SHARP INTERFACE LIMITS FOR NON LOCAL ANISOTROPIC INTERACTIONS

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ABSTRACT. We prove a convexity property of the surface tension of non local, anisotropic free energy functionals which implies that the Wulff shape is strictly convex and smooth. We also prove that the transport coefficients of the limit anisotropic motion by mean curvature obtained in [33] are strictly positive and equal to the stiffness parameters determined by the surface tension.

1. INTRODUCTION

In this paper we tie up a few loose ends in the theory of sharp interface limits for the van der Waals free energy functionals of the form

$$F(m) = \int dr\, [f(m(r)) - f^*] + \frac{1}{4} \int \int dr\, dr'\, J(r - r')\, [m(r) - m(r')]^2 \tag{1.1}$$

where $m \in L^\infty(\mathbb{R}^n)$ is an order parameter valued function, for instance $m(r)$ is a magnetization density. $f(m)$ is a double well shaped (mean field) free energy density, f^* its minimum attained at $\pm m^*$, $m^* > 0$; further assumptions on $f(m)$ are specified in the beginning of the next section. Finally, the interaction $J(r)$ is a non negative, smooth, even function supported by the unit ball, with integral equal to 1.

The idea already present in the original van der Waals approach, is to build a bridge from statistical mechanics to continuum mechanics. A rigorous connection with the former, however, was first established only in the 1960's, in the works of Kac, Uhlenbeck and Hemmer, [31]; the connection with continuum mechanics, in particular with the physics of surfaces, which is the case we are interested here, is even more recent. For what follows, the two relevant results are, from one side, by Alberti and Bellettini, [2], [3] and, from the other one, by Katsoulakis and Souganidis, [33] and Barles and Souganidis, [6].

In the papers [2], [3] it is proved that, in the macroscopic continuum limit, the free energy cost of an interface ∂E boundary of a finite perimeter set E is given by

$$\int_{\partial E} dH^{n-1}(r)\, \sigma(\nu(r)) \tag{1.2}$$

where dH^{n-1} is the $(n-1)$-dimensional surface area on ∂E, $\nu(r)$ is the outward unit normal to ∂E at r, almost everywhere defined, and $\sigma(e)$, e a unit vector, is the surface tension of a flat surface whose normal is e. Here we are interested in anisotropic interactions $J(r)$ when $\sigma(e)$ depends effectively on e, the isotropic case had been solved earlier and there are no loose ends there! In the next sections we will come back to the derivation result, here we concentrate on the properties of a body whose surface tension is $\sigma(e)$.

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The shape of a droplet at fixed volume is given by the Wulff construction, whose procedure only requires knowledge of $\sigma(e)$ and the resulting droplet is a convex set. It may have facets and corners, though, the former a possible manifestation of a crystalline structure, the latter of microscopic wrinkles and of forbidden orientations of the equilibrium surface.

We will prove that there are neither facets nor forbidden orientations, hence the Wulff shape in systems with free energy (1.1) is smooth and strictly convex. Thus if we want to model systems with wrinkles and/or facets, we should modify the starting free energy functional F, either by taking anisotropic interactions $J(\cdot)$ which are somewhere negative, and/or introducing many body interactions, as for instance in the non local functionals introduced by Lebowitz, Mazel and Presutti, [34] to study phase transitions for continuum particle models.

Concerning the geometry of Wulff shapes, the relevant quantity when making the above description quantitative, is the stiffness matrix $S(e)$. For any e in the unit sphere S^{n-1}, $S(e)$ is the real, $(n \times n)$ symmetric matrix which has e in its kernel, and is such that, for any unit vector u orthogonal to e,

$$\langle u, S(e)u \rangle = \sigma(e) + \partial_u^2 \sigma(e) \tag{1.3}$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$, ∂_u is the derivative in the direction of u, while by ∂_u^2 we are shorthanding the following: let $r(\theta)$ be the maximal circle on S^{n-1}, parametrized by its angle θ, chosen so that $r(0) = e$ and $r'(0) = u$. Then we set

$$\partial_u^2 \sigma(e) := \left. \frac{d^2 \sigma(r(\theta))}{d\theta^2} \right|_{\theta=0} \tag{1.4}$$

An equivalent definition of $S(e)$ can be given as follows: let $\tilde{\sigma}(r) := |r| \sigma(r/|r|)$ be the 1-homogeneous extension of $\sigma(e)$ to $\mathbb{R}^n \setminus \{0\}$; then

$$S(e) = \left(D^2 \tilde{\sigma} \right)(e) \tag{1.5}$$

where $D^2 \tilde{\sigma}$ is the Hessian matrix of $\tilde{\sigma}$.

The stability of an interface orientation is determined by $S(e)$, see for instance [30], the interface being unstable if (1.3) is negative for some directions e and u. Physically it is the same existence of a macroscopic limit for the surface energy which implies the absence of instabilities and indeed, as a consequence of the convergence result in [3], $S(e)$ is non negative. Moreover, the larger are the eigenvalues of $S(e)$ the more stable is the orientation e and, correspondingly, more rigid is the Wulff shape where the normal is e. The extreme case is when a facet appears, and the stiffness matrix becomes infinite at the value of e corresponding to the normal to the facet, see for instance [36]. On the opposite side, if all eigenvalues of $S(e)$ are small then the curvature of the Wulff shape where the normal is e becomes very large and if $S(e)$ vanishes, corners and forbidden orientations may arise. For instance, in the $d = 2$ Ising model at 0 temperature, the Wulff shape is a coordinate square, thus excluding all surface elements whose normal is not directed along the coordinate axes. These effects remind of the phase transitions in bulk thermodynamics, where intervals of values of an order parameter are forbidden if there is a phase transition. The analogy becomes quantitative in SOS models, as explained by Spohn in [39]. As already mentioned, all these effects are ruled out in systems whose free energy is given by (1.1); we will prove in fact that $\langle u, S(e)u \rangle > 0$ whenever u is orthogonal to e.

Katsoulakis and Souganidis [33] and Barles and Souganidis [6] have instead studied the dynamical aspects of the problem, starting from the gradient flow associated to the functional (1.1) (see the next sections for precise statements). They have shown that in a sharp interface limit the interface $\partial E(t)$ evolves according to anisotropic motion by mean curvature: namely the points of $\partial E(t)$ have normal velocity

$$v(r) = -\mu(\nu(r))\sigma(\nu(r)) \left\{ \sum_{i=1}^{n-1} \langle u_i, A(\nu(r))u_i\rangle \, \kappa_i(r) \right\} \nu(r) \tag{1.6}$$

$\nu(r)$ is the outward unit normal to $\partial E(t)$ at r, $\kappa_i(r)$ are the principal curvatures at r (positive for a sphere) and u_i are the corresponding principal curvature directions; $\mu(e)$ are strictly positive coefficients (interpreted as the mobility of the surface in the direction e); finally $A(e)$ is a real, symmetric $(n \times n)$ matrix with e in its kernel. (A notational warning: our $A(e)$ here is equal to $(Id - e \otimes e)\hat{A}(e)$, $\hat{A}(e)$ the matrix in [33], and our $\mu(e)$ is equal to $\hat{\mu}(e)\sigma(e)^{-1}$, $\hat{\mu}(e)$ the coefficient in [33]).

The situation is similar to the previous one: the evolution defined by the equation (1.6) is well posed if $A(e)$ is non negative and it is the same convergence result to (1.6) which gives automatically non negativity. Thus $A(e)$ is a non negative matrix. But again, if $A(e)u = 0$ for some unit vector u orthogonal to e, there are degenerate effects, it may even happen that non flat surfaces do not move at all, with no tendency, therefore, to straighten up. We prove that this is not possible, more precisely we show that $\langle u, A(e)u\rangle > 0$ for all unit $u \in \mathbb{R}^n$ orthogonal to e. Thus the normal velocity is obtained by averaging the principal curvatures κ_i with the positive weights $\langle u_i, A(e)u_i\rangle$. In the isotropic case all these weights are equal and the sum of all the curvatures gives the mean curvature; in the anisotropic case we just have to weight the average in a direction dependent way, but the average effectively involves all the curvatures because the weights are strictly positive.

The last loose end to tie up is about where the surface tension enters in (1.6), as indeed we know, on physical grounds, that the interface motions are determined by the surface tension. Also this issue is settled here in a completely satisfactory fashion, as we prove that $A(e) = S(e)$. The result confirms a conjecture proposed by Spohn, [39] and already verified in the isotropic case, see the next sections for more details, and agrees with the balance of forces computations of continuum mechanics, see [30].

In Section 2 we state the main definitions and results. In Section 3 we recall with some more details the results in [2], [3], [33] and [6]. In Section 4 we prove that $A(e) = S(e)$. In Section 5 we prove that $\langle u, A(e)u\rangle > 0$ whenever $\langle u, e\rangle = 0$; surprisingly enough, the key step in the proof is related to classical inequalities for random walks in random environments. In Section 6 we discuss the asymptotic shape of either growing or shrinking droplets, where the mobility, so far quite disregarded, has instead an important role. In particular we will prove that it is not true that the Wulff shape is always preserved by the limit interface dynamics given by (1.6). Finally, in the Appendix, we recall known properties of the one dimensional case which have been used in the proofs.

