\Upsilon_{\ell m}}(\alpha) Y_{\ell m}(\mathbf{x}) = \frac{\sqrt{2\ell+1}}{(\sqrt{3})^{\ell}\ell!} \left(\Phi_{\sigma_2}(\alpha, \mathbf{x})\right)^{\ell} \tag{4.9}$$

then we have:

$$\mathbf{U}_{\sigma_2}\Psi(\mathbf{x}) = \lim_{M\mapsto\infty} \left(\int_{|\alpha|\le M} \sum_{\ell=0}^{\infty} \frac{\sqrt{2\ell+1}}{\ell!} (\mathbf{x} \bullet \alpha)^{\ell} \right) \Psi(\alpha) d\mu_2(\alpha), \quad \Psi \in \mathcal{E}_3 \tag{4.10}$$

Let us summarize our conclusions of this section in the following

Theorem 4.1 *The following triangle of unitary transformations commutes:*

$$\begin{tikzcd} \mathcal{B}_{S^2} \arrow[d, "\mathbf{U}_{\rho_2}"'] & L^2(S^2) \arrow[l, "\mathbf{B}_{S^2}"'] \\ \mathcal{E}_3 \arrow[ur, "\mathbf{U}_{\sigma_2}"'] & \end{tikzcd}$$

This triangle is the quantum version of the commuting triangle of canonical transformations in theorem (3.2).

IV.2 Quantizations of ρ_3 and σ_3

As in the previous section, let $d\mu_4(\alpha)$ be the measure on $\mathbf{Q}_4 - \{0\}$ obtained from the measure $d\nu_4(\mathbf{z}) = \frac{1}{\pi^4}\exp(-|\mathbf{z}|^2)\prod_{j=1}^{4} dx_j dy_j$ through the map ρ_3 (again, here we are regarding the map ρ_3 as a function from $\mathbf{C}^4$ onto $\mathbf{Q}_4$).

Now, given $n \ge 1$ a natural number, let $\mathcal{Z}_n$ be the space of homogeneous polynomials of degree $n-1$ in four complex variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with the restriction $\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2 = 0$. We shall consider the following inner product in $\mathcal{Z}_n$

$$\langle F, G\rangle = \int_{\alpha\in\mathbf{Q}_4} F(\alpha)\overline{G(\alpha)} d\mu_4(\alpha) \ . \tag{4.11}$$

The space $\mathcal{Z}_n$ has dimension n^2 and the following particular orthonormal basis

$$\Upsilon_{n\ell m} = y_{n\ell m}(\mathbf{z}(\alpha)) \tag{4.12}$$

where, since the functions $y_{n\ell m}$ depend on $\mathbf{z}$ through the variables $z_1z_3,\ z_1z_4,\ z_2z_3,\ z_2z_4$, we mean by $\mathbf{z}(\alpha)$ the expressions:

$$z_1z_3 = \frac{-\imath\alpha_1+\alpha_2}{2},\quad z_1z_4 = \frac{\imath\alpha_3+\alpha_4}{2},\quad z_2z_3 = \frac{\imath\alpha_3-\alpha_4}{2},\quad z_2z_4 = \frac{\imath\alpha_1+\alpha_2}{2}. \tag{4.13}$$

Let $\mathcal{E}_4$ be the Hilbert space generated by the spaces $\mathcal{Z}_n$. The quantization of ρ_3 is the unitary operator $\mathbf{U}_{\rho_3}$ given by the linear extension of the assignment $y_{n\ell m} \mapsto \Upsilon_{n\ell m}$. That is,

$$\begin{aligned} \mathbf{U}_{\rho_3} &: \mathcal{F}_4 \mapsto \mathcal{E}_4 \\ \mathbf{U}_{\rho_3}\Psi(\alpha) &= \int_{\mathbf{z}\in\mathbf{C}^4}\left(\sum_{n=1}^{\infty}\sum_{\ell=o}^{n-1}\sum_{m=-\ell}^{\ell}\overline{y_{n\ell m}(\mathbf{z})}\Upsilon_{n\ell m}(\alpha\right)\Psi(\mathbf{z})d\nu(\mathbf{z}),\quad \Psi\in\mathcal{F}_4. \end{aligned} \tag{4.14}$$

Since, for $n \geq 1$ a natural number,

$$\sum_{\ell=o}^{n-1}\sum_{m=-\ell}^{\ell}\overline{y_{n\ell m}(\mathbf{z})}\Upsilon_{n\ell m}(\alpha) = \frac{1}{((n-1)!)^2}\left(\Phi_{\rho_3}(\mathbf{z},\alpha)\right)^{n-1}\ , \tag{4.15}$$

where

$$\Phi_{\rho_3}(\mathbf{z},\alpha) = \sum_{\ell=o}^{1}\sum_{m=-\ell}^{\ell}\overline{y_{2\ell m}(\mathbf{z})}\Upsilon_{2\ell m}(\alpha) = \frac{\alpha\bullet\rho_3(\mathbf{z})}{2} \tag{4.16}$$

then we have

$$\mathbf{U}_{\rho_3}\Psi(\alpha) = \int_{\mathbf{z}\in\mathbf{C}^4}\left(\sum_{n=1}^{\infty}\frac{1}{((n-1)!)^2}\frac{(\alpha\bullet\rho_3(\mathbf{z}))^{n-1}}{2^{n-1}}\right)\Psi(\mathbf{z})d\nu_4(\mathbf{z}),\quad \Psi\in\mathcal{F}_4\ . \tag{4.17}$$

The quantization of σ_3 is the Unitary map $\mathbf{U}_{\sigma_3} : \mathcal{E}_4 \mapsto L^2(S^3)$ given by the linear extension of the assignment $\Upsilon_{n\ell m} \mapsto Y_{n\ell m}$:

$$\mathbf{U}_{\sigma_3}\Psi(\mathbf{x}) = \lim_{M\mapsto\infty}\int_{|\alpha|\leq M}\left(\sum_{n=1}^{\infty}\sum_{\ell=o}^{n-1}\sum_{m=-\ell}^{\ell}\overline{\Upsilon_{n\ell m}(\alpha)}Y_{n\ell m}(\mathbf{x})\right)\Psi(\alpha)d\mu_4(\alpha) \tag{4.18}$$

Since, for $n \geq 1$ a natural number,

$$\sum_{\ell=o}^{n-1}\sum_{m=-\ell}^{\ell}\overline{\Upsilon_{n\ell m}(\alpha)}Y_{n\ell m}(\mathbf{x}) = \frac{\sqrt{n}}{(n-1)!}\left(\frac{1}{2}\right)^{\frac{n-1}{2}}\left(\Phi_{\sigma_3}(\alpha,\mathbf{x})\right)^{n-1} \tag{4.19}$$

with

$$\Phi_{\sigma_3}(\alpha,\mathbf{x}) = \sum_{\ell=o}^{1}\sum_{m=-\ell}^{\ell}\overline{\Upsilon_{2\ell m}(\alpha)}Y_{2\ell m}(\mathbf{x}) = \sqrt{2}(\mathbf{x}\bullet\alpha) \tag{4.20}$$

then

$$\mathbf{U}_{\sigma_3}\Psi(\mathbf{x}) = \lim_{M\mapsto\infty}\int_{|\alpha|\leq M}\left(\sum_{n=1}^{\infty}\frac{\sqrt{n}}{(n-1)!}(\mathbf{x}\bullet\alpha)^{n-1}\right)\Psi(\alpha)d\mu_4(\alpha),\quad \Psi\in\mathcal{E}_4\ . \tag{4.21}$$

Theorem 4.2 *The following triangle of unitary transformations commutes:*

$$\begin{tikzcd}
\mathcal{F}_4 \arrow[d, "\mathbf{U}_{\rho_3}"'] & L^2(S^3) \arrow[l, "\mathbf{B}_{S^3}"'] \\
\mathcal{E}_4 \arrow[ur, "\mathbf{U}_{\sigma_3}"'] &
\end{tikzcd}$$

This triangle is the quantum version of the commuting triangle of canonical transformations in theorem (3.3).

