Stochastic phase field equations: existence and uniqueness

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Abstract

We consider a conservative system of stochastic PDE's, namely a one dimensional phase field model perturbed by an additive space–time white noise. We prove a global existence and uniqueness result in a space of continuous functions on $\mathbb{R}_+ \times \mathbb{R}$. This result is obtained by extending previous results of Doering [3] on the stochastic Allen–Cahn equation to the phase field model.

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1. Introduction and results

We consider a phase field system with additive stochastic noise in one space dimension, which is formally written as

$$
\begin{aligned}
&\partial_t m(t) = \frac{1}{2}\Delta m(t) - V'(m(t)) + \lambda h(t) + a\eta(t) \\
&\partial_t \left[h(t) + m(t)\right] = \frac{1}{2}\Delta h(t)
\end{aligned}
\tag{1.1}
$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, $m(t) = m(t,x)$, $h(t) = h(t,x)$ are two scalar random fields, λ is a positive parameter, Δ is the Laplacian on $\mathbb{R}$, $V(m) = m^4/4 - m^2/2$ is a double well potential, $a = a(x) \in C_b(\mathbb{R})$, i.e. a bounded continuous function, and $\eta(t) = \eta(t,x)$ is a space–time white noise, i.e. $\mathbb{E}\left(\eta(t,x)\,\eta(s,y)\right) = \delta(t-s)\,\delta(x-y)$. In particular a translationally covariant noise is obtained for $a = 1$.

The deterministic system obtained by setting $a = 0$ in (1.1), usually referred to as *phase field* equations, describes the kinetic of phase segregation when the presence of the *latent heat* is taken into account. The first equation describes in fact the evolution of the order parameter m which is coupled to the external field h (which can be thought as the excess temperature measured from the melting temperature); h is however itself a dynamic variable which diffuses and, via the coupling λ, introduces a feedback into m whose effect is to slow down the phase segregation process. We stress that $q = m + h$ is locally conserved as apparent from the second equation in (1.1). Scaling limit of the deterministic phase field equations as $\lambda \to 0$ have been considered in [2] and [6].

Due to the phenomenological character of the phase field model, the introduction of a random forcing term appears natural and makes possible to discuss statistical properties of solutions. We also mention that the stochastic system (1.1) is very similar to the so–called, in the physical literature on critical phenomena, model C of Hohenberg and Halperin [5]. The specific choice of an additive white noise only in the first equation has been done to keep the model as simple as possible: the random forcing term is non–conservative, whereas the conservation law is still linear and not perturbed. Moreover, for λ small, m and h are weakly coupled so that we may refer to (1.1) as a *weakly conservative system*. This simplifying feature might help in developing a mathematical theory for phase segregation in conservative models. Indeed, in the companion paper [1], front fluctuations for (1.1) are analyzed in a suitable scaling limit as $\lambda \to 0$ and $a = O(\lambda)$. The need of an existence and uniqueness result for the stochastic system (1.1) in [1] is the main motivation for the present paper.

Referring to [1] for a more exhaustive discussion on the stochastic phase field equations, we next state precisely our results. For $\alpha > 0$ and $\gamma \in (0,1]$, let us define the following norms on $C(\mathbb{R})$:

$$
\begin{aligned}
\|\varphi\|_{C_\alpha(\mathbb{R})} &:= \sup_{x\in\mathbb{R}} e^{-\alpha|x|}\,|\varphi(x)| \\
\|\varphi\|_{C_\alpha^\gamma(\mathbb{R})} &:= \|\varphi\|_{C_\alpha(\mathbb{R})} + \sup_{x\neq y} e^{-\alpha(|x|+|y|)}\,\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\gamma}
\end{aligned}
$$

and also the following ones on $C(\mathbb{R}_+ \times \mathbb{R})$

$$\begin{aligned} \|f\|_{C_\alpha(\mathbb{R}_+\times\mathbb{R})} &:= \sup_{t\in\mathbb{R}_+} e^{-\alpha^2 t/2} \|f(t)\|_{C_\alpha(\mathbb{R})} \\ \|f\|_{C^\gamma_\alpha(\mathbb{R}_+\times\mathbb{R})} &:= \sup_{t\in\mathbb{R}_+} e^{-\alpha^2 t/2} \|f(t)\|_{C^\gamma_\alpha(\mathbb{R})} \end{aligned}$$

we shall denote by $C_\alpha(\mathbb{R}), \dots,$ the corresponding Banach spaces.

Let $p_t = e^{t\Delta/2}$ be the heat semigroup, namely the integral operator with kernel $p_t(x,y) = (2\pi t)^{-1/2} \exp\{-(x-y)^2/2t\}$. We introduce the process $Z(t) = Z(t,x)$ given by $Z(t) = \int_0^t ds\, p_{t-s}[a\eta(s)]$. Then Z is the mean zero Gaussian process with covariance

$$\mathbb{E}\left(Z(t,x)Z(s,y)\right) = \Gamma(t,s;x,y) := \int_0^{t\wedge s} du \int dz\, p_{t-u}(x-z)p_{s-u}(y-z)a(z)^2 \tag{1.2}$$

where $t \wedge s := \min\{t,s\}$. In the next lemma we state some properties of the paths of Z, which follow, by standard Gaussian arguments, from the properties of the covariance Γ, see e.g. [3] and references therein.