2. Main definitions and results

Throughout the sequel we will suppose that $f(m)$ is a double well shaped function in $m \in (-1,1)$ with two absolute minimizers, $\pm m^*$, $m^* \in (0,1)$; we call $m_0 \in (-m^*, m^*)$ the local maximizer, and $f^* = f(\pm m^*)$. The next assumption is more stringent, we will in fact require that

$$f(m) = -\frac{1}{2}m^2 + \Phi(m) \tag{2.1}$$

with Φ a real valued C^∞ function on $(-1,1)$ which is strictly convex. It is known that non convex Φ's give rise to anomalous behaviors, [7], [8] and will not be considered here.

The expression (2.1) arises in the analysis of Ising models with Kac interactions, see [35]: $\Phi(m)$ is then the free energy of the "reference system" without Kac interaction. The assumption of strict convexity on $\Phi(m)$ yields absence of phase transitions in the reference system, so that the phase transition (i.e. the occurrence of two distinct minimizers for the functional $F(m)$) is brought in by the Kac potential alone: namely, the loss of convexity of $f(m)$ is due to the first term on the r.h.s. of (2.1) which has "the wrong sign".

The most studied case is when the reference interaction is 0, then

$$\Phi(m) = -\beta^{-1}\left\{-\frac{1+m}{2}\log\frac{1+m}{2} - \frac{1-m}{2}\log\frac{1-m}{2}\right\} \tag{2.2}$$

where $\beta = 1/kT$, T the temperature, k the Boltzmann constant. The curly bracket in (2.2) is the entropy of a Bernoulli process where the variables are spins taking values ± 1 and m is the average spin. The case of a double well arises when $\beta > 1$, the two minimizers, $\pm m^*$, are obtained by solving the mean field equation

$$m^* = \tanh\{\beta m^*\} \tag{2.3}$$

In the next section we will discuss sharp interface limits and Γ-convergence, here we just give the expression for the surface tension $\sigma(e)$ found in [2]. Given a unit vector e and $\xi \in \mathbb{R}$, we define

$$J_e(\xi) := \int_{e^\perp} dH^{n-1}(r)\, J(r + \xi e) \tag{2.4}$$

where

$$e^\perp := \{r \in \mathbb{R}^n : r_e = 0\}, \qquad r_e := \langle r, e\rangle \qquad \forall\, r \in \mathbb{R}^n$$

We introduce the set

$$\mathcal{M} := \{m \in L^\infty(\mathbb{R}, (-1,1)) \;:\; x(m(x) - m_0) > 0 \text{ for } x \neq 0\}$$

and call $F_e(\cdot)$ the functional (1.1) in the one dimensional case $n = 1$, with J replaced by J_e. Then the surface tension $\sigma(e)$ is defined as

$$\sigma(e) := \inf_{m \in \mathcal{M}} F_e(m) \tag{2.5}$$

Notice that σ is even, i.e. $\sigma(e) = \sigma(-e)$. It is proved in [2] that the inf is a minimum and the unique minimizer $\bar{m}_e$ (sometimes called instanton in the direction e) is a strictly increasing solution in $\mathcal{M}$ of the non local mean field equation

$$\Phi'(\bar{m}_e(\xi)) = (J_e * \bar{m}_e)(\xi), \qquad \xi \in \mathbb{R} \tag{2.6}$$

where $(J_e * \bar{m}_e)(\xi) = \int d\xi' \, J(\xi' - \xi)\bar{m}_e(\xi')$, and $\lim_{\xi \to \pm\infty} \bar{m}_e(\xi) = \pm m^*$. When $\Phi(m)$ is given by (2.2), (2.6) takes the well known form

$$\bar{m}_e(\xi) = \tanh\{\beta \, (J_e * \bar{m}_e)\,(\xi)\} \tag{2.7}$$

The existence of the instanton solution $\bar{m}_e$ gives

$$\sigma(e) = F_e(\bar{m}_e) \tag{2.8}$$

which proves the validity of the original van der Waals expression for the surface tension, its rigorous justification, as already mentioned, will be discussed in Section 3. Note that $\sigma(e) \in (0, +\infty)$.

We next turn to dynamics: again we postpone to Section 3 a discussion on the works of Katsoulakis and Souganidis and Barles and Souganidis, and just give their final expression for the transport coefficients matrix $A(e)$. As already mentioned $A(e)$ is a real, symmetric, $(n \times n)$ matrix with $A(e)e = 0$ and such that, if u is a unit vector orthogonal to e and ∂_u denotes derivative in the direction of u, then

$$\langle u, A(e)u \rangle = \int d\xi \int dr \; J(r)\bar{m}_e'(\xi) \left[\bar{m}_e'(\xi + r_e)\, \frac{r_u^2}{2} + (\partial_u \bar{m}_e)(\xi + r_e)\, r_u \right] \tag{2.9}$$

The main result of this paper, proved in Sections 4 and 5, reads as follows.

Theorem 2.1. *Let $f(m)$ be as in* (2.1) *and let $\Phi(m)$ satisfy the conditions below* (2.1). *Then $A(e) = S(e)$. Moreover, for any unit vector e and any unit vector u orthogonal to e, the r.h.s. of* (1.3) *is well defined and strictly positive.*

3. Γ-convergence and motion by mean curvature

We start from equilibrium, where the main question is to estimate the "free energy cost" of a profile $m(\cdot)$, when both phases occupy a non negligible fraction of the total volume (phases are minimizers of F, which in the present case are the two profiles $m(r) \equiv m^*$ and $m(r) \equiv -m^*$, as for both $F = 0$). A scaling parameter $\varepsilon > 0$ is introduced as the ratio between microscopic and macroscopic units: the macroscopic limit (sharp interface limit) is attained when $\varepsilon \to 0$. As explained in [4], after rescaling and normalization, the free energy functional (1.1) is replaced by

$$F^{(\varepsilon)}(m) = \int_{\mathcal{T}} dr \, \varepsilon^{-1}[f(m(r)) - f(m^*)] + \frac{\varepsilon}{4} \int_{\mathcal{T}\times\mathcal{T}} dr\, dr' \,\{\varepsilon^{-n} J(\varepsilon^{-1}(r - r'))\}\,[m(r) - m(r')]^2 \tag{3.1}$$

where $\mathcal{T}$, the unit torus in $\mathbb{R}^n$, is the "macroscopic" region where the system is confined, and $m \in L^\infty(\mathcal{T}, (-1,1))$ is a magnetization profile. Periodic boundary conditions are chosen for simplicity. An analogous expression arises starting from the Ginzburg-Landau functional, see for instance [20] and [37], where ε multiplies the term with the gradient and ε^{-1} the energy, as in (3.1).

Let $\hat{m} \in BV(\mathcal{T}, \{\pm m^*\})$ denote a given macroscopic profile where the $+$ phase is confined in the region

$$E = \{r \in \mathcal{T} : \hat{m}(r) = m^*\} \tag{3.2}$$

and the $-$ phase in the complement $E^c = \mathcal{T} \setminus E$. Its δ-cost, $\delta > 0$, is defined as

$$\inf_{\|m-\hat{m}\|_1 < \delta} F^{(\varepsilon)}(m) \tag{3.3}$$

$\|\cdot\|_1$ being the L^1-norm in $\mathcal{T}$. The definition is meant to optimize the free energy (3.1) with the relaxed constraint of the actual profile "almost" verifying the desired macroscopic request of having the $+$ and $-$ phases in E and its complement. δ is the corresponding accuracy parameter which is eventually taken to 0, but only after the macroscopic limit $\varepsilon \to 0$, and, since the limit might not exist, we should consider liminfs and limsups. This is indeed the definition of Γ-convergence [20]: in fact $F^{(\varepsilon)}$ Γ-converges as $\varepsilon \to 0$ to F at $\hat{m}$ iff

$$\lim_{\delta \to 0} \liminf_{\varepsilon \to 0} \inf_{\|m-\hat{m}\|_1 < \delta} F^{(\varepsilon)}(m) = \lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \inf_{\|m-\hat{m}\|_1 < \delta} F^{(\varepsilon)}(m) =: F^{(0)}(\hat{m}) \tag{3.4}$$

We refer, for instance, to [19] for an organic presentation of Γ-convergence and to [4] for relating such a notion to large deviations for Gibbs measures in Ising systems with Kac potentials. These systems have been further studied in [9], [11], [12]; the notion of Γ-convergence also plays an essential role in the analysis of the Wulff shape in independent percolation, [17] and in $d \geq 3$ n.n. Ising models, [13], [18], [14]. Γ-convergence for a more general class of van der Waals functionals is proved in [3].

In [4], [2], and [3] it is proved that if $J(r)$ is spherically symmetric, then (3.4) holds with

$$F^{(0)}(\hat{m}) = \sigma P(E) \tag{3.5}$$

where $P(\cdot)$ is the perimeter functional, i.e. $P(E)$ is the $(n-1)$-dimensional Hausdorff measure of the (reduced) boundary of the set E in (3.2), which is well defined because of the assumption that $\hat{m}$ has bounded variation, see for instance [29]. $\sigma > 0$ is the "surface tension", which is given by (2.8); by the assumption that J is spherically symmetric, σ is independent of e. Then, by (1.3), $\langle u, S(e)u\rangle = \sigma$ is trivially positive, for any $e, u \in S^{n-1}$ mutually orthogonal.