V. THE BARGMANN TRANSFORM AS A COHERENT STATE TRANSFORM

It is known that the Bargmann transform $\mathbf{B}_{\mathbf{R}^n}$ can be identified as a coherent state transform. That is, for any $F \in L^2(\mathbf{R}^n)$, the function $\mathbf{B}_{\mathbf{R}^n} F(\mathbf{z})$ can be expressed as the inner product of F with a canonical coherent state, labeled by $\mathbf{z} \in \mathbf{C}^n$, of the n-dimensional harmonic oscillator.

In this section we show that both Bargmann transforms $\mathbf{B}_{S^2}$ and $\mathbf{B}_{S^3}$ can also be written as coherent states transforms. Here sets of coherent states in $L^2(S^2)$ ($L^2(S^3)$) appear for each irreducible representation of $SO(3)$ ($SO(4)$) considered when we defined the Bargmann transform $\mathbf{B}_{S^2}$ ($\mathbf{B}_{S^3}$).

V.1 $\mathbf{B}_{\mathbf{R}^n}$ as a coherent state transform

Let us note that the kernel of $\mathbf{B}_{\mathbf{R}^n}$ can be expressed in the following way: let us take $\mathbf{z} = (\mathbf{q} - \imath\mathbf{p})/\sqrt{2}$ with $\mathbf{x}, \mathbf{p} \in \mathbf{R}^n$, then

$$A(\mathbf{x},\mathbf{z}) = \exp\left(-\frac{1}{2}(\mathbf{z}^2 + \mathbf{x}^2) + \sqrt{2}\mathbf{z} \bullet \mathbf{x}\right) = \exp\left(-\frac{(\mathbf{x}-\mathbf{q})^2}{2} + \frac{\imath\mathbf{p} \bullet \mathbf{q}}{2} - \imath\mathbf{p} \bullet \mathbf{x} + \frac{\mathbf{q}^2 + \mathbf{p}^2}{4}\right) \tag{5.1}$$

Now let us denote by

$$\begin{aligned} \Phi^{(\mathbf{z})}(\mathbf{x}) &= \frac{1}{\pi^{n/4}}\overline{A(\mathbf{x},\mathbf{z})} \\ &= \frac{1}{\pi^{n/4}}\exp\left(-\frac{(\mathbf{x}-\mathbf{q})^2}{2} - \frac{\imath\mathbf{p} \bullet \mathbf{q}}{2} + \imath\mathbf{p} \bullet \mathbf{x} + \frac{\mathbf{q}^2 + \mathbf{p}^2}{4}\right), \end{aligned} \tag{5.2}$$

The functions $\Phi^{(\mathbf{z})} \in L^2(\mathbf{R}^n)$, labeled by $\mathbf{z} \in \mathbf{C}^n$, are the canonical coherent states of the n-dimensional harmonic oscillator.

Since

$$\frac{1}{\pi^{n/2}}\int_{\mathbf{x}\in\mathbf{R}^n} A(\mathbf{x},\mathbf{w})\overline{A(\mathbf{x},\mathbf{z})}d\mathbf{x} = \exp(\overline{\mathbf{z}}\mathbf{w}), \quad \mathbf{z},\mathbf{w} \in \mathbf{C}^n \tag{5.3}$$

then we have that the Bargmann transform sends the coherent states of the n-dimensional harmonic oscillator over into the coherent states defined in section II:

$$\mathbf{B}_{\mathbf{R}^n}\Phi^{(\mathbf{z})}(\mathbf{w}) = \Psi_{\mathbf{z}}(\mathbf{w}), \tag{5.4}$$

Thus the Bargmann transform $\mathbf{B}_{\mathbf{R}^n}$ acting on a function $F \in L^2(\mathbf{R}^n)$ can be expressed as a coherent state transform:

$$\mathbf{B}_{\mathbf{R}^n}F(\mathbf{z}) = \langle F, \Phi^{(\mathbf{z})}\rangle = \int_{\mathbf{x}\in\mathbf{R}^n} F(\mathbf{x})\overline{\Phi^{(\mathbf{z})}(\mathbf{x})}d\mathbf{x} \tag{5.5}$$

It is also known that the set of coherent states $\Phi^{(\mathbf{z})}$ gives a resolution of the identity in the sense that every function $F \in L^2(\mathbf{R}^n)$ can be written as

$$F(\mathbf{x}) = \lim_{M\mapsto\infty}\int_{|\mathbf{z}|\leq M}\langle F, \Phi^{(\mathbf{z})}\rangle_{\mathbf{R}^n}\Phi^{(\mathbf{z})}(\mathbf{x})d\nu_n(\mathbf{z}) \tag{5.6}$$

where $d\nu_n(\mathbf{z})$ is the measure given in equation (2.1).

V.2 $\mathbf{B}_{S^2}$ as a coherent state transform.

In section III we showed that the generating function Φ_{S^2} can be written as the inner product of the vectors $\rho_2(\mathbf{z}) \in \mathbf{Q}_3$ and $\mathbf{x} \in S^2$ (see equation(3.13)). Thus we are able to establish the following

Theorem 5.1 *The Bargmann transform $\mathbf{B}_{S^2}$ is a coherent state transform in the sense that $\mathbf{B}_{S^2}$ acting on a function $\Psi \in L^2(S^2)$ can be written in the following way*

$$\mathbf{B}_{S^2}\Psi(\mathbf{z}) = \sum_{\ell=0}^{\infty} \frac{\sqrt{2\ell+1}}{\ell!} \langle \Psi, \Phi_{\rho_2(\mathbf{z}),\ell} \rangle_{S^2} \tag{5.7}$$

where, for each $\alpha \in \mathbf{Q}_3 - \{0\}$, we define

$$\Phi_{\alpha,\ell}(\mathbf{x}) = (\mathbf{x} \bullet \alpha)^{\ell} \tag{5.8}$$

The functions $\Phi_{\alpha,\ell}$ are coherent states in the sense described below.

For each integer number $\ell \geq 1$, the states $\Phi_{\alpha,\ell}$, with $\alpha \in \mathbf{Q}_3 - \{0\}$, are a system of coherent states for the eigenspace V_ℓ of the Laplacian Δ_{S^2} (see reference[8]) in the following sense:
(a) The map σ_2 establishes a bijective identification between $\mathbf{Q}_3 - \{0\}$ and $T^*S^2 - \{0\}$. Thus the states $\{\Phi_{\alpha,\ell}\}$ are labeled by elements in the cotangent bundle of the configuration space S^2 (minus the zero section).
(b) For large ℓ, the state $\Phi_{\alpha,\ell}$ is concentrated around the geodesic in S^2 (classical orbit of the geodesic flow) generated by the vectors $\Re(\alpha)/\|\Re(\alpha)\|$ in S^2 and $-\Im(\alpha)$ in T^*S^2 at the point $\Re(\alpha)/\|\Re(\alpha)\|$. The concentration is in the sense that most of the $L^2(S^2)$-norm of $\Phi_{\alpha,\ell}$ comes from a small neighborhood of the geodesic (see reference[7]).
(c) The states $\{\Phi_{\alpha,\ell}\}$ give a resolution of the identity in the space V_ℓ. Namely, for each function $\Psi \in V_\ell$, the following equation holds:

$$\Psi(\mathbf{x}) = r(\ell) \int_{\mathbf{Q}_3 - \{0\}} \langle \Psi, \Phi_{\alpha,\ell} \rangle_{S^2} \Phi_{\alpha,\ell}(\mathbf{x}) d\mu_2(\alpha) \tag{5.9}$$

for some constant $r(\ell)$ which will be evaluated below.