Lemma 1.1. *For each $\alpha > 0$ and $\gamma \in (0,1)$ we have that $Z \in C^\gamma_\alpha(\mathbb{R}_+ \times \mathbb{R})$, $\mathbb{P}$–a.s.*

We shall denote by $\mathcal{F}_t$ the filtration given by $\mathcal{F}_t := \sigma\{Z(s,x);\ (s,x) \in [0,t] \times \mathbb{R}\}$. In the following we consider a fixed realization of $Z \in C^\gamma_\alpha(\mathbb{R}_+ \times \mathbb{R})$.

Let $q := m + h$ and $U'(m) := V'(m) - \lambda m$; we formulate the problem (1.1) in the integral form

$$\begin{aligned} m(t) &= p_t m(0) + \int_0^t ds\, p_{t-s}\left[-U'(m(s)) + \lambda q(s)\right] + Z(t) \\ q(t) &= p_t q(0) - \frac{1}{2}\int_0^t ds\, \Delta p_{t-s} m(s) \end{aligned} \tag{1.3}$$

note that the integral on the r.h.s. of the second equation is well defined (the result is a continuous function) provided $x \mapsto m(s,x)$ is Hölder continuous.

Our main result is the following existence and uniqueness result for the system (1.3) on the space of Hölder continuous functions.

Theorem 1.2. *Let $m(0), q(0) \in C^\gamma_\alpha(\mathbb{R})$ for any $\alpha > 0$ and $\gamma \in (0,1)$. Then there exists a unique $\mathcal{F}_t$–adapted process $(m,q) \in C^\gamma_\alpha(\mathbb{R}_+ \times \mathbb{R}) \times C^\gamma_\alpha(\mathbb{R}_+ \times \mathbb{R})$ for any $\alpha > 0$ and $\gamma \in (0,1)$ which solves (1.3).*

We introduce the linear operators G, $\tilde{G}$ on $C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ given by

$$\begin{aligned} G[F]\,(t) &:= \int_0^t ds\, p_{t-s}F(s) \\ \tilde{G}[F]\,(t) &:= \int_0^t ds\, (t-s)\frac{1}{2}\Delta p_{t-s}F(s) \end{aligned} \tag{1.4}$$

and set $R(t) := p_t m(0) + \lambda t\, p_t q(0) + Z(t)$. By plugging the second equation in (1.3) into the first one we get that m solves the problem

$$m = R - G[U'(m)] - \lambda\, \tilde{G}[m] \tag{1.5}$$

By standard estimates on the heat kernel, it is easy to verify that for each $\alpha > 0$ and $\gamma \in (0,1]$ there exists $\alpha' > 0$ and a constant $C = C(\alpha, \gamma)$ such that

$$\|Gf\|_{C_\alpha^\gamma(\mathbb{R}_+\times\mathbb{R})} \le C\|f\|_{C_{\alpha'}(\mathbb{R}_+\times\mathbb{R})}$$
$$\|\tilde{G}f\|_{C_\alpha^\gamma(\mathbb{R}_+\times\mathbb{R})} \le C\|f\|_{C_{\alpha'}(\mathbb{R}_+\times\mathbb{R})}$$

Furthermore, for each $\alpha > 0$ and $\gamma \in (0,1)$, there exists $\alpha' > 0$, $\gamma' \in (0,1)$ and a constant $C = C(\alpha, \gamma)$ such that

$$\left\| \int_0^t ds\, \Delta p_{t-s} f(s) \right\|_{C_\alpha^\gamma(\mathbb{R}_+\times\mathbb{R})} \le C\|f\|_{C_{\alpha'}^{\gamma'}(\mathbb{R}_+\times\mathbb{R})}$$

Therefore Theorem 1.2 can be easily deduced from the following existence and uniqueness result for the problem (1.5) on the space of continuous functions.

Theorem 1.3. *Let $m(0), q(0) \in C_\alpha(\mathbb{R})$ for any $\alpha > 0$. Then there exists a unique $\mathcal{F}_t$–adapted process $m \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ for any $\alpha > 0$ which solves (1.5).*

In the rest of the paper we prove Theorem 1.3 by adapting the techniques developed to treat the (one space dimensional) stochastic Allen–Cahn equation, i.e. the first equation in (1.1) with $\lambda = 0$. We shall follow closely the proof in [3] referring to that paper for some technical Lemmata; see also [4] for the analogous result in a bounded domain. The main difficulties of the phase field model are due to the non local term (although linear) λh in (1.1). Since we cannot estimate the L^p norm of h by the L^p norm of m, the necessary *a priori* bounds (needed both for the existence and the uniqueness) are not a straightforward extension of the analogous ones for the stochastic Allen–Cahn equation. To overcame this problem we shall bound an appropriate negative Sobolev norm of h in terms of the L^p norm of m, see Lemmata 2.4 and 2.5 below.

We finally remark that the quartic double well potential, $V(m) = m^4/4 - m^2/2$ has been chosen only for notation simplicity; the proof works for any polynomial of even degree with positive leading coefficient.

Since the parameter λ will be kept fixed throughout all the paper we omit to indicate the dependence on it. We shall denote by C a generic positive constant whose numerical value may change from line to line.