Γ-convergence has been proved in [3] also in the anisotropic case, where $J(r)$ is no longer a function of $|r|$. In such a case the limit functional $F^{(0)}$ is as in (1.2), i.e.

$$F^{(0)}(\hat{m}) = \int_{\partial E} dH^{n-1}(r)\, \sigma(\nu(r)) \tag{3.6}$$

where $\sigma(\nu)$ is the surface tension given by (2.8). By its very definition the limit functional $F^{(0)}$ is L^1-lower semicontinuous, namely if $E_j \to E$ as $j \to \infty$ in $L^1(\mathbb{R}^n)$ (with reference to characteristic functions of the sets) and E_j, E have finite perimeter, then

$$\liminf_{j \to \infty} F^{(0)}(E_j) \geq F^{(0)}(E)$$

By a general theorem [5], this implies that $\langle u, S(e)u\rangle \geq 0$ for any unit e and u mutually orthogonal. Such an "abstract" result does not allow to decide whether the inequality is strict, for this we have to rely on a very specific analysis, as we will see in Section 5.

Stiffness naturally arises when we consider the first variation of the functional (3.6), and this will also lead to anisotropic motion by mean curvature. Let us start from some known facts, see [10]. Let $\psi : \mathbb{R}^n \times (-1,1) \to \mathbb{R}^n$ be a smooth deformation of $\mathbb{R}^n$ into itself, set $\psi_\lambda(r) := \psi(r,\lambda)$ and assume that $\psi_0(r) = r$ and that $\psi_\lambda(r) - r$ is compactly supported. Setting

$$\left.\frac{\partial \psi_\lambda(r)}{\partial \lambda}\right|_{\lambda=0} =: g(r) \tag{3.7}$$

we then have

$$\left.\frac{d}{d\lambda} F^{(0)}(\psi_\lambda(E))\right|_{\lambda=0} = \int_{\partial E} dH^{n-1}(r)\, K_\sigma(r)\, \langle g(r), \nu(r)\rangle \tag{3.8}$$

where

$$K_\sigma(r) = \sum_{i=1}^{n-1} \langle u_i, S(\nu(r))u_i\rangle\, \kappa_i \tag{3.9}$$

Thus by (3.8) and (3.9) and the strict positivity of $\langle u_i, S(\nu(r))u_i\rangle$, as stated in Theorem 2.1, if E is a bounded convex set and $g(r)$ is everywhere directed towards the interior of E, then the variation of the functional is strictly negative. This translates the physical intuition that the free energy of a convex droplet must decrease when the droplet shrinks, due to the action of the surface tension. Strict positivity of the stiffness matrix is thus essential for a correct physical interpretation of the model; its failure corresponds to phenomena analogous to lack of convexity for the thermodynamical free energy, related to phase transition behaviors, as clear in the so called SOS model, see [39]. Such phenomena may and are even expected to arise when the interaction is no longer ferromagnetic.

K_σ has also a natural geometric interpretation, but this is beyond our purposes here and we thus just refer to [10]; we only recall that, for the Wulff shape, K_σ is constant. We instead use (3.8) to define the anisotropic motion by mean curvature as a gradient flow for the functional (1.2), which is simply done by taking the normal velocity $v(r)$ of the interface $\partial E(t)$ at r as

$$v(r) = -\mu(\nu(r))\sigma(\nu(r))K_\sigma(r)\nu(r) \tag{3.10}$$

The choice optimizes the decrease of free energy, modulo the factor $\mu(\nu)$ which is a mobility coefficient. We should mention here that another convention is often used in the literature, as for instance in [30] and [39] where the factor $\sigma(\nu(r))$ in (3.10) is dropped. Our choice (see [10]) corresponds to requiring that the Wulff shape shrinks in a self-similar way when the mobility coefficient μ in (3.10) is a constant.

In either cases the mobility coefficient may be determined independently by measuring the mobility of a flat interface under the action of an external force, we will be back on this in Section 6. We have now a limit interface dynamics, (3.10).

On the other hand, dynamics (at a mesoscopic level) is defined as the gradient flow associated to the van der Waals functional $F(m)$, which yields

$$\frac{\partial m}{\partial t} = J * m - \Phi'(m) \tag{3.11}$$

When $\Phi(m)$ is given by (2.2) another dynamics is also of interest, namely

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta J * m\} \tag{3.12}$$

which is obtained from the Glauber dynamics on the spins by a scaling limit procedure, [24]. Like for (3.11), also the evolution defined via (3.12) makes $F(m)$ decrease.

Given the scaling parameter $\varepsilon > 0$, we consider as initial datum the ε-dependent function $m(r,0)$ equal to m^* in $\varepsilon^{-1}E$ and to $-m^*$ in the complement. Unlike in equilibrium, here we do not need an accuracy parameter δ, the evolution itself will adjust the datum in the proper way. Let $m(r,t)$ be the solution of (3.11) [or (3.12)] with initial datum as above; let

$$m^{(\varepsilon)}(r,t) := m(\varepsilon^{-1}r, \varepsilon^{-2}t) \tag{3.13}$$

We then say that there is convergence to (1.6) if

$$\lim_{\varepsilon\to 0} m^{(\varepsilon)}(r,t) = m^* \tag{3.14}$$

if r is in the interior of $E(t)$, while if r is in the exterior of $E(t)$, then

$$\lim_{\varepsilon\to 0} m^{(\varepsilon)}(r,t) = -m^* \tag{3.15}$$

In the isotropic case such a convergence result has been proved in [23] provided the interface motion is smooth; the analogous result for the Ising system with Kac potentials and Glauber dynamics was proved in [24]. Buttà in [16] has proved that the extinction time of a droplet in Ising converges to the extinction time computed by the limit motion by mean curvature. Convergence past singularities under the condition that the viscosity solution has no fattening, was proved in [32] in the isotropic case (for same models) and then extended in [33] to the anisotropic case. The extension uses a general theorem of [6] which also shows that if convergence holds in the smooth case, then the result automatically extends past singularities (for viscosity solutions) if there is no fattening.

In the isotropic case Buttà has proved in [15] the identity between the stiffness elements $\langle u, S(e)u\rangle = \sigma$ (recall $\partial_u^2\sigma = 0$) and the transport coefficients $\langle u, A(e)u\rangle$ (which, by isotropy, is independent of $u \in e^\perp$). Our Theorem 2.1 extends the result to the anisotropic case, establishing that $A(e) = S(e)$. Hence, if the mobility in (3.10) is chosen equal to the one found by Katsoulakis and Souganidis, then the macroscopic limit of (3.11) [or (3.12)] in the sense of (3.14) and (3.15) is given by (3.10) and (3.9). The result can therefore be interpreted as a justification of an inversion of limits, since it states that the limit of the gradient flow of F is the gradient flow (3.10) of the limit functional $F^{(0)}$.

In Section 6 we will show that the mobility of Katsoulakis and Souganidis is the same as the one obtained by studying the motion of a flat interface under the action of an external driving force.

4. Proof that $S(e) = A(e)$

By (1.3), we need to prove

$$\partial_u^2 \sigma(e) = \langle u, A(e)u\rangle - \sigma(e) \tag{4.1}$$

We recall that σ is of class C^∞, see the Appendix. As the formula (2.8) for the surface tension is rather implicit, we first rewrite $\sigma(e)$ in a more convenient form.

Lemma 4.1. *The surface tension $\sigma(e)$ defined in (2.8) can be written as:*

$$\sigma(e) = \int d\xi \, [G(\bar m_e) - G(m^*)](\xi) \tag{4.2}$$

$$\sigma(e) = \frac{1}{2}\int d\xi \int dr\, J(r)\bar m_e'(\xi)\bar m_e(\xi + r_e)\, r_e \tag{4.3}$$

where, for any $w \in L^\infty(\mathbb{R}, (-1,1))$,

$$\begin{aligned} G(w)(\xi) &:= \frac{1}{2} w(\xi)(J_e * w)(\xi) + \Phi(w(\xi)) - w(\xi)\Phi'(w(\xi)) \\ &= \frac{1}{2}\int dr\, J(r) w(\xi) w(\xi + r_e) + \Phi(w(\xi)) - w(\xi)\Phi'(w(\xi)) \end{aligned} \tag{4.4}$$

Proof. Recalling (2.4), for any $w \in L^\infty(\mathbb{R}, (-1,1))$ we have

$$(J_e * w)(\xi) = \int d\xi' \int_{e^\perp} dH^{n-1}(y)\, J(y + (\xi - \xi')e) w(\xi')$$

so that, setting $r = -y + (\xi' - \xi)e$ (hence $\xi' - \xi = r_e$), and using that J is even, we get

$$(J_e * w)(\xi) = \int dr\, J(r) w(\xi + r_e) \tag{4.5}$$

which proves the second equality in (4.4).