This equation is a consequence of the following three facts: (i) the operator on the right hand side of equation (5.9) commutes with all the rotations in $SO(3)$ acting as operators on V_ℓ, (ii) the irreducibility of the representation of $SO(3)$ acting on V_ℓ and (iii) the invariance of the measure $d\mu_2(\alpha)$ under rotations in $SO(3)$, which we state as follows:

Proposition 5.2 *Let $R \in SO(3)$, then for any $d\mu_2-$integrable function $f(\alpha)$ we have:*

$$\int_{\mathbf{Q_3} - \{0\}} f(R\alpha) d\mu_2(\alpha) = \int_{\mathbf{Q_3} - \{0\}} f(\alpha) d\mu_2(\alpha) \tag{5.10}$$

here $R\alpha$ means $R\Re(\alpha) + \imath R\Im(\alpha)$.

Proof By definition of the measure $d\mu_2$

$$\int_{\mathbf{Q_3} - \{0\}} f(R\alpha) d\mu_2(\alpha) = \int_{\mathbf{z} \in \mathbf{C}^2} f(R\rho_2(\mathbf{z})) \exp(-|\mathbf{z}|^2) d\nu_2(\mathbf{z}). \tag{5.11}$$

Since for any rotation $R \in SO(3)$ there exists $U \in SU(2)$ such that $R\rho_2(\mathbf{z}) = \rho_2(U\mathbf{z})$, we just need to make the change of variables $\mathbf{w} = U\mathbf{z}$ and use the invariance of the measure $d\nu_2$ under the unitary transformation U. $\square$

For each ℓ, the image of the coherent states $\Phi_{\alpha,\ell}$ under the transformation $\mathbf{U}_{\sigma_2}^{-1} = \mathbf{B}_{S^2} \circ \mathbf{U}_{\rho_2}$ provide a reproducing kernel for the space $\mathcal{X}_\ell$. The proof of this fact is based on the two following propositions:

Proposition 5.3 *The inner product of two coherent states $\Phi_{\beta,\ell}$ and $\Phi_{\alpha,k}$ in $L^2(S^2)$ is:*

$$\langle \Phi_{\beta,\ell}, \Phi_{\alpha,k} \rangle_{S^2} = \frac{\gamma_\ell}{2^{\ell+1}} (\alpha \bullet \beta)^\ell \delta_{\ell,k} \tag{5.12}$$

with $\gamma_\ell \equiv \int_0^\pi \sin^{2\ell+1}(\theta) d\theta = 2\frac{(2\ell)!!}{(2\ell+1)!!}$ *(here the notation* $(2\ell)!!$ *means the product of all positive integer even numbers lesser or equal than* 2ℓ*, the notation for* $(2\ell+1)!!$ *is similar).*

Proof:
Given $\beta \in \mathbf{Q_3} - 0$, there exists $\lambda > 0$ and a rotation $R \in SO(3)$ such that $\beta = \lambda R(\mathbf{e}_1 - \imath \mathbf{e}_2)$ (with $\{\mathbf{e}_j,\ j = 1,2,3\}$ the canonical orthonormal basis of $\mathbf{R}^3$). Thus

$$\langle \Phi_{\beta,\ell}, \Phi_{\alpha,\ell} \rangle_{S^2} = \lambda^\ell \int_{\mathbf{x} \in S^2} (\mathbf{x} \bullet R(\mathbf{e}_1 - \imath \mathbf{e}_2))^\ell \, \overline{(\mathbf{x} \bullet \alpha)}^\ell d\sigma_2(\mathbf{x}) \tag{5.13}$$

By making the change of variable $\mathbf{y} = R^{-1}\mathbf{x}$, using spherical coordinates for the variable $\mathbf{y}$ and the orthogonality of the matrix R, the result follows by straight computation. □

Proposition 5.4 *The Bargmann Transform $\mathbf{B}_{S^2}$ of the state $\Phi_{\beta,\ell}$ is*

$$\mathbf{B}_{S^2} \Phi_{\beta,\ell}(\mathbf{z}) = \frac{\sqrt{2\ell+1}}{\ell!} \frac{\gamma_\ell}{2^{\ell+1}} \left(\rho_2(\mathbf{z}) \bullet \beta\right)^\ell \ . \tag{5.14}$$

Proof This result is an immediate consequence of the previous proposition and the expression for $\mathbf{B}_{S^2}$ given in equation(5.7). □

Thus the image of the coherent state $\Phi_{\beta,\ell}$ under the transformation $\mathbf{U}_{\sigma_2}^{-1}$ is

$$\mathbf{U}_{\sigma_2}^{-1} \Phi_{\beta,\ell}(\alpha) = \frac{\sqrt{2\ell+1}}{\ell!} \frac{\gamma_\ell}{2^{\ell+1}} \left(\alpha \bullet \beta\right)^\ell \tag{5.15}$$

The factor $(\alpha \bullet \beta)^\ell$ appearing in the last equation provides us with a reproducing kernel for the space $\mathcal{X}_\ell$ as the following theorem establishes it:

Theorem 5.5 *For each integer number $\ell \geq 1$, the function $K_{\mathbf{Q_3},\ell} : \mathbf{Q_3} - \{0\} \times \mathbf{Q_3} - \{0\} \mapsto \mathbf{C}$ given by*

$$K_{\mathbf{Q_3},\ell}(\alpha, \beta) \equiv \frac{2^\ell}{(2\ell)!} \left(\beta \bullet \alpha\right)^\ell \tag{5.16}$$

is a reproducing kernel for the space $\mathcal{X}_\ell$, that is:

$$\int_{\mathbf{Q_3} - \{0\}} f(\alpha) K_{\mathbf{Q_3},\ell}(\alpha, \beta) d\mu_2(\alpha) = f(\beta), \quad f \in \mathcal{X}_\ell \ . \tag{5.17}$$

Proof:
It is enough to prove equation(5.17) when $f = \Upsilon_{\ell,m}$. From the definition of $d\mu_2$ we have:

$$\int_{\mathbf{Q_3} - \{0\}} \Upsilon_{\ell,m}(\alpha) K(\alpha, \beta) d\mu_2(\alpha) = \int_{\mathbf{z} \in \mathbf{C}^2} y_{\ell,m}(\mathbf{z}) \frac{2^\ell}{(2\ell)!} \overline{(\rho_2(\mathbf{z}) \bullet \beta)}^\ell d\nu_2(\mathbf{z}) \ . \tag{5.18}$$

Now let us define the function

$$\begin{aligned} \Phi_{\rho_2}(\mathbf{z}, \alpha) &\equiv \overline{y_{1,1}(\mathbf{z})} \Upsilon_{1,1}(\alpha) + \overline{y_{1,0}(\mathbf{z})} \Upsilon_{1,0}(\alpha) + \overline{y_{1,-1}(\mathbf{z})} \Upsilon_{1,-1}(\alpha) \\ &= \alpha \bullet \rho_2(\mathbf{z}) \ . \end{aligned} \tag{5.19}$$

