A Remark on the Dirichlet-Neumann Decoupling and the Integrated Density of States

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Abstract

We prove an estimate on the difference of the number of eigenvalues for Schrödinger operators with Dirichlet and Neumann boundary conditions in large boxes. In the proof, we use Krein's theory of the spectral shift function. As an application, we show that the integrated density of states is independent of the choice of boundary conditions.

1 Introduction

We consider a Schrödinger operator

$$H = (p - A(x))^2 + V(x) \qquad \text{on } L^2(\mathbb{R}^d)$$

with $d \geq 1$, where $p = -i\partial_x$ is the momentum operator, $A(x)$ is the vector potential and $V(x)$ is the scalar potential. We always assume $A(x)$ is an $\mathbb{R}^d$-valued C^1-class function, and $V(x)$ is real-valued. The magnetic field is given by

$$B_{ij}(x) = \partial_{x_i} A_j(x) - \partial_{x_j} A_i(x), \qquad i \neq j, \quad x \in \mathbb{R}^d.$$

For $L \in \mathbb{N}$, we set

$$\Omega_L = \bigl\{ x \in \mathbb{R}^d \bigm| 0 \leq x_j \leq L, j = 1, \dots, d \bigr\}.$$

We denote the Dirichlet and the Neumann Hamiltonian on Ω_L by H_L^D and H_L^N, respectively. Namely, H_L^D and H_L^N are the same differential operator as H, and the form domains are given by

$$\mathcal{Q}(H_L^D) = \mathcal{D}(|H_L^D|^{1/2}) = H_0^1(\Omega_L),$$
$$\mathcal{Q}(H_L^N) = \mathcal{D}(|H_L^D|^{1/2}) = H^1(\Omega_L).$$

In this paper, we suppose

Assumption A. $V(x)$ and $\{B_{ij}(x)\}$ are uniformly bounded.

Theorem 1. *Suppose Assumption A. Then for any* $f \in C_0^1(\mathbb{R})$*, there exists* $C > 0$ *such that*

$$\left|\mathrm{Tr}[f(H_L^N) - f(H_L^D)]\right| \le CL^{d-1}, \qquad L \in \mathbb{N}.$$

In particular,

$$\lim_{L\to\infty}\left[\frac{1}{L^d}\mathrm{Tr} f(H_L^N) - \frac{1}{L^d}\mathrm{Tr} f(H_L^D)\right] = 0.$$

Remark . Even though we suppose Ω_L is a box, we may consider more general domains. For example, it is easy to extend our result to the case Ω_L is a ball. In fact, the shape may be different for each L. The constant C in the theorem depends only on the constants appearing in Lemmas 7 and 10, and they are easily extended to domain with piecewise C^1-boundary. Roughly speaking, $\mathrm{Tr}(f(H_L^N) - f(H_L^D))$ is bounded by the area of $\partial\Omega_L$ in general, provided $\partial\Omega_L$ is not too wild.

Now we apply this result to the integrated density of states (IDS for short). We suppose

Assumption B. For almost all $\lambda \in \mathbb{R}$,

$$k^\flat(\lambda) = \lim_{L\to\infty}\frac{1}{L^d}\#\{\text{eigenvalues of } H_L^\flat \le \lambda\}$$

exists, where $\flat = D$ or N.

If H is an ergodic (random) operator, Assumption B is proved for almost all samples by the subadditive ergodic theorem (see [4] Section 7.3). Since $L^{-d}\#\{\text{e.v.'s of } H_L^\flat \le \lambda\}$ is uniformly bounded, it is easy to see that

$$\lim_{L\to\infty}\frac{1}{L^d}\mathrm{Tr} f(H_L^\flat) = -\int f'(\lambda)k^\flat(\lambda)d\lambda = \int f(\lambda)dk^\flat(\lambda),$$

where $\flat = D$ or N. Then the next corollary follows from Theorem 1 and the fundamental lemma of variations.

Corollary 2. *Suppose Assumptions A and B. Then* $k^D(\lambda) = k^N(\lambda)$ *for almost all* $\lambda \in \mathbb{R}$*. Moreover, if either* k^D *or* k^N *is continuous at* λ*, then* $k^D(\lambda) = k^N(\lambda)$*.*

Proof. It remains only to show the latter statement. Suppose k^D is continuous at λ_0 and $k^D(\lambda_0) \ne k^N(\lambda_0)$. Since $k^N(\lambda) \ge k^D(\lambda)$ for any λ, $k^N(\lambda_0) < k^D(\lambda_0)$. By the assumption, there is $\varepsilon > 0$ such that $k^D(\lambda) < k^N(\lambda_0)$ for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. On the other hand, $k^N(\lambda) \ge k^N(\lambda_0)$ for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$ since $k^N(\lambda)$ is monotone nondecreasing. Thus we have $k^N(\lambda) > k^D(\lambda)$ on $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, a contradiction. The other case is proved similarly. ☐

By the same argument, we can also prove that the integrated density of states is independent of the choice of boundary conditions. Let $H_L^{\min}$ be the operator defined by

$$H_L^{\min}\varphi = [(p-A)^2 + V]\varphi, \qquad \varphi \in \mathcal{D}(H_L^{\min}) = H_0^2(\Omega_L).$$

It is well-known that if H_L is a self-adjoint extension of $H_L^{\min}$, then $H_L^N \leq H_L \leq H_L^D$.