2. Finite volume approximations

Let $C_K^\infty(\mathbb{R})$ be the space of infinitely differentiable functions on $\mathbb{R}$ with compact support and introduce $\mathcal{L} = \{\Lambda \in C_K^\infty(\mathbb{R}) \,:\, 0 \le \Lambda \le 1\}$. For $\Lambda \in \mathcal{L}$, we introduce the following finite volume approximations of the problem (1.5)

$$m_\Lambda = \Lambda R - G[\Lambda U'(m_\Lambda)] - \lambda\, \tilde{G}[\Lambda m_\Lambda] \tag{2.1}$$

In this section we establish a global existence result for (2.1) together with some bounds uniform for $\Lambda \in \mathcal{L}$. To this end we need to introduce some more notation.

For $\alpha > 0$, let $\varrho(x) := e^{-\alpha|x|}$. We shall denote by ϱ also the measure $\varrho(x)dx$ on $\mathbb{R}$. We introduce the following finite measures on $\mathbb{R}_+ \times \mathbb{R}$, omitting the dependence on $\alpha > 0$ from the notation

$$\begin{aligned}
\mu(dt, dx) &:= e^{-\alpha^2 t/2} \varrho(x)\, dx\, dt \\
\mu_T(dt, dx) &:= \chi_{[0,T]}(t)\, \mu(dt, dx) \\
\mu_\Lambda(dt, dx) &:= \Lambda(x)\, \mu(dt, dx) \\
\mu_{T,\Lambda}(dt, dx) &:= \chi_{[0,T]}(t)\, \Lambda(x)\, \mu(dt, dx)
\end{aligned}$$

where $\chi_{[0,T]}$ denotes the characteristic function of $[0, T]$. For ν a measure and f a function use the notation $\nu(f) = \int d\nu\, f$.

For $p \in (1, \infty)$, we introduce the Sobolev space $H_1^p(\varrho)$ obtained by completing $C_K^\infty(\mathbb{R})$ with respect to the norm

$$\|\varphi\|^p_{H_1^p(\varrho)} := \|\nabla\varphi\|^p_{L^p(\varrho)} + \|\varphi\|^p_{L^p(\varrho)}$$

where ∇ denotes the derivative with respect to x. Since $H_1^p(\varrho) \subset L^p(\varrho)$ for $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$ we introduce the dual space $H_{-1}^q(\varrho)$ by completing $L^q(\varrho)$ with respect to the norm

$$\|\ell\|_{H_{-1}^q(\varrho)} := \sup_{\varphi\,:\, \|\varphi\|_{H_1^p(\varrho)} \le 1} \varrho(\ell\, \varphi)$$

For $f = f(t, x)$ we shall use the notation

$$\|f\|^p_{H_{-1}^p(\mu_T)} := \int_0^T dt\, e^{-\alpha^2 t/2}\, \|f(t)\|^p_{H_{-1}^p(\varrho)}$$

omitting to write T on the l.h.s. if $T = \infty$.

From Lemma 1.1 it follows that for each $\alpha > 0$ and $p \in [1, \infty)$ we have $Z \in L^p(\mu)$. Furthermore for each $T > 0$ and $\Lambda \in \mathcal{L}$ we have $\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} |\Lambda(x) Z(t, x)| < \infty$. Moreover, by the assumptions on the initial data, the same holds for $R(t) = p_t m(0) + \lambda t\, p_t q(0) + Z(t)$.

We next state a local existence result for the problem (2.1). Since $\tilde{G}$ has a kernel which can be estimated as the one of G, the proof of the next result, which is based on Picard iterations, is the same as [3, Prop. 1] and we omit it.

Lemma 2.1. *For each $\Lambda \in \mathcal{L}$ and each realization of Z in $C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ there is a time $T_0 > 0$ such that there exists a unique $\mathcal{F}_t$–adapted continuous and bounded solution of (2.1) on $[0, T_0] \times \mathbb{R}$.*

To show the existence of a global solution, we prove that if m_Λ is a continuous solution of (2.1) on $[0, T^*) \times \mathbb{R}$ then

$$\sup_{t\in[0,T^*)} \sup_{x\in\mathbb{R}} |m_\Lambda(t, x)| < \infty \tag{2.2}$$

A standard argument permits then to conclude. The key ingredient for proving (2.2) is the following *a priori* bound on the $L^p(\mu_{T^*,\Lambda})$ norm of solutions.

Proposition 2.2. *Let m_Λ be a continuous solution of (2.1) on $[0,T^*)\times\mathbb{R}$. For each $\alpha>0$ and $p\in[1,\infty)$ there exists a constant $C=C\left(\alpha,p,\|R\|_{L^p(\mu)}\right)<\infty$, independent of $T^*>0$ and $\Lambda\in\mathcal{L}$, such that*

$$\|m_\Lambda\|_{L^p(\mu_{T^*,\Lambda})}\le C \tag{2.3}$$

The proof of the proposition is split in several Lemmata, the first one, which is proven integrating by parts, is [3, Lemma 7].