Since

$$(\Phi_{\rho_2}(\mathbf{z},\beta))^\ell = \frac{(2\ell)!}{2^\ell} \sum_{m=-\ell}^{\ell} \overline{y_{\ell,m}}(\mathbf{z}) \Upsilon_{\ell,m}(\beta) \ , \tag{5.20}$$

the result follows. □

We can use this theorem, in particular, to evaluate the constant $r(\ell)$ appearing in equation (5.9) in the following manner: Let us apply equation (5.9) to the state $\Phi_{\beta,\ell}$:

$$\begin{aligned} \Phi_{\beta,\ell}(\mathbf{x}) &= r(\ell)\int_{\alpha\in\mathbf{Q_3}} \langle\Phi_{\beta,\ell}\Phi_{\alpha,\ell}\rangle_{S^2}\Phi_{\alpha,\ell}(\mathbf{x})d\mu_2(\alpha) \\ &= r(\ell)\frac{\gamma_\ell}{2^{\ell+1}}\int_{\alpha\in\mathbf{Q_3}} (\alpha\bullet\beta)^\ell\,\Phi_{\alpha,\ell}(\mathbf{x})d\mu_2(\alpha) \\ &= r(\ell)\frac{\gamma_\ell(2\ell)!}{2^{2\ell+1}}\overline{\int_{\alpha\in\mathbf{Q_3}} (\alpha\bullet\mathbf{x})^\ell\, K_{\mathbf{Q_3},\ell}(\alpha,\beta)d\mu_2(\alpha)} = r(\ell)\frac{\gamma_\ell(2\ell)!}{2^{2\ell+1}}\Phi_{\beta,\ell}(\mathbf{x}) \end{aligned} \tag{5.21}$$

Thus we must have

$$r(\ell) = \frac{2^{2\ell+1}}{\gamma_\ell(2\ell)!} \tag{5.22}$$

From theorem (5.5) we also obtain a reproducing kernel for the space $\mathcal{E}_3$:

Theorem 5.6 *The function*

$$K_{\mathbf{Q_3}}(\alpha,\beta) \equiv \sum_{\ell=0}^{\infty} \frac{2^\ell}{(2\ell)!}(\beta\bullet\alpha)^\ell = \frac{1}{2}\left(\exp(\sqrt{2\beta\bullet\alpha}) + \exp(-\sqrt{2\beta\bullet\alpha})\right) \tag{5.23}$$

is an analytic function in the space $\mathcal{E}_3$ (for α fixed) and provides a reproducing kernel for this space:

$$\int_{\mathbf{Q_3}-\{0\}} f(\alpha)K_{\mathbf{Q_3}}(\alpha,\beta)d\mu_2(\alpha) = f(\beta), \quad f\in\mathcal{E}_3 \ . \tag{5.24}$$

Note that in equation (5.23) we can take any branch of the square root function and we obtain an analytic function in any case.

Proof:

The series $\sum_{\ell=0}^{\infty}\frac{2^\ell}{(2\ell)!}z^\ell$ is absolutely convergent for all values of $z\in\mathbf{C}$, so $K_{\mathbf{Q_3}}(\alpha,\beta)$ is an analytic function on the variable β (α fixed). Moreover, each function $2^\ell(\beta\bullet\alpha)^\ell/(2\ell)!$ is in the space $\mathcal{X}_\ell$ and its norm $a_\ell^2 \equiv \|2^\ell(\beta\bullet\alpha)^\ell/(2\ell)!\|^2$ is equal to $\frac{2^{3\ell+1}}{(2\ell+1)\gamma_\ell}\left(\frac{\ell!}{(2\ell)!}\right)^2|\alpha|^{2\ell}$. Since $\sum_{\ell=0}^{\infty}a_\ell^2$ is finite, then we conclude that the kernel $K_{\mathbf{Q_3}}$ is in $\mathcal{E}_3$ (α fixed). The equality in equation (5.23) can be checked directly by using the Taylor series of the exponential function. □

The kernel $K_{\mathbf{Q_3}}$ and the function ρ_2 give an expected reproducing kernel for the space $\mathcal{B}_{S^2}$:

Theorem 5.7 *The function $\frac{1}{2}(\exp(\mathbf{z}\bullet\mathbf{w})+\exp(-\mathbf{z}\bullet\mathbf{w}))$ is a reproducing kernel of the space $\mathcal{B}_{S^2}$. This kernel function is in the space $\mathcal{B}_{S^2}$ for $\mathbf{w}$ fixed.*

REMARK 1

V. Bargmann[20] has also analyzed the space $\mathcal{E}_3$ of analytic functions on the quadric $\mathbf{Q_3}$. He gives to such space a structure of a Hilbert space endowed with a inner product with respect to a measure like the one we have $(d\mu_2)$. He also considers a map $\mathbf{A} : \mathcal{E}_3 \mapsto \mathcal{B}_2$ which is very similar to the transformation $\mathbf{U}_{\rho_2}^{-1} : \mathcal{E}_3 \mapsto \mathcal{B}_{S^2}$ given by:

$$\mathbf{U}_{\rho_2}^{-1}\Psi(\mathbf{z}) = \Psi(\rho_2(\mathbf{z})), \quad \Psi \in \mathcal{E}_3 \tag{5.25}$$

The difference between $\mathbf{A}$ and $\mathbf{U}_{\rho_2}^{-1}$ is that Bargmann uses a slightly different expression for ρ_2 and that $\mathbf{A}$ is not onto. He shows that $\mathbf{A}$ preserves the inner product. In particular he also finds the adjoint of the multiplicative operator by the coordinate $\alpha_j, \quad j = 1, 2, 3$. He does it by an integration by parts whereas we can do it by using our unitary transformation $\mathbf{U}_{\rho_2}$, the chain rule and taking into account that the adjoint, in the space $\mathcal{B}_{S^2}$, of the multiplicative operators by $z_1^2, z_2^2, z_1 z_2$ are the operators $\frac{\partial^2}{\partial z_1^2}, \frac{\partial^2}{\partial z_2^2}, \frac{\partial^2}{\partial z_2 \partial z_1}$ respectively. Thus we find that the adjoint operator:

$$\alpha_j^* = \frac{\partial}{\partial \alpha_j} + 2\sum_{k=1}^{3} \alpha_k \frac{\partial}{\partial \alpha_k}\frac{\partial}{\partial \alpha_j} - \alpha_j \Delta \,, \tag{5.26}$$

where $\Delta = \sum_{k=1}^{3} \frac{\partial^2}{\partial \alpha_k^2}$. Our expression for α_j^* differs from the one of Bargmann by a factor of $\frac{1}{2}$ because he uses a different expression for ρ_2.