Corollary 3. *Suppose Assumption A, and let H_L be a self-adjoint extension of H_L^{min} for each L. Suppose*

$$k(\lambda) = \lim_{L\to\infty} \frac{1}{L^d} \#\{\text{eigenvalues of } H_L \leq \lambda\}$$

exists for almost all λ. Then $k^D(\lambda)$ and $k^N(\lambda)$ exist for almost all λ and $k^D(\lambda) = k^N(\lambda) = k(\lambda)$ (a.e. λ). Moreover, if $k(\lambda)$ is continuous at λ, then $k^D(\lambda) = k^N(\lambda) = k(\lambda)$.

It is widely believed that the IDS is independent of the boundary conditions, and often it is implicitly assumed. However, it is proved only for Schrödinger operators with ergodic potentials (without magnetic field) using stochastic method (cf. [4], [2]). It appears that operator theoretical proof has not been known, and our result can be also applied to deterministic Schrödinger operators, provided that the IDS can be defined in the sense of Corollary 3. This work is partly motivated by an application to Schrödinger operator with random magnetic field (cf. [9], [5], [6]). We note that a similar idea can be applied to discrete magnetic Schrödinger operators, and in fact it is much simpler for the discrete case. We refer Appendix of [5] for a proof of $k^D = k^N$ which employ this idea.

This paper is organized as follows: We give a proof of Theorem 1 in Section 2. A key estimate is Proposition 5, and its proof is given in Section 3. The proof of a technical lemma (Lemma 9) is given in Appendix A. As a simple (almost trivial) application of Corollary 3, we compute the IDS for the Landau Hamiltonian on $\mathbb{R}^2$ in Appendix B.

Notation: We denote the trace ideal of order r by $\mathfrak{I}_r$. The inner product of L^2-spaces is denoted by $\langle\varphi,\psi\rangle = \int \overline{\varphi}\psi dx$, and the norm $\|\cdot\|$ without any subscript is the L^2-norm. $\#A$ denotes the cardinality of a set A.

2 Proof of Theorem 1

At first, we recall the Krein's theory of the spectral shift function (cf. [1], [10]).

Theorem 4 (Krein). *Let A_1 and A_2 be self-adjoint operators such that $W = A_1 - A_2 \in \mathfrak{I}_1$ is a trace class operator. Then there exists an integrable function $\xi(\lambda)$ on $\mathbb{R}$ such that*

$$\mathrm{Tr}(f(A_1) - f(A_2)) = \int f'(\lambda)\xi(\lambda)d\lambda$$

for any $f \in C_0^1(\mathbb{R})$. Moreover,

$$\|\xi\|_{L^1(\mathbb{R})} \leq \|W\|_{\mathfrak{I}_1},$$

where $\|\cdot\|_{\mathfrak{I}_1}$ denotes the trace norm.

The function $\xi(\lambda)$ in the Theorem 4 is called the spectral shift function. We often denote $\xi(\lambda) = \xi(\lambda; A_1, A_2)$. Let

$$M \geq \sup(-V(x)) + 1,$$

and let $m = 2a + 1$ with $a > d/4$, $a \in \mathbb{N}$. We note that H_L^D, $H_L^N \geq -M + 1$, and we set

$$A_1 = -(H_L^D + M)^{-m}, \qquad A_2 = -(H_L^N + M)^{-m} \quad \text{on } L^2(\Omega_L).$$

We prove the next proposition in Section 3.

Proposition 5. *$A_1 - A_2$ is a trace class operator. Moreover, there exists $C_1 > 0$ such that*

$$\|A_1 - A_2\|_{\mathfrak{I}_1} \leq C_1 L^{d-1}, \qquad L \in \mathbb{N}.$$

Combining Theorem 4 with Proposition 5, we now prove Theorem 1.

Proof of Theorem 1. We may suppose $f \in C_0^1((-M + 1/2, \infty))$. Then there exists $g \in C_0^1((-1, 0])$ such that

$$f(\lambda) = g(-(\lambda + M)^{-m}),$$

and hence

$$f(H_L^D) = g(A_1), \qquad f(H_L^N) = g(A_2).$$

Then by Theorem 4, we have

$$\begin{aligned} \operatorname{Tr}\big(f(H_L^D) - f(H_L^N)\big) &= \operatorname{Tr}(g(A_1) - g(A_2)) \\ &= \int g'(\mu)\xi(\mu; A_1, A_2)d\mu, \end{aligned}$$

and also

$$\|\xi\|_{L^1} \leq \|A_1 - A_2\|_{\mathfrak{I}_1} \leq C_1 L^{d-1}$$

by Proposition 5. These imply

$$\big|\operatorname{Tr}\big(f(H_L^D) - f(H_L^N)\big)\big| \leq \sup_\mu |g'(\mu)| \cdot C_1 L^{d-1},$$

completing the proof of Theorem 1. ☐

3 Proof of Proposition 5

In this section, C_* denotes generic constant, which is independent of L. Let γ be the trace to the boundary of $\partial\Omega_L$, i.e., γ is the operator from $H^1(\Omega_L)$ to $L^2(\partial\Omega_L)$ defined by

$$\gamma\varphi(x) = \varphi(x) \qquad \text{for } x \in \partial\Omega_L \text{ and } \varphi \in C^1(\Omega_L),$$

and extended continuously to $H^1(\Omega_L)$. We recall the Krein's formula:

Lemma 6. *Let $z \in \rho(H_L^D) \cap \rho(H_L^N)$. Then*

$$(H_L^D - z)^{-1} - (H_L^N - z)^{-1} = (H_L^N - z)^{-1}\gamma^*\gamma\, \partial_\nu (H_L^D - z)^{-1},$$

where ν denotes the outer unit normal vector, and ∂_ν is the outer normal derivative on the boundary.

Proof. This is well-known, but we prove it for the completeness. Let $\varphi_1, \psi_1 \in L^2(\Omega_L)$ and we set

$$\varphi = (H_L^N - \bar{z})^{-1}\varphi_1 \in \mathcal{D}(H_L^N), \qquad \psi = (H_L^D - z)^{-1}\psi_1 \in \mathcal{D}(H_L^D).$$

Then it suffices to show

$$\langle (H_L^N - \bar{z})\varphi, \psi\rangle - \langle \varphi, (H_L^D - z)\psi\rangle = \langle \gamma\varphi, \gamma\partial_\nu\psi\rangle. \tag{3.1}$$

The left hand side of (3.1) is

$$\begin{aligned}
&\int_{\Omega_L} \overline{(p-A)^2\varphi(x)}\, \psi(x)dx - \int_{\Omega_L} \overline{\varphi(x)}(p-A)^2\psi(x)dx \\
&\quad = -\int_{\Omega_L} \overline{\triangle\varphi}\, \psi dx - i\int_{\Omega_L} \overline{(A\cdot\partial + \partial\cdot A)\varphi}\, \psi dx \\
&\qquad + \int_{\Omega_L} \overline{\varphi}\, \triangle\psi dx - i\int_{\Omega_L} \overline{\varphi}\, (A\cdot\partial + \partial\cdot A)\psi dx \\
&\quad = -\int_{\partial\Omega_L} \overline{\partial_\nu\varphi}\, \psi dS + \int_{\partial\Omega_L} \overline{\varphi}\, \partial_\nu\psi dS - 2i\int_{\partial\Omega_L} \overline{\varphi}\, (\nu\cdot A)\psi dS
\end{aligned}$$

by the Green formula and the Gauss formula. Here we denote the surface measure by dS. The first and the third terms vanishes since $\psi \in \mathcal{D}(H_L^D)$, and the first term is the right hand side of (3.1). □

By Lemma 6, we have

$$A_1 - A_2 = \sum_{j=1}^{m} (H_L^N + M)^{-m+1+j}\gamma^*\gamma\, \partial_\nu (H_L^D + M)^{-j} \tag{3.2}$$

We have to estimate the trace norm of the right hand side terms.

Lemma 7. *$\gamma(H_L^N + M)^{-1}$ and $\gamma\partial_\nu(H_L^D + M)^{-1}$ are bounded from $L^2(\Omega_L)$ to $L^2(\partial\Omega_L)$, uniformly in L.*

Proof. We prove the statement for $\gamma\partial_\nu(H_L^D + M)^{-1}$ only. The other case is simpler. We note that the proof is quite straightforward if $A = 0$.

For $O \subset \mathbb{R}^d$, we denote the space of the $H^1_{\mathrm{loc}}(O)$ functions with the norm

$$\|\varphi\|^2_{H^{1,A}(O)} = \|(p - A(x))\varphi\|^2 + \|\varphi\|^2$$

by $H^{1,A}(O)$. We note

$$\gamma\, \partial_\nu (H_L^D + M)^{-1} = i\gamma\, [\nu\cdot(p - A(x))](H_L^D + M)^{-1},$$

and $(p-A(x))(H_L^D+M)^{-1}$is uniformly bounded from $L^2(\Omega_L)$ to $H^{1,A}(\Omega_L)$ since V is bounded. Let

$$Q_0=\bigl\{x\in\mathbb{R}^d \bigm| 0\le x_j\le 1,\ j=1,\cdots,d\bigr\},$$
$$Z_L=\bigl\{n\in\mathbb{Z}^d \bigm| 0\le n_j< L,\ j=1,\cdots,d\bigr\},$$

and we set $Q_n=Q_0+n$ for $n\in Z_L$. We decompose

$$L^2(\Omega_L)=\bigoplus_{n\in Z_L}L^2(Q_n),$$

and we denote the identification map by $\pi=(\pi_n)$, i.e., $\pi_n\varphi$ is the restriction of $\varphi\in L^2(\Omega_L)$ to Q_n. It is easy to see that π is also an isometry from $H^{1,A}(\Omega_L)$ to $\bigoplus_n H^{1,A}(Q_n)$. Let γ_n be the trace operator from $H^{1,A}(Q_n)$ to $L^2(\partial\Omega_L\cap Q_n)$. Then we have

$$\gamma\partial_\nu(H_L^D+M)^{-1}=\sum_{n\in Z_L}i\gamma_n\pi_n\bigl[\nu\cdot(p-A)(H_L^D+M)^{-1}\bigr].$$