Lemma 2.3. *If both $f(t,x)$ and $(\partial_t-\frac{1}{2}\Delta)f$ are continuous on $(0,T)\times\mathbb{R}$, $f(0)=0$ and $|f|^{2n+2}$ and $|\nabla f|^{2n+2}$ are in $L^1(\mu_T)$, then, for any $n=0,1,\ldots$,*

$$\mu_T\left(f^{2n+1}\big(\partial_t-\frac{1}{2}\Delta\big)f\right)\ge(2n+1)\mu_T\left(f^{2n}|\nabla f|^2\right) \tag{2.4}$$

Let m_Λ be a continuous solution of (2.1). We define

$$u_\Lambda:=m_\Lambda-\Lambda R,\qquad\qquad q_\Lambda:=\big(\partial_t-\frac{1}{2}\Delta\big)\tilde{G}[\Lambda m_\Lambda] \tag{2.5}$$

From (2.1), if m_Λ is continuous and bounded on $[0,T]\times\mathbb{R}$, then $u_\Lambda\in C^\infty((0,T]\times\mathbb{R})$ by the regularizing properties of G and $\tilde{G}$. Moreover, for $t\in[0,T]$, $u_\Lambda(t,x)$ (together with its derivative) is exponentially decaying as $x\to\infty$.

Lemma 2.4. *Let m_Λ be a continuous solution of (2.1) on $[0,T^*)\times\mathbb{R}$; then for each $n=0,1,\ldots$ and $\beta>0$ there exists a constant $C=C(n,\beta)$, independent of $T^*>0$ and $\Lambda\in\mathcal{L}$, such that*

$$\mu_{T^*,\Lambda}\left(e^{-(2n+2)\beta t}u_\Lambda^{2n+1}U'(m_\Lambda)\right)\le C\|e^{-\beta t}q_\Lambda\|^{2n+2}_{H^{2n+2}_{-1}(\mu_{T^*})} \tag{2.6}$$

Proof. We apply Lemma 2.3 with $f=e^{-\beta t}u_\Lambda$ and $T<T^*$. By using equation (2.1), we get

$$\big(\partial_t-\frac{1}{2}\Delta\big)e^{-\beta t}u_\Lambda=-\beta e^{-\beta t}u_\Lambda-e^{-\beta t}\left[\Lambda U'(m_\Lambda)+\lambda q_\Lambda\right]$$

hence, by (2.4)

$$\begin{aligned}&\mu_T\left(e^{-(2n+2)\beta t}\left[(2n+1)u_\Lambda^{2n}(\nabla u_\Lambda)^2+\beta u_\Lambda^{2n+2}+u_\Lambda^{2n+1}\Lambda U'(m_\Lambda)\right]\right)\\&\qquad\le-\lambda\mu_T\left(e^{-(2n+2)\beta t}u_\Lambda^{2n+1}q_\Lambda\right)\end{aligned} \tag{2.7}$$

Let $p,q\in(1,\infty)$ such that $p^{-1}+q^{-1}=1$; then for each $\gamma>0$ there exists a constant $C=C(\gamma,p)$ such that for any $a,b\in\mathbb{R}$

$$|a\,b|\le\gamma|a|^p+C|b|^q \tag{2.8}$$

therefore, by the duality between $H^p_1(\varrho)$ and $H^q_{-1}(\varrho)$, we have

$$|\varrho(fg)| \le \gamma \|f\|^p_{H^p_1(\varrho)} + C\|g\|^q_{H^q_{-1}(\varrho)} \tag{2.9}$$

By applying (2.9) we get

$$\begin{aligned}
\left|\mu_T\left(e^{-(2n+2)\beta t} u_\Lambda^{2n+1} q_\Lambda\right)\right| &\le \int_0^T dt\, e^{-\alpha^2 t/2-(2n+2)\beta t} \left|\varrho\left(u_\Lambda^{2n+1}(t) q_\Lambda(t)\right)\right| \\
&\le \int_0^T dt\, e^{-\alpha^2 t/2-(2n+2)\beta t} \Big[\gamma(2n+1)^{\frac{2n+2}{2n+1}} \varrho\left(u_\Lambda(t)^{2n\frac{2n+2}{2n+1}} \left(\nabla u_\Lambda(t)\right)^{\frac{2n+2}{2n+1}}\right) \\
&\qquad + \gamma\varrho\left(u_\Lambda^{2n+2}(t)\right) + C\|q_\Lambda(t)\|^{2n+2}_{H^{2n+2}_{-1}(\varrho)}\Big] \\
&\le \int_0^T dt\, e^{-\alpha^2 t/2-(2n+2)\beta t} \Big[\gamma(2n+1)^{\frac{2n+2}{2n+1}} c_1(n) \varrho\left(u_\Lambda(t)^{2n}\left(\nabla u_\Lambda(t)\right)^2 + u_\Lambda(t)^{2n+2}\right) \\
&\qquad + \gamma\varrho\left(u_\Lambda^{2n+2}(t)\right) + C\|q_\Lambda(t)\|^{2n+2}_{H^{2n+2}_{-1}(\varrho)}\Big] \\
&= \gamma(2n+1)^{\frac{2n+2}{2n+1}} c_1(n) \mu_T\left(e^{-(2n+2)\beta t} u_\Lambda^{2n}\left(\nabla u_\Lambda\right)^2\right) \\
&\quad + \gamma\left[(2n+1)^{\frac{2n+2}{2n+1}} c_1(n) + 1\right] \mu_T\left(e^{-(2n+2)\beta t} u_\Lambda^{2n+2}\right) + C\|e^{-(2n+2)\beta t} q_\Lambda\|^{2n+2}_{H^{2n+2}_{-1}(\mu_T)}
\end{aligned} \tag{2.10}$$