Next, recalling (2.1) and that $\int dr\, J(r) = 1$,

$$F_e(\bar m_e) = \int d\xi\, [\eta(\bar m_e) - \eta(m^*)], \qquad \eta(w) := \Phi(w) - \frac{1}{2} w\, J_e * w \tag{4.6}$$

From (2.6), $\bar m_e J_e * \bar m_e = \bar m_e \Phi'(\bar m_e)$, so that

$$\eta(\bar m_e) = \frac{1}{2}\bar m_e J_e * \bar m_e + \Phi(\bar m_e) - \bar m_e J_e * \bar m_e = G(\bar m_e)$$

Then (4.2) follows from (4.6).

Finally (2.6) implies

$$(J_e * \bar m_e')(\xi) = \Phi''(\bar m_e(\xi))\bar m_e'(\xi) \tag{4.7}$$

so that, by using the first equality in (4.4),

$$\frac{d}{d\xi} G(\bar m_e) = \frac{1}{2}\left[\bar m_e' J_e * \bar m_e - \bar m_e J_e * \bar m_e'\right] \tag{4.8}$$

Integrating by parts in (4.2) we get

$$\begin{aligned}\sigma(e) &= -\int d\xi\, \xi \frac{d}{d\xi} G(\bar{m}_e) \\ &= \frac{1}{2}\int d\xi \int d\xi'\, J_e(\xi-\xi')\bar{m}_e'(\xi)\bar{m}_e(\xi')(\xi'-\xi)\end{aligned}$$

and using (2.4) we obtain (4.3). □

Proof of (4.1). Let u be a unit vector orthogonal to e, ∂_u the derivative along u, see the notation introduced before (1.4). See also the Appendix for a sketch of proof of differentiability of the instanton. By (4.2) and the second equality in (4.4),

$$\begin{aligned}\partial_u\sigma(e) &= \frac{1}{2}\int d\xi \int dr\, J(r)\left[(\partial_u\bar{m}_e)(\xi)\bar{m}_e(\xi+r_e)+\bar{m}_e(\xi)(\partial_u\bar{m}_e)(\xi+r_e)\right] \\ &\quad +\frac{1}{2}\int d\xi \int dr\, J(r)\bar{m}_e(\xi)\bar{m}_e'(\xi+r_e)\,\partial_u r_e - \int d\xi\, \bar{m}_e(\xi)\Phi''(\bar{m}_e(\xi))(\partial_u\bar{m}_e)(\xi) \\ &= \int d\xi\, \bar{m}_e(\xi)\left[\int dr\, J(r)(\partial_u\bar{m}_e)(\xi+r_e) - \Phi''(\bar{m}_e(\xi))(\partial_u\bar{m}_e)(\xi)\right] \\ &\quad +\frac{1}{2}\int d\xi \int dr\, J(r)\bar{m}_e(\xi)\bar{m}_e'(\xi+r_e)\,\partial_u r_e \end{aligned}\tag{4.9}$$

where in the last equality we used the identity

$$\int d\xi \int dr\, J(r)(\partial_u\bar{m}_e)(\xi)\bar{m}_e(\xi+r_e) = \int d\xi \int dr\, J(r)\bar{m}_e(\xi)(\partial_u\bar{m}_e)(\xi+r_e)$$

which holds because J is even (changing variables first $\xi\to\xi-r_e$ and then $r\to -r$ in one of the integrals one gets the other one).

By differentiating (2.6) along u and using (4.5) we have

$$\Phi''(\bar{m}_e(\xi))(\partial_u\bar{m}_e)(\xi) = \int dr\, J(r)\left[(\partial_u\bar{m}_e)(\xi+r_e)+\bar{m}_e'(\xi+r_e)\,\partial_u r_e\right]\tag{4.10}$$

so that, from (4.9),

$$\partial_u\sigma(e) = -\frac{1}{2}\int d\xi \int dr\, J(r)\bar{m}_e(\xi)\bar{m}_e'(\xi+r_e)\,\partial_u r_e$$

hence, after an integration by parts,

$$\partial_u\sigma(e) = \frac{1}{2}\int d\xi \int dr\, J(r)\bar{m}_e'(\xi)\bar{m}_e(\xi+r_e)\,\partial_u r_e\tag{4.11}$$

Since

$$\partial_u r_e = \langle \partial_u e, r\rangle = r_u, \qquad \partial_u^2 r_e = -r_e$$

(see the notation leading to (1.4)), by differentiating (4.11) along u we get

$$\begin{aligned}\partial_u^2\sigma(e) &= \frac{1}{2}\int d\xi \int dr\, J(r)\bar m_e'(\xi)\,\big[\bar m_e'(\xi+r_e)\,r_u^2+(\partial_u\bar m_e)(\xi+r_e)\,r_u\big]\\ &\quad+\frac{1}{2}\int d\xi\int dr\, J(r)\bar m_e(\xi+r_e)\,(\partial_u\bar m_e')(\xi)\,r_u\\ &\quad-\frac{1}{2}\int d\xi\int dr\, J(r)\bar m_e'(\xi)\bar m_e(\xi+r_e)\,r_e \end{aligned}\tag{4.12}$$

By (4.3), the third integral on the r.h.s. of (4.12) is equal to $-\sigma(e)$. In the second one we integrate by parts w.r.t. ξ, then we substitute $\xi\to\xi-r_e$ and finally $r\to -r$. Using that J is even, we have

$$\int d\xi\int dr\, J(r)\bar m_e(\xi+r_e)\,(\partial_u\bar m_e')(\xi)\,r_u=\int d\xi\int dr\, J(r)\bar m_e'(\xi)(\partial_u\bar m_e)(\xi+r_e)\,r_u$$

Using the last two identities on the r.h.s. of (4.12) and recalling the espression of $A(e)$ in (2.9), we complete the proof of (4.1). $\square$

5. Strict positivity of $A(e)$

The r.h.s. of (2.9) is the sum of two terms, while the first one is clearly positive, the sign of the second one is not evident. Indeed, positivity of the r.h.s. of (2.9) could not be established directly by Katsoulakis and Souganidis, but only as a corollary of the convergence to anisotropic motion by mean curvature together with a general criterion proved by Barles and Souganidis, see Section 3. We will instead prove strict positivity by direct manipulations on (2.9) inspired by a different proof of convergence to motion by mean curvature, based on the analysis in [23] of the isotropic case. The advantage of this approach is a new formula for $\langle u, A(e)u\rangle$ which is at first sight non negative and for which it is easier to prove strict positivity. Less obvious is checking directly that the new expression is equal to the one given by (2.9): it must be by construction, since they both lead to the same anisotropic motion by mean curvature. We do not want, though, to go through still another proof of convergence to motion by curvature and prefer to establish directly the identity between the two formulas.

Let e and u be any pair of mutually orthogonal unit vectors and for any $f\in L^\infty(\mathbb{R}^n)$ we define the bounded operator

$$(L_e f)(r)=\int dr'\, J(r'-r)f(r')-\Phi''\big(\bar m_e(r_e)\big)f(r)\tag{5.1}$$

and the semigroup $e^{L_e t}$. Then, for any $t>0$, we set

$$a_{e,u}(t):=\frac{1}{2}\int d\xi\,\bar m_e'(\xi)^2\frac{1}{\bar m_e'(\xi)}\left\{e^{L_e t}\left[\bar m'(r_e)r_u^2\right]\right\}(\xi e)\tag{5.2}$$

We also introduce the new generator

$$\mathcal{L}_e f = \frac{1}{\bar m'_e}\, L_e(\bar m'_e f) \tag{5.3}$$

where $\bar m'_e = \bar m'_e(r_e)$, so that

$$a_{e,u}(t) = \frac{1}{2}\int d\xi\, \bar m'_e(\xi)^2 \left\{e^{\mathcal{L}_e t} r_u^2\right\}(\xi e) \tag{5.4}$$

We will show that $\mathcal{L}_e$ is the generator of a Markov semigroup, hence $a_{e,u}(t) > 0$. We will then prove that

$$\langle u, A(e)u\rangle = \lim_{t\to\infty} \frac{1}{t} a_{e,u}(t) \tag{5.5}$$

thus concluding that the l.h.s. is non negative. Our last step will then be an explicit lower bound on the r.h.s. of (5.5) which will prove its strict positivity.

We observe that, using (5.3), the r.h.s. of (5.5) can be interpreted as the diffusion coefficient of a random walk with internal degrees of freedom: r_u is the position of the random walker and r_e its "internal degree of freedom". One important point will be that the Markov process of the random walker is reversible (w.r.t. a suitable measure) and this makes available a general formula for the diffusion coefficient, [21], which turns out to be the expression (2.9) for $\langle u, A(e)u\rangle$, in this way we will prove (5.5). To prove strict positivity, we will use an inequality similar to those obtained for random walks in random environments, based on freezing the environment. As we do not want to suppose all readers familiar with these procedures, we present a self-consistent proof both to avoid excessive use of probability and because, in our specific context, there are several simplifying features which make the whole quite easier.