Now Lemma 7 follows from the next lemma. □

Lemma 8. *γ_n is bounded from $H^{1,A}(Q_n)$ to $L^2(\partial Q_n)$ uniformly in n and L.*

Proof. By Assumption A, we can find a gauge $\tilde{A}$ such that $\sup|\tilde{A}_j(x)|\le d\sup|B(x)|$ for $j=1,\cdots,d$ and $x\in Q_n$. Namely, there exists $\Psi(x)$ such that

$$p-\tilde{A}=e^{i\Psi}(p-A)e^{-i\Psi}.$$

Then, for $\varphi\in H^{1,A}(Q_n)$, we have

$$\begin{aligned}\|\gamma_n\varphi\|_{L^2(\partial Q_n)}&=\|e^{i\Psi}\gamma_n e^{-i\Psi}\varphi\|_{L^2(\partial Q_n)}\\&\le C\|e^{-i\Psi}\varphi\|_{H^1(Q_n)}\le C\bigl(\|pe^{-i\Psi}\varphi\|+\|\varphi\|\bigr)\\&\le C'\bigl(\|(p-\tilde{A})e^{-i\Psi}\varphi\|+\|\varphi\|\bigr)\\&=C'\bigl(\|e^{-i\Psi}(p-A)\varphi\|+\|\varphi\|\bigr)\le C''\|\varphi\|_{H^{1,A}(Q_n)},\end{aligned}$$

where all constants are independent of n and L. □

Definition 1. Let $J=J(x)$ be a smooth function on Ω_L with L being a parameter. We write $J\in\mathfrak{S}$ if for any α,

$$|\partial_x^\alpha J(x)|\le C_\alpha,\qquad x\in\Omega_L, L\in\mathbb{N},$$

and $\operatorname{supp} J\subset\bigl\{x\in\Omega_L \bigm| \operatorname{dist}(x,\partial\Omega_L)\le 1\bigr\}$.

For $f\in\mathfrak{S}$, we denote the multiplication operator by the same symbol. We choose $J_0\in\mathfrak{S}$ such that $J_0=1$ in a neighborhood of $\partial\Omega_L$.

Lemma 9. *Let $J \in \mathfrak{S}$ and let $\ell \in \mathbb{N}$. Then there exist functions*

$$\{J_{ij} \mid i = 1, \cdots, \ell, j = 1, \cdots, K\} \subset \mathfrak{S}$$

and uniformly bounded family of operators

$$\{B_{ij} \mid i = 1, \cdots, \ell, j = 1, \cdots, K\} \subset B(L^2(\Omega_L))$$

such that

$$J^\ell (H_L^\flat + M)^{-\ell} = \sum_{j=1}^{K} \prod_{i=1}^{\ell} [J_{ij}(H_L^\flat + M)^{-1} B_{ij}],$$

where $\flat = B$ or N. In particular, $J_{1j} = J$.

The proof is somewhat long, and we give it in Appendix.

Lemma 10. *There exists an extension operator*

$$\beta \,:\, H^{2,A}(\Omega_L) \mapsto H^{2,A}(\mathbb{R}^d)$$

such that the operator norm is uniformly bounded in L.

Proof. This result is well-known if $A = 0$. See [8] Chapter IV, Section 3.2. We sketch the necessary modification.

We construct an extension operator in a neighborhood of lattice points in $\partial\Omega_L$ with a fixed size. Then we paste them using the partition of unity. In each open set, we change the gauge so that A is uniformly bounded (as in the proof of Lemma 8). Then the extension operator is constructed in the usual way in this open set, and the operator norm is uniformly bounded. Then we change the gauge back, and paste together to construct the global extension operator. We note that the derivative of the partition of unity can be also taken to be uniformly bounded. ☐

Lemma 11. *Let $a \in \mathbb{N}$ such that $a > d/4$ (as in the last section), and let $J \in \mathfrak{S}$. Then $J(H_L^\flat + M)^{-1} \in \mathfrak{I}_{2a}$ and*

$$\|J(H_L^\flat + M)^{-1}\|_{\mathfrak{I}_{2a}} \le C L^{(d-1)/2a}, \qquad L \in \mathbb{N},$$

where $\flat = D$ or N.