REMARK 2

V. Guillemin[21] has introduced a transformation p_* which is related to our unitary transformation $\mathbf{U}_{\sigma_2} : \mathcal{E}_3 \mapsto L^2(S^2)$. The transformation p_* can be described in our terms as follows: The unit cotangent bundle $S^*(S^2)$ of S^2 can be identified with the following subspace of the quadric $\mathbf{Q}_3$:

$$X \equiv \{\mathbf{x} + \imath\mathbf{y} | \|\mathbf{x}\| = \|\mathbf{y}\| = 1, \;\; \mathbf{x} \bullet \mathbf{y} = 0\} \,. \tag{5.27}$$

Let V_ℓ be the subspace of $L^2(S^2)$ generated by the spherical harmonics of order ℓ as above. Let $f \in V_\ell$. Let us denote by $F_\ell(x_1, x_2, x_3)$ the unique polynomial in three real variables $(x_1, x_2, x_3) \in \mathbf{R}^3$ which is homogeneous of degree ℓ, harmonic ($\Delta_{\mathbf{R}^3} F_\ell \equiv \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} F_\ell = 0$) and whose restriction to S^2 is equal to f. Let $\tilde{F}_\ell(w_1, w_2, w_3)$ be the polynomial in three complex variables $(w_1, w_2, w_3) \in \mathbf{C}^3$ obtained from F_ℓ by replacing (x_1, x_2, x_3) by (w_1, w_2, w_3). Let us denote by $\mathbf{S}_1$ the assignment $f \mapsto \tilde{F}_\ell$. Now let $\mathbf{S}_2$ be the operator which assigns to $\tilde{F}_\ell$ its restriction $\tilde{F}_\ell|_X$ to the space X. Then finally let us introduce the transformation p_* which applied to $\tilde{F}_\ell|_X$ gives us a function in the space V_ℓ:

$$p_* \tilde{F}_\ell|_X(\mathbf{x}) \equiv \int_{\mathbf{y} \in \mathcal{C}_\mathbf{x}} \tilde{F}_\ell|_X(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y}) \tag{5.28}$$

where, for fixed $\mathbf{x}$, the integration is on the unit circle $\mathcal{C}_\mathbf{x}$ in the cotangent space of $\mathbf{x} \in S^2$ and $d\mu$ is the usual normalized measure on that circle.

It can be shown that the operator $p_* \mathbf{S}_2 \mathbf{S}_1$ commutes with all the rotation operators in the representation of $SO(3)$ acting on the space $L^2(S^2)$. Thus, by Schur's lemma, there exist a constant λ_ℓ such that

$$p_* \mathbf{S}_2 \mathbf{S}_1 f = \lambda_\ell f, \quad f \in V_\ell \,. \tag{5.29}$$

The constant λ_ℓ can be evaluated by considering the polynomial $(x_1 + \imath x_2)^\ell$ and the point $\mathbf{x} = (1, 0, 0)$ (see reference [21]). In this way we have $\lambda_\ell = \frac{(2\ell - 1)!!}{\ell!}$.

Since the elements of $\mathcal{X}_\ell$ are homogeneous polynomials of degree ℓ in the quadric $\mathbf{Q}_3$, the exposition above for the operator $p_* \mathbf{S}_2 \mathbf{S}_1$ suggest to consider the following operator:

Let $f \in V_\ell$ and consider the function $\mathbf{U}_{\sigma_2}^{-1} f$ defined on $\mathbf{Q}_3$. Let G_ℓ be the unique homogeneous polynomial of order ℓ in three complex variables (z_1, z_2, z_3) which is harmonic

($\Delta_{\mathbf{C}^3} F_\ell \equiv \sum_{k=1}^{3} \frac{\partial^2}{\partial z_k^2} F_\ell = 0$) and whose restriction to $\mathbf{Q}_3$ is equal to $\mathbf{U}_{\sigma_2}^{-1} f$ (denote the assignment $\mathbf{U}_{\sigma_2}^{-1} f \mapsto G_\ell$ by $\mathbf{T}$). Let us now consider the operator $p_* \mathbf{S}_2 \mathbf{T} \mathbf{U}_{\sigma_2}^{-1}$. It can be shown that the operator $p_* \mathbf{S}_2 \mathbf{T} \mathbf{U}_{\sigma_2}^{-1}$ also commutes with all the rotation operators in the representation of $SO(3)$ acting on $L^2(S^2)$. Thus, by Schur's lemma, there exist a constant μ_ℓ such that

$$p_* \mathbf{S}_2 \mathbf{T} \mathbf{U}_{\sigma_2}^{-1} f = \mu_\ell f, \quad f \in V_\ell \ . \tag{5.30}$$

To evaluate the constant μ_ℓ, we also consider the polynomial $(x_1 + \imath x_2)^\ell$ in V_ℓ and the point $\mathbf{x} = (1, 0, 0)$, to obtain $\mu_\ell = \frac{1}{\ell! \sqrt{2\ell+1}}$.

Thus we finally have, for $F \in \mathcal{X}_\ell$:

$$p_* \mathbf{S}_2 \mathbf{T} F = \frac{1}{\ell! \sqrt{2\ell + 1}} \mathbf{U}_{\sigma_2} F \ . \tag{5.31}$$

Therefore, we can think the operator $\mathbf{U}_{\sigma_2}$ as the unitary-ization of the operator $p_* \mathbf{S}_2 \mathbf{T}$.

V.3 **$\mathbf{B}_{S^3}$ as a coherent state transform.**

This section is the analog of the previous section IV.2 concerning the Bargmann Transform $\mathbf{B}_{S^3}$. Namely, this transform can also be written as a coherent state transform (with the coherent states defined in the same way as above) and we also have a reproducing kernel for the spaces $\mathcal{Z}_n$, $\mathcal{E}_4$ and $\mathcal{F}_4$.

From the definition of $\mathbf{B}_{S^3}$ (2.47) we obtain:

Theorem 5.8

$$\mathbf{B}_{S^3} \Psi(\mathbf{z}) = \sum_{n=1}^{\infty} \frac{\sqrt{n} 2^{n-1}}{(n-1)!} \langle \Psi, \Phi_{\rho_3(\mathbf{z}), n-1} \rangle_{S^3}, \quad \Psi \in L^2(S^3) \ , \tag{5.32}$$

where the coherent states in $L^2(S^3)$ are defined in the same way as before

$$\Phi_{\alpha, n-1}(\mathbf{x}) = (\mathbf{x} \bullet \alpha)^{n-1}, \quad \alpha \in \mathbf{Q}_4 - \{0\} \ \ n \geq 1 \ . \tag{5.33}$$

These states provide a resolution of the identity:

For $\Psi \in \mathcal{M}_n$, the following equation holds:

$$\Psi(\mathbf{x}) = \frac{n}{((n-1)!)^2} \int_{\mathbf{Q}_4 - \{0\}} \langle \Psi, \Phi_{\alpha, n-1} \rangle_{S^3} \Phi_{\alpha, n-1}(\mathbf{x}) d\mu_4(\alpha) \ , \tag{5.34}$$

This equation is a consequence of the following four facts:

(a)

Proposition 5.9 *The inner product of two coherent states in $L^2(S^3)$ is:*

$$\langle \Phi_{\beta, k}, \Phi_{\alpha, \ell} \rangle_{S^3} = \frac{1}{(k+1) 2^k} (\alpha \bullet \beta)^k \delta_{\ell, k}, \quad k \geq 0 \ . \tag{5.35}$$

(b) the invariance of the measure $d\mu_4$ under rotations $R \in SO(4)$:

Proposition 5.10 *Let $R \in SO(4)$. for any $d\mu_4-$integrable function $f(\alpha)$ the following equation holds:*

$$\int_{\mathbf{Q_4} - \{0\}} f(R\alpha) d\mu_4(\alpha) = \int_{\mathbf{Q_4} - \{0\}} f(\alpha) d\mu_4(\alpha) \ . \tag{5.36}$$

(c)