We set

$$K_e(r,r') := \frac{1}{c_e(r_e)}\,\left\{\frac{1}{\bar m'_e(r_e)} J(r'-r)\bar m'_e(r'_e)\right\} \tag{5.6}$$

$$c_e(\xi) := \Phi''(\bar m_e(\xi)), \quad c_e(r,r') := c_e(r_e)\, K_e(r,r') \tag{5.7}$$

Lemma 5.1. *$K_e(r,r')$ is a transition probability density and*

$$(\mathcal{L}_e f)(r) = \int dr'\, c_e(r,r')[f(r') - f(r)] \tag{5.8}$$

$\mathcal{L}_e$ is then the generator of a jump Markov process with jump intensities $c_e(r,r')$ and it is self-adjoint w.r.t. the σ-finite measure $\bar m'_e(r_e)^2 dr$. Moreover $\mathcal{L}_e$ leaves invariant the set of functions which depend only on r_e.

Proof. $K_e(r,r')$ is evidently non negative and, by (4.7) and (4.5), $\int dr'\, K_e(r,r') = 1$. The other statements are trivial. □

By Lemma 5.1 the action of $\mathcal{L}_e$ on functions which depend only on r_e is still a function of only r_e, so that we can define a generator $\mathcal{L}_e^{(1)}$ on $L^\infty(\mathbb{R})$ by setting, for $f \in L^\infty(\mathbb{R})$ and denoting $\hat{f}(r) = f(r_e)$,

$$\big(\mathcal{L}_e^{(1)} f\big)(\xi) := \big(\mathcal{L}_e \hat{f}\big)(\xi e) \tag{5.9}$$

The jump intensity of $\mathcal{L}_e^{(1)}$ is $c_e(\xi) K_e^{(1)}(\xi,\xi')$, where

$$K_e^{(1)}(\xi,\xi') = \frac{1}{c_e(\xi)}\,\frac{1}{\bar{m}_e'(\xi)}\, J_e(\xi'-\xi)\,\bar{m}_e'(\xi') \tag{5.10}$$

Obviously, for any $t \geq 0$,

$$\big\{e^{\mathcal{L}_e^{(1)} t} f\big\}(\xi) = \big\{e^{\mathcal{L}_e t}\hat{f}\big\}(\xi e) \tag{5.11}$$

Moreover, by Lemma 5.1, $\mathcal{L}_e^{(1)}$ is the generator of a jump Markov semigroup on $\mathbb{R}$ with finite, positive, invariant measure $\bar{m}_e'(\xi)^2 d\xi$ with respect to which it is self-adjoint. We denote by $(\cdot,\cdot)_e$ the scalar product in $L^2(\mathbb{R}, \bar{m}_e'(\xi)^2 d\xi)$.

With reference to (2.9) we define

$$b_{e,u} := \frac{1}{2}\int d\xi \int dr\; J(r)\bar{m}_e'(\xi)\bar{m}_e'(\xi + r_e)\, r_u^2 \tag{5.12}$$

so that

$$\langle u, A(e)u\rangle = b_{e,u} + \int d\xi \int dr'\; J(r')\bar{m}_e'(\xi)(\partial_u \bar{m}_e)(\xi + r_e')\, r_u' \tag{5.13}$$

Lemma 5.2. *Let*

$$w(\xi) := (\mathcal{L}_e r_u)(\xi e) \tag{5.14}$$

Then

$$a_{e,u}(t) = t b_{e,u} - \int_0^t ds \int_0^s ds'\, \big(w, e^{\mathcal{L}_e^{(1)} s'} w\big)_e \tag{5.15}$$

Proof. By (5.4),

$$a_{e,u}(t) = \frac{1}{2}\int d\xi\, \bar{m}_e'(\xi)^2 \int_0^t ds\, \left\{e^{\mathcal{L}_e s}\mathcal{L}_e r_u^2\right\}(\xi e) \tag{5.16}$$

We have

$$\mathcal{L}_e r_u^2 = \frac{1}{\bar{m}_e'(r_e)}\int dr'\; J(r')\bar{m}_e'(r_e' + r_e)\,\left[(r_u')^2 + 2 r_u r_u'\right] \tag{5.17}$$

Recalling (5.14),

$$w(\xi) = \frac{1}{\bar{m}_e'(\xi)}\int dr'\; J(r')\bar{m}_e'(\xi + r_e')\, r_u' \tag{5.18}$$

so that the second term on the r.h.s. of (5.17) reduces to $2 r_u w(r_e)$. On the other hand, the first term on the r.h.s. is a function which depends only on r_e. Hence by using (5.11) and self-adjointness of $\mathcal{L}_e^{(1)}$, its contribution in (5.16) is equal to $tb_{e,u}$. We have thus proved that

$$a_{e,u}(t) = tb_{e,u} + \int d\xi\, \bar{m}_e'(\xi)^2 \int_0^t ds\, \left\{ e^{\mathcal{L}_e s}\left[r_u w(r_e)\right]\right\}(\xi e) \tag{5.19}$$

We postpone the proof that

$$\int d\xi\, \bar{m}_e'(\xi)^2 \left\{ e^{\mathcal{L}_e s}\left[r_u w(r_e)\right]\right\}(\xi e) = -\int d\xi\, \bar{m}_e'(\xi)^2 w(\xi) \left\{ e^{\mathcal{L}_e s} r_u \right\}(\xi e) \tag{5.20}$$

By arguing as in (5.17) and using (5.14) we find $(\mathcal{L}_e r_u)(r) = w(r_e)$ for all $r \in \mathbb{R}^n$. Then

$$\left\{ e^{\mathcal{L}_e s} r_u \right\}(\xi e) = \int_0^s ds' \left\{ e^{\mathcal{L}_e s'} w(r_e)\right\}(\xi e)$$

so that (5.15) follows from (5.19) using (5.11) and (5.20).

The proof of (5.20) follows from the fact that the Markov process with generator $\mathcal{L}_e$ is reversible w.r.t. the measure $\bar{m}_e'(r_e)^2 dr$, here are some details. Let $\ell > 0$ and $\Omega_\ell := \mathbb{R} \times \mathcal{T}_\ell$, $\mathcal{T}_\ell$ the torus in $\mathbb{R}^{n-1}$ of side ℓ. We think of Ω_ℓ as imbedded in $\mathbb{R}^n$ by taking the axis $\mathbb{R}$ through the origin along the direction e. Call $\mathcal{L}_{e,\ell}$ the generator $\mathcal{L}_e$ as acting on Ω_ℓ; then, since the intensities $\mathcal{L}_e$ is bounded and $J(\cdot)$ is supported by the unit ball, we have

$$\text{l.h.s. of (5.20)} = \lim_{\ell \to \infty} \int d\xi\, \bar{m}_e'(\xi)^2 \left\{ e^{\mathcal{L}_{e,\ell} s}\left[r_u w(r_e)\right]\right\}(\xi e)$$

For $\ell > 0$ and $r \in \Omega_\ell$, we set $\rho_\ell(r) := \ell^{-(n-1)} \bar{m}_e'(r_e)^2$. Then, by translational invariance,

$$\begin{aligned}&\int d\xi\, \bar{m}_e'(\xi)^2 \left\{ e^{\mathcal{L}_{e,\ell} s}\left[r_u w(r_e)\right]\right\}(\xi e)\\ &\qquad = \int_{\Omega_\ell} dr'\, \rho_\ell(r') \left[\left\{ e^{\mathcal{L}_{e,\ell} s}\left[r_u w(r_e)\right]\right\}(r') - r_u' \left\{ e^{\mathcal{L}_{e,\ell} s} w(r_e)\right\}(r')\right]\end{aligned}$$

By reversibility (w.r.t. the measure $\rho_\ell(r')dr'$) the r.h.s. can be rewritten as

$$\int_{\Omega_\ell} dr'\, \rho_\ell(r') \left[r_u' w(r_e') - w(r_e') \left\{ e^{\mathcal{L}_{e,\ell} s} r_u \right\}(r')\right]$$

which, again by translational invariance, becomes

$$-\int d\xi\, \bar{m}_e'(\xi)^2 w(\xi) \left\{ e^{\mathcal{L}_{e,\ell} s} r_u \right\}(\xi e)$$

By taking the limit as $\ell \to \infty$ we then get (5.20). $\square$

We now conclude the proof of (5.5). After recalling that $\partial_u r_e = r_u$ and using (5.18) we rewrite (4.10) as

$$(L_e \partial_u \bar{m}_e)(\xi e) = -\int dr\, J(r) \bar{m}_e'(\xi + r_e)\, r_u = -\bar{m}_e'(\xi) w(\xi)$$

Hence

$$\left(\mathcal{L}_e \frac{\partial_u \bar{m}_e}{\bar{m}_e'} \right)(\xi e) = -w(\xi)$$

which we shorthand as

$$\mathcal{L}_e^{(1)} f = w, \qquad f = -\frac{\partial_u \bar m_e}{\bar m_e'} \tag{5.21}$$