where we used, in the third step, Hölder inequality in the form (as follows from (2.8))

$$\varrho(fg) \le c_1(n)\left[\varrho\left(|f|^{\frac{2n+1}{n+1}}\right) + \varrho\left(|g|^{\frac{2n+1}{n}}\right)\right]$$

with $f = [u_\Lambda(t)^n \nabla u_\Lambda(t)]^{\frac{2n+2}{2n+1}}$ and $g = u_\Lambda(t)^{n\frac{2n+2}{2n+1}}$.

Choosing $\gamma = \gamma(n, \lambda, \beta)$ small enough and taking $T \uparrow T^*$ the lemma now follows from (2.7) and (2.10). $\square$

Lemma 2.5. *For each $p \in (1, \infty)$ there exists a constant $C = C(p)$ such that, for any $\alpha, \beta, T > 0$ and $F \in L^p(\mu_T)$,*

$$\left\| e^{-\beta t}\left(\partial_t - \frac{1}{2}\Delta\right)\tilde{G}[F] \right\|_{H^p_{-1}(\mu_T)} \le C\left(1 + \frac{1}{\alpha}\right)\left\|e^{-\beta t} F\right\|_{L^p(\mu_T)} \tag{2.11}$$

Proof. We can write

$$\tilde{G}[F](t) = \int_0^t ds\, p_{t-s} \int_0^s ds'\, \frac{1}{2}\Delta p_{s-s'} F(s') \tag{2.12}$$

so that (in distribution sense)

$$\left(\partial_t - \frac{1}{2}\Delta\right)\tilde{G}[F]\,(t) = \int_0^t ds\, \frac{1}{2}\Delta p_{t-s} F(s) \tag{2.13}$$

We thus get

$$\left\| \left(\partial_t - \frac{1}{2}\Delta\right)\tilde{G}[F]\,(t) \right\|_{H^p_{-1}(\varrho)} \leq \frac{1}{2}\int_0^t ds\ \|\Delta p_{t-s}F(s)\|_{H^p_{-1}(\varrho)} \tag{2.14}$$

For $q^{-1}+p^{-1}=1$ and $\varphi \in C_K^\infty(\mathbb{R})$ we now have

$$\begin{aligned}(\varphi, \Delta p_{t-s}F(s))_\varrho &= (\varrho\varphi, \Delta p_{t-s}F(s))_{L^2(dx)} \\ &= -\left(\varrho\left[\nabla\varphi + \frac{\nabla\varrho}{\varrho}\right], \nabla p_{t-s}F(s)\right)_{L^2(dx)} \\ &\leq \left[\|\nabla\varphi\|_{L^q(\varrho)} + \|\varphi\|_{L^q(\varrho)}\sup_{x\in\mathbb{R}}|\nabla \log\varrho(x)|\right]\|\nabla p_{t-s}F(s)\|_{L^p(\varrho)} \end{aligned} \tag{2.15}$$

since $\sup_{x\in\mathbb{R}}|\nabla\log\varrho(x)| = \alpha$ we conclude

$$\|\Delta p_{t-s}F(s)\|_{H^p_{-1}(\varrho)} \leq (1+\alpha)\,\|\nabla p_{t-s}F(s)\|_{L^p(\varrho)} \tag{2.16}$$

As $\exp\{-\alpha|x|\} \leq \exp\{-\alpha|y|\}\exp\{\alpha|x-y|\}$,

$$|\nabla p_{t-s}F(s)\,(x)|\,e^{-\alpha|x|/p} \leq \int dy\, e^{-\alpha|y|/p}|F(s,y)|\,|\nabla p_{t-s}(x-y)|\,e^{\alpha|x-y|/p}, \tag{2.17}$$

by applying Young's inequality for convolutions we then obtain

$$\|\nabla p_{t-s}F(s)\|_{L^p(\varrho)} \leq \|F(s)\|_{L^p(\varrho)}\int dx\ |\nabla p_{t-s}(x)|\,e^{\alpha|x|/p} \tag{2.18}$$

We define

$$h_{\alpha,p}(t) := \int dx\ |\nabla p_t(x)|\ e^{\alpha|x|/p} \leq \frac{2}{\sqrt{2\,\pi\,t}} + \frac{2\alpha}{p}\exp\left\{\frac{\alpha^2 t}{2p^2}\right\} \tag{2.19}$$

Then, from (2.14), (2.16) and (2.18),

$$\begin{aligned}&\chi_{[0,T]}(t)\,e^{-\frac{\alpha^2 t}{2p}-\beta t}\left\|\left(\partial_t - \frac{1}{2}\Delta\right)\tilde{G}[F]\,(t)\right\|_{H^p_{-1}(\varrho)} \\ &\quad\leq \frac{1+\alpha}{2}\int ds\,\chi_{[0,T]}(s)e^{-\frac{\alpha^2 s}{2p}-\beta s}\,\|F(s)\|_{L^p(\varrho)}\,\chi_{[0,T]}(t-s)e^{-\frac{\alpha^2(t-s)}{2p}-\beta(t-s)}h_{\alpha,p}(t-s)\end{aligned}$$