Proposition 5.11 *The Bargmann transform $\mathbf{B}_{S^3}$ maps the coherent states as follows:*

$$\mathbf{B}_{S^3}\Phi_{\beta,k}(\mathbf{z}) = \frac{\sqrt{k+1}}{(k+1)!2^k}\left(\rho_3(\mathbf{z})\bullet\beta\right)^k, \quad k \geq 0 \tag{5.37}$$

and then the image of the coherent states $\Phi_{\beta,k}$ under the unitary transformation $\mathbf{U}_{\sigma_3}^{-1}$ is

$$\mathbf{U}_{\sigma_3}^{-1}\Phi_{\beta,k}(\alpha) = \frac{\sqrt{k+1}}{(k+1)!2^k}\left(\alpha\bullet\beta\right)^k, \quad k \geq 0\,. \tag{5.38}$$

(d)

Theorem 5.12 *For each integer number $n \geq 1$, the function $K_{\mathbf{Q_4},n} : \mathbf{Q_4}\times\mathbf{Q_4} \mapsto \mathbf{C}$ given by*

$$K_{\mathbf{Q_4},n-1}(\alpha,\beta) \equiv \frac{1}{((n-1)!)^2 2^{n-1}}\left(\beta\bullet\alpha\right)^{n-1} \tag{5.39}$$

is a reproducing kernel for the space $\mathcal{Z}_n$, that is:

$$\int_{\mathbf{Q_4}-\{0\}} f(\alpha)K_{\mathbf{Q_4},n-1}(\alpha,\beta)d\mu_4(\alpha) = f(\beta), \quad f\in\mathcal{Z}_n\,. \tag{5.40}$$

The proofs of these facts are similar to those given in section IV.2. We just point out that to prove equation(5.36) we need to use the fact that for any rotation $R \in SO(4)$ there exist a couple of matrices U and V in $SU(2)$ such that $R\rho_3(\mathbf{z}) = \rho_2(M_{U,V}\mathbf{z})$ where $M_{U,V}$ is a 4×4 matrix with the 2×2 matrices U and V on the diagonal and zeros elsewhere.

Theorem 5.12 provides us with a reproducing kernel for the space $\mathcal{E}_4$:

Theorem 5.13 *The function*

$$K_{\mathbf{Q_4}}(\alpha,\beta) \equiv \sum_{n=1}^{\infty}\frac{1}{((n-1)!)^2 2^{n-1}}\left(\beta\bullet\alpha\right)^{n-1} \tag{5.41}$$

belongs to the space $\mathcal{E}_4$ (for α fixed) and it is a reproducing kernel of this space:

$$\int_{\mathbf{Q_4}-\{0\}} f(\alpha)K_{\mathbf{Q_4}}(\alpha,\beta)d\mu_4(\alpha) = f(\beta), \quad f\in\mathcal{E}_4\,. \tag{5.42}$$

From this last theorem and the map ρ_3 we obtain the following

Theorem 5.14 *The function*

$$\tilde{K}_{\mathbf{Q_4}}(\mathbf{z},\mathbf{w}) \equiv \sum_{n=1}^{\infty}\frac{1}{((n-1)!)^2}\left((\overline{w_1}z_1+\overline{w_2}z_2)(\overline{w_3}z_3+\overline{w_4}z_4)\right)^{n-1} \tag{5.43}$$

belongs to the space $\mathcal{F}_4$ (for $\mathbf{w}$ fixed) and it is a reproducing kernel of this space:

$$\int_{\mathbf{C_4}} f(\mathbf{w})\tilde{K}_{\mathbf{Q_4}}(\mathbf{z},\mathbf{w})d\nu_4(\mathbf{w}) = f(\mathbf{z}), \quad f\in\mathcal{F}_4. \tag{5.44}$$

REMARK 3

V. Bargmann[20] also considers a transformation $A : \mathcal{E}_4 \mapsto \mathcal{B}_4$ which is very similar to our transformation $\mathbf{U}_{\rho_3}^{-1} : \mathcal{E}_4 \mapsto \mathcal{F}_4$ given by

$$\mathbf{U}_{\rho_3}^{-1}\Psi(\mathbf{z}) = \Psi(\rho_3(\mathbf{z})), \quad \Psi\in\mathcal{E}_4 \tag{5.45}$$

V. Bargmann uses a slightly different expression for the map ρ_3. He proves that A preserves the inner product and that the image of $\mathcal{E}_4$ under the map A is the set: $A(\mathcal{E}_4) = \{f \in \mathcal{B}_4 | Mf = f\}$ where Mf is the mean value of f:

$$Mf(z_1, z_2, z_3, z_4) \equiv \int_0^\pi f(z_1 \exp(\imath\psi), z_2 \exp(\imath\psi), z_3 \exp(-\imath\psi), z_4 \exp(-\imath\psi)) \frac{d\psi}{2\pi} \tag{5.46}$$

This motivates to establish the following

Proposition 5.15 *The spaces $A(\mathcal{E}_4)$ and $\mathcal{F}_4$ are the same.*

One can prove this proposition by following the proof of proposition (2.1).

As noted by V. Bargmann[20], the function $\tilde{K}_{\mathbf{Q_4}}$ can be expressed in terms of the modified Bessel function of the first kind $I_0(z) \equiv \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$:

$$\tilde{K}_{\mathbf{Q_4}}(\mathbf{z}, \mathbf{w}) = I_0 \left(2\sqrt{(\overline{w_1} z_1 + \overline{w_2} z_2)(\overline{w_3} z_3 + \overline{w_4} z_4)} \right). \tag{5.47}$$

REMARK 4 We can also relate our transformation $\mathbf{U}_{\sigma_3} : \mathcal{E}_4 \mapsto L^{(2)}(S^3)$ with the transformation p_* of V. Guillemin[21]: For $F \in \mathcal{Z}_n$

$$p_* \mathbf{S}_2 \mathbf{T} F = \frac{1}{n! \sqrt{n}} \mathbf{U}_{\sigma_3} F \tag{5.48}$$

where the operators $\mathbf{S}_2$ and $\mathbf{T}$ are defined in a similar way as above. The proof of this last equation follows the same steps as the one to prove equation (5.31). Thus we can think the operator $\mathbf{U}_{\sigma_3}$ as the unitary-ization of $p_* \mathbf{S}_2 \mathbf{T}$.

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Appendix A

The Kernel of the Bargmann transform for S^2.

The purpose of this appendix is to prove that the kernel $K_{S^2}(\mathbf{x},\mathbf{z})$ can be expressed as a power series of the function $\Phi_{S^2}(\mathbf{x},\mathbf{z})$. That is

$$K_{S^2}(\mathbf{x},\mathbf{z}) = \sum_{k=0}^{\infty} \frac{\sqrt{2k+1}}{k!} \left(\frac{1}{\sqrt{3}}\right)^k (\Phi_{S^2}(\mathbf{x},\mathbf{z}))^k$$

We want to do it by showing that the following equation holds for all $k \geq 0$:

$$\mathcal{K}_k(\mathbf{x},\mathbf{z}) = \frac{\sqrt{2k+1}}{k!} \left(\frac{1}{\sqrt{3}}\right)^k (\Phi_{S^2})^k (\mathbf{x},\mathbf{z})$$

with

$$\mathcal{K}_k(\mathbf{x},\mathbf{z}) = \sum_{m=-k}^{k} Y^*_{km}(\mathbf{x}) y_{km}(\mathbf{z})$$

The main idea of the proof is based on the fact that the following function $G_k(\mathbf{x},\mathbf{z}) \equiv \mathcal{K}_k(\mathbf{x},\mathbf{z})/c(k)\,(\Phi_{S^2}(\mathbf{x},\mathbf{z}))^k$ (with $c(k)$ some constant) satisfies an homogeneous system of first order differential equations with initial condition $G_k(\mathbf{x}_0,\mathbf{z}_0) = 1$. Then we show that the gradient of G_k must be zero, which implies that the solution of the system must be $G_k(\mathbf{x},\mathbf{z}) \equiv 1$, i.e. $\mathcal{K}_k(\mathbf{x},\mathbf{z}) = c(k)\,(\Phi_{S^2}(\mathbf{x},\mathbf{z}))^k$. Next, we evaluate the constant $c(k)$.