We have

$$\int_0^s ds' \,\big(w, e^{\mathcal{L}_e^{(1)}s'} w\big)_e = \int_0^s ds' \,\big(w, e^{\mathcal{L}_e^{(1)}s'} \mathcal{L}_e^{(1)} f\big)_e = \big(w, e^{\mathcal{L}_e^{(1)}s} f\big)_e - \big(w, f\big)_e$$

and

$$\begin{aligned}\int_0^t ds \int_0^s ds' \,\big(w, e^{\mathcal{L}_e^{(1)}s'} w\big)_e &= -t\big(w, f\big)_e + \int_0^t ds\,\big(w, e^{\mathcal{L}_e^{(1)}s} f\big)_e \\ &= -t\big(w, f\big)_e - \big(f, f\big)_e + \big(f, e^{\mathcal{L}_e^{(1)}t} f\big)_e\end{aligned}$$

Since

$$\big(f, e^{\mathcal{L}_e^{(1)}t} f\big)_e \le \big(f, f\big)_e = \int d\xi\, (\partial_u \bar m_e)(\xi)^2 < \infty$$

because $\partial_u \bar m_e$ is in $L^2(\mathbb{R}, d\xi)$ (see the Appendix) we conclude from (5.15) that

$$\lim_{t\to\infty} \frac{a_{e,u}(t)}{t} = b_{e,u} + \big(w, f\big)_e \tag{5.22}$$

We finally write

$$\begin{aligned}\big(w, f\big)_e &= -\int d\xi\, \bar m_e'(\xi)^2 \left\{\frac{1}{\bar m_e'(\xi)} \int dr'\, J(r') \bar m_e'(\xi + r_e')\, r_u'\right\} \left\{\frac{(\partial_u \bar m_e)(\xi)}{\bar m_e'(\xi)}\right\} \\ &= -\int d\xi \int dr'\, J(r') \bar m_e'(\xi + r_e') (\partial_u \bar m_e)(\xi)\, r_u' \\ &= -\int d\xi \int dr'\, J(r') \bar m_e'(\xi) (\partial_u \bar m_e)(\xi - r_e')\, r_u' \\ &= \int d\xi \int dr'\, J(r') \bar m_e'(\xi) \partial_u \bar m_e(\xi + r_e')\, r_u'\end{aligned}$$

which is equal to the second term on the r.h.s. of (5.13). Thus (5.22) proves (5.5).

We will next prove a lower bound on $a_{e,u}(t)/t$ which is strictly positive and independent of t, thus completing the proof that $\langle u, A(e)u\rangle$ is strictly positive. To this end, it is convenient to consider the whole Markov process generated by $\mathcal{L}_e$, or, in other words, to introduce a multiple integral representation for $e^{\mathcal{L}_e t}$. Given $f(r)$, call $f(r,t) = \{e^{\mathcal{L}_e t} f\}(r)$. Then

$$\frac{\partial f(r,t)}{\partial t} = -c_e(r_e) f(r,t) + \int dr'\, c_e(r_e) K_e(r, r') f(r', t)$$

which can be written in integral form as

$$f(r,t) = e^{-c_e(r_e)t} f(r) + \int_0^t ds\, c_e(r_e) e^{-c_e(r_e)s} \int dr'\, K_e(r, r') f(r', t-s) \tag{5.23}$$

The multiple integral representation mentioned before refers to the series obtained by solving the integral equation by iterations.

For any $t > 0$ and $k \geq 0$ let $d\mu_{k,t} = \mu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k; dz_1, ..., dz_k)$ be the following measure on $\mathbb{R}^k \times \mathbb{R}^{k+1} \times (\mathbb{R}^{n-1})^k$

$$
\begin{aligned}
d\mu_{k,t} &= \prod_{i=1}^{k} \left\{ c_e(\xi_{i-1})\, \mathbf{1}_{t_{i-1} \leq t_i} e^{-c_e(\xi_{i-1})[t_i - t_{i-1}]}\, dt_i \right\} \mathbf{1}_{t_k \leq t} e^{-c_e(\xi_k)[t-t_k]} \\
&\quad \times \left\{ \bar{m}_e'(\xi_0)^2\, d\xi_0 \right\} \prod_{i=1}^{k} \left\{ K_e^{(1)}(\xi_{i-1}, \xi_i)\, d\xi_i \right\} \prod_{i=1}^{k} \left\{ P_e(z_i \,|\, \xi_{i-1}, \xi_i)\, dz_i \right\} \qquad (5.24)
\end{aligned}
$$

where $t_0 := 0$, $K_e^{(1)}(\xi, \xi')$ is defined in (5.10) and

$$
P_e(z \,|\, \xi, \xi') = \frac{J(z + e(\xi' - \xi))}{J_e(\xi' - \xi)} \qquad (5.25)
$$

with $z \in \mathbb{R}^{n-1}$ embedded in $\mathbb{R}^n$ as an element of $e^{\perp}$, so that

$$
K_e\,(r, r') = K_e^{(1)}\,(r_e, r_e')\; P_e\big(r' - r - (r_e' - r_e)e \,\big|\, r_e, r_e'\big) \qquad (5.26)
$$

We also define measures on $\mathbb{R}^k \times \mathbb{R}^{k+1}$ by setting

$$
\nu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k) = \int_{(z_1, ..., z_k) \in \mathbb{R}^{(n-1)k}} \mu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k; dz_1, ..., dz_k) \qquad (5.27)
$$

i.e. by integrating the measure $\mu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k; dz_1, ..., dz_k)$ over all the variables z_i. Since the measures $P_e(z \,|\, \xi, \xi')dz$ have unit mass, $\nu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k)$ has density w.r.t. Lebesgue obtained from the density of $\mu_{k,t}$ simply by dropping all the factors $P_e(z_i \,|\, \xi_{i-1}, \xi_i)$.

Lemma 5.3. *Let $f(r)$ be a bounded measurable function. Then*

$$
\begin{aligned}
&\int d\xi\, \bar{m}_e'(\xi)^2 \left\{ e^{\mathcal{L}_e t} f \right\} (\xi e) \\
&\qquad = \sum_{k \geq 0} \int \mu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k; dz_1, ..., dz_k)\, f\left(\xi_k e + (z_1 + ... + z_k)\right) \qquad (5.28)
\end{aligned}
$$

Moreover if $g = g(\xi)$ is a bounded measurable function,

$$
\sum_{k \geq 0} \int \nu_{k,t}(dt_1, ..., dt_k; d\xi_0, ..., d\xi_k)\, g(\xi_k) = \int d\xi\, \bar{m}_e'(\xi)^2\, g(\xi) \qquad (5.29)
$$

Proof. (5.28) is obtained by solving by iterations (5.23). Let $f(r) = g(r_e)$ and use (5.28). By (5.11) and the invariance of the measure $\bar{m}_e'(\xi)^2 d\xi$, the l.h.s. of (5.28) is equal to the r.h.s. of (5.29). On the other hand, recalling (5.27), the integral over the z variables only, of the r.h.s. of (5.28) gives the l.h.s. of (5.29), which is therefore proved. □

By a density argument which is omitted, we can use (5.28) with $f(r) = r_u^2$. Average and variance of the probability $P_e(z\,|\,\xi,\xi')dz$ are denoted by

$$w(\xi,\xi') = \int_{\mathbb{R}^{n-1}} dz\, P_e(z\,|\,\xi,\xi')\, z_u \tag{5.30}$$

$$\omega(\xi,\xi') = \int_{\mathbb{R}^{n-1}} dz\, P_e(z\,|\,\xi,\xi')\, [z_u - w(\xi,\xi')]^2 \tag{5.31}$$

where $z_u = \langle z, u\rangle$ is meanful recalling $z \in \mathbb{R}^{n-1}$ is embedded in $\mathbb{R}^n$ as an element of $e^\perp$

Then calling $\eta_i := [(z_i)_u - w(\xi_{i-1},\xi_i)]$, from (5.4) and (5.28),

$$\begin{aligned}
a_{e,u}(t) &= \sum_{k\geq 0}\int \mu_{k,t}(dt_1,...,dt_k;d\xi_0,...,d\xi_k;dz_1,...,dz_k)\left\{\sum_{i=1}^k [\eta_i + w(\xi_{i-1},\xi_i)]\right\}^2 \\
&\geq \sum_{k\geq 0}\int \mu_{k,t}(dt_1,...,dt_k;d\xi_0,...,d\xi_k;dz_1,...,dz_k)\sum_{i=1}^k \eta_i \sum_{i=1}^k [\eta_i + 2w(\xi_{i-1},\xi_i)] \\
&= \sum_{k\geq 0}\int \mu_{k,t}(dt_1,...,dt_k;d\xi_0,...,d\xi_k;dz_1,...,dz_k)\sum_{i=1}^k \eta_i^2 \\
&= \sum_{k\geq 0}\sum_{i=1}^k \alpha_{i,k}(t), \quad \alpha_{i,k}(t) := \int \nu_{k,t}(dt_1,...,dt_k;d\xi_0,...,d\xi_k)\,\omega(\xi_{i-1},\xi_i)
\end{aligned} \tag{5.32}$$

where we have used that integrals where some η_i appears linearly, i.e. not squared, do vanish. We have thus proved that