Proof. Let β as in Lemma 10. Then

$$J(H_L^\flat + M)^{-1} = J(H+M)^{-1}\big[(H+M)\beta(H_L^\flat + M)^{-1}\big] \tag{3.3}$$

and $(H+M)\beta(H_L^\flat + M)^{-1}$ is uniformly bounded. By the diamagnetic inequality (cf. [3] (1.8)), we have

$$|J(H+M)^{-1}\varphi| \le |J|(H_0 + 1)^{-1}|\varphi| \qquad \text{for } \varphi \in L^2(\mathbb{R}^d),$$

where $H_0 = -\triangle$. By Theorem 4.1 of [7], we learn $|J|(H_0+1)^{-1} \in \mathfrak{I}_{2a}$ and

$$\||J|(H_0+1)^{-1}\|_{\mathfrak{I}_{2a}} \le \|J\|_{L^{2a}} \|(\xi^2+1)^{-1}\|_{L^{2a}} \le C L^{(d-1)/2a}.$$

Then we apply Theorem 2.13 of [7] to observe $J(H+M)^{-1} \in \mathfrak{I}_{2a}$ and

$$\|J(H+M)^{-1}\|_{\mathfrak{I}_{2a}} \leq CL^{(d-1)/2a}.$$

This and (3.3) imply the conclusion. $\square$

Proof of Proposition 5. Let $1 \leq k \leq m$ and we consider

$$P_k = (H_L^N + M)^{-m-1+k} \gamma^* \gamma \partial_\nu (H_L^D + M)^{-k}.$$

By (3.2), it suffices to estimate $\|P_k\|_{\mathfrak{I}_1}$. By Lemma 9,

$$\begin{aligned} \gamma\partial_\nu (H_L^D + M)^{-k} &= \gamma \partial_\nu J_0^k (H_L^D + M)^{-k} \\ &= \gamma\partial_\nu (H_L^D + M)^{-1} \sum_{j=1}^{K} B_{1j} \prod_{i=2}^{k} \big[J_{ij} (H_L^D + M)^{-1} B_{ij} \big]. \end{aligned}$$

Hence, using Lemma 11 and [7] Theorem 2.8, we find

$$\left\| \prod_{i=2}^{k} \big[J_{ij} (H_L^D + M)^{-1} B_{ij} \big] \right\|_{\mathfrak{I}_p} \leq CL^{(d-1)/p},$$

where $p^{-1} = (k-1)/2a$. Similarly, we have

$$\gamma (H_L^N + M)^{-m-1+k} = \gamma (H_L^N + M)^{-1} \sum_{j=1}^{a'} B'_{1j} \prod_{i=2}^{m+1-k} \big[J'_{ij} (H_L^N + M)^{-1} B'_{ij} \big]$$

and

$$\left\| \prod_{i=2}^{k} \big[J'_{ij} (H_L^N + M)^{-1} B'_{ij} \big] \right\|_{\mathfrak{I}_q} \leq CL^{(d-1)/q},$$

where $q^{-1} = (m-k)/2a$. Recall that we have chosen m so that

$$\frac{1}{p} + \frac{1}{q} = \frac{m-1}{2a} = 1.$$

Hence, combining these with Lemma 7 and [7] Theorem 2.8, we conclude $P_k \in \mathfrak{I}_1$ and

$$\|P_k\|_{\mathfrak{I}_1} \leq CL^{(d-1)/q} \times CL^{(d-1)/p} = C' L^{d-1}.$$

This competes the proof. $\square$

Appendix A. Proof of Lemma 9

In this appendix, we consider a Schrödinger operator $H = (p - A(x))^2 + V(x)$ on $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$, such that A is continuous and V is a Kato-class potential (cf. e.g., [3] Section 1.2). Let H be a self-adjoint realization of the differential operator in the quadratic form sense.

Let $M > -\inf \sigma(H)$, and fix $J_0 \in C^\infty(\Omega)$ such that $\sup |\partial_x^\alpha J_0(x)| < \infty$ for any multi-index α. Let $\mathfrak{S}$ be the linear subspace of $C^\infty(\Omega)$ spanned by the derivatives of J_0, i.e.,

$$\mathfrak{S} = \mathrm{Span}\big\{\partial_x^\alpha J_0 \;\big|\; \alpha : \text{multi-index}\big\} \subset C^\infty(\Omega).$$

For simplicity, we denote $R = (H+M)^{-1}$, $T = p - A$. We write

$$Q_1 = R, \quad Q_2 = TR, \quad Q_3 = RT, \quad Q_4 = TRT.$$

Let $\mathfrak{B}$ be the subspace of $B(L^2(\Omega))$ spanned by the polynomials of $Q_1, \cdots, Q_4$ and elements of $\mathfrak{S}$. We also write

$$\mathfrak{B}_k = \left\{ \sum_{j=1}^{m} \prod_{i=1}^{k} J_{ij} R B_{ij} \;\middle|\; J_{ij} \in \mathfrak{S}, B_{ij} \in \mathfrak{B} \right\}$$

for $k \in \mathbb{N}$. Our goal is the following.

Proposition 12. *For any $k \in \mathbb{N}$ and $J \in \mathfrak{S}$, $J^k R^k \in \mathfrak{B}_k$.*

Lemma 9 follows immediately from Proposition 12.