We know that the Bargmann transform $\mathbf{B}_{S^2}$ intertwines the operators L_i and L'_i, $i = 1,2,3$. Thus for any smooth function $\Psi(\mathbf{x})$

$$L'_i \mathbf{B}_{S^2} \Psi = \mathbf{B}_{S^2} \mathrm{L}_i \Psi$$

Now we want to show that for any $k \geq 0$

$$(L_i + L'_i)\mathcal{K}_k(\mathbf{x},\mathbf{z}) = 0 \quad i = 1,2,3$$

Let P_k be the projector on the span of $\{y_{km}(\mathbf{z})/\ m = -k,\ldots,k\}$. P_k acts on the Bargmann Space B_2. Thus we have for any smooth function $\Psi(\mathbf{x})$

$$P_k \mathbf{B}_{S^2} \Psi(\mathbf{z}) = \int_{\mathbf{x}\in S^2} \mathcal{K}_k(\mathbf{x},\mathbf{z}) \Psi(\mathbf{x}) d\Omega(\mathbf{x})$$

which implies

$$\begin{aligned} P_k \mathbf{B}_{S^2} L_i \Psi(\mathbf{z}) &= \int_{\mathbf{x}\in S^2} \mathcal{K}_k(\mathbf{x},\mathbf{z}) L_i \Psi(\mathbf{x}) d\Omega(\mathbf{x}) \\ &= -\int_{\mathbf{x}\in S^2} (L_i \mathcal{K}_k(\mathbf{x},\mathbf{z})) \Psi(\mathbf{x}) d\Omega(\mathbf{x}) \end{aligned}$$

where we have used the selfadjointness of L_i.

In the other hand, if we apply P_k to both sides of the equation $L'_i \mathbf{B}_{S^2} \Psi = \mathbf{B}_{S^2} \mathrm{L}_i \Psi$ and since P_k commutes with $L'_i,\ \ i = 1,2,3$, we get

$$P_k \mathbf{B}_{S^2} L_i \Psi(\mathbf{z}) = \int_{\mathbf{x}\in S^2} L'_i \mathcal{K}_k(\mathbf{x},\mathbf{z}) \Psi(\mathbf{x}) d\Omega(\mathbf{x})$$

Since the last two equations hold for a dense set in $L^2(S^2)$, we conclude that $(L_i + L_i')\mathcal{K}_k(\mathbf{x},\mathbf{z}) = 0 \quad i = 1,2,3$.

The function $\mathcal{K}_k$ satisfies another differential equation which is a consequence that $\mathcal{K}_k$ is an homogeneous polynomial of degree $2k$ in the variables (z_1, z_2). Thus we must have for all $k \geq 0$:

$$(H_0 - 2k)\mathcal{K}_k = 0$$

with

$$H_0 = z_1\partial_1 + z_2\partial_2$$

The analysis above must hold, in particular, for $k = 1$. Thus we have:

$$(L_i + L_i')\Phi_{S^2}(\mathbf{x},\mathbf{z}) = 0 \quad i = 1,2,3$$

and

$$(H_0 - 2k)\Phi_{S^2} = 0$$

Since L_i, L_i' and H_0 are first order linear differential operators, we have for all $k \geq 0$

$$\begin{array}{c} (L_i' + L_i)\Phi_{S^2}^{\ell} = 0 \\ (H_0 - 2\ell)\Phi_{S^2} = 0 \end{array}$$

Now consider a particular point $(\mathbf{x}_0, \mathbf{z}_0)$ where $\Phi_{S^2}(\mathbf{x}_0,\mathbf{z}_0) \neq 0$. Let

$$c(k) = \frac{\mathcal{K}_k(\mathbf{x}_0,\mathbf{z}_0)}{\Phi_{S^2}^k(\mathbf{x}_0,\mathbf{z}_0)}$$

Define the function

$$G_k(\mathbf{x},\mathbf{z}) = \frac{\mathcal{K}_k(\mathbf{x},\mathbf{z})}{c(k)\Phi_{S^2}^k(\mathbf{x},\mathbf{z})}$$

This function satisfies the following system of homogeneous differential equations:

$$\begin{array}{rcl} (L_i' + L_i)G_k(\mathbf{x},\mathbf{z}) & = & 0 \quad i = 1,2,3 \\ H_0 G_k(\mathbf{x},\mathbf{z}) & = & 0 \end{array}$$

with initial condition

$$G_k(\mathbf{x}_0,\mathbf{z}_0) = 1$$

The system of equations can also be written as

$$\mathbf{M}\nabla G_k = 0$$

with $\nabla = \left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)$ and $\mathbf{M}$ a 4×4 matrix given by

$$\mathbf{M} = \begin{bmatrix} \imath\sin(\phi) & \imath\cot(\theta)\cos(\phi) & \frac{1}{2}z_2 & \frac{-1}{2}z_1 \\ -\imath\cos(\phi) & \imath\cot(\theta)\sin(\phi) & -\frac{1}{2\imath}z_2 & \frac{1}{2\imath}z_1 \\ 0 & -\imath & \frac{1}{2}z_1 & \frac{1}{2}z_2 \\ 0 & 0 & -z_1 & z_2 \end{bmatrix}$$

The determinant of the matrix $\mathbf{M}$ is:

$$det\mathbf{M} = \frac{1}{\sqrt{3}\sin(\theta)}\Phi_{S^2}(\mathbf{x},\mathbf{z})$$

Since $\Phi_{S^2}(\mathbf{x}_0,\mathbf{z}_0) \neq 0$, then $\mathbf{M}$ is invertible on a neighborhood $\mathcal{V}$ of the point $(\mathbf{x}_0,\mathbf{z}_0)$. Therefore, we must have

$$G_k(\mathbf{x},\mathbf{z}) = 1 \text{ for all } (\mathbf{x},\mathbf{z}) \in \mathcal{V}$$

and then

$$\mathcal{K}_k = c(k)\Phi_{S^2}^k(\mathbf{x},\mathbf{z}) \text{ for all } (\mathbf{x},\mathbf{z}) \in \mathcal{V}$$

Since $\mathcal{K}_k$ and $\Phi_{S^2}^k$ are analytical functions of $(\mathbf{x},\mathbf{z})$, then the last equation must actually hold for all $(\mathbf{x},\mathbf{z})$.