Again by Young's inequality for convolutions we get

$$\begin{aligned}\left\|e^{-\beta t}\left(\partial_t - \frac{1}{2}\Delta\right)\tilde{G}[F]\right\|_{H^p_{-1}(\mu_T)} &= \left\{\int_0^T dt\, e^{-\frac{\alpha^2 t}{2}-\beta p t}\left\|\left(\partial_t - \frac{1}{2}\Delta\right)\tilde{G}[F]\,(t)\right\|^p_{H^p_{-1}(\varrho)}\right\}^{1/p} \\ &\leq \frac{1+\alpha}{2}\left\{\int_0^T dt\, e^{-\frac{\alpha^2 t}{2p}}h_{\alpha,p}(t)\right\}\left\{\int_0^T dt\, e^{-\frac{\alpha^2 t}{2}-\beta p t}\,\|F(t)\|^p_{L^p(\varrho)}\right\}^{1/p}\end{aligned}$$

By using the estimate in (2.19) and recalling $p > 1$, it is easy to show there is a constant $C = C(p) > 0$ so that

$$\frac{1+\alpha}{2} \int_0^T dt\, e^{-\frac{\alpha^2 t}{2p}} h_{\alpha,p}(t) \le C\left(1 + \frac{1}{\alpha}\right)$$

The lemma is proved. □

Proof of Proposition 2.2. Recalling (2.5) and that $U'(m) = m^3 - (1-\lambda)m$, by expanding the l.h.s. of (2.6), using Lemma 2.5, and $\Lambda^2 \le \Lambda$, we get there exists a constant $C = C(\alpha, n, \beta)$ such that

$$\begin{aligned}\mu_{T^*,\Lambda}\left(e^{-(2n+2)\beta t} u_\Lambda^{2n+4}\right) \quad &\le \quad C\mu_{T^*,\Lambda}\Big(e^{-(2n+2)\beta t}\big\{|u_\Lambda|^{2n+1}\left[u_\Lambda^2|\Lambda R| + |u_\Lambda|(\Lambda R)^2 + |\Lambda R|^3\right] \\ &\qquad + u_\Lambda^{2n+2} + |u_\Lambda|^{2n+1}|\Lambda R| + |\Lambda R|^{2n+2}\big\}\Big) \qquad (2.20)\end{aligned}$$

Let

$$M := \left(\mu_{T^*,\Lambda}\left(e^{-(2n+2)\beta t} u_\Lambda^{2n+4}\right)\right)^{1/(2n+4)}$$

by using repeatedly Hölder inequality in (2.20) we get

$$\begin{aligned}M^{2n+4} \quad \le \quad & C\Big\{M^{2n+3}\|\Lambda R\|_{L^{2n+4}(\mu)} + M^{2n+2}\left[1 + \|\Lambda R\|^2_{L^{2n+4}(\mu)}\right] \\ & + M^{2n+1}\|\Lambda R\|_{L^{2n+4}(\mu)}\left[1 + \|\Lambda R\|^2_{L^{2n+4}(\mu)}\right] + \|\Lambda R\|^{2n+2}_{L^{2n+4}(\mu)}\Big\}\end{aligned}$$

we then conclude that M is bounded by some constant $C = C\left(\alpha, n, \beta, \|\Lambda R\|_{L^{2n+4}(\mu)}\right)$.

Recalling $m_\Lambda = u_\Lambda + \Lambda R$, by using triangular and Cauchy–Schwartz inequalities, we have

$$\|m_\Lambda\|_{L^p(\mu_{\Lambda,T^*})} \le \|\Lambda R\|_{L^p(\mu_{\Lambda,T^*})} + \|e^{\beta t}\|_{L^{2p}(\mu_{\Lambda,T^*})}\, \|e^{-\beta t}u_\Lambda\|_{L^{2p}(\mu_{\Lambda,T^*})}$$

Since, for each $p \in [1,\infty)$, $\|\Lambda R\|_{L^p(\mu_{\Lambda,T^*})} \le \|R\|_{L^p(\mu)} < \infty$, by choosing $\beta = \beta(\alpha, p)$ small enough and $n = n(p)$ large enough, the proposition follows. □

Proof of (2.2). Let $Y^\star := \sup\{|x| : \Lambda(x) > 0\}$; writing explicitly the kernels in the integral equation (2.1) we have, for $t < T^*$,

$$\begin{aligned}|m_\Lambda(t,x)| \le\, & |\Lambda(x)\, R(t,x)| + e^{\frac{\alpha^2}{4}T^* + \frac{\alpha}{2}Y^\star} \\ & \times \Big\{\int ds \int dy\, \chi_{[0,T^*)}(t-s)\, p_{t-s}(x-y)\chi_{[0,T^*)}(s)\, |U'(m_\Lambda(s,y))|\, \Lambda(y) e^{-\frac{\alpha^2}{4}s - \frac{\alpha}{2}|y|} \\ & \quad + \lambda \int ds \int dy\, \chi_{[0,T^*)}(t-s)\, \frac{1}{2}\, |\Delta p_{t-s}(x-y)|\, \chi_{[0,T^*)}(s)\Lambda(y)\, |m_\Lambda(s,y)|\, e^{-\frac{\alpha^2}{4}s - \frac{\alpha}{2}|y|}\Big\}\end{aligned}$$