We prove the assertion by induction. Suppose the assertion holds for $k \le K$, and we show $J^{K+1}R^{K+1} \in \mathfrak{B}_{K+1}$. By the assumption, we can write

$$J^K R^K = \sum_{j=1}^{m} \prod_{i=1}^{K} J_{ij} R B_{ij} = \sum_{j=1}^{m} P_j \in \mathfrak{B}_K$$

with some $J_{ij} \in \mathfrak{S}$ and $B_{ij} \in \mathfrak{B}$. Since $J^{K+1}R^{K+1} = J\big(\sum P_j\big)R$, it suffices to show

$$JPR \in \mathfrak{B}_{K+1} \quad \text{for} \quad P = \prod_{i=1}^{K} J_i R B_i \in \mathfrak{B}_K. \tag{A.1}$$

We use the following notations:

$$\begin{aligned} \tilde{\mathfrak{B}}_1 &= \big\{\textstyle\sum B_i P_i \;\big|\; B_i \in \mathfrak{B}, P_i \in \mathfrak{B}_1\big\} \\ \tilde{\mathfrak{S}} &= \big\{\textstyle\sum B_i J_i \;\big|\; B_i \in \mathfrak{B}, J_i \in \mathfrak{S}\big\}. \end{aligned}$$

Lemma 13. *Let J, $J_1 \in \mathfrak{S}$. Then $JJ_1R \in \mathfrak{B}_1\tilde{\mathfrak{S}} + \mathfrak{B}_2$.*

Proof. In the following computations, J_* denotes some element of $\mathfrak{S}$. Then we have

$$\begin{aligned} J_1 R &= RJ_1 + R[H, J_1]R = RJ_1 + R(J_2 + TJ_3)R \\ &= RJ_1 + RJ_2R + RTRJ_3 + RTR[H, J_3]R \\ &= RJ_1 + RJ_2R + RTRJ_3 + RTRJ_4R + RTRTJ_5R \\ &= R\big(J_1 + Q_2J_3 + (J_2R + Q_2J_4R + Q_4J_5R)\big) \end{aligned} \tag{A.2}$$

and hence

$$\begin{aligned} JJ_1R &= (JR)(J_1 + Q_2J_3) + (JR)(J_2R + Q_2J_4R + Q_4J_5R) \\ &\in \mathfrak{B}_1 \cdot \tilde{\mathfrak{S}} + \mathfrak{B}_2. \end{aligned}$$

□

Lemma 14. *Let* $J_1 \in \mathfrak{S}$. *Then* $J_1 Q_i \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$ *for* $i = 1, \cdots, 4$.

Proof. We note

$$RJ = JR + RJ_aR + RTJ_bR \in \tilde{\mathfrak{B}}_1 \quad \text{if} \quad J \in \mathfrak{S}. \tag{A.3}$$

$J_1 Q_1 \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$ follows immediately from (A.2). Using (A.2), we also have

$$\begin{aligned} J_1 Q_2 &= J_1 T R = T J_1 R + J_6 R \\ &= (TR)(J_1 + Q_2 J_3) + (TR)(J_2 R + Q_2 J_4 R + Q_4 J_5 R) + J_6 R \\ &\in \mathfrak{G} + \tilde{\mathfrak{B}}_1. \end{aligned}$$

About $J_1 Q_3$, we have

$$\begin{aligned} J_1 Q_3 = J_1 RT &= R\big(J_1 + Q_2 J_3 + (J_2 R + Q_2 J_4 R + Q_4 J_5 R)\big)T \\ &= (RT)J_1 + RJ_7 + (RT)(RT)J_3 + (RT)(RJ_8) \\ &\quad + (RJ_2)(RT) + (RT)(RJ_4)(RT) + RTRTJ_5RT. \end{aligned}$$

Now the right hand side terms are elements of $\mathfrak{G} + \tilde{\mathfrak{B}}_1$ by (A.3) except for the last term. The last term is

$$RTRTJ_5RT = (RT)(RJ_5)(TRT) + (RT)(RJ_9)(RT) \in \tilde{\mathfrak{B}}_1.$$

Thus $J_1 Q_3 \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$. Similarly, we compute

$$J_1 Q_4 = J_1 TRT = TJ_1RT + J_{10}RT.$$

The second term in the right hand side is $J_{10} Q_3 \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$. By (A.2), the first term is

$$\begin{aligned} TJ_1RT &= TR\big(J_1 + Q_2 J_3 + (J_2 R + Q_2 J_4 R + Q_4 J_5 R)\big)T \\ &= (TRT)J_1 + (TR)J_{11} + (TRT)(RT)J_3 + (TR)(TR)J_{12} \\ &\quad + (TR)(J_2 Q_3 + Q_2 J_4 Q_3 + Q_4 J_5 Q_3) \in \mathfrak{G} + \tilde{\mathfrak{B}}_1, \end{aligned}$$

where we have used $J_* Q_3 \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$. ☐