$$a_{e,u}(t) \geq \sum_{k\geq 0}\sum_{i=1}^k \alpha_{i,k}(t) \tag{5.33}$$

Defining

$$\beta_{i,k}(t) := \int \nu_{k,t}(dt_1,...,dt_k;d\xi_0,...,d\xi_k)\,\omega(\xi_{i-1},\xi_i)c_e(\xi_k) \tag{5.34}$$

$$\gamma_k(t) := \int \nu_{k,t}(dt_1,...,dt_k;d\xi_0,...,d\xi_k)\left\{\int d\xi\, c_e(\xi_k)K_e^{(1)}(\xi_k,\xi)\omega(\xi_k,\xi)\right\} \tag{5.35}$$

we have, for $1 \leq i < k$,

$$\frac{d\alpha_{i,k}(t)}{dt} = -\beta_{i,k}(t) + \beta_{i,k-1}(t) \tag{5.36}$$

and

$$\frac{d\alpha_{k,k}(t)}{dt} = -\beta_{k,k}(t) + \gamma_{k-1}(t) \tag{5.37}$$

Thus, after a telescopic cancellation,

$$a_{e,u}(t) \geq \int_0^t ds \sum_{k\geq 0}\gamma_k(s) \tag{5.38}$$

By applying (5.29) with g equal to the curly bracket in (5.35), we get

$$a_{e,u}(t) \geq t \int d\xi_0\, \bar{m}_e'(\xi_0)^2 \left\{ \int d\xi\, c_e(\xi_0) K_e^{(1)}(\xi_0, \xi) \omega(\xi_0, \xi) \right\} \tag{5.39}$$

which completes the proof that the limit of $a_{e,u}(t)/t$ as $t \to \infty$ is strictly positive.

6. Mobility coefficient

Two issues are discussed here:

- Is there any way to determine the mobility coefficient other than it being the correct value for convergence to (1.6)?
- Does the Wulff shape evolve in a self-similar way? Is a droplet shape asymptotically Wulff?

For the first question we follow again Spohn, [39]. The procedure leads to a problem involving a one dimensional travelling front equation and the computations become identical to those for the isotropic case done in [15]. We consider the functional defined by (1.1) with the mean field free energy density $f(m)$ replaced by $f(m) - \varepsilon h m$, ε the scaling parameter used in the sharp interface limit, $h > 0$ an external magnetic field. The Γ-convergence result extends to this case, and the first variation of the new limit functional is given by the r.h.s. of (3.8) with K_σ replaced by $K_\sigma^{(h)}$, where

$$K_\sigma^{(h)}(r) := \sum_{i=1}^{n-1} \langle u_i, S(\nu(r)) u_i \rangle\, \kappa_i - 2m^* h \tag{6.1}$$

The analogue of (3.10) is then

$$v(r) = -\mu(\nu(r))\sigma(\nu(r)) K_\sigma^{(h)}(r) \nu(r) \tag{6.2}$$

By (6.2) we can deduce the equations of motion of a flat interface with normal e:

$$v = \mu(e)\sigma(e) 2m^* h \tag{6.3}$$

Thus the velocity of a flat interface is proportional to h and the proportionality factor is $\mu(e)\sigma(e)$ times $2m^*$.

We now go back to (1.1) with the modified free energy density. The corresponding gradient flow is, similarly to (3.11),

$$\frac{\partial m}{\partial t} = J * m - \Phi'(m) + \varepsilon h \tag{6.4}$$

and, when $\Phi(m)$ is given by (2.2), the limit Glauber dynamics yields

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta[J * m + \varepsilon h]\} \tag{6.5}$$

We then look for travelling fronts in the direction e. This is a one dimensional problem with J replaced by J_e, see (2.4), which has been solved in [22], [28], [38]. The velocity of the front, $V(\varepsilon h)$, written in macroscopic units where space is scaled by ε and time by ε^2, becomes $V^{(\varepsilon)} := \varepsilon^{-1} V(\varepsilon h)$. We have

$$\lim_{\varepsilon \to 0} V^{(\varepsilon)} = v$$

with v the same as in (6.3). The computations are the same as in [15] and are omitted.

Let us now turn to the second question raised in the beginning of this section. First of all we observe that the Wulff shape is relevant also in a dynamical context. In fact, if an interface is stationary w.r.t. the evolution defined by (the rescaled limit of) (6.4), then the sum of the $A(e)$-weighted curvatures is constant. Since $A(e) = S(e)$ the Wulff shape is stationary. Notice however that in the argument the mobility has not played any role, which is not the case when studying asymptotic shapes. The shape of a growing droplet is described asymptotically by (6.2) without the curvature term, as it vanishes in the limit when the droplet becomes very large. We then get

$$v(r) = -\mu(\nu(r))\sigma(\nu(r)) 2m^* h \nu(r) \tag{6.6}$$

The condition for the Wulff shape to be a self-similar solution of (6.6) is that $\mu(e)$ is independent of e. The same condition appears again when we require that the motion without h has a self-similar, shrinking solution given by the Wulff shape. The question then is whether the mobility coefficient happens to be independent of e. There is no apparent reason for such a property, but there may be also here hidden symmetries like those leading to the equality $A(e) = S(e)$. We will show that this is not the case, at least in general, as we will construct an example where $\mu(e)$ does depend on e. The explicit expression for the mobility, see [6] and [33], for the evolution (3.11) is

$$\mu(e) = \left[\sigma(e) \int d\xi \, \bar{m}_e'(\xi)^2 \right]^{-1} \tag{6.7}$$

while, for (3.12)

$$\mu(e) = \left[\sigma(e) \int d\xi \, \Phi''(\bar{m}_e(\xi)) \bar{m}_e'(\xi)^2 \right]^{-1} \tag{6.8}$$

To construct our example we use [8] where it is shown that there is a class of non local functionals whose instanton solutions are discontinuous and the corresponding mobility vanishes. Such functionals are not those considered here, but they can be approximated by ours. In this way we will prove that the mobility may be made arbitrarily small in some directions and therefore it will not be independent of e.

For notational simplicity we consider $n = 2$ dimensions and $\Phi(m)$ given by (2.2) with $\beta > 1$. Let $j(\xi)$, $\xi \in \mathbb{R}$, be a non negative, smooth, even function, supported by the interval $[-1, 1]$, with $\int d\xi \, j(\xi) = 1$. We also suppose that $j'(\xi) \le 0$ for $\xi \ge 0$.

Let $\alpha \in (0, 1)$ be such that

$$\beta\alpha > 1$$

and, for $\varepsilon > 0$,

$$J(r; \varepsilon) = [\alpha j_\varepsilon(x) + (1-\alpha) j(x)]\, j(y), \qquad r = (x, y), \quad j_\varepsilon(x) := \varepsilon^{-1} j(\varepsilon^{-1} x) \tag{6.9}$$

We denote by $\mu(e; \varepsilon)$ the corresponding mobility. Let e_1 and e_2 be the unit vectors respectively along the x and y axes. It is clear that $J_{e_1}(\xi) = \alpha j_\varepsilon(\xi) + (1-\alpha) j(\xi)$ and $J_{e_2}(\xi) = j(\xi)$.

Obviously $\mu(e_2;\varepsilon)$ is positive and independent of ε. We will show that $\mu(e_1;\varepsilon) \to 0$ as $\varepsilon \to 0$ thus concluding the argument.

Call m_α the positive solution of $m_\alpha = \tanh\{\beta\alpha m_\alpha\}$, which exists because $\beta\alpha > 1$. Since the graph of $\tanh\{\beta\alpha x\}$, $0 < x < m_\alpha$, is strictly above the diagonal, there exist $c \in (0, m_\alpha)$ and $\zeta > 0$ so that

$$c < \tanh\{\beta\alpha(c - \zeta)\} \tag{6.10}$$

Calling $\bar m_\varepsilon$ the instanton in the direction e_1 for the given value of ε, we denote by $x_\varepsilon(m)$ the inverse of $\bar m_\varepsilon$. We claim that

$$x_\varepsilon(c) - x_\varepsilon(c - \zeta) \le \varepsilon \tag{6.11}$$

Since the function $\Phi(m)$ given by (2.2) is even, the corresponding instanton $\bar m_\varepsilon$ is odd, see [26]. Since j is even and monotone decreasing on $\mathbb{R}_+$ and $\bar m_e(\xi)$ is odd and positive for $\xi > 0$, we have $(j * \bar m_\varepsilon)(\xi) \ge 0$ for $\xi > 0$. Then, using (2.7) we have, for $\xi > 0$,

$$\bar m_\varepsilon(\xi) \ge \tanh\{\beta\alpha(j_\varepsilon * \bar m_\varepsilon)(\xi)\} \tag{6.12}$$

If by contradiction our claim (6.11) is false, then $x_\varepsilon(c - \zeta) < x_\varepsilon(c) - \varepsilon$, and therefore $\bar m_\varepsilon(x) > c - \zeta$ for all $|x - x_\varepsilon(c)| \le \varepsilon$. Taking $\xi = x_\varepsilon(c)$ in (6.12) we then get

$$c = \bar m_\varepsilon(x_\varepsilon(c)) > \tanh\{\beta\alpha(c - \zeta)\}$$

which contradicts (6.10). Thus claim (6.11) is proved.