Then, from Young's inequalities for convolutions,

$$\begin{aligned}\sup_{t\in[0.T^*)}\sup_{x\in\mathbb{R}} |m_\Lambda(t,x)| \le & \sup_{t\in[0,T^*)}\sup_{x\in\mathbb{R}} |\Lambda(x) R(t,x)| \\ & + e^{\frac{\alpha^2}{4}T^* + \frac{\alpha}{2}Y^\star} \times \Bigg\{\left[\int_0^{T^*} dt \int dz\, p_t(z)^2\right]^{\frac{1}{2}} \left[\int \mu_{T^*,\Lambda}(dt,dx)\, U'\left(m_\Lambda(t,y)\right)^2\right]^{\frac{1}{2}} \\ & + \lambda\left[\int_0^{T^*} dt \int dz\, p_t(z)^2 \left[\frac{z^2}{2t} + \frac{1}{2}\right]^2\right]^{\frac{1}{2}} \left[\int \mu_{T^*,\Lambda}(dt,dx)\, m_\Lambda(t,y)^2\right]^{\frac{1}{2}}\Bigg\} \qquad (2.21)\end{aligned}$$

where we used also that $\Lambda^2 \leq \Lambda$. By using that, as follows from Proposition 2.2, $U'(m_\Lambda)$ is in $L^2(\mu_{T^*,\Lambda})$, and the properties of the heat kernel, it is easy to see that the r.h.s. of (2.21) is bounded. $\square$

The following lemma states an *a priori* uniform bound for the solutions of (2.1), which by (2.2) are defined for any $t \geq 0$.

Lemma 2.6. *Let m_Λ be a continuous solution of (2.1) on $\mathbb{R}_+ \times \mathbb{R}$. Then for each $\alpha > 0$, $p \in [1, \infty)$ there exists a constant $C = C\left(\alpha, p, \|R\|_{L^p(\mu)}\right) < \infty$ independent of $\Lambda \in \mathcal{L}$ such that $\|m_\Lambda\|_{L^p(\mu)} \leq C$.*

Proof. Since the continuous solution of equation (2.1) exist globally in time, the inequality (2.3) may be extended for $T^* \uparrow \infty$; we get

$$\|\Lambda m_\Lambda\|_{L^p(\mu)} \leq \|m_\Lambda\|_{L^p(\mu_\Lambda)} \leq C < \infty \tag{2.22}$$

Note that for $p > 1$ both G and $\tilde{G}$ are bounded operators in $L^p(\mu)$ (see [3, Lemma 9] for G and the same proof also works for $\tilde{G}$). From the integral equation (2.1), the bound (2.22), and $R \in L^p(\mu)$ the lemma follows. $\square$

3. Infinite volume equation

In this section we conclude the proof of Theorem 1.3 by removing the truncation Λ in (2.1); following [3], we prove first that, provided β is chosen large enough, $\exp(-\beta t)\,\Lambda\, m_\Lambda$ converges in $L^p(\mu)$.

Lemma 3.1. *Let m_Λ be a solution of (2.1). For each $p \in [1, \infty)$ there exists a positive constant $k_p < \infty$ such that, for any $\alpha > 0$ and $\beta \geq k_p(1+\alpha^{-p})$, $\{\exp(-\beta t)\,\Lambda\, m_\Lambda\,,\ \Lambda \in \mathcal{L}\}$ is Cauchy in $L^p(\mu)$ as $\Lambda \uparrow 1$.*

Proof. Recall that u_Λ and q_Λ have been defined in (2.5). We apply Lemma 2.3 to the function $f = e^{-\beta t}[u_\Lambda - u_{\bar{\Lambda}}]$. We have

$$\left(\partial_t - \frac{1}{2}\Delta\right) f = -\beta f - e^{-\beta t}\left[\Lambda U'(m_\Lambda) - \bar{\Lambda} U'(m_{\bar{\Lambda}})\right] - \lambda e^{-\beta t}\left[q_\Lambda - q_{\bar{\Lambda}}\right]$$

Therefore, by the same computations as in Lemma 2.4 (see equations (2.7) and (2.10)), but now choosing $\gamma = \gamma(n, \lambda)$ independent of β and taking $T \uparrow \infty$, there exists a constant $c_1 = c_1(n)$ independent of α, β such that

$$\begin{aligned}\mu\left(e^{-(2n+2)\beta t}\left[u_\Lambda - u_{\bar{\Lambda}}\right]^{2n+1}\left[\Lambda U'(m_\Lambda) - \bar{\Lambda} U'(m_{\bar{\Lambda}})\right]\right)& \\ + (\beta - 1)\mu\left(e^{-(2n+2)\beta t}\left[u_\Lambda - u_{\bar{\Lambda}}\right]^{2n+2}\right) \leq c_1 \|e^{-\beta t}\left(q_\Lambda - q_{\bar{\Lambda}}\right)\|^{2n+2}_{H^{2n+2}_{-1}(\mu)}&\end{aligned} \tag{3.1}$$