Proof of Proposition 12 (continued). By Lemma 13, we have

$$J(J_1 RB) = (B_1 G_1 + B_2)B, \quad \text{with} \quad B_1 \in \mathfrak{B}_1, B_2 \in \mathfrak{B}_2, G_1 \in \mathfrak{G},$$

if J, $J_1 \in \mathfrak{S}$ and $B \in \mathfrak{B}$. By Lemma 14, $G_1 Q_j \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$ ($j = 1, \cdots, 4$), and by iterating this formula, we learn $G_1 B \in \mathfrak{G} + \tilde{\mathfrak{B}}_1$. Hence we obtain

$$J(J_1 RB) \in \mathfrak{B}_1 \mathfrak{G} + \mathfrak{B}_2.$$

Iterating this formula again, we learn

$$J\Big\{\prod_{i=1}^{K} J_i R B_i\Big\} \in \mathfrak{B}_K \mathfrak{G} + \mathfrak{B}_{K+1},$$

and hence $J\big\{\prod_{i=1}^{K} J_i R B_i\big\} R \in \mathfrak{B}_{K+1}$. This completes the induction step (A.1), and Proposition 12 is proved. ☐

Appendix B. Integrated density of states for the Landau Hamiltonian on $\mathbb{R}^2$.

In this appendix, we compute the IDS for the Landau Hamiltonian on $\mathbb{R}^2$. It appears that the result (and the following computation) is well-known among physicists, and in fact the author was informed of this by T. Koma (Dept. Physics, Gakushuin University). However, the justification requires the fact that the IDS is independent of the choice of the boundary condition.

We consider $H = (p - A)^2$ on $L^2(\mathbb{R}^2)$, where $A = (0, Bx)$ for $(x, y) \in \mathbb{R}^2$ with $B \neq 0$. For simplicity, we suppose $B > 0$. We define

$$H_L = (p - A)^2 \qquad \text{on } L^2(\Omega_L)$$

with

$$\begin{aligned}&\mathcal{Q}(H_L) = \mathcal{D}((H_L)^{1/2})\\ &\quad = \big\{\varphi \in H^1(\Omega_L) \bigm| \varphi(x,0) = \varphi(x,L), \varphi(0,y) = \varphi(L,y) = 0, \text{a.e. } x, y\big\},\end{aligned}$$

i.e., $\varphi \in \mathcal{D}(H_L)$ satisfies the Dirichlet boundary condition on $\{x = 0, L\}$, and the periodic boundary condition on $\{y = 0, L\}$. We set

$$N_L(E) = \#\{\text{eigenvalues of } H_L \leq E\},$$

and we compute

$$k(E) = \lim_{L\to\infty} L^{-2} N_L(E).$$

$k(E)$ is the IDS by virtue of Corollary 3. In order to compute the eigenvalues of H_L, we use the separation of variables. A complete system of eigenfunctions of $-\partial_y^2$ is given by

$$f_n(y) = c\, e^{i2\pi ny/L}, \qquad y \in [0, L], n \in \mathbb{Z},$$

where $c = L^{-1/2}$ is a normalization constant. Let

$$\mathfrak{H}_n = \big\{g(x) f_n(y) \in L^2(\Omega_L) \bigm| g \in L^2([0, L])\big\}.$$

Then $L^2(\Omega_L) = \oplus_{n\in\mathbb{Z}} \mathfrak{H}_n$, and we have

$$H_L\, \varphi_n = \left[-\partial_x^2 + \Big(\frac{2\pi n}{L} - Bx\Big)^2\right]\varphi_n \qquad \text{for } \varphi_n \in \mathfrak{H}_n.$$

Hence we need only to compute the eigenvalues of

$$H_{L,n} = -\partial_x^2 + \Big(\frac{2\pi n}{L} - Bx\Big)^2 \qquad \text{on } L^2([0, L])$$

with $\mathcal{Q}(H_{L,n}) = H_0^1([0, L])$. Since

$$\Big(\frac{2\pi n}{L} - Bx\Big)^2 = B^2\Big(x - \frac{2\pi n}{BL}\Big)^2,$$

$H_{L,n}$ is a one-dimensional harmonic oscillator with the center at $2\pi n/BL$, restricted to $[0, L]$. If

$$0 << \frac{2\pi n}{BL} << L,$$

then

$$\sigma(H_{L,n}) \sim \sigma(-\partial_x^2 + B^2x^2) = \{(2\ell+1)B \,|\, \ell = 0, 1, 2, \dots\}.$$

On the other hand, if

$$\frac{2\pi n}{BL} << 0 \quad \text{or} \quad \frac{2\pi n}{BL} >> L,$$

then $\inf \sigma(H_{L,n}) >> 0$. Hence we expect

$$\begin{aligned} N_L(E) &\sim L\Big(\frac{2\pi}{BL}\Big)^{-1} \#\big\{\ell \geq 0 \,\big|\, (2\ell+1)B \leq E\big\} \\ &= L^2 \frac{B}{2\pi} F(E/B), \end{aligned}$$

where we set

$$F(\lambda) = \begin{cases} 0, & \text{if } \lambda < 1, \\ \ell, & \text{if } 2\ell - 1 \leq \lambda < 2\ell + 1 \text{ with } \ell \in \mathbb{N}. \end{cases}$$

Thus we arrive at a conjecture

$$k(E) = \frac{B}{2\pi} F(E/B), \qquad E \in \mathbb{R}.$$

In fact we can prove the following.