By the Cauchy-Schwartz inequality and (6.11) it follows

$$\zeta = \int_{x_\varepsilon(c-\zeta)}^{x_\varepsilon(c)} dx\, \bar m_\varepsilon'(x) \le \sqrt{\varepsilon \int dx\, \bar m_\varepsilon'(x)^2}$$

which proves that the integral of $\bar m_\varepsilon'(x)^2$ diverges as $\varepsilon \to 0$.

It remains to prove that the surface tension along e_1 stays away from 0 as $\varepsilon \to 0$. By considering the first term on the r.h.s. of (1.1), it is sufficient to prove that there is $a > 0$ so that, for all $\varepsilon > 0$,

$$\bar m_\varepsilon(x) < m^* \qquad \text{for all } x \in [0, a] \tag{6.13}$$

Since $\alpha < 1$ and $\tanh\{\beta m^*\} = m^*$ we can choose $\zeta > 0$ in such a way that

$$\tanh\{\beta\alpha m^* + \zeta\} < m^* \tag{6.14}$$

We have

$$\bar m_\varepsilon(x) \le \tanh\{\beta\alpha m^* + (1 - \alpha)(j * \bar m_\varepsilon)(x)\}$$

Since $(j * \bar m_\varepsilon)(0) = 0$ (recall that $\bar m_\varepsilon$ is odd) and $|\bar m_\varepsilon(x)| \le m^*$, there is $a > 0$ such that, for any $\varepsilon > 0$,

$$(1 - \alpha)(j * \bar m_\varepsilon)(x) < \zeta \qquad \text{for all } x \in [0, a]$$

Hence

$$\bar m_\varepsilon(x) \le \tanh\{\beta\alpha m^* + \zeta\} \qquad \text{for all } x \in [0, a]$$

Using (6.14) we then get (6.13).

Observe that the same argument applies to (6.8), corresponding to the evolution (3.12), which is the continuum limit of the Glauber dynamics in the underlying Ising model.

Appendix A.

For the reader's convenience we collect here the main properties of the one dimensional case used in the text. For a while we will just consider the one dimensional case by itself and drop the subfix e from the notation.

Existence of instantons. *The variational problem* (2.5) *has a unique minimizer* $\bar{m}(\xi)$, $\xi \in \mathbb{R}$, *which is a strictly increasing function of* ξ. *This is the unique solution of the non local mean field equation* (2.6) *in the class* $\mathcal{M}^*$ *of measurable functions* $m : \mathbb{R} \to [-m^*, m^*]$ *such that*

$$\lim_{\xi \to \pm\infty} m(\xi) = \pm m^*$$

This is proved in [1], [2], and [3] for a more general class of functionals. The statements about (2.6) are proved in [8]. When Φ is given by (2.2), uniqueness holds in a class larger than $\mathcal{M}^*$, [26]. Under our assumptions on f, one can also prove that the instanton is of class $C^\infty(\mathbb{R})$.

Space asymptotics. *There are* $\alpha_\pm > 0$ *and* $M_\pm > 0$ *so that*

$$\lim_{\xi \to \pm\infty} e^{\alpha_\pm |\xi|}\, \bar{m}'(\xi) = M_\pm$$

$\alpha_\pm$ *are the positive solutions of*

$$\Phi''(\pm m^*) = \int d\xi\, J(\xi)\, e^{\alpha_\pm \xi}$$

This is proved in [25], [22], [27], for $\Phi(m)$ given by (2.2), but the proof extends to the more general class considered here.

The two lines argument, taken from [25], for proving a lower bound on the convergence of $\bar{m}(\xi)$ to m^* as $\xi \to +\infty$, goes as follows. Let Ψ be the inverse of Φ' (which is strictly increasing by the convexity assumption on Φ). Since $\bar{m}(\xi)$ is increasing and $J(\xi) = 0$ for $|\xi| \geq 1$, we have, for $\xi > 0$ (hence $\bar{m}(\xi) > m_0$)

$$\bar{m}(\xi + n) \geq \Psi^n(\bar{m}(\xi))$$

Thus $\bar{m}(\xi) \to m^*$ at least as fast as the iterates of the map $\Psi(\cdot)$. The proof of exponential convergence and the computation of the exponents of such a convergence, as done in [22] and [27], are based on an analysis of the equation

$$\bar{m}'(\xi) = \Phi''(\bar{m}(\xi))^{-1} (J * \bar{m}')(\xi)$$

in the interval $[a, \infty)$ with a positive and large.

Spectral gap. *The operator* L *defined by*

$$Lv = J * v - \Phi''(\bar{m}(\cdot))v$$

and acting either on $L^2(\mathbb{R}, d\xi)$ or on $L^\infty(\mathbb{R})$ has 0 as a simple eigenvalue with eigenvector $\bar m'$ while the rest of the spectrum is contained in a closed subset of the negative axis away from 0 (spectral gap). L as an operator on $L^2(\mathbb{R}, d\xi)$ is self-adjoint.

This is proved in [8], earlier proofs relative to Φ as in (2.2) are given in [25], [22] and [27] where also spaces with weighted sup norms are considered.

Dependence on the direction. *Let $J \in C^\infty(\mathbb{R}^n)$. Then $\bar m_e(\xi)$ is a C^∞ function of e. In particular, σ and μ are C^∞ functions of e.*

The proof of the statement, sketched in [33], is based on the implicit function theorem. Fix $e \in S^{n-1}$, and define

$$W_e := \{w \in L^\infty(\mathbb{R}) \,:\, \pi_e(w) = 0\}, \qquad \pi_e(w) := \int d\xi\, \bar m'_e(\xi) w(\xi)$$

Since the instanton is a monotone increasing function there exists a unique function $d_e : S^{n-1} \to \mathbb{R}$ so that

$$\tilde m_\nu(\xi) := \bar m_\nu(\xi - d_e(\nu)) \in W_e$$

$d_e(e) = 0$ and $\bar m_e = \tilde m_e \in W_e$. We are now ready for the implicit function theorem. Let $G : W_e \times S^{n-1} \to W_e$ be given by

$$G(w, \nu) := g(w, \nu) - \pi_e\left(g(w, \nu)\right) \bar m'_e, \qquad g(w, \nu) := J_\nu * w - \Phi'(w)$$

The equation $G(w, \nu) = 0$ in W_e is solved by $\tilde m_\nu$. The partial derivative (recall $\pi_e \circ L_e = 0$)

$$D_w G(\tilde m_e, e) w = L_e w, \qquad w \in W_e$$

as an operator on W_e has bounded inverse by the spectral gap property, so that in a neighborhood of e the solution $\tilde m_\nu$ is a differentiable function of ν.

Let us compute $(\partial_u \tilde m_e)(\xi)$, for $u \in S^{n-1}$ orthogonal to e. The derivative of G along the direction τ orthogonal to ν computed at $(\tilde m_\nu, \nu)$ is given by

$$(\partial_\tau G)(\tilde m_\nu, \nu) = \int dr\, J(r) \tilde m'_\nu(\xi + r_\nu)\, r_\tau - \int d\xi \int dr\, J(r) \bar m'_e(\xi) \tilde m'_\nu(\xi + r_\nu)\, r_\tau \tag{A.1}$$

where we used $\partial_\tau r_\nu = r_\tau$. Since $\tilde m_e = \bar m_e$ and the function $r \mapsto \int d\xi\, \bar m'_e(\xi) \bar m'_e(\xi + r_e)$ is even, the second term on the r.h.s. of (A.1) vanishes when computed at $(\tilde m_e, e)$. We conclude that $(\partial_u \tilde m_e)(\xi)$ is the unique solution (in W_e) of the equation

$$(L_e \partial_u \tilde m_e)(\xi) = -\int dr\, J(r) \bar m'_e(\xi + r_e)\, r_u$$

Next we turn back to $\bar m_e$. We are going to prove the function $d_e(\cdot)$ is differentiable, so that, from the above discussion, also $\bar m_\nu$ is smooth in a neighborhood of e and

$$\partial_u \bar m_e = \partial_u \tilde m_e + b_e \bar m'_e, \qquad b_e = \big(\partial_u d_e(\cdot)\big)(e)$$

We observe that $d_e(\nu) = x_\nu$, where x_ν is the unique number such that $\tilde m_\nu(x_\nu) = m_0$. Since $\tilde m_\nu$ is smooth in ν and $\tilde m'_\nu(\xi)$ is strictly positive, using again the implicit function theorem, x_ν is differentiable and

$$\partial_\tau x_\nu = -\frac{\partial_\tau \tilde m_\nu(x_\nu)}{\tilde m'_\nu(x_\nu)}$$

By starting with the functions $\bar{m}_e(\xi) - m^*\text{sign}(\xi)$ we could have worked in $L^2(\mathbb{R}, d\xi)$ instead of $L^\infty(\mathbb{R})$. In particular, for any unit vector u orthogonal to e, the derivative $\partial_u \bar{m}_e$ is a bounded function in $L^2(\mathbb{R}, d\xi)$.

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