By using Lemma 2.5 there is a constant $c_2 = c_2(n)$ such that

$$\|e^{-\beta t}\left(q_\Lambda - q_{\bar{\Lambda}}\right)\|^{2n+2}_{H^{2n+2}_{-1}(\mu)} \leq c_2\left(1 + \alpha^{-(2n+2)}\right)\mu\left(e^{-(2n+2)\beta t}\left[\Lambda m_\Lambda - \bar{\Lambda} m_{\bar{\Lambda}}\right]^{2n+2}\right) \tag{3.2}$$

On the other hand, since for any $a,b \in \mathbb{R}$ we have $(a^3-b^3)(a-b) \ge (a-b)^4/4$, there exists a constant $c_3 = c_3(\lambda)$ such that

$$\begin{aligned}
&(u_\Lambda - u_{\bar\Lambda})\left[\Lambda U'(m_\Lambda) - \bar\Lambda U'(m_{\bar\Lambda})\right] \\
&\quad = \left(m_\Lambda - m_{\bar\Lambda} - (\Lambda-\bar\Lambda)R\right)\left(\Lambda m_\Lambda^3 - \bar\Lambda m_{\bar\Lambda}^3\right) - (1-\lambda)\left(u_\Lambda - u_{\bar\Lambda}\right)\left(\Lambda m_\Lambda - \bar\Lambda m_{\bar\Lambda}\right) \\
&\quad \ge \frac{1}{4}\left(\Lambda m_\Lambda - \bar\Lambda m_{\bar\Lambda}\right)^4 - c_3(u_\Lambda - u_{\bar\Lambda})^2 \\
&\qquad - c_3\left(|1-\Lambda| + |1-\bar\Lambda|\right)\left[m_\Lambda^2(1+m_\Lambda^2) + m_{\bar\Lambda}^2(1+m_{\bar\Lambda}^2) + R^2(1+R^2)\right]
\end{aligned} \tag{3.3}$$

By plugging the bounds (3.2) and (3.3) into (3.1) and using Hölder inequality, we find there exists $c_4 = c_4(n,\alpha,\lambda)$ such that

$$\begin{aligned}
&\frac{1}{4}\mu\left(e^{-(2n+2)\beta t}\left(\Lambda m_\Lambda - \bar\Lambda m_{\bar\Lambda}\right)^{2n+4}\right) \\
&\qquad + \left(\beta - 1 - c_3 - c_1c_2\left(1+\alpha^{-(2n+2)}\right)\right)\mu\left(e^{-(2n+2)\beta t}\left(u_\Lambda - u_{\bar\Lambda}\right)^{2n+2}\right) \\
&\le c_4\left(\|1-\Lambda\|_{L^2(\mu)} + \|1-\bar\Lambda\|_{L^2(\mu)}\right)\left\{1 + \|m_\Lambda\|_{L^{4n+8}(\mu)}^{2n+4} + \|m_{\bar\Lambda}\|_{L^{4n+8}(\mu)}^{2n+4} + \|R\|_{L^{4n+8}(\mu)}^{2n+4}\right\}
\end{aligned}$$

Given $p \ge 2$, let $n = n(p) = [p/2]-1$ and k_p such that $1 + c_3 + c_1c_2(1+\alpha^{-(2n+2)}) \le k_p(1+\alpha^{-p})$ for any $\alpha > 0$. The lemma now follows from Lemma 2.6 and $R \in L^p(\mu)$ for any $p \in [1,\infty)$. $\square$

The proof of Theorem 1.3 can now be completed as in [3], we shall just sketch the argument.

Proof of Theorem 1.3. To prove existence of a continuous solution of (1.5), we first note that, for each $p \in [1,\infty)$ and α so large that $\alpha^2 > k_p(1+\alpha^{-p})$, m_Λ is Cauchy in $L^p(\mu)$. This follows from Lemmata 2.6, 3.1, and Hölder inequality; moreover the limit m satisfies equation (1.5). Since G and $\tilde G$ map $L^2(\mu)$ into $C_\alpha(\mathbb{R}_+ \times \mathbb{R})$, see [3, Lemma 12], we also have $m \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$.

In order to show $m \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ for any $\alpha > 0$, we note that, by Lemma 2.6, m_Λ is uniformly bounded in $L^p(\mu)$; we can thus find a weakly convergent subsequence $m_{\Lambda_k} \to m'$. On the other hand, by Lemma 3.1, $e^{-\beta t}\Lambda_k m_{\Lambda_k}$ converges strongly in $L^p(\mu)$ for $\beta \ge k_p(1+\alpha^{-p})$, hence $m = m'$ μ–a.s. Since $m' \in L^p(\mu)$ for any $\alpha > 0$ and $p \in [1,\infty)$, by the same argument as above, we get $m' \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ and $m = m'$.

To prove uniqueness, let m_1 and m_2 be two continuous solutions of (1.5). By applying Lemma 2.3 to the function $f = e^{-\beta t}[m_1 - m_2]$ and repeating the same computations as in Lemma 3.1 it is easy to show $m_1 = m_2$. $\square$

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