Proposition 15. *Let* $E \notin \{(2\ell+1)B \,|\, \ell = 0, 1, \dots\}$. *Then*

$$k(E) = \lim_{L\to\infty} L^{-2} N_L(E) = \frac{B}{2\pi} F(E/B).$$

Proof. We set $\ell_0 \in \mathbb{Z}$ so that $E \in ((2\ell_0 - 1)B, (2\ell_0 + 1)B)$. Let

$$0 < \varepsilon < \operatorname{dist}(E, \{(2\ell_0 \pm 1)B\}),$$

and we denote the j-th eigenvalue of $H_{L,n}$ by $e^j_{L,n}$. Then there exists $M_1 > 0$ (independent of L) such that if $2\pi n/BL \in [M_1, L - M_1]$ then

$$\big|e^j_{L,n} - (2j-1)B\big| < \varepsilon \qquad \text{for } j = 1, \dots, \ell_0 + 1.$$

This condition implies

$$\#\,\{\text{eigenvalues of } H_{L,n} \leq E\} = \ell_0 \quad \text{if} \quad \frac{2\pi n}{BL} \in [M_1, L - M_1]. \quad \text{(B.1)}$$

On the other hand, there exists $M_2 > 0$ such that

$$\inf \sigma(H_{L,n}) \geq E + 1 \qquad \text{if} \quad \frac{2\pi n}{BL} \notin [-M_2, L + M_2]. \qquad \text{(B.2)}$$

We also note that

$$\#\left\{\text{e.v.'s of } H_{L,n} \le E\right\} \le \#\left\{\text{e.v.'s of } -\partial_x^2 + B^2x^2 \le E\right\} = \ell_0 \quad \text{(B.3)}$$

for any $n \in \mathbb{Z}$. Combining these, we compute

$$\begin{aligned} N_L(E) &= \sum_{n\in\mathbb{Z}} \#\left\{\text{eigenvalues of } H_{L,n} \le E\right\} \\ &= \sum_{n\in I_1} \#\{\cdots\} + \sum_{n\in I_2} \#\{\cdots\} + \sum_{n\in I_3} \#\{\cdots\} = N_1 + N_2 + N_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \left\{ n \;\middle|\; \frac{2\pi n}{BL} \in [M_1, L - M_1] \right\}, \\ I_2 &= \left\{ n \;\middle|\; \frac{2\pi n}{BL} \in (-\infty, -M_2] \cup [L + M_2, \infty) \right\}, \\ I_3 &= \left\{ n \;\middle|\; \frac{2\pi n}{BL} \in (-M_2, M_1) \cup (L - M_1, L + M_2) \right\}. \end{aligned}$$

Since

$$\# I_1 = \left(\frac{2\pi}{BL}\right)^{-1} (L - 2M_1) + \mathcal{O}(1) = \frac{BL^2}{2\pi} + \mathcal{O}(L),$$

we learn from (B.1) that

$$N_1 = \ell_0 \frac{BL^2}{2\pi} + \mathcal{O}(L).$$

By (B.2), it is easy to see that $N_2 = 0$. At last, using (B.3), we have

$$N_3 \le \ell_0 \times \# I_3 = \ell_0 \left\{ \left(\frac{2\pi}{BL}\right)^{-1} (M_1 + M_2) + \mathcal{O}(1) \right\} = \mathcal{O}(L).$$

Thus we obtain

$$N_L(E) = L^2 \frac{B}{2\pi} \ell_0 + \mathcal{O}(L),$$

and this completes the proof. □

Remark . The IDS for the free Schrödinger operator is given by

$$k_0(E) = \frac{E}{4\pi}, \qquad E > 0.$$

If we compare $k(E)$ with $k_0(E)$, we find that $k(E) \ge k_0(E)$ does not always hold. Namely, the IDS is *not* decreasing function of B (see Figure 1).

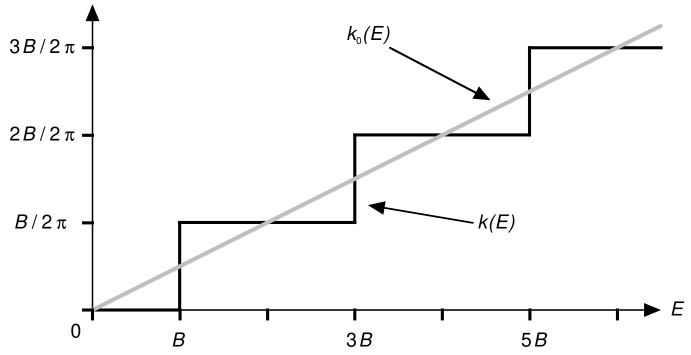

Figure 1: Comparison of $k(E)$ and $k_0(E)$

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