Finally, we want to evaluate the constant $c(k)$ appearing in the last equation. By using the equations

$$Y_{11}(\mathbf{x}) = -\sqrt{\frac{3}{2}}\sin(\theta)\exp(\imath\phi), \quad Y_{10}(\mathbf{x}) = \sqrt{3}\cos(\theta),$$
$$Y_{1-1}(\mathbf{x}) = \sqrt{\frac{3}{2}}\sin(\theta)\exp(-\imath\phi)$$
$$y_{11}(\mathbf{z}) = \frac{z_1^2}{\sqrt{2}}, \quad y_{10}(\mathbf{z}) = z_1 z_2, \quad y_{1-1}(\mathbf{z}) = \frac{z_2^2}{\sqrt{2}},$$

we obtain

$$\Phi_{S^2}(\mathbf{x},\mathbf{z}) = \frac{\sqrt{3}}{2}\left(-\sin(\theta)\exp(-\imath\phi)z_1^2 + 2\cos(\theta)z_1 z_2 + \sin(\theta)\exp(\imath\phi)z_2^2\right)$$

Now consider the relation

$$Y_{km}(\mathbf{x}) = \sqrt{\frac{(2k+1)(k-m)!}{(k+m)!}}P_k^m(\cos(\theta))\exp(\imath m\phi)$$

with P_k^m the associated Legendre polynomials. By taking $\theta = 0$ and since $P_k^m(1) = 1\delta_{m0}$,

$$\sqrt{(2k+1)}\frac{z_1^k z_2^k}{k!} = c(k)\left(\sqrt{3}z_1 z_2\right)^k$$

which in turn implies

$$c(k) = \frac{\sqrt{2k+1}}{k!}\left(\frac{1}{\sqrt{3}}\right)^k$$

Appendix B

The Kernel of the Bargmann transform for S^3.

The purpose of this appendix is to prove the equation

$$K_{S^3} = \sum_{n=1}^{\infty} \frac{\sqrt{2n}}{(n-1)!} \frac{1}{2^{n/2}} \left(\Phi_{S^3}\right)^{n-1}$$

We want to do it by showing that for all $n \geq 1$:

$$\tilde{\mathcal{K}}_n(\mathbf{x}, \mathbf{z}) = \frac{\sqrt{2n}}{(n-1)!} \frac{1}{2^{n/2}} \left(\Phi_{S^3}\right)^{n-1} (\mathbf{x}, \mathbf{z})$$

with

$$\tilde{\mathcal{K}}_n(\mathbf{x}, \mathbf{z}) = \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{m=\ell} Y^*_{n\ell m}(\mathbf{x}) y_{n\ell m}(\mathbf{z})$$

The idea of the proof is exactly the same as the one we used in appendix A. Thus for each $n \geq 1$, let us define the function

$$F_n(\mathbf{x}, \mathbf{z}) = \tilde{\mathcal{K}}_n(\mathbf{x}, \mathbf{z}) / \left(c(n) \left(\Phi_{S^3}\right)^{n-1} (\mathbf{x}, \mathbf{z}) \right)$$

(in a neighborhood $\mathcal{V}$ of a point $(\mathbf{x}_0, \mathbf{z}_0)$ where $(\Phi_{S^3})(\mathbf{x}_0, \mathbf{z}_0) \neq 0$) such that F_n is a solution of the system of differential equations:

$$\mathbf{M} \nabla F_n = 0$$

with $\nabla = (\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4})$ and $\mathbf{M}$ the 7×7 matrix

$$\begin{bmatrix} \imath \sin(\phi) & \imath \cot(\theta) \cos(\phi) & 0 & \frac{z_2}{2} & \frac{z_1}{2} & \frac{z_4}{2} & \frac{z_3}{2} \\ -\imath \cos(\phi) & \imath \cot(\theta) \sin(\phi) & 0 & -\frac{z_2}{2\imath} & \frac{z_1}{2\imath} & -\frac{z_4}{2\imath} & \frac{z_3}{2\imath} \\ 0 & -\imath & 0 & \frac{z_1}{2} & -\frac{z_2}{2} & \frac{z_3}{2} & -\frac{z_4}{2} \\ f(\phi, \lambda) & \frac{-\cot(\lambda)\sin(\phi)}{\imath \sin(\theta)} & -\imath \sin(\theta) \cos(\phi) & \frac{z_2}{2} & \frac{z_1}{2} & -\frac{z_4}{2} & -\frac{z_3}{2} \\ g(\phi, \lambda) & \frac{\cot(\lambda)\cos(\phi)}{\imath \sin(\theta)} & -\imath \sin(\theta) \sin(\phi) & -\frac{z_2}{2\imath} & \frac{z_1}{2\imath} & -\frac{z_4}{2\imath} & \frac{z_3}{2\imath} \\ \imath \sin(\theta) \cot(\lambda) & 0 & -\imath \cos(\theta) & \frac{z_1}{2} & -\frac{z_2}{2} & -\frac{z_3}{2} & \frac{z_4}{2} \\ 0 & 0 & 0 & -z_1 & -z_2 & -z_3 & -z_4 \end{bmatrix}$$

with $f(\phi, \lambda) = -\imath \cot(\lambda) \cos(\theta) \cos(\phi)$ and $g(\phi, \lambda) = -\imath \cot(\lambda) \cos(\theta) \sin(\phi)$. The function F_n also satisfies the initial condition $F_n(\mathbf{x}_0, \mathbf{z}_0) = 1$.

The determinant of $\mathbf{M}$ is

$$det\mathbf{M} = \frac{1}{4 \sin^2(\lambda) \sin(\theta)} \left(\Phi_{S^3}(\mathbf{x}, \mathbf{z})\right)^2$$

Since $\Phi_{S^3}(\mathbf{x}_0, \mathbf{z}_0) \neq 0$, we proceed as in appendix A to conclude that

$$\tilde{\mathcal{K}}_n(\mathbf{x}, \mathbf{z}) = c(n) \left(\Phi_{S^3}\right)^{n-1} (\mathbf{x}, \mathbf{z})$$

It is just left to evaluate the constant $c(n)$. Since

$$\begin{aligned}
Y_{100} &= 1, \quad Y_{200} = 2\cos(\lambda), \quad Y_{211} = -\imath\sqrt{2}\sin(\theta)\sin(\lambda)e^{\imath\phi}, \\
Y_{210} &= 2\imath\cos(\theta)\sin(\lambda), \quad Y_{21-1} = \imath\sqrt{2}\sin(\theta)\sin(\lambda)e^{-\imath\phi} \\
y_{100} &= 1, \quad y_{200} = \frac{1}{\sqrt{2}}[z_1z_4 - z_2z_3], \quad y_{211} = z_1z_3, \\
y_{210} &= \frac{1}{\sqrt{2}}[z_1z_4 + z_2z_3], \quad y_{21-1} = z_2z_4
\end{aligned}$$

we find

$$\begin{aligned}
\Phi_{S^3}(\mathbf{x},\mathbf{z}) &= \sqrt{2}\,[z_1z_4(\cos(\lambda) - \imath\cos(\theta)\sin(\lambda)) - z_2z_3(\cos(\lambda) + \imath\cos(\theta)\sin(\lambda)) \\
&\quad + \imath z_1z_3\sin(\theta)\sin(\lambda)\exp(-\imath\phi) - \imath z_2z_4\sin(\theta)\sin(\lambda)\exp(\imath\phi)]
\end{aligned}$$

Now consider $\lambda = 0$. Since $Y_{n\ell m}(\theta,\phi,\lambda = 0) = n\delta_{\ell 0}$ we get

$$ny_{n00}(\mathbf{z}) = c(n)\left(\sqrt{2}(z_1z_4 - z_2z_3)\right)^{n-1}$$

By taking the norm in both sides of last equation, we have

$$n = c(n)2^{(n-1)/2}\sqrt{n}(n-1)!$$

where we used $\|(z_1z_4 - z_2z_3)^{n-1}\| = \sqrt{n}(n-1)!$. Thus we conclude

$$c(n) = \frac{\sqrt{2n}}{(n-1)!}\frac{1}{2^{n/2}}$